07 Absolute Value of Dirichlet Beta Function

7.1 Dirichlet Beta Function

7.1.1 Definition

Dirichlet Beta Function $\beta(z)$ is defined in the half plane $Re\{\beta(z)\}>0$ as follows.

$$\beta(z) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(2r-1)^z} \tag{1.0}$$

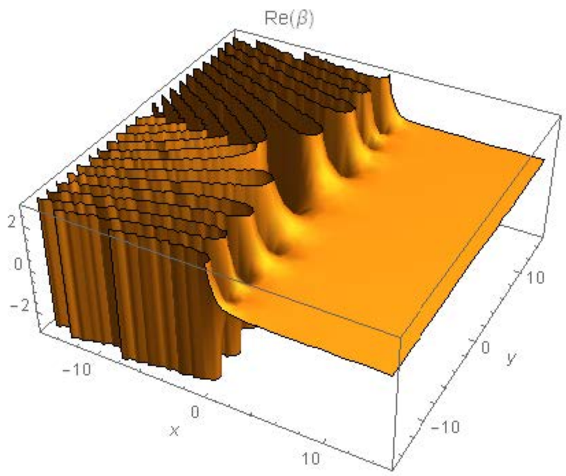
This series is analytically continued to the whole complex plane by applying some kind of acceleration method. The easiest of these is the Euler transformation as follows.

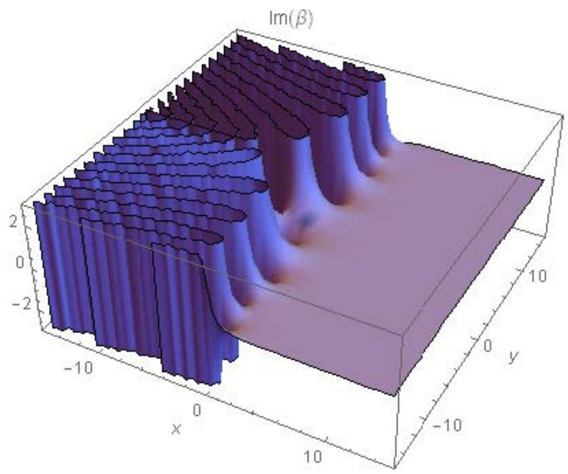
$$\beta(z) = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \sum_{r=1}^{k} \binom{k}{r} \frac{(-1)^{r-1}}{(2r-1)^z} \tag{1.1}$$

(1.0) and (1.1) are the same in $Re\{\beta(z)\}>0$. Although (1.0) can not express the left side of line the of convergence, (1.1) can express also the left side of this. Therefore, we can define Dirichlet Beta Function $\beta(z)$ by (1.1) .

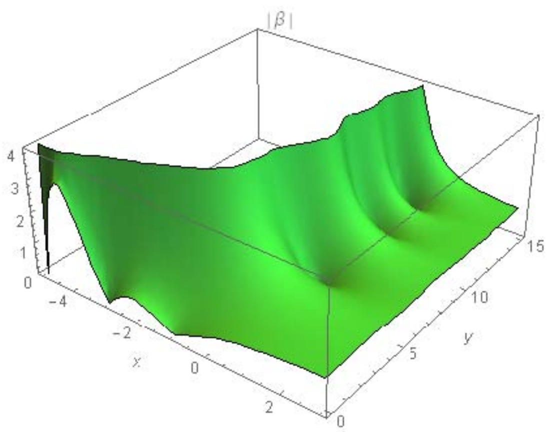
7.1.2 Overview

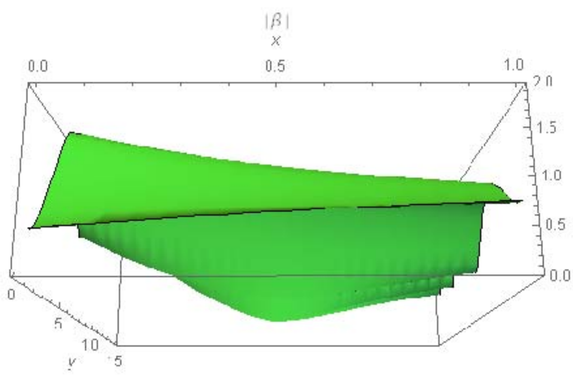
The 3D figures of the real part and the imaginary part of Dirichlet Beta Function $\beta(x+iy)$ are as follows.





Further, the 3D figure of the absolute value is as follows. In the left figure, trivial zeros of $\beta(z)$ are observed along the x axis. The right figure is a view of the left figure from the bottom. We can see that zeros of $\beta(z)$ are located along $x=1/2$. Unlike Dirichlet eta function, there is no zero point on $x=1$.





As seen in these figures, Dirichlet Beta Function $\beta(z)$ has two kinds of zeros as follows.

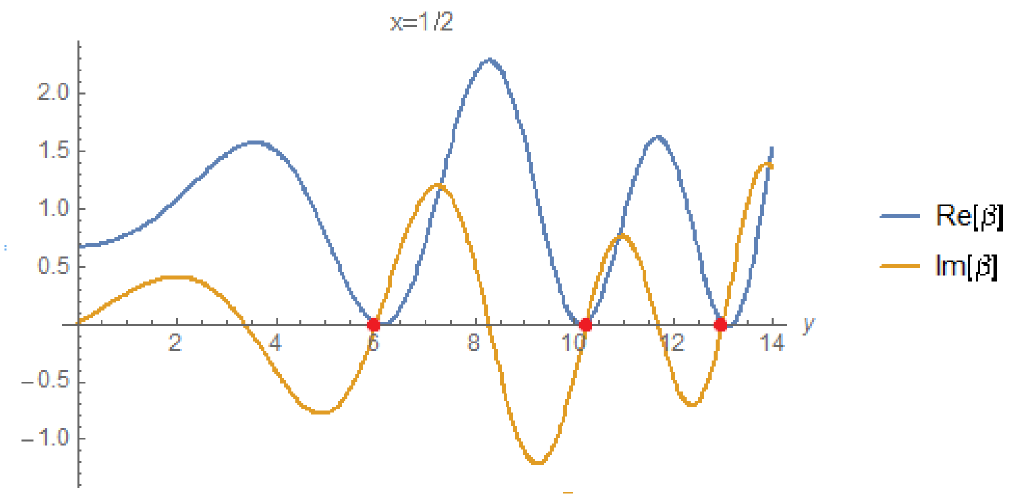
(1) Trivial zeros $-1, -3, -5, -7, \cdots$

(2) Non-trivial zeros $1/2 \pm i\,6.020\cdots$, $1/2 \pm i\,10.2437\cdots$, $1/2 \pm i\,12.9880\cdots$, $\cdots$

Non-trivial zeros (1) exist in $0 < x < 1$ called **Critical Strip** . Moreover, it is proved that they have to exist symmetrically with respect to $x = 1/2$. (" 04 Completed Dirichlet Beta " , Theorem 4.2.1) . And, fortunately, this critical strip is included within the convergence range of the series (1.0) .

7.1.3 Non-trivial zeros

The figures of the real part and the imaginary part of Dirichlet Beta Function $\beta(z)$ at $x = 1/2$ are as follows.



The red points are non-trivial zeros. We can see the followings from the figure.

(1) The real part resembles a negative cosine curve and the imaginary part resembles a sine curve.

