04 Absolute Value of Dirichlet Eta Function

4.1 Dirichlet Eta Function

4.1.1 Definition

Dirichlet Eta Function $\eta(z)$ is defined in the half plane $Re\{\eta(z)\}>0$ as follows.

$$\eta(z) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^z} \tag{1.0}$$

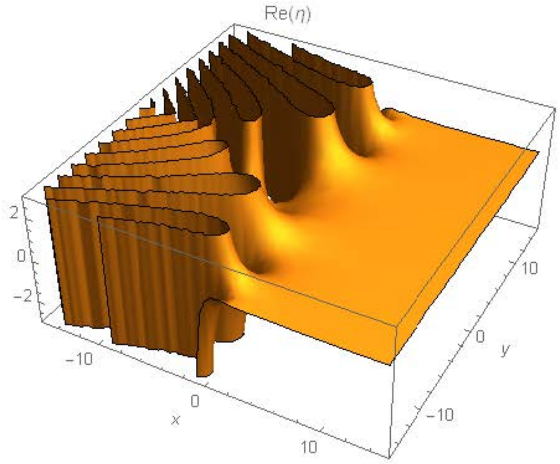
This series is analytically continued to the whole complex plane by applying some kind of acceleration method. The easiest of these is the Euler transformation as follows.

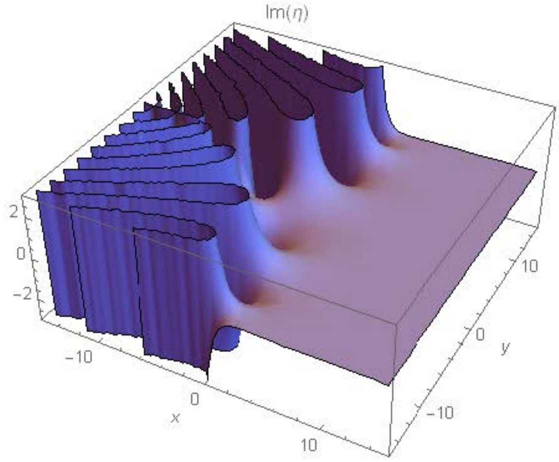
$$\eta(z) = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \sum_{r=1}^{k} \binom{k}{r} \frac{(-1)^{r-1}}{r^z} \tag{1.1}$$

(1.0) and (1.1) are the same in $Re\{\eta(z)\}>0$. Although (1.0) can not express the left side of line the of convergence, (1.1) can express also the left side of this. Therefore, we can define Dirichlet Eta Function $\eta(z)$ by (1.1) .

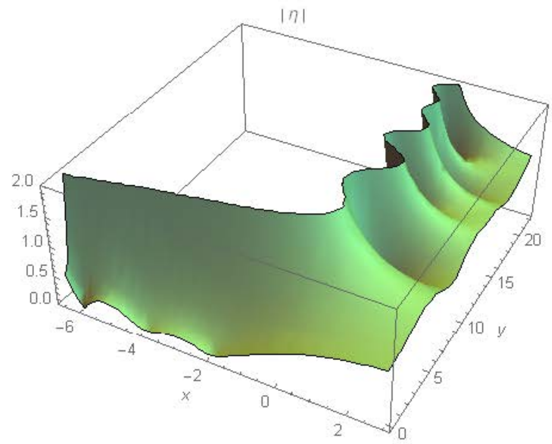
4.1.2 Overview

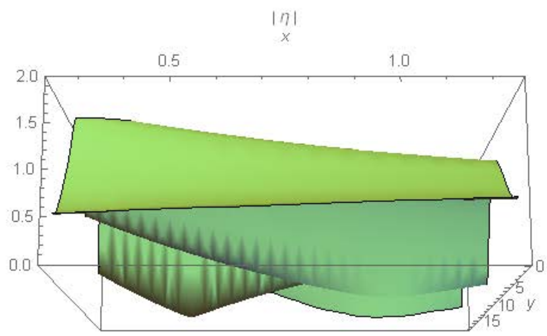
The 3D figures of the real part and the imaginary part of Dirichlet Eta Function $\eta(x+iy)$ are as follows.





Further, the 3D figure of the absolute value is as follows. In the left figure, trivial zeros of $\eta(z)$ are observed along the x axis. The right figure is a view of the left figure from the bottom. We can see that zeros of $\eta(z)$ are located in two lines along $x=1/2$ and $x=1$.





As seen in these figures, Dirichlet Eta Function $\eta(z)$ has three kinds of zeros as follows.

(1) Trivial zeros $-2, -4, -6, -8, \cdots$

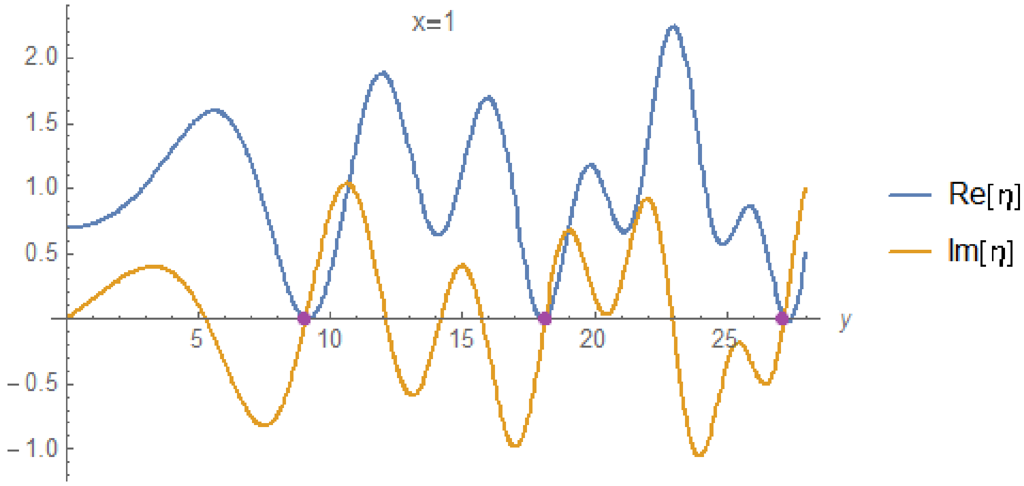
(2) Non-trivial zeros $1/2 \pm i\,14.1347\cdots$, $1/2 \pm i\,21.0220\cdots$, $1/2 \pm i\,25.0108\cdots$, $\cdots$

(3) $\eta(z)$ specific zeros $1 \pm 2\pi i/\log 2$, $1 \pm 4\pi i/\log 2$, $1 \pm 6\pi i/\log 2$, $\cdots$

Among these, (1) and (2) are common to the zeros of Riemann zeta function. Further, it is well known that non-trivial zeros (2) exist in $0 < x < 1$ called **Critical Strip** . Moreover, it is proved that they have to exist symmetrically with respect to $x = 1/2$. And, fortunately, this critical strip is included within the convergence range of the series (1.0) .

4.1.3 η specific zeros

The figures of the real part and the imaginary part of Dirichlet Eta Function $\eta(z)$ at $x = 1$ are as follows.



The purple points are η specific zeros. We can see the followings from the figure.

(1) The real part resembles a negative cosine curve and the imaginary part resembles a sine curve.

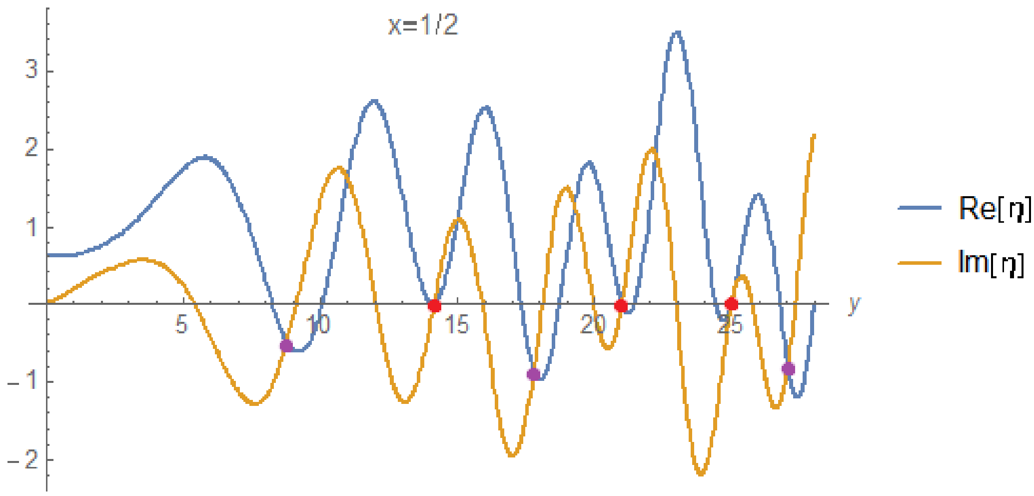
(2) The extrema points of the real part are close to the zeros of the imaginary part but not zeros.

(3) η specific zeros are close to the local minimum points of the real part, but are not so.

(4) η specific zeros are the uphill zeros (end of cycles) of the imaginary part. However, the reverse is not true.

4.1.4 Non-trivial zeros

The figures of the real part and the imaginary part of Dirichlet Eta Function $\eta(z)$ at $x = 1/2$ are as follows.



The red points are non-trivial zeros. We can see the followings from the figure.

(1) The real part resembles a negative cosine curve and the imaginary part resembles a sine curve.

(2) The extrema points of the real part are close to the zeros of the imaginary part but not zeros.

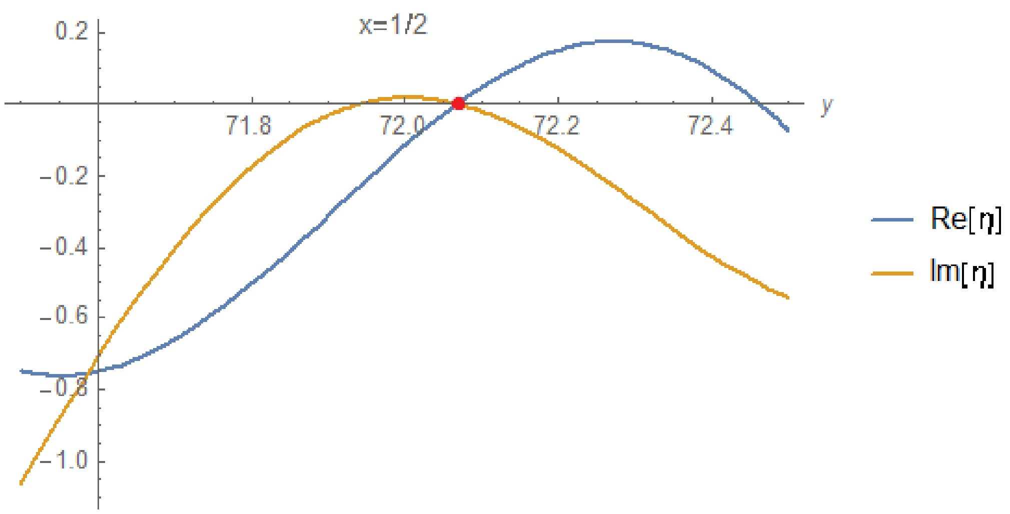
(3) Non-trivial zeros are close to the local minimum points of the real part, but are not so.

(4) Non-trivial zeros are the uphill zeros (end of cycles) of the imaginary part. However, the reverse is not true.

(5) The purple points which were η specific zeros, have moved below the y -axis.

Downhill zeros

Exceptions exist in (3) and (4). For example, the figure in the vicinity of $y=72$ is as follows. Non-trivial zero point $72.0671\cdots$ (red point) is near the local maximum point of the real part and is the downhill zero point of the imaginary part.



4.1.5 Feature of Dirichlet Eta Series

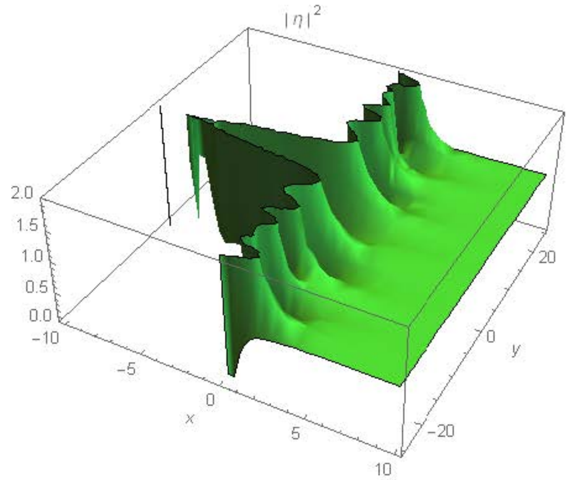
Observing both 4.1.3 and 4.1.4 , we can see that this Dirichlet series contains two periodic functions. Among these, the periodic function giving η specific zeros is constant in period and amplitude, but the periodic function giving non-trivial zeros is not constant in period and amplitude.

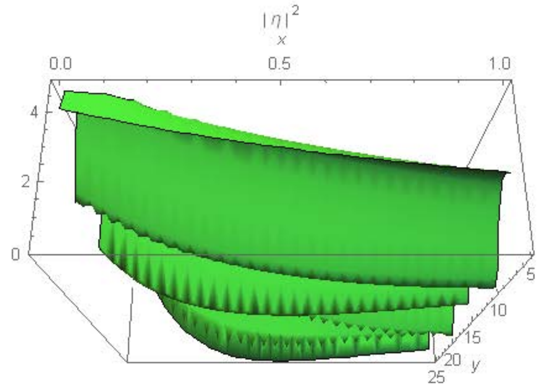
4.2 Squared Absolute Value of Dirichlet Eta

Squared absolute value of Dirichlet eta function is

$$g(x,y) = |\eta(x,y)|^2 \tag{2.0}$$

This is a real-valued function with two variables. And it is shown in the figure as follows.

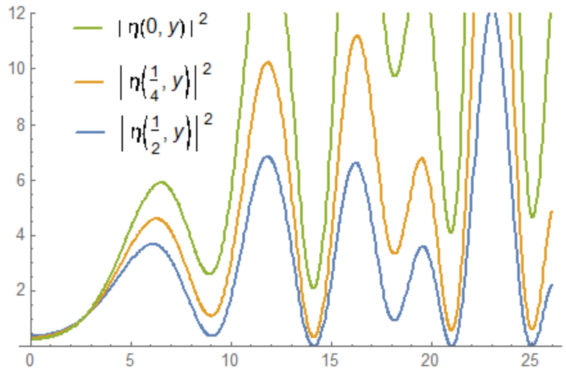


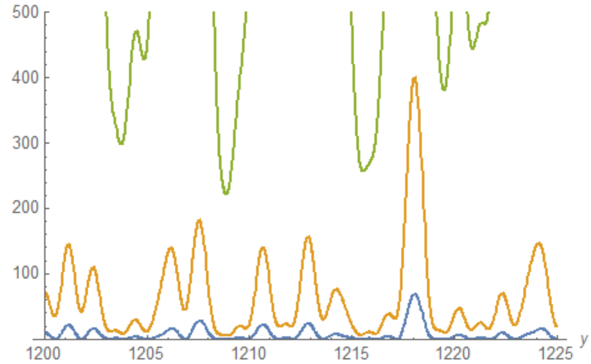


In the left figure, dents are observed along $x=1/2$ and $x=1$. The right figure is a view of the left figure from the bottom. We can see that zeros of $\eta(z)$ are located in two lines along $x=1/2$ and $x=1$. The zeros on the $x=1/2$ correspond to the zeros of $\zeta(z)$ function and the zeros on the $x=1$ are η specific zeros. On the other hand, there is no zero on the $x=0$.

Features in $0 \le x \le 1/2$

Let us focus on space $0 \le x \le 1/2$. The figures of section in $x=0$, $1/4$, $1/2$ are drawn as follows.