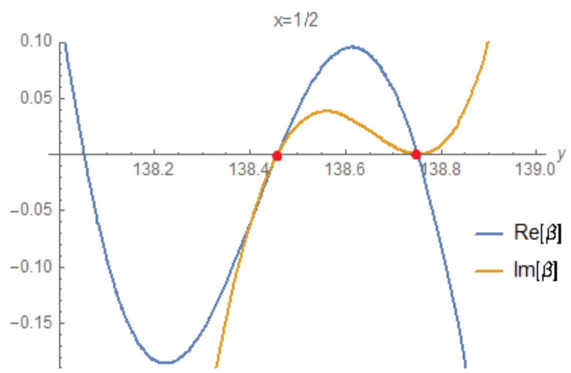
(2) The extrema points of the real part are close to the zeros of the imaginary part but not zeros.

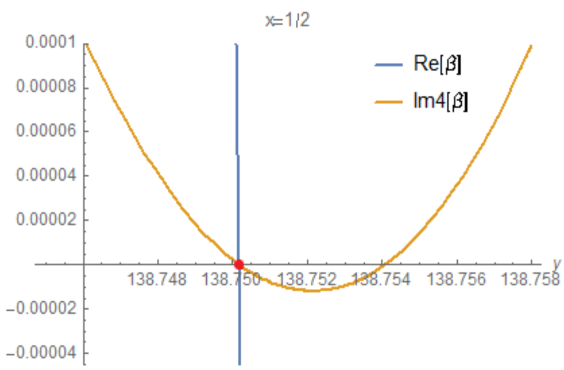
(3) Non-trivial zeros are close to the local minimum points of the real part, but are not so.

(4) Non-trivial zeros are the uphill zeros (end of cycles) of the imaginary part.

Downhill Zeros

Exceptions exist in (3) and (4). For example, The left figure is near $y = 139$. The right figure is an enlarged view around the zero point (red point) on the right side of the left figure. Non-trivial zero point $138.7501\cdots$ is near the local maximum point of the real part and is the downhill zero point of the imaginary part.





7.1.4 Feature of Dirichlet Beta Series

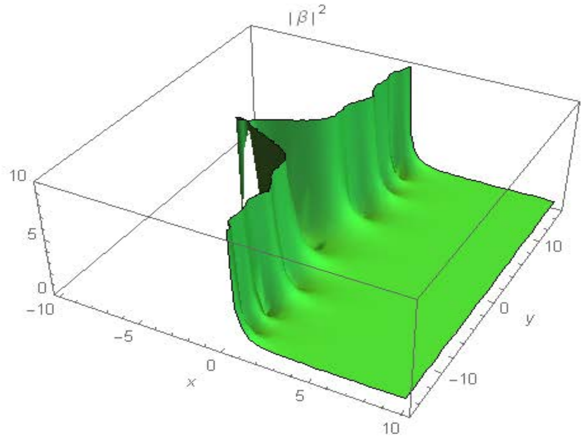
Observing 7.1.3 , we can see that this Dirichlet beta series consists of one periodic function which gives nontrivial zeros. However, its amplitude and period are not constant.

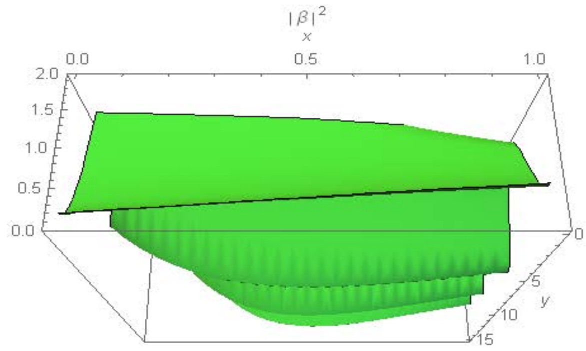
7.2 Squared Absolute Value of Dirichlet Beta

Squared absolute value of Dirichlet Beta function is

$$f(x,y) = |\beta(x,y)|^2 \tag{2.0}$$

This is a real-valued function with two variables. And it is shown in the figure as follows.

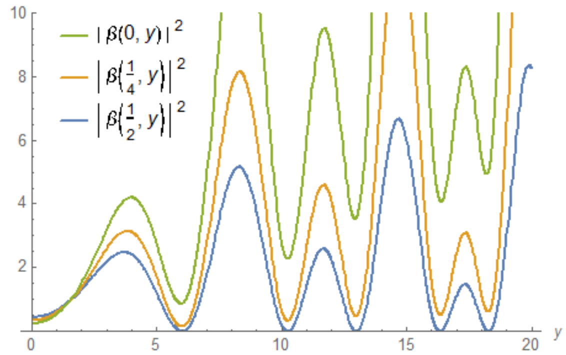


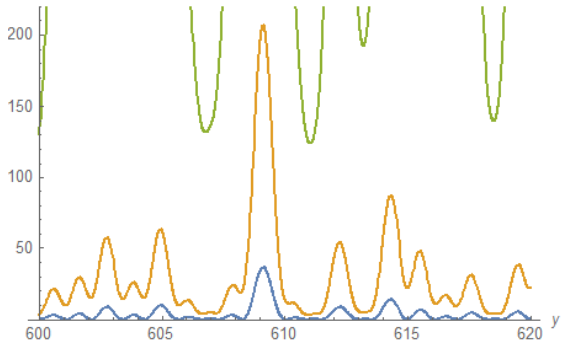


On the left figure, dents are observed along $x=1/2$. The right figure is a view from the bottom of a part $(0 \le x \le 1, y \ge 0)$ of the left figure. In the right figure, zeros are observed along $x=1/2$. No zero point is observed on lines other than $x=1/2$.

Features in $0 \le x \le 1/2$

Let us focus on space $0 \le x \le 1/2$. The figures of section in $x=0$, $1/4$, $1/2$ are drawn as follows.

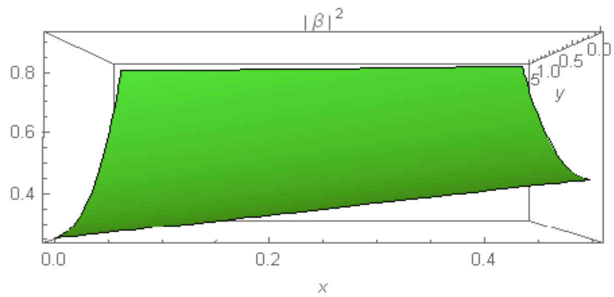


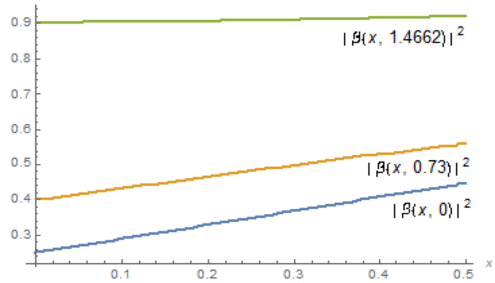


Looking at this, it looks like $|\beta(0,y)|^2 > |\beta(1/4,y)|^2 > |\beta(1/2,y)|^2$ in $1.5639 \le y \le 20$. It is the same also in $600 \le y \le 620$. Below, we observe this in more detail.

(1) $0 \le y \le 1.4662$

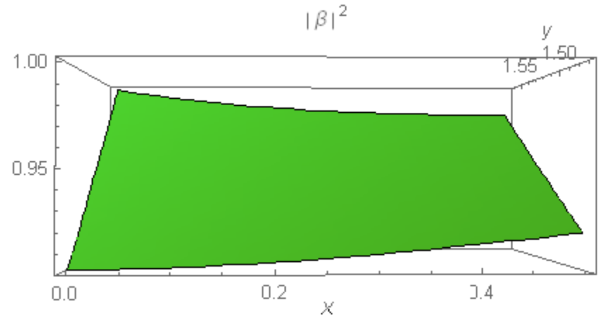
The front view of 3D in this interval is the left figure. The cutaway view at $y=0,\ 0.73,\ 1.4662$ of this is the right figure. In this interval, $|\beta(x,y)|^2$ seems to be monotonically increasing with respect to x.