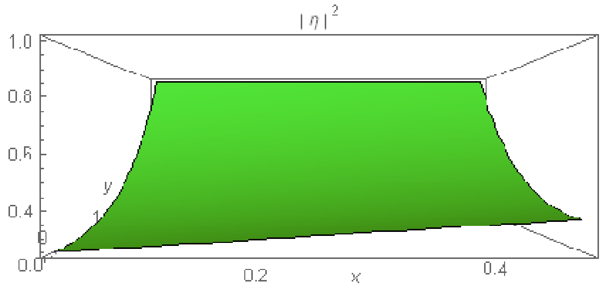


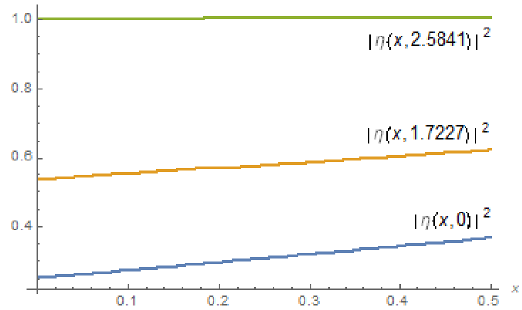


Looking at this, it looks like $|\eta(0,y)|^2 > |\eta(1/4,y)|^2 > |\eta(1/2,y)|^2$ in $2.6222 \le y \le 25$. It is the same also in $1200 \le y \le 1225$. Below, we observe this in more detail.

(1) $0 \le y \le 2.5841$

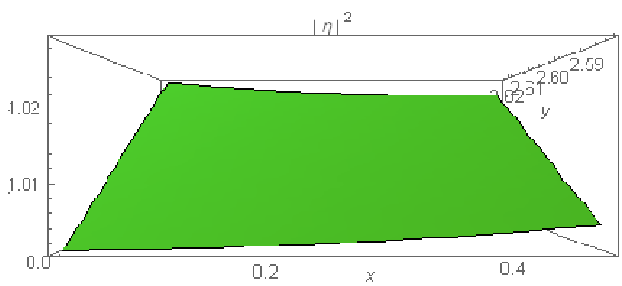
The front view of 3D in this interval is the left figure. The cutaway view at $y=0,\ 1.7227,\ 2.5841$ of this is the right figure. In this interval, $|\eta(x,y)|^2$ seems to be monotonically increasing with respect to x.

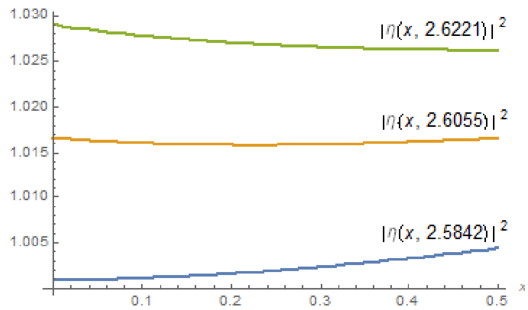




(2) $2.5841 < y < 2.6222$

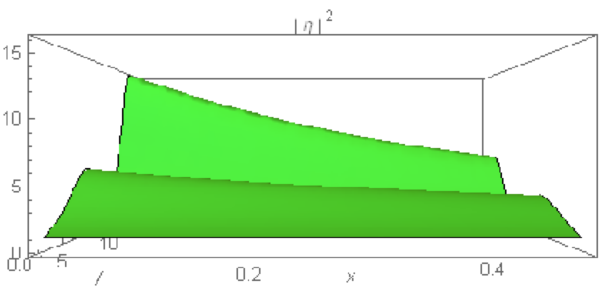
The front view of 3D in this interval is the left figure. The cutaway view at $y = 2.5842,\ 2.6055,\ 2.6221$ of this is the right figure. In this interval, $|\eta(x,y)|^2$ is not monotonic with respect to x. In the right figure, although the curve of $y = 2.5842$ looks like monotonically increasing, it is decreasing at the left end when it is seen enlarged. Although the curve of $y = 2.6221$ looks like monotonically decreasing, it is increasing at the right end when it is seen enlarged.

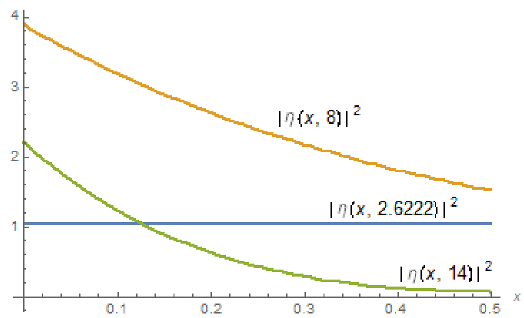




(3) $y \geq 2.6222$

The front view of 3D in this interval is the left figure. The cutaway view at $y = 2.6222,\ 8,\ 14$ of this is the right figure. In this interval, $|\eta(x,y)|^2$ seems to be monotonically decreasing with respect to x.





Based on the observations above, I present the next hypothesis equivalent to the Riemannian hypothesis.

Hypothesis 4.2.1

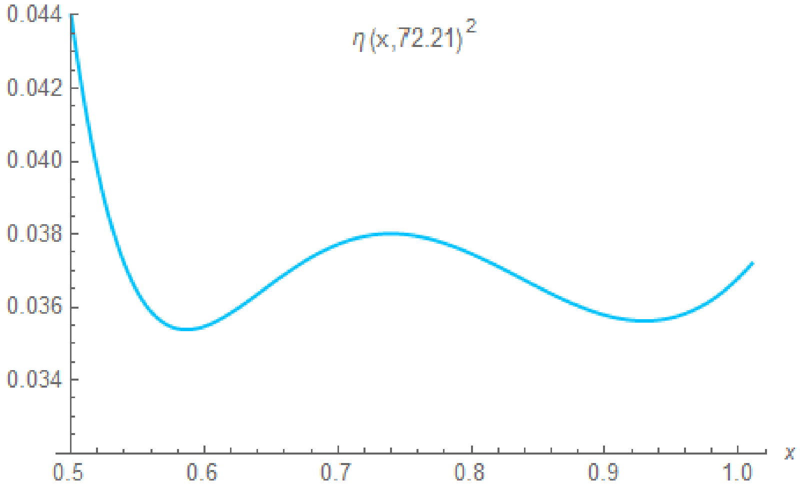
When $\eta(x,y)$ is the Dirichlet eta function on the complex plane, the squared absolute value $|\eta(x,y)|^2$ is a monotonically decreasing function in the region $0 < x < 1/2,\ y \geq 3$.

Note1

The zeros common to the Riemann zeta function exist in $0 < x < 1$ called critical strip. Moreover, it is proved that they have to exist symmetrically with respect to $x=1/2$. So, if $|\eta(x,y)|^2$ is monotonically decreasing with respect to x in the region $0 < x < 1/2, y \geq 3$, zeros do not exist in the region and the opposite region $1/2 < x < 1, y \geq 3$. This is equivalent to the Riemann hypothesis.

Note2

Incidentally, in the opposite region $1/2 < x < 1, y \geq 3$, $|\eta(x,y)|^2$ is not necessarily a monotone function with respect to x. For example, $|\eta(x,72.21)|^2$ is as follows.



4.3 Expression of Squared Absolute Value by Series

4.3.1 Expression of Dirichlet Eta Function by Series

As seen in **4.1.1** , Dirichlet Eta Function $\eta(z)$ was defined as follows.

$$\eta(z) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^z} \qquad Re(z) > 0$$

When $Re(z) > 0$, let $z = x + iy$. Then,

$$\eta(x,y) = \sum_{r=1}^{\infty} (-1)^{r-1} r^{-x-iy} \qquad x > 0$$

If this is represented by an exponential function,

$$\eta(x,y) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-(x+iy)\log r} = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-x\log r - iy\log r}$$

If this is represented by a trigonometric function,

$$\eta(x,y) = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\cos(y\log r)}{r^x} - i\sum_{r=1}^{\infty} (-1)^{r-1} \frac{\sin(y\log r)}{r^x} \tag{3.1}$$

4.3.2 Expression of $|\eta|^2$ by Double Series

Squared absolute value of Dirichlet eta function $|\eta(x,y)|^2$ is expressed using (3.1) as follows.

$$|\eta(x,y)|^2 = \left\{ \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\cos(y\log r)}{r^x} \right\}^2 + \left\{ \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\sin(y\log r)}{r^x} \right\}^2$$

Although it looks like a very complicated, it becomes an unexpectedly simple expression when it is expanded and organized.