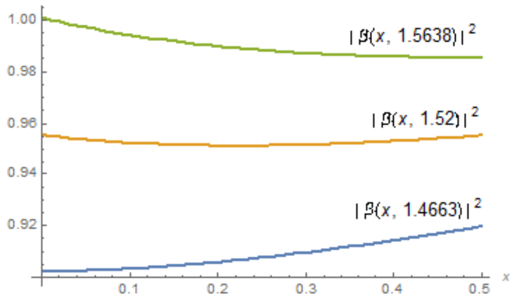




(2) $1.4662 < y < 1.5639$

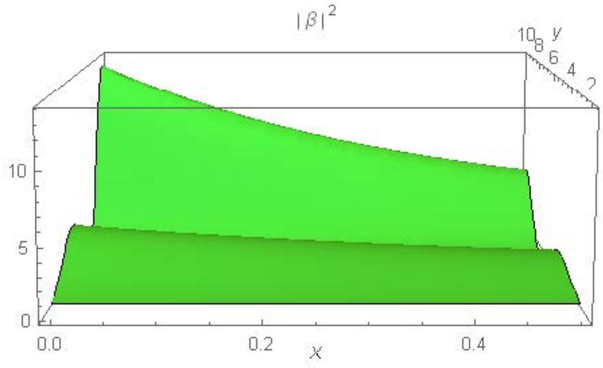
The front view of 3D in this interval is the left figure. The cutaway view at $y = 1.4663,\ 1.52,\ 1.5638$ of this is the right figure. In this interval, $|\beta(x,y)|^2$ is not monotonic with respect to x. In the right figure, although the curve of $y = 1.4663$ looks like monotonically increasing, it is decreasing at the left end when it is seen enlarged. Although the curve of $y = 1.5638$ looks like monotonically decreasing, it is increasing at the right end when it is seen enlarged.

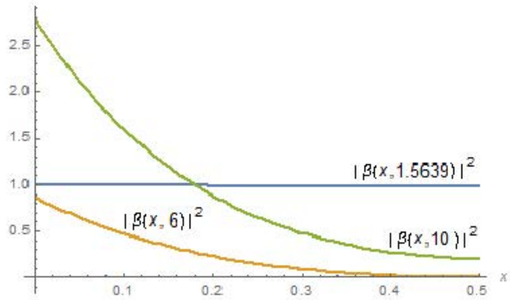




(3) $y \geq 1.5639$

The front view of 3D in this interval is the left figure. The cutaway view at $y = 0,\ 1.5639,\ 6,\ 10$ of this is the right figure. In this interval, $|\beta(x,y)|^2$ seems to be monotonically decreasing with respect to x.





Based on the observations above, I present the next hypothesis equivalent to the Riemannian hypothesis.

Hypothesis 7.2.1

When $\beta(x,y)$ is the Dirichlet Beta function on the complex plane, the squared absolute value $|\beta(x,y)|^2$ is a monotonically decreasing function in the region $0 < x < 1/2$, $y \geq 2$.

Remark

The zeros common to the Riemann Zeta function exist in $0 < x < 1$ called critical strip. Moreover, it is proved that they have to exist symmetrically with respect to $x = 1/2$. So, if $|\beta(x,y)|^2$ is monotonically decreasing with respect to x in the region $0 < x < 1/2$, $y \geq 2$, zeros do not exist in the region and the opposite region $1/2 < x < 1$, $y \geq 2$. This is equivalent to the Riemann hypothesis.

Incidentally, in the opposite region $1/2 < x < 1$, $y \geq 2$, $|\beta(x,y)|^2$ is not necessarily a monotone function with respect to x.

7.3 Expression of Squared Absolute Value by Series

7.3.1 Expression of Dirichlet Beta Function by Series

As seen in **7.1.1** , Dirichlet Beta Function $\beta(z)$ was defined as follows.

$$\beta(z) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(2r-1)^z} \qquad Re(z) > 0$$

When $Re(z) > 0$, let $z = x + iy$. Then,

$$\beta(x,y) = \sum_{r=1}^{\infty} (-1)^{r-1} (2r-1)^{-x-iy} \qquad x > 0$$

If this is represented by an exponential function,

$$\beta(x,y) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-(x+iy)log(2r-1)} = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-xlog(2r-1)-iylog(2r-1)}$$

If this is represented by a trigonometric function,

$$\beta(x,y) = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{cos\{ylog(2r-1)\}}{(2r-1)^x} - i \sum_{r=1}^{\infty} (-1)^{r-1} \frac{sin\{ylog(2r-1)\}}{(2r-1)^x} \tag{3.1}$$

7.3.2 Expression of $|\beta|^2$ by Double Series

Squared absolute value of Dirichlet Beta function $|\eta(x,y)|^2$ is expressed using (3.1) as follows.

$$|\beta(x,y)|^2 = \left[\sum_{r=1}^{\infty} (-1)^{r-1} \frac{cos\{ylog(2r-1)\}}{(2r-1)^x} \right]^2 + \left[\sum_{r=1}^{\infty} (-1)^{r-1} \frac{sin\{ylog(2r-1)\}}{(2r-1)^x} \right]^2$$

Although it looks like a very complicated, it becomes an unexpectedly simple expression when it is expanded and organized.

Formula 7.3.2

When $\beta(x,y)$ is the Dirichlet Beta Function,

$$|\beta(x,y)|^2 = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^x} cos\left(y \, log \frac{2s-1}{2r-1} \right) \tag{3.2}$$

Proof

Let $(-1)^{r-1} cos\{ylog(2r-1)\} = C_{2r-1}$. Then,

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1} cos\{ylog(2r-1)\}}{(2r-1)^x} = \frac{C_1}{1^x} + \frac{C_3}{3^x} + \frac{C_5}{5^x} + \frac{C_7}{7^x} + \cdots$$

This square is

$$\frac{C_1}{1^x} + \frac{C_3}{3^x} + \frac{C_5}{5^x} + \frac{C_7}{7^x} + \frac{C_9}{9^x} + \frac{C_{11}}{11^x} + \cdots$$

$$\times \frac{C_1}{1^x} + \frac{C_3}{3^x} + \frac{C_5}{5^x} + \frac{C_7}{7^x} + \frac{C_9}{9^x} + \frac{C_{11}}{11^x} + \cdots$$

$$= \frac{C_1}{1^x}\frac{C_1}{1^x} + \frac{C_1}{1^x}\frac{C_3}{3^x} + \frac{C_1}{1^x}\frac{C_5}{5^x} + \frac{C_1}{1^x}\frac{C_7}{7^x} + \cdots$$

$$+ \frac{C_3}{3^x}\frac{C_1}{1^x} + \frac{C_3}{3^x}\frac{C_3}{3^x} + \frac{C_3}{3^x}\frac{C_5}{5^x} + \frac{C_3}{3^x}\frac{C_7}{7^x} + \cdots$$

$$+\frac{C_5}{5^x}\frac{C_1}{1^x}+\frac{C_5}{5^x}\frac{C_3}{3^x}+\frac{C_5}{5^x}\frac{C_5}{5^x}+\frac{C_5}{5^x}\frac{C_7}{7^x}+\cdots$$

$$+\frac{C_7}{7^x}\frac{C_1}{1^x}+\frac{C_7}{7^x}\frac{C_3}{3^x}+\frac{C_7}{7^x}\frac{C_5}{5^x}+\frac{C_7}{7^x}\frac{C_7}{7^x}+\cdots$$

$$\vdots$$

$$=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{C_{2r-1}C_{2s-1}}{(2r-1)^x(2s-1)^x}$$

i.e.

$$\left[\sum_{r=1}^{\infty}\frac{(-1)^{r-1}cos\{y\,log(2r-1)\}}{(2r-1)^x}\right]^2=\left\{\sum_{r=1}^{\infty}\frac{C_{2r+1}}{(2r-1)^x}\right\}^2=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{C_{2r+1}C_{2s+1}}{(2r-1)^x(2s-1)^x}$$

Let $(-1)^{r-1}sin\{y\,log(2r-1)\}=S_{2r-1}$. Then, in a similar way, we obtain

$$\left[\sum_{r=1}^{\infty}\frac{(-1)^{r-1}sin\{y\,log(2r-1)\}}{(2r-1)^x}\right]^2=\left\{\sum_{r=1}^{\infty}\frac{S_{2r-1}}{(2r-1)^x}\right\}^2=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{S_{2r-1}S_{2s-1}}{(2r-1)^x(2s-1)^x}$$

Then,

$$|\beta(x,y)|^2=\left[\sum_{r=1}^{\infty}(-1)^{r-1}\frac{cos\{y\,log(2r-1)\}}{(2r-1)^x}\right]^2+\left[\sum_{r=1}^{\infty}(-1)^{r-1}\frac{sin\{y\,log(2r-1)\}}{(2r-1)^x}\right]^2$$

$$=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{C_{2r-1}C_{2s-1}+S_{2r-1}S_{2s-1}}{\{(2r-1)(2s-1)\}^x}$$

Returning to the original symbol ,

$$|\beta|^2=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{\{(2r-1)(2s-1)\}^x}\Big[(-1)^{r-1}cos\{y\,log(2r-1)\}(-1)^{s-1}cos\{y\,log(2s-1)\}$$
$$+(-1)^{r-1}sin\{y\,log(2r-1)\}(-1)^{s-1}sin\{y\,log(2s-1)\}\Big]$$

i.e.

$$|\beta|^2=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{cos\{ylog(2r-1)\}cos\{ylog(2s-1)\}+sin\{ylog(2r-1)\}sin\{ylog(2s-1)\}}{\{(2r-1)(2s-1)\}^x}$$

Here,

$$cos\{ylog(2r-1)\}cos\{ylog(2s-1)\}+sin\{ylog(2r-1)\}sin\{ylog(2s-1)\}=cos\left(ylog\frac{2s-1}{2r-1}\right)$$

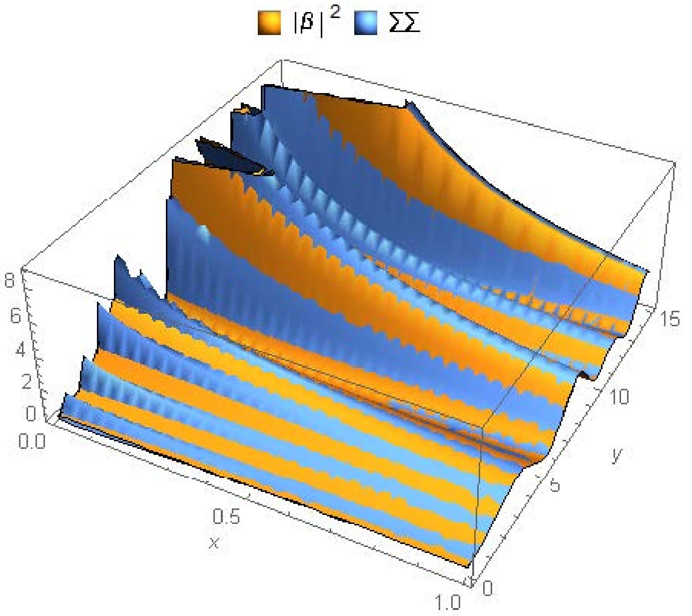
Using this,

$$|\beta(x,y)|^2=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^x}cos\left(y\,log\frac{2s-1}{2r-1}\right)\tag{3.2}$$

The first few lines are as follows.

$$|\beta(x,y)|^2=\frac{1}{(1\cdot 1)^x}cos\left(y\,log\frac{1}{1}\right)-\frac{1}{(1\cdot 3)^x}cos\left(y\,log\frac{3}{1}\right)+\frac{1}{(1\cdot 5)^x}cos\left(y\,log\frac{5}{1}\right)-+\cdots$$
$$-\frac{1}{(3\cdot 1)^x}cos\left(y\,log\frac{1}{3}\right)+\frac{1}{(3\cdot 3)^x}cos\left(y\,log\frac{3}{3}\right)-\frac{1}{(3\cdot 5)^x}cos\left(y\,log\frac{5}{3}\right)+-\cdots$$
$$+\frac{1}{(5\cdot 1)^x}cos\left(y\,log\frac{1}{5}\right)-\frac{1}{(5\cdot 3)^x}cos\left(y\,log\frac{3}{5}\right)+\frac{1}{(5\cdot 5)^x}cos\left(y\,log\frac{5}{5}\right)-+\cdots$$
$$\vdots$$

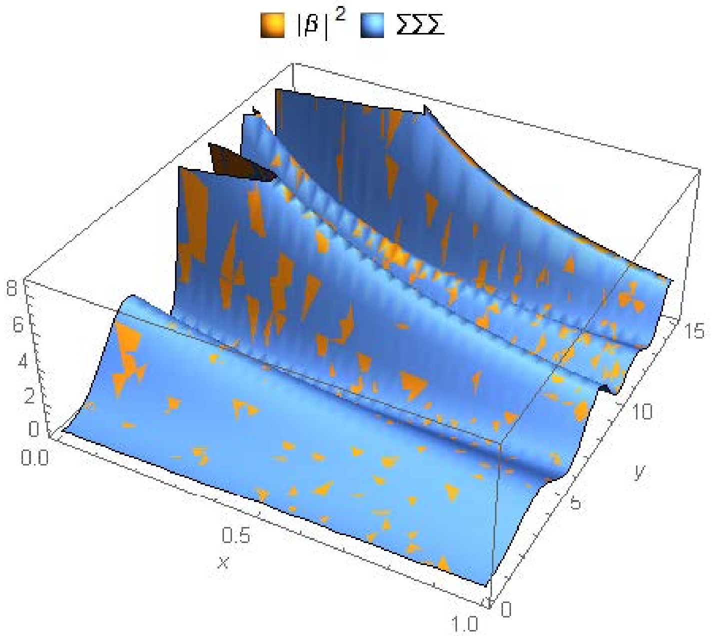
When both sides of (3.2) are illustrated together, it is as follows. Here, the upper limit of $\Sigma\Sigma$ is 150 x 150. Both sides almost coincide near $x=1$, but do not coincide near $x=0$.



Then, we attach the parallel accelerator (See " **13 Convergence Acceleration of Multiple Series** " (A la carte)) to the right side of (3.2) .

$$f(x,y,q)=\sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{s=1}^{k}\frac{q^{k-r-s}}{(q+1)^{k+1}}\binom{k}{r+s}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^{x}}\cos\left(y\log\frac{2s-1}{2r-1}\right) \tag{3.2'}$$

When this is illustrated at $q=1/3$ and $m=30$, it is as follows. (m is the upper limit of Σ. Same as below.) Both sides overlap exactly and look like spots.