Formula 4.3.2

When $\eta(x,y)$ is the Dirichlet Eta Function,

$$|\eta(x,y)|^2 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{(-1)^{r+s}}{(rs)^x} \cos\left(y\log\frac{s}{r}\right) \qquad \{\ := g(x,y)\ \} \tag{3.2}$$

Proof

Let $(-1)^{r-1}\cos(y\log r) = C_r$. Then,

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1}\cos(y\log r)}{r^x} = \frac{C_1}{1^x} + \frac{C_2}{2^x} + \frac{C_3}{3^x} + \frac{C_4}{4^x} + \cdots$$

This square is

$$\frac{C_1}{1^x} + \frac{C_2}{2^x} + \frac{C_3}{3^x} + \frac{C_4}{4^x} + \frac{C_5}{5^x} + \frac{C_6}{6^x} + \cdots$$

$$\times \frac{C_1}{1^x} + \frac{C_2}{2^x} + \frac{C_3}{3^x} + \frac{C_4}{4^x} + \frac{C_5}{5^x} + \frac{C_6}{6^x} + \cdots$$

$$= \frac{C_1}{1^x}\frac{C_1}{1^x} + \frac{C_1}{1^x}\frac{C_2}{2^x} + \frac{C_1}{1^x}\frac{C_3}{3^x} + \frac{C_1}{1^x}\frac{C_4}{4^x} + \cdots$$

$$+ \frac{C_2}{2^x}\frac{C_1}{1^x} + \frac{C_2}{2^x}\frac{C_2}{2^x} + \frac{C_2}{2^x}\frac{C_3}{3^x} + \frac{C_2}{2^x}\frac{C_4}{4^x} + \cdots$$

$$+\frac{C_3}{3^x}\frac{C_1}{1^x}+\frac{C_3}{3^x}\frac{C_2}{2^x}+\frac{C_3}{3^x}\frac{C_3}{3^x}+\frac{C_3}{3^x}\frac{C_4}{4^x}+\cdots$$

$$+\frac{C_4}{4^x}\frac{C_1}{1^x}+\frac{C_4}{4^x}\frac{C_2}{2^x}+\frac{C_4}{4^x}\frac{C_3}{3^x}+\frac{C_4}{4^x}\frac{C_4}{4^x}+\cdots$$

$$\vdots$$

$$=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{C_r C_s}{r^x s^x}$$

i.e.

$$\left\{\sum_{r=1}^{\infty}\frac{(-1)^{r-1}\cos(y\log r)}{r^x}\right\}^2=\left\{\sum_{r=1}^{\infty}\frac{C_r}{r^x}\right\}^2=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{C_r C_s}{(rs)^x}$$

Let $(-1)^{r-1}\sin(y\log r)=S_r$. Then, in a similar way, we obtain

$$\left\{\sum_{r=1}^{\infty}\frac{(-1)^{r-1}\sin(y\log r)}{r^x}\right\}^2=\left\{\sum_{r=1}^{\infty}\frac{S_r}{r^x}\right\}^2=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{S_r S_s}{(rs)^x}$$

Then,

$$|\eta(x,y)|^2=\left\{\sum_{r=1}^{\infty}\frac{(-1)^{r-1}\cos(y\log r)}{r^x}\right\}^2+\left\{-\sum_{r=1}^{\infty}\frac{(-1)^{r-1}\sin(y\log r)}{r^x}\right\}^2$$

$$=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{C_r C_s+S_r S_s}{(rs)^x}$$

Returning to the original symbol ,

$$|\eta(x,y)|^2=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{(rs)^x}\left[(-1)^{r-1}\cos(y\log r)(-1)^{s-1}\cos(y\log s)\right.$$
$$\left.+(-1)^{r-1}\sin(y\log r)(-1)^{s-1}\sin(y\log s)\right]$$

i.e.

$$|\eta(x,y)|^2=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\cos(y\log r)\cos(y\log s)+\sin(y\log r)\sin(y\log s)}{(rs)^x}$$

Here,

$$\cos(y\log r)\cos(y\log s)+\sin(y\log r)\sin(y\log s)=\cos\left(y\log\frac{s}{r}\right)$$

Using this,

$$|\eta(x,y)|^2=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \tag{3.2}$$

The first few lines are as follows.

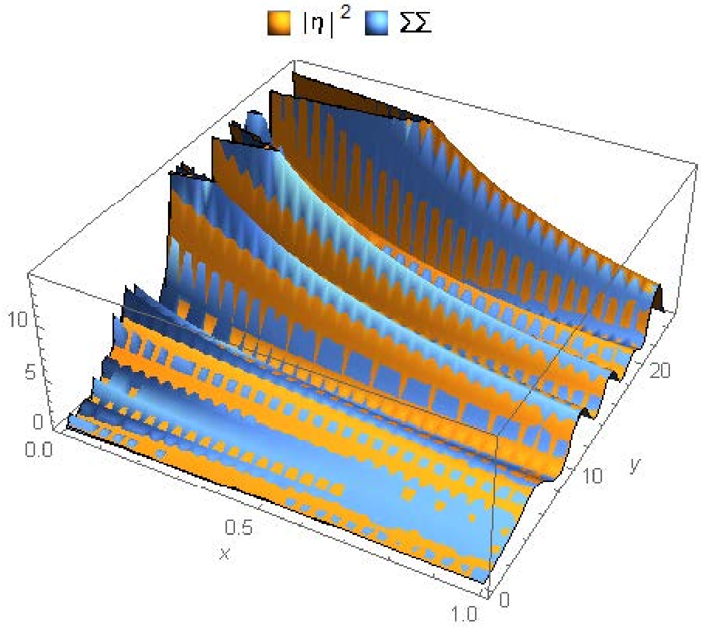
$$|\eta(x,y)|^2=\frac{1}{(1\cdot 1)^x}\cos\left(y\log\frac{1}{1}\right)-\frac{1}{(1\cdot 2)^x}\cos\left(y\log\frac{2}{1}\right)+\frac{1}{(1\cdot 3)^x}\cos\left(y\log\frac{3}{1}\right)-+\cdots$$

$$-\frac{1}{(2\cdot 1)^x}\cos\left(y\log\frac{1}{2}\right)+\frac{1}{(2\cdot 2)^x}\cos\left(y\log\frac{2}{2}\right)-\frac{1}{(2\cdot 3)^x}\cos\left(y\log\frac{3}{2}\right)+-\cdots$$

$$+\frac{1}{(3\cdot 1)^x}\cos\left(y\log\frac{1}{3}\right)-\frac{1}{(3\cdot 2)^x}\cos\left(y\log\frac{2}{3}\right)+\frac{1}{(3\cdot 3)^x}\cos\left(y\log\frac{3}{3}\right)-+\cdots$$

$$\vdots$$

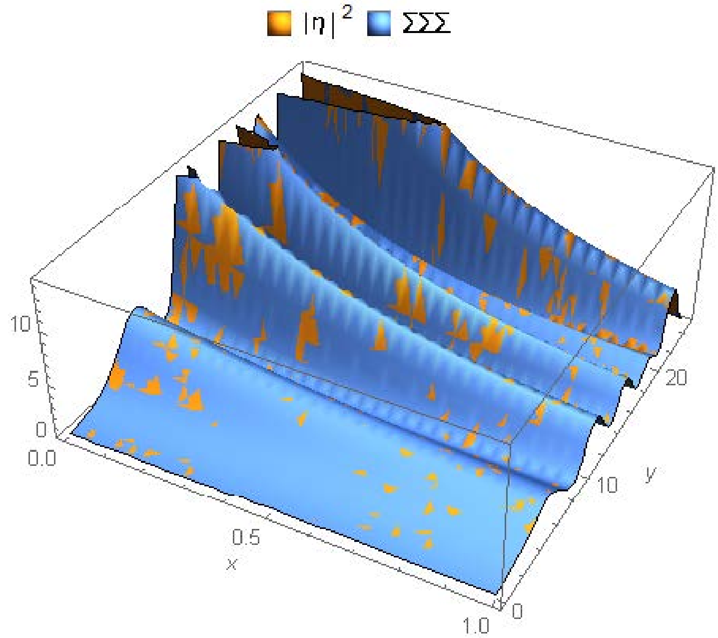
When both sides of (3.2) are illustrated together, it is as follows. Here, the upper limit of $\Sigma\Sigma$ is 300 x 300. Both sides almost coincide near $x=1$, but do not coincide near $x=0$.