Incidently, when (3.2') is calculated at $x=-3$, $y=0$, $q=1$, $m=50$, it is as follows.

```
N[f[-3, 0, 1, 50]]
-4.44089×10^-16
```

Since $x=-3, y=0$ is a trivial zero point of $\beta(x,y)$, this means that the right side of (3.2) is analytically continued in the negative direction beyond the line of convergence $x=0$.

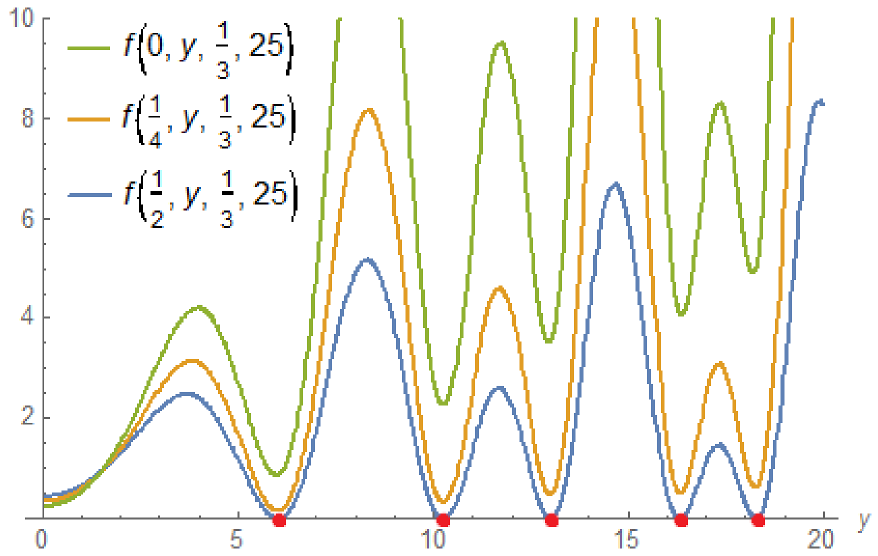
Using Formula 7.3.2 , a hypothesis equivalent to the Riemann hypothesis can also be described as follows.

Hypothesis 7.3.3

When $\beta(x,y)$ is the Dirichlet Beta function on the complex plane, the following inequality holds.

$$|\beta(x,y)|^2 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^x}\cos\left(y\log\frac{2s-1}{2r-1}\right) > 0 \quad for \quad \begin{array}{c} 0 < x < 1/2 \\ y \geq 2 \end{array} \tag{3.3}$$

If this is illustrated, it is as follows. Green, orange, and blue are each a cutaway view of $|\beta(x,y)|^2$ at $x=0$, $1/4$ and $1/2$, and the red points are zeros on the critical line $(x=1/2)$.



Other than the blue line is not in contact with the y-axis. This means that $|\beta(x,y)|^2$ has no zero point at $x<1/2$. And, it is equivalent to that the Dirichlet Beta Function has no zero point except at the critical line.

7.4 Theorems at Zeros

As seen in the previous section , squared absolute value of Dirichlet Beta function was expressed as follows.

$$|\beta(x,y)|^2 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^x}\cos\left(y\log\frac{2s-1}{2r-1}\right) \tag{3.2}$$

The following theorems hold for the zeros of this double series.

Theorem 7.4.0

When $\beta(x,y)$ is Dirichlet Beta Function, if $\beta(a,b)=0$,

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^a}\cos\left(b\log\frac{2s-1}{2r-1}\right) = 0 \tag{4.0c}$$

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^a}\sin\left(b\log\frac{2s-1}{2r-1}\right) = 0 \tag{4.0s}$$

Proof

Since the left side of (4.0c) is the absolute value of $\beta(a,b)$, (4.0c) is natural. (4.0s) is proved at the end of this section.

Interestingly, at a zero point (a,b) of β , each of these rows have to be all 0. Below, we state this as a theorem.

Theorem 7.4.1

When $\beta(x,y)$ is Dirichlet Beta Function, if $\beta(a,b)=0$,

$$\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^a}\cos\left(b\log\frac{2s-1}{2r-1}\right) = 0 \qquad for\ \ r=1,2,3,\cdots \tag{4.1c}$$

$$\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^a}\sin\left(b\log\frac{2s-1}{2r-1}\right) = 0 \qquad for\ \ r=1,2,3,\cdots \tag{4.1s}$$

Proof

Let c_r be the r th row of the double series (4.1c) . Then,

$$c_r = \sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^a}\cos\left(b\log\frac{2s-1}{2r-1}\right)$$
$$= \frac{(-1)^r}{(2r-1)^a}\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}\cos\left(b\log\frac{2s-1}{2r-1}\right)$$

Here,

$$\cos\left(b\log\frac{2s-1}{2r-1}\right) = \cos\{b\log(2r-1)\}\cos\{b\log(2s-1)\} + \sin\{b\log(2r-1)\}\sin\{b\log(2s-1)\}$$

Using this,

$$c_r = \frac{(-1)^r}{(2r-1)^a}\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}[\cos\{b\log(2r-1)\}\cos\{b\log(2s-1)\}$$
$$+\sin\{b\log(2r-1)\}\sin\{b\log(2s-1)\}]$$

i.e.

$$c_r = \frac{(-1)^r}{(2r-1)^a}\left[cos\{blog(2r-1)\}\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}cos\{blog(2s-1)\} \right.$$

$$\left. + sin\{blog(2r-1)\}\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}sin\{blog(2s-1)\} \right]$$

At a zero point (a,b) of β ,

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}cos\{blog(2s-1)\} = 0 \quad , \quad \sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}sin\{blog(2s-1)\} = 0$$

Therefore, $c_r(a,b) = 0 \quad for\ r=1,2,3,\cdots$.

In a similar way, let S_r be the r th row of the double series (4.1s) . Then,

$$S_r = \frac{(-1)^r}{(2r-1)^a}\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}sin\left(b\ log\frac{2s-1}{2r-1} \right)$$

Here,

$$sin\left(blog\frac{2s-1}{2r-1} \right) = cos\{blog(2r-1)\}sin\{blog(2s-1)\} - sin\{blog(2r-1)\}cos\{blog(2s-1)\}$$

Using this,

$$S_r = \frac{(-1)^r}{(2r-1)^a}\left[cos\{blog(2r-1)\}\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}sin\{blog(2s-1)\} \right.$$

$$\left. - sin\{blog(2r-1)\}\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}cos\{blog(2s-1)\} \right]$$

For the same reason as the above, $S_r(a,b) = 0 \quad for\ r=1,2,3,\cdots$ at a zero point (a,b) of β .

From this, the following corollary follows.

Corollary 7.4.1'

When $\beta(x,y)$ is Dirichlet Beta Function, if $\beta(a,b)=0$,

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}cos\left(b\ log\frac{2s-1}{2r-1} \right) = 0 \qquad for\ \ r=1,2,3,\cdots \tag{4.1c'}$$

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}sin\left(b\ log\frac{2s-1}{2r-1} \right) = 0 \qquad for\ \ r=1,2,3,\cdots \tag{4.1s'}$$

Putting $\theta = -b\ log\, r$ in this corollary, we obtain the following.

Corollary 7.4.1''

When $\beta(x,y)$ is Dirichlet Beta Function, if $\beta(a,b)=0$, the following expressions hold for arbitrary real number θ .

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}cos\{b\ log(2s-1)+\theta\} = 0 \tag{4.1c''}$$

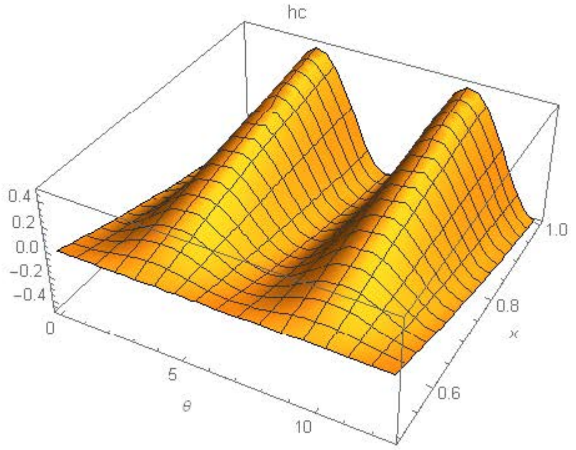
$$\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}sin\{b\ log(2s-1)+\theta\} = 0 \tag{4.1s''}$$

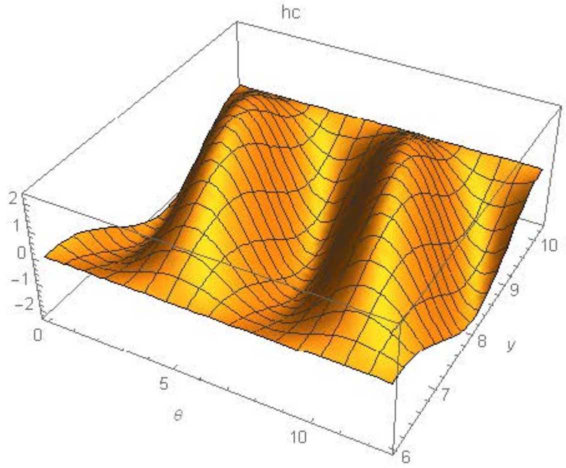
Example

We illustrate (4.1c"). Since the convergence of this left side is slow and it is difficult to draw an accurate figure, we apply Knopp transformation (See " **10 Convergence Acceleration & Summation Method** " (A la carte)) to this as follows.

$$h_c(x,y,\theta,q,m) = \sum_{k=1}^{m}\sum_{s=1}^{k}\frac{q^{k-s}}{(q+1)^{k+1}}\binom{k}{s}\frac{(-1)^{s-1}}{(2s-1)^x}\cos\{y\log(2s-1)+\theta\}$$

Here, this is illustrated at $q=1/3$ and $m=25$.





The left figure is a cutaway view at $x=1/2$ when $y=6.0209\cdots$. We can see that $h_c=0$ for any θ in this cutting surface.

The right figure is a cutaway view at $y=6.0209\cdots$ and $y=10.2437\cdots$ when $x=1/2$. We can see that $h_c=0$ for any θ in these cutting surfaces. It is surprising that the contour lines appear innumerably in such a twisted figure.

Using Corollary 7.4.1' , we obtain the following theorem.

Theorem 7.4.2

When $\beta(x,y)$ is Dirichlet Beta Function and $c(r)$ is arbitrary real valued function, if $\beta(a,b)=0$, the following expressions hold.

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{c(r)}{\{(2r-1)(2s-1)\}^a}\cos\left(b\log\frac{2s-1}{2r-1}\right) = 0 \tag{4.2c}$$

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{c(r)}{\{(2r-1)(2s-1)\}^a}\sin\left(b\log\frac{2s-1}{2r-1}\right) = 0 \tag{4.2s}$$

Proof

From Corollary 7.4.1' (4.1c') ,

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}\cos\left(b\log\frac{2s-1}{2r-1}\right) = 0 \qquad for\ \ r=1,2,3,\cdots$$

Multiplying both sides by $(-1)^r c(r)/(2r-1)^a$,

$$\sum_{s=1}^{\infty}(-1)^{r+s}\frac{c(r)}{\{(2r-1)(2s-1)\}^a}\cos\left(b\log\frac{2s-1}{2r-1}\right) = 0 \qquad for\ \ r=1,2,3,\cdots$$

Adding up about r , we obtain (4.2c) . In a similar way, (4.2s) is obtained.

Proof of Theorem 7.4.0 (4.0s)

Particularly placed $c(r)=1$ in Theorem 7.4.2 (4.2s) , (4.0s) is obtained.

7.5 Partial Derivative of Squared Absolute Value

7.5.1 First order Partial Derivative

Formula 7.5.1

When squared absolute value of Dirichlet beta function is

$$f(x,y)=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^{x}}\cos\left(y\log\frac{2s-1}{2r-1}\right)\qquad \left(=|\beta(x,y)|^{2}\right) \tag{3.2}$$

The 1st order partial derivatives are givern as follows.

$$f_x=-2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^{x}}\cos\left(y\log\frac{2s-1}{2r-1}\right) \tag{5.1x}$$

$$f_y=\ 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^{x}}\sin\left(y\log\frac{2s-1}{2r-1}\right) \tag{5.1y}$$

Proof

Differentiating (3.2) with respect to x

$$f_x=-\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log\{(2r-1)(2s-1)\}}{\{(2r-1)(2s-1)\}^{x}}\cos\left(y\log\frac{2s-1}{2r-1}\right)$$

$$=-\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^{x}}\cos\left(y\log\frac{2s-1}{2r-1}\right)$$

$$-\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2s-1)}{\{(2r-1)(2s-1)\}^{x}}\cos\left(y\log\frac{2s-1}{2r-1}\right)$$

Swapping r and s in the 2nd term on the right side,

$$-\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2s-1)}{\{(2r-1)(2s-1)\}^{x}}\cos\left(y\log\frac{2s-1}{2r-1}\right)$$

$$=-\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{s+r}\frac{\log(2r-1)}{\{(2s-1)(2r-1)\}^{x}}\cos\left(y\log\frac{2r-1}{2s-1}\right)$$

$$=-\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^{x}}\cos\left(y\log\frac{2s-1}{2r-1}\right)\qquad\{\because \cos(-z)=\cos z\}$$

Substituting this for the 2nd term on the right side, we obtain (5.1x) .

Next, differentiating (3.2) with respect to y

$$f_y=-\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log\{(2s-1)/(2r-1)\}}{\{(2r-1)(2s-1)\}^{x}}\sin\left(y\log\frac{2s-1}{2r-1}\right)$$

$$=-\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2s-1)}{\{(2r-1)(2s-1)\}^{x}}\sin\left(y\log\frac{2s-1}{2r-1}\right)$$

$$+\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^{x}}\sin\left(y\log\frac{2s-1}{2r-1}\right)$$

Swapping r and s in the 1st term on the right side,

$$-\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2s-1)}{\{(2r-1)(2s-1)\}^{x}}\sin\left(y\log\frac{2s-1}{2r-1}\right)$$

$$= -\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{s+r}\frac{\log(2r-1)}{\{(2s-1)(2r-1)\}^x}\sin\left(y\log\frac{2r-1}{2s-1}\right)$$

$$= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{s+r}\frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^x}\sin\left(y\log\frac{2s-1}{2r-1}\right) \quad \{\because \sin(-z) = -\sin z\}$$

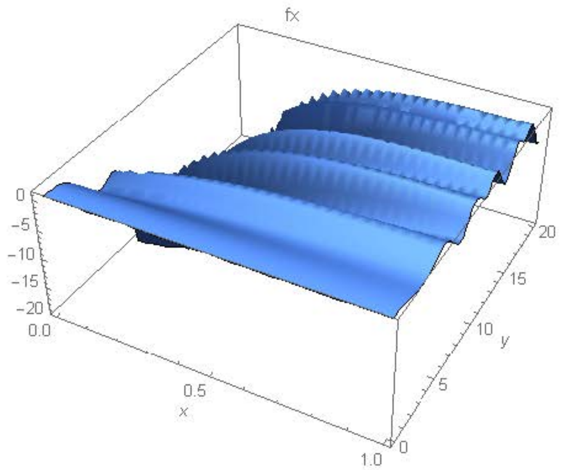
Substituting this for the 1st term on the right side, we obtain (5.1y) .