Then, we attach the parallel accelerator (See " **13 Convergence Acceleration of Multiple Series** " (A la carte)) to the right side of (3.2) .

$$g(x,y,q) = \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{s=1}^{k}\frac{q^{k-r-s}}{(q+1)^{k+1}}\binom{k}{r+s}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \tag{3.2'}$$

When this is illustrated at $q=1/2$ and $m=40$, it is as follows. (m is the upper limit of Σ. Same as below.) Both sides overlap exactly and look like spots.

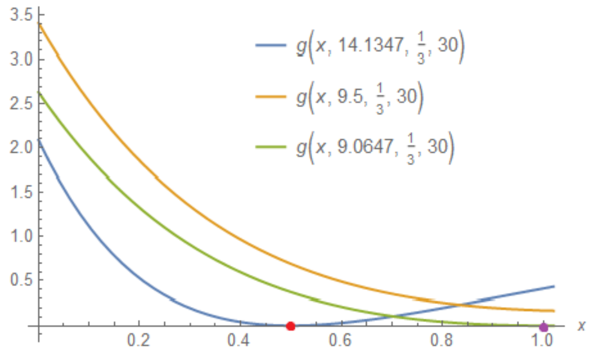


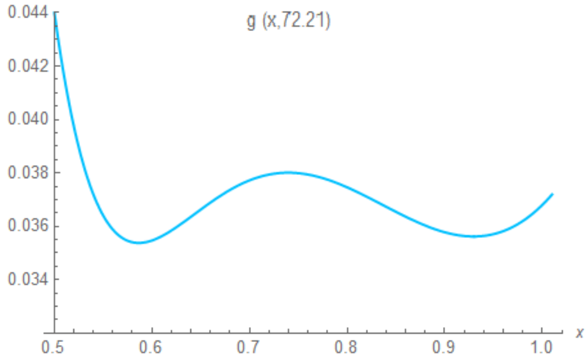
(3.2) and (3.2') are the same. (3.2) may be used where y is large, but (3.2') is required where y is small. .. Hereinafter, (3.2) is used for theoretical explanation, and (3.2') is used for drawing and calculation. Further, $|\eta(x,y)|^2$ is described as $g(x,y)$.

4.3.3 2D figures of $g(x,y)$

(1) Cutting figure at y

The left is a cut figure at $y = 14.1347\cdots$, 9.5 , $9.0647\cdots$ of $g(x,y)$, and the right is a cut figure at $y = 72.21$. The red points are zeros on the critical line and the purple points are η specific zeros.

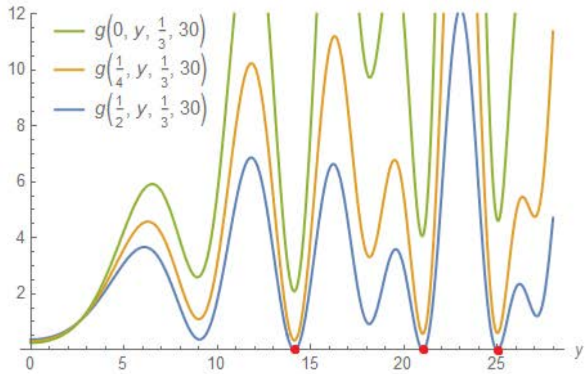


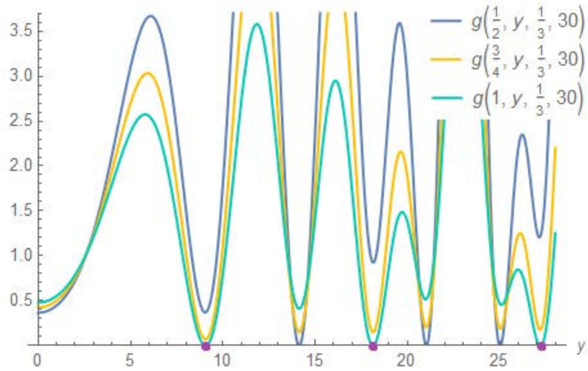


When observing both figures, it looks like $g(x,y) > 0$ at $x < 1/2$.

(2) Cutting figure at x

The left is a cut figure at $x = 0$, $1/4$, $1/2$ of $g(x,y)$, and the right is a cut figure at $x = 1/2$, $3/4$, 1 . The red points are zeros on the critical line and the purple points are η specific zeros.





In $x \le 1/2$, $y \ge 3$ (left figure) , $g(x,y)$ is inversely proportional to x , maximum at $x = 0$ and minimum ($= 0$) at $x = 1/2$. That is, $g(x,y)$ has no zero at $x < 1/2$, $y \ge 3$.

If the right figures of (1) and (2) are expressed analytically, the following hypothesis equivalent to the Riemann hypothesis is obtained.

Hypothesis 4.3.3

When $\eta(x,y)$ is the Dirichlet eta function on the complex plane, the following inequality holds.

$$g(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{(-1)^{r+s}}{(rs)^x} \cos\left(y \log \frac{s}{r}\right) > 0 \qquad for \quad \begin{array}{c} 0 < x < 1/2 \\ y \ge 3 \end{array} \tag{3.3}$$

Note

Since $g(x,y) \ge 0$, this proof only excludes the equal sign. This may be the easiest in this chapter.

4.4 Theorems at Zeros

As seen in the previous section , squared absolute value of Dirichlet eta function was expressed as follows.

$$|\eta(x,y)|^2 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \quad \{ := g(x,y) \} \tag{3.2}$$

The following theorems hold for the zeros of this double series.

Theorem 4.4.0

When $\eta(x,y)$ is Dirichlet Eta Function, if $\eta(a,b)=0$,

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^a}\cos\left(b\log\frac{s}{r}\right) = 0 \tag{4.0c}$$

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^a}\sin\left(b\log\frac{s}{r}\right) = 0 \tag{4.0s}$$

Proof

Since the left side of (4.0c) is the absolute value of $\eta(a,b)$, (4.0c) is natural. (4.0s) is proved at the end of this section.

Interestingly, at a zero point (a,b) of $\eta(x,y)$, each of these rows have to be all 0. Below, we state this as a theorem.

Theorem 4.4.1

When $\eta(x,y)$ is Dirichlet Eta Function, if $\eta(a,b)=0$,

$$\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^a}\cos\left(b\log\frac{s}{r}\right) = 0 \qquad for\ r=1,2,3,\cdots \tag{4.1c}$$

$$\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^a}\sin\left(b\log\frac{s}{r}\right) = 0 \qquad for\ r=1,2,3,\cdots \tag{4.1s}$$

Proof

Let C_r be the r th row of the double series (4.1c) . Then,

$$C_r = \sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^a}\cos\left(b\log\frac{s}{r}\right)$$

$$= \frac{(-1)^r}{r^a}\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\cos\left(b\log\frac{s}{r}\right)$$

Here,

$$\cos\left(b\log\frac{s}{r}\right) = \cos(b\log r)\cos(b\log s) + \sin(b\log r)\sin(b\log s)$$

Using this,

$$C_r = \frac{(-1)^r}{r^a}\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\{\cos(b\log r)\cos(b\log s)+\sin(b\log r)\sin(b\log s)\}$$

i.e.

$$C_r = \frac{(-1)^r}{r^a}\left\{\cos(b\log r)\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\cos(b\log s) + \sin(b\log r)\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\sin(b\log s)\right\}$$

At a zero point (a,b) of η ,

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\cos(b\log s) = 0 \quad , \quad \sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\sin(b\log s) = 0$$

Therefore, $C_r(a,b) = 0 \quad for\ r=1,2,3,\cdots$.

In a similar way, let S_r be the r th row of the double series (4.1s) . Then,

$$S_r = \sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^a}\sin\left(b\log\frac{s}{r}\right)$$

$$= \frac{(-1)^r}{r^a}\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\sin\left(b\log\frac{s}{r}\right)$$

Here,

$$\sin\left(b\log\frac{s}{r}\right) = \cos(b\log r)\sin(b\log s) - \sin(b\log r)\cos(b\log s)$$

Using this,

$$S_r = \frac{(-1)^r}{r^a}\left\{\cos(b\log r)\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\sin(b\log s) - \sin(b\log r)\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\cos(b\log s)\right\}$$

For the same reason as the above, $S_r(a,b) = 0 \quad for\ r=1,2,3,\cdots$ at a zero point (a,b) of $\eta(x,y)$.

From this, the following corollary follows.

Corollary 4.4.1'

When $\eta(x,y)$ is Dirichlet Eta Function, if $\eta(a,b)=0$,

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\cos\left(b\log\frac{s}{r}\right) = 0 \qquad for\ \ r=1,2,3,\cdots \tag{4.1c'}$$

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\sin\left(b\log\frac{s}{r}\right) = 0 \qquad for\ \ r=1,2,3,\cdots \tag{4.1s'}$$

Putting $\theta = -b\log r$ in this corollary, we obtain the following.

Corollary 4.4.1"

When $\eta(x,y)$ is Dirichlet Eta Function, if $\eta(a,b)=0$, the following expressions hold for arbitrary real number θ .

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\cos(b\log s+\theta) = 0 \tag{4.1c"}$$

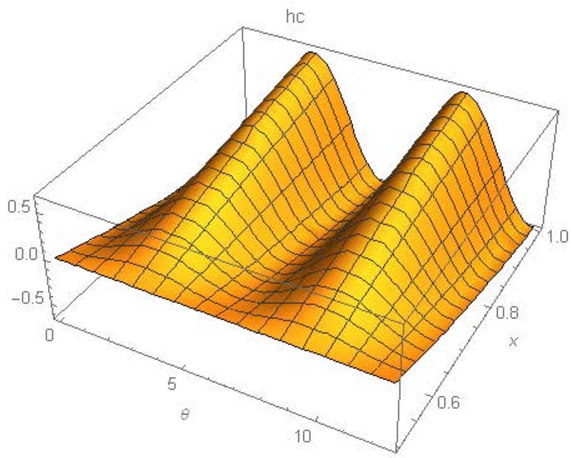
$$\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\sin(b\log s+\theta) = 0 \tag{4.1s"}$$

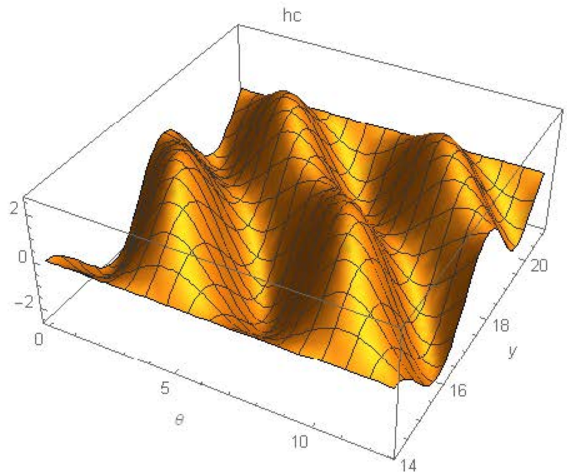
Example

We illustrate (4.1c"). Since the convergence of this left side is slow and it is difficult to draw an accurate figure, we apply Knopp transformation (See " **10 Convergence Acceleration & Summation Method** " (A la carte)) to this as follows.

$$h_c(x,y,\theta,q,m) = \sum_{k=1}^{m}\sum_{s=1}^{k}\frac{q^{k-s}}{(q+1)^{k+1}}\binom{k}{s}\frac{(-1)^s}{s^x}\cos(y\log s+\theta)$$

Here, this is illustrated at $q=1/3$ and $m=20$.





The left figure is a cutaway view at $y=14.1347\cdots$ when $x=1/2$. We can see that $h_c=0$ for any θ in this cutting surface.

The right figure is a cutaway view at $y=14.1347\cdots$ and $y=21.0220\cdots$ when $x=1/2$. We can see that $h_c=0$ for any θ in these cutting surfaces. It is surprising that the contour lines appear innumerably in such a twisted figure.

Note

However, Corollary 4.4.1" can be obtained directly and easily using the angle sum identities of trigonometric functions. When R is a set of real numbers, the following equation holds for any $\theta \in R$.

$$\cos(\theta+b\log s) = \cos\theta\cos(b\log s) - \sin\theta\sin(b\log s)$$

Multiplying $(-1)^{s-1}/s^a$ on both sides and summing up from 1 to ∞ for s,

$$\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^a}\cos(\theta+b\log s) = \sum_{s=1}^{\infty}\frac{(-1)^{s}}{s^a}\{\cos\theta\cos(b\log s) - \sin\theta\sin(b\log s)\}$$

$$= \cos\theta\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^a}\cos(b\log s) - \sin\theta\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^a}\sin(b\log s)$$

Since, (a,b) is a zero of $\eta(x,y)$,

$$\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^a}\cos(b\log s) = 0 \quad \& \quad \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^a}\sin(b\log s) = 0$$

Substituting this for the rght side

$$\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^a}\cos(\theta+b\log s) = 0 \qquad \forall\theta \in R \tag{4.1c"}$$

Replacing θ with $\theta+\pi/2$,

$$\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^a}\sin(\theta+b\log s) = 0 \qquad \forall\theta \in R \tag{4.1s"}$$

Using Corollary 4.4.1', we obtain the following important theorem.

Theorem 4.4.2

When $\eta(x,y)$ is Dirichlet Eta Function and $c(r)$ is arbitrary real valued function, if $\eta(a,b)=0$, the following expressions hold.

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{c(r)}{(rs)^{a}}\cos\left(b\log\frac{s}{r}\right)=0 \tag{4.2c}$$

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{c(r)}{(rs)^{a}}\sin\left(b\log\frac{s}{r}\right)=0 \tag{4.2s}$$

Proof

From Corollary 4.4.1' (4.1c') ,

$$\sum_{s=1}^{\infty}\frac{(-1)^{s}}{s^{a}}\cos\left(b\log\frac{s}{r}\right)=0 \qquad for\ \ r=1,2,3,\cdots$$

Multiplying both sides by $(-1)^{r}c(r)/r^{a}$,

$$\sum_{s=1}^{\infty}(-1)^{r+s}\frac{c(r)}{(rs)^{a}}\cos\left(b\log\frac{s}{r}\right)=0 \qquad for\ \ r=1,2,3,\cdots$$

summing up for r , we obtain (4.2c) . In a similar way, (4.2s) is obtained.

Proof of Theorem 4.4.0 (4.0s)

Particularly placed $c(r)=1$ in Theorem 4.4.2 (4.2s) , (4.0s) is obtained.

4.5 Partial Derivatives of Squared Absolute Value

4.5.1 First order Partial Derivatives

Formula 4.5.1

When squared absolute value of Dirichlet eta function is

$$g(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \qquad \left(=|\eta(x,y)|^2\right) \tag{3.2}$$

The 1st order partial derivatives are givern as follows.

$$g_x = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \tag{5.1x}$$

$$g_y = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) \tag{5.1y}$$

Proof

Differentiating (3.2) with respect to x

$$g_x = -\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(rs)}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \tag{5.0x}$$

$$= -\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) - \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log s}{(rs)^x}\cos\left(y\log\frac{s}{r}\right)$$

Swapping r and s in the 2nd term on the right side,

$$-\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log s}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) = -\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{s+r}\frac{\log r}{(sr)^x}\cos\left(y\log\frac{r}{s}\right)$$

$$= -\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right)$$

$$\{\because \cos(-z) = \cos z\}$$

Substituting this for the 2nd term on the right side, we obtain (5.1x) .

Next, differentiating (3.2) with respect to y

$$g_y = -\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(s/r)}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) \tag{5.0y}$$

$$= -\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log s}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) + \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\sin\left(y\log\frac{s}{r}\right)$$

Swapping r and s in the 1st term on the right side,

$$-\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log s}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) = -\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{s+r}\frac{\log r}{(sr)^x}\sin\left(y\log\frac{r}{s}\right)$$

$$= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\sin\left(y\log\frac{s}{r}\right)$$

$$\{\because \sin(-z) = -\sin z\}$$

Substituting this for the 1st term on the right side, we obtain (5.1y) .