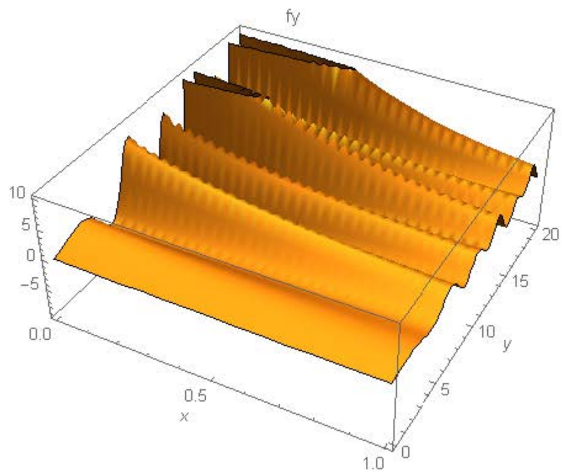
If Formula 7.5.1 is drawn as it is, the vicinity of the origin can not be drawn beautifully. So, we attach the parallel accelerator (See " **13 Convergence Acceleration of Multiple Series** " (A la carte)) to the right sides.

$$f_x(x,y,q) = -2\sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{s=1}^{k}\frac{q^{k-r-s}}{(q+1)^{k+1}}\binom{k}{r+s}(-1)^{r+s}\frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^x}\cos\left(y\log\frac{2s-1}{2r-1}\right) \tag{5.1x'}$$

$$f_y(x,y,q) = 2\sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{s=1}^{k}\frac{q^{k-r-s}}{(q+1)^{k+1}}\binom{k}{r+s}(-1)^{r+s}\frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^x}\sin\left(y\log\frac{2s-1}{2r-1}\right) \tag{5.1y'}$$

When these are illustrated at $q=1/3$ and $m=25$, it is as follows. Blue is f_x and orange is f_y .
It looks like $f_x \le 0$.





In addition, it is better to draw using (5.1x) or (5.1y) in the area where y is large (approximately 30 or more).

7.5.2 Necessary Condition for Local Minimum

Theorem 7.5.2 (Stationary Condition)

When $\beta(x,y)$ is Dirichlet Beta Function, if $\beta(a,b)=0$, the following expressions hold .

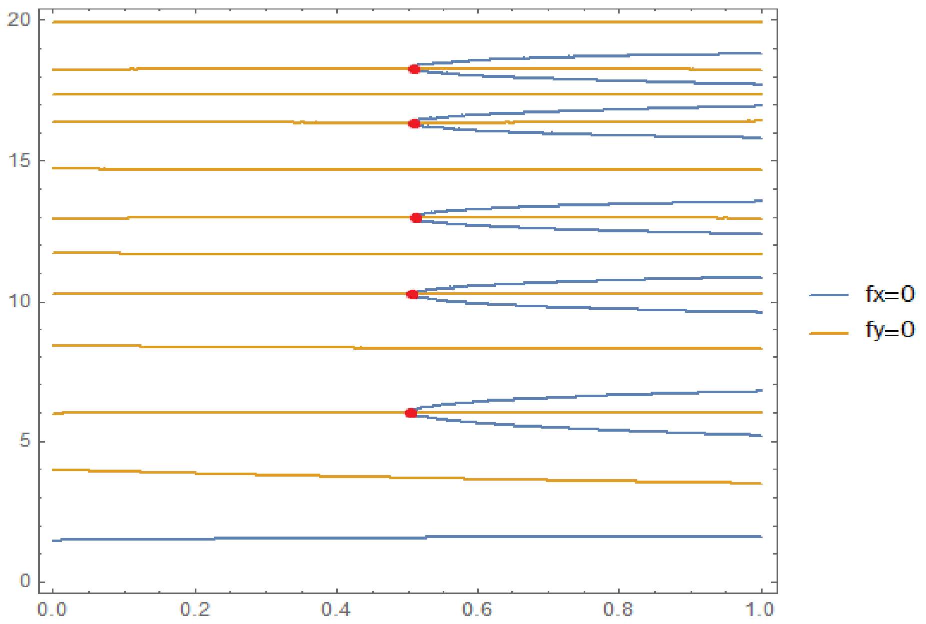
$$f_x(a,b) = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^a}\cos\left(b\log\frac{2s-1}{2r-1}\right) = 0 \tag{5.2x}$$

$$f_y(a,b) = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^a}\sin\left(b\log\frac{2s-1}{2r-1}\right) = 0 \tag{5.2y}$$

Proof

The problem of searching for the zeros of $\beta(x,y)$ reduce to the problem of finding the local minimums of $f(x,y)\ \{=|\beta(x,y)|^2\}$. (5.2x) and (5.2y) are well known as stationary conditions of two-variable real valued function.

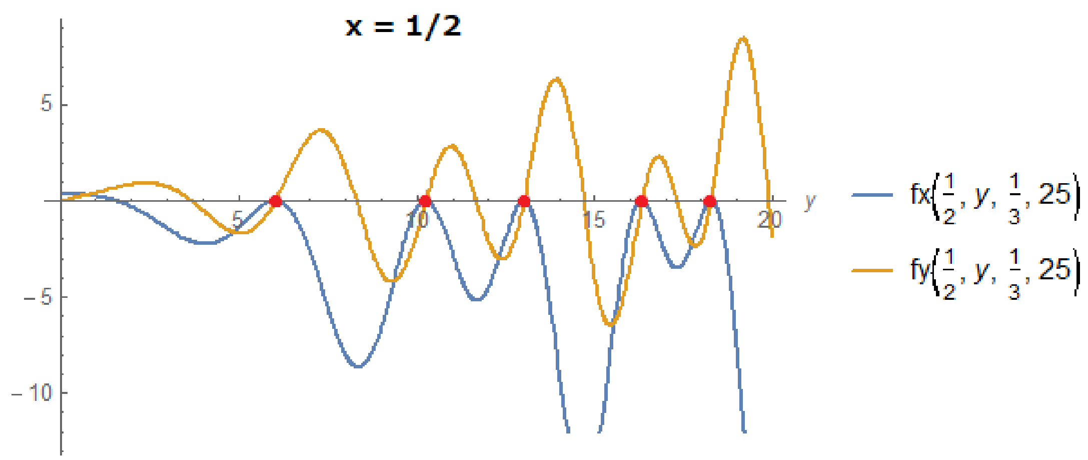
The point (a,b) satisfying the stationary condition (5.2x) and (5.2y) is called a **stationary point**. When the contour plot is drawn in $0 \le x \le 1$, $0 \le y \le 20$ using the function *ContourPlot* of *Mathematica*, it is as follows.



The red points on $x = 1/2$ are non-trivial zeros. Since $f(a,b) = |\beta(a,b)|^2 = 0$, these are all loacl minimums of $f(x,y)$. As far as observing this figure, it seems that there is no stationary point other than the local minimum point.

7.5.3 Non-trivial zeros

The figures of f_x and f_y at $x=1/2$ are as follows.



Blue is f_x, orange is f_y and red points are non-trivial zeros. We can see the followings from the figure.

(1) f_x resembles a cosine curve and f_y resembles a sine curve.

(2) The extrema points of f_x are not generally zeros of f_y. However, exceptions exist.

(3) Non-trivial zeros are the local maximum points of f_x and are zeros of f_y..