4.5.2 Second order Partial Derivatives

Formula 4.5.2

When squared absolute value of Dirichlet eta function is

$$g(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \qquad \left(= |\eta(x,y)|^2\right) \tag{3.2}$$

The 2nd order partial derivatives are givern as follows.

$$g_{xx} = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r\log s + \log^2 r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \tag{5.xx}$$

$$g_{xy} = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2 r}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) \qquad (= g_{yx}) \tag{5.xy}$$

$$g_{yy} = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r\log s - \log^2 r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \tag{5.yy}$$

Proof

The first-order partial derivatives in the proof of Formula 4.5.1 are

$$g_x = -\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(rs)}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \tag{5.0x}$$

$$g_y = -\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(s/r)}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) \tag{5.0y}$$

Differentiating these with respect to x,y ,

$$\begin{aligned}
g_{xx} &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2(rs)}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \qquad \left(=\frac{\partial^2}{\partial x^2}|\eta|^2\right)\\
&= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2 r + 2\log r\log s + \log^2 s}{(rs)^x}\cos\left(y\log\frac{s}{r}\right)\\
&= 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r\log s + \log^2 r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \qquad \{\because \cos(-z) = \cos z\}
\end{aligned}$$

$$\begin{aligned}
g_{xy} &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log(rs)\log(s/r)}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) \qquad \left(=\frac{\partial^2}{\partial x\partial y}|\eta|^2\right)\\
&= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2 s}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) - \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2 r}{(rs)^x}\sin\left(y\log\frac{s}{r}\right)\\
&= \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{s+r}\frac{\log^2 r}{(sr)^x}\sin\left(y\log\frac{r}{s}\right) - \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2 r}{(rs)^x}\sin\left(y\log\frac{s}{r}\right)\\
&= -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2 r}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) \qquad \{\because \sin(-z) = -\sin z\}
\end{aligned}$$

$$g_{yy} = -\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2(s/r)}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \qquad \left(=\frac{\partial^2}{\partial y^2}|\eta|^2\right)$$

$$= -\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{log^2 r - 2log\,r\,log\,s + log^2 s}{(rs)^x}cos\left(y\,log\frac{s}{r}\right)$$

$$= 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{log\,r\,log\,s - log^2 r}{(rs)^x}cos\left(y\,log\frac{s}{r}\right) \qquad \{\because cos(-z) = cos\,z\}$$

4.6 Partial Differential Coefficients at Zeros

In this section, we consider what the value of the partial derivative obtained in the previous section is at the zeros of Dirichlet Eta Function $\eta(x,y)$.

4.6.1 1st-order Partial Differential Coefficients

Theorem 4.6.1

When (a,b) is a zero of Dirichlet eta function $\eta(x,y)$, the following equations hold for the partial derivatives in Formula 4.5.1 .

$$g_x(a,b) = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^a}\cos\left(b\log\frac{s}{r}\right) = 0 \tag{6.1x}$$

$$g_y(a,b) = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^a}\sin\left(b\log\frac{s}{r}\right) = 0 \tag{6.1y}$$

Proof

Putting $c(r)=\log r$ in Theorem 14.4.2 , we obtain the desired expressions immediately.

Note

This theorem is called "stationary condition" in the extreme value problem.

4.6.2 2nd-order Partial Differential Coefficients

Theorem 4.6.2

When (a,b) is a zero of Dirichlet eta function $\eta(x,y)$, the following equations hold for the partial derivatives in Formula 4.5.2 .

$$g_{xx}(a,b) = g_{yy}(a,b) = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r\log s}{(rs)^a}\cos\left(b\log\frac{s}{r}\right) > 0 \tag{6.xx}$$

$$g_{xy}(a,b) = g_{yx}(a,b) = 0 \tag{6.xy}$$

Proof

Substituting zero point (a,b) for (5.xx) , (5.xy) , (5.yy) in Formula 4.5.2 ,

$$g_{xx}(a,b) = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r\log s+\log^2 r}{(rs)^a}\cos\left(b\log\frac{s}{r}\right)$$

$$g_{xy}(a,b) = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2 r}{(rs)^a}\sin\left(b\log\frac{s}{r}\right) \qquad \{\ = g_{yx}(a,b)\ \}$$

$$g_{yy}(a,b) = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r\log s-\log^2 r}{(rs)^a}\cos\left(b\log\frac{s}{r}\right)$$

Since $\log^2 r$ is a real valued function with respect to r , from Theorem 4.4.2 ,

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2 r}{(rs)^a}\cos\left(b\log\frac{s}{r}\right) = 0$$

Substituting this for the aboves,

$$g_{xy}(a,b) = 0 \qquad \{\ = g_{yx}(a,b)\ \} \tag{6.xy}$$

$$g_{xx}(a,b) = g_{yy}(a,b) = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r\log s}{(rs)^a}\cos\left(b\log\frac{s}{r}\right)$$

Here, zero (a,b) of $\eta(x,y)$ is the minimum value ($=0$) of the following expression.

$$g(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{(-1)^{r+s}}{(rs)^x} cos\left(y\,log\frac{s}{r}\right) \tag{3.2}$$

So, $g_x(x,b)$ changes from negative to positive before and after a , and $g_y(a,y)$ changes from negative to positive before and after b . Thus, it has to be $g_{xx}(a,b)>0$, $g_{yy}(a,b)>0$.

Extreme Value Judgment by Hessian Matrix

Generally, in the extreme value problem, the extreme value is determined by the Hessian matrix as follows.

$$g_{xx}(a,b) \gtrless 0$$

$$g_{xx}(a,b)\, g_{yy}(a,b) - \{g_{xy}(a,b)\}^2 \gtrless 0$$

In the case of (3.2), if $g(a,b)>0$ this judgment is necessary. However, if $g(a,b)=0$ this judgment is unnecessary. Because, $g(a,b)=0$ is the minimum value of $g(x,y)\ge 0$.

Even so, if we dare to make a judgment, these are as follows from Theorem 4.6.2 .

$$g_{xx}(a,b) > 0$$

$$g_{xx}(a,b)\, g_{yy}(a,b) - \{g_{xy}(a,b)\}^2 = g_{xx}(a,b)^{\,2} > 0$$

That is, $g(a,b)$ is determined to be the local minimum of $g(x,y)$.

Example $a=1/2$, $b=14.134725\cdots$

$$g_{xx}(a,b) = g_{yy}(a,b) = 7.089093\cdots$$

Note

The two theorems in this section generally hold for holomorphic complex functions with zeros. The proof can be easily done using Cauchy-Riemann equation and Laplace'equation.

4.7 Geometric Relationships between Functions

In this section, we consider the geometrical relationship between squared absolute value of Dirichlet eta function and the partial derivatives a little analytically.

4.7.1 $g(x,y)$ and $g_x(x,y)$

Theorem 4.7.1

Let real valued function with two variables $g(x,y)$, $g_x(x,y)$ are as follows respectively.

$$g(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \qquad \{ = |\eta(x,y)|^2 \} \tag{3.2}$$

$$g_x(x,y) = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \tag{5.1x}$$

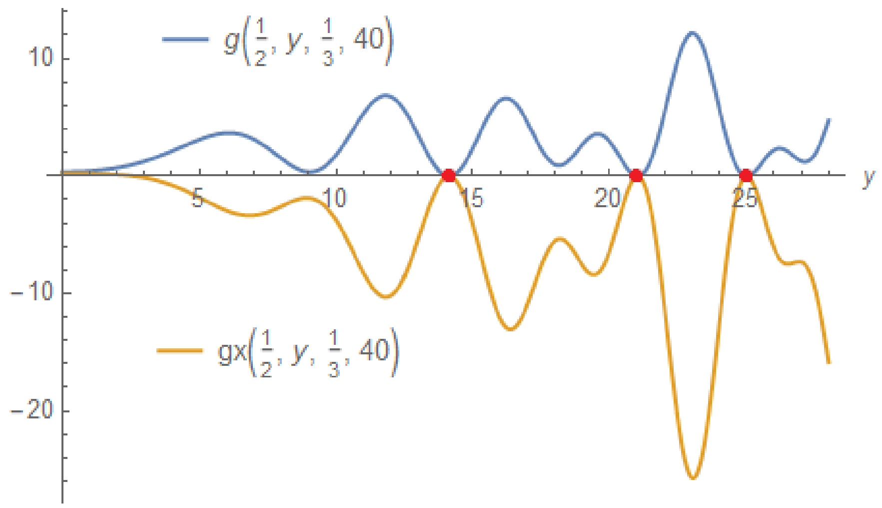
Then, when (a,b) is a zero of Dirichlet eta function $\eta(x,y)$,

(1) b is the common root of $g(a,y)=0$, $g_x(a,y)=0$.