(4) Non-trivial zeros are the uphill zeros (end of cycles) of f_y.

(5) In f_x, f_y, the end of each cycle is a non-trivial zero point. That is, there is no useless cycle.

When these results are compared with **7.1.3** , the parts of the blue are very different. For (1), the sign of the cosine curve is different. As an example of (3) , when the local maximum point of f_x and the zero point of f_y around $y=6$ are calculated, they are as follows. Both coincide with the first non-trivial zero $6.02094\cdots$.

```
FindMaximum[fx[1/2, y, 1/3, 35], {y, 6}]        FindRoot[fy[1/2, y, 1/3, 30], {y, 6}]
{-1.08773×10^-11, {y → 6.02095}}                 {y → 6.02095}
```

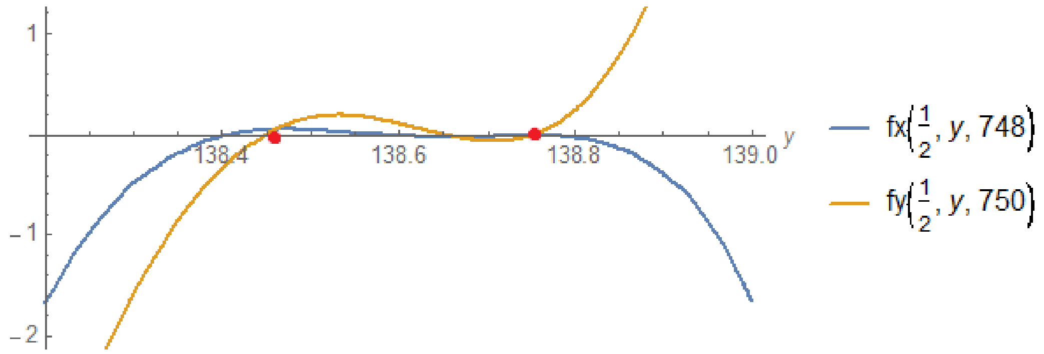
Absence of downhill zeros

In **7.1.3** , there was a downhill zero point of imaginary part around $y=139$. What about f_x , f_y ?
To see this, we rearrange (5,1x) and (5.1y) along the diagonal as follows.

$$f_x = -2\sum_{r=1}^{\infty}\sum_{s=1}^{r}(-1)^{r+1}\frac{\log(2r+1-2s)}{\{(2r+1-2s)(2s-1)\}^x}\cos\left(y\log\frac{2s-1}{2r+1-2s}\right) \qquad (5,1x')$$

$$f_y = 2\sum_{r=1}^{\infty}\sum_{s=1}^{r}(-1)^{r+1}\frac{\log(2r+1-2s)}{\{(2r+1-2s)(2s-1)\}^x}\sin\left(y\log\frac{2s-1}{2r+1-2s}\right) \qquad (5,1y')$$

When the figure around $y=139$ is drawn using these , it is as follows.



Non-trivial zero point $138.7501\cdots$ (right red point) is the local maxium point of f_x and is uphill zero point of f_y. The left zero point $138.4572\cdots$ is not drawn correctly. but if correctly drawn, it should also be the uphill zero point of f_y. That is, In the case of f_x , f_y , there are no exceptions of (3) and (4) .

7.5.4 Second order Partial Derivative

The 2nd order partial derivatives of squared absolute value of Dirichlet Beta function $f(x,y)$ is expressed as follows respectively,.

$$f_{xx} = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2r-1)\log(2s-1)+\log^2(2r-1)}{\{(2r-1)(2s-1)\}^x}\cos\left(y\log\frac{2s-1}{2r-1}\right)$$

$$f_{xy} = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2(2r-1)}{\{(2r-1)(2s-1)\}^x}\sin\left(y\log\frac{2s-1}{2r-1}\right) \qquad (=f_{yx})$$

$$f_{yy} = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2r-1)\log(2s-1)-\log^2(2r-1)}{\{(2r-1)(2s-1)\}^x}\cos\left(y\log\frac{2s-1}{2r-1}\right)$$

As well known, the discriminant for determining the stationary point (a,b) of $f(x,y)$ to be the minimum point is as follows.

$$f_{xx}(a,b) > 0$$

$$f_{xx}(a,b)\, f_{yy}(a,b) - \{f_{xy}(a,b)\}^2 > 0$$

As a result of some numerical calculations, all the stationary points satisfied two inequality without exception. As far as this function $f(x,y)\ \{=|\beta(x,y)|^2\}$ is concerned, it seems that there is no stationary point that does not satisfy these inequalities.

7.5.5 hypothesis equivalent to the Riemann hypothesis

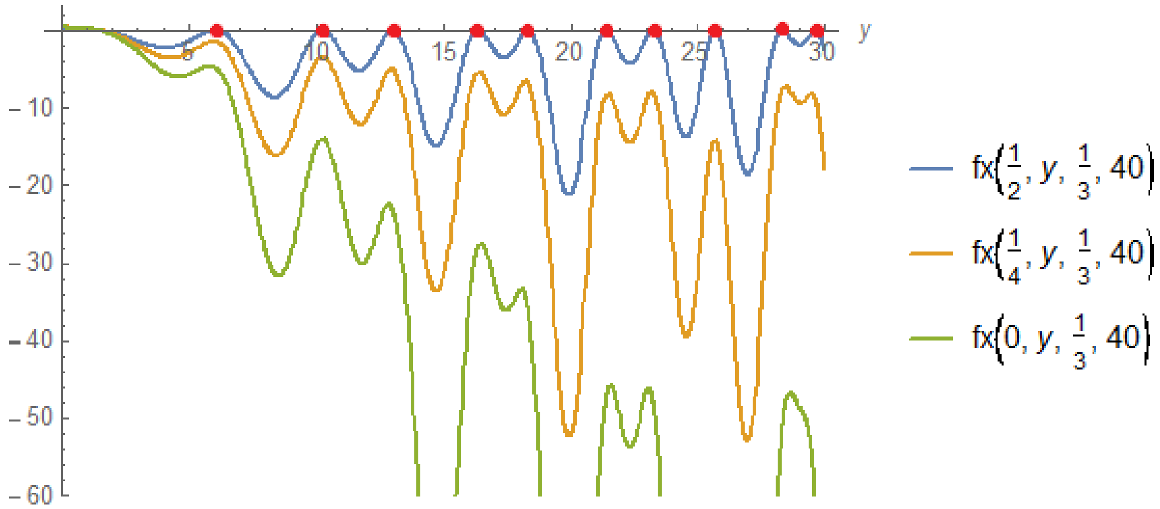
Last, using Formula 7.5.1 , we present a hypothesis equivalent to the Riemann hypothesis.

Hypothesis 7.5.5

When $\beta(x,y)$ is the Dirichlet Beta function on the complex plane, the following inequality holds.

$$f_x = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^x}\cos\left(y\log\frac{2s-1}{2r-1}\right) < 0 \quad for \quad \begin{array}{c} 0 < x < 1/2 \\ y \ge 2 \end{array} \qquad (5.5)$$

If this is illustrated, it is as follows. Blue is $f_x(1/2, y)$, orange is $f_x(1/4, y)$, green is $f_x(0, y)$ and the red points are zeros on the critical line $(x = 1/2)$. Other than the blue line is not in contact with the



Other than the blue line is not in contact with the y-axis. This means that $f_x(x,y)$ has no zero point at $x < 1/2$, $y \ge 2$. And, it is equivalent to that the Dirichlet Beta Function has no zero point except at the critical line.

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Kano Kono

Alien's Mathematics