(2) b is at least a multiple root in both $g(a,y)=0$, $g_x(a,y)=0$.

(3) $g(a,y)$ and $g_x(a,y)$ are almost symmetric with respect to the y -axis.

g and g_x on the critical line (Red point is b)



Proof

(1) $g(a,b) = |\eta(a,b)|^2 = 0$. On the other hand, $g_x(a,b) = 0$ according to Theorem 4.6.1 . So, (a,b) is the common root of $g(x,y)$ and $g_x(x,y)$,

(2) Since $g(x,y) = |\eta(x,y)|^2$, the root of $g(a,y)=0$ is at least multiple root. On the other hand, since $g_{xy}(a,b)=0$ from Theorem 4.6.2 , $g_x(a,b)$ has to be an extremum with respect to y .
Therefore, the root b of $g_x(a,y)=0$ is also at least a multiple root.

(3) Expanding $\cos\{y\log(s/r)\}$ to the Maclaurin series,

$$\cos\left(y\log\frac{s}{r}\right) = \sum_{t=0}^{\infty}(-1)^t\frac{\log^{2t}(s/r)}{(2t)!}y^{2t}$$

Using this, $g(x,y)$, $g_x(x,y)$ are expressed as follows, respectively.

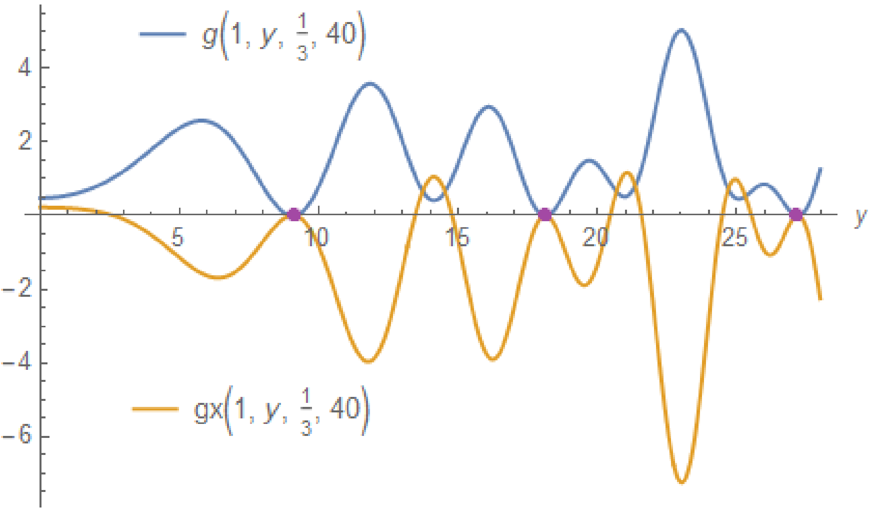
$$g(x,y) = \sum_{t=0}^{\infty}\left\{\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s+t}\frac{1}{(rs)^x}\frac{\log^{2t}(s/r)}{(2t)!}\right\}y^{2t} \tag{7.1}$$

$$g_x(x,y) = -2\sum_{t=0}^{\infty}\left\{\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s+t}\frac{\log r}{(rs)^x}\frac{\log^{2t}(s/r)}{(2t)!}\right\}y^{2t} \tag{7.1x}$$

Both (7.1) and (7.1x) are even functions with respect to y , and have similar shapes except for the sign. However, the two have different signs. So, $g_x(a,b)$ has to be a local maximum with respect to the y . If (1),(2) are added to this, $g(a,y)$ and $g_x(a,y)$ are almost symmetric with respect to the y -axis.

Dirichlet eta function has η specific zeros at $x=1$. The 2D figure of this is as follows. The difference from the zeros on the critical line is that b is not the maximum point of $g_x(a,y)$ but the local maximum point.

g and g_x on $x=1$ (Purple point is b)



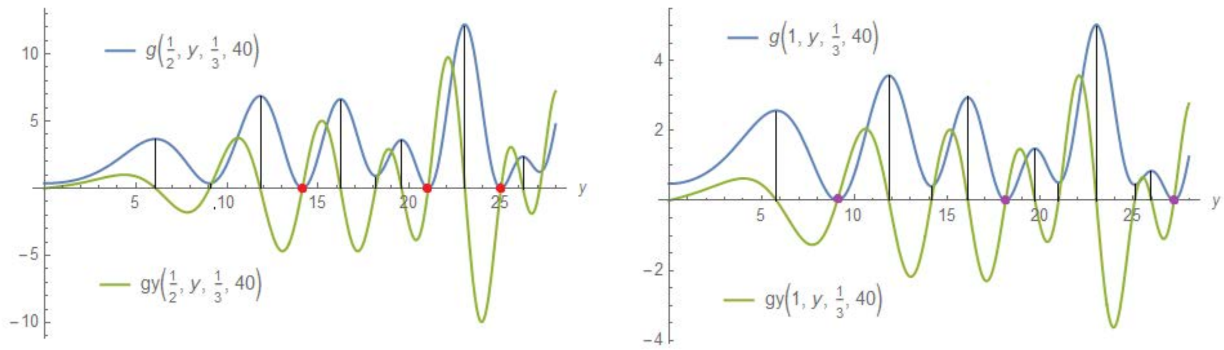
4.7.2 $g(x,y)$ and $g_y(x,y)$

These are the functions expressed by the following equations.

$$g(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \qquad \{=|\eta(x,y)|^2\} \tag{3.2}$$

$$g_y(x,y) = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) \tag{5.1y}$$

These cutting figure at $x=1/2$ & $x=1$ are as follows. The left is a cut figure at $x=1/2$ and the left is a cut figure at $x=1$. The red points are zeros on the critical line and the purple points are η specific zeros.



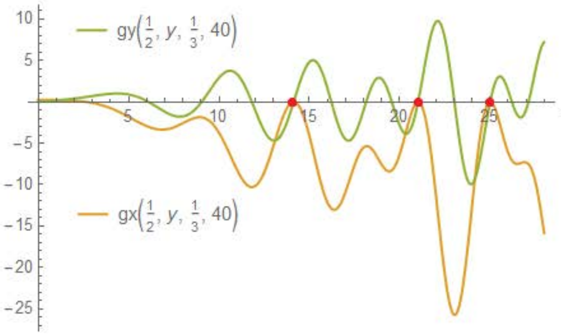
The importance of this pair is that g_y is a derivative of g with respect to y. That is, on the y -axis, the extremum of g corresponds to the zero of g_y. The black vertical lines show that. Therefore, they have to match at zero (a,b) of $\eta(x,y)$. It is according to Theorem 4.6.1 . Further, the local minimum points of g are the uphill zeros of g_y. Theorem 4.6.2 $g_{yy}(a,b) > 0$ supports this.

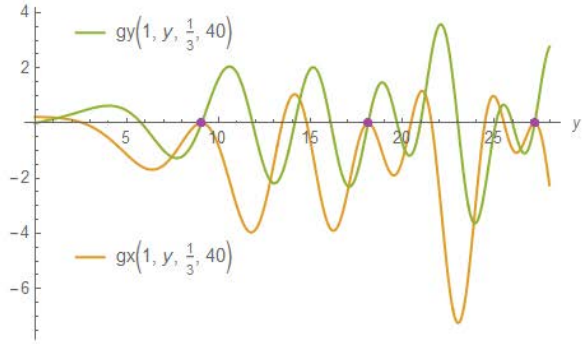
4.7.3 $g_x(x,y)$ and $g_y(x,y)$

$$g_x(x,y) = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \tag{5.1x}$$

$$g_y(x,y) = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) \tag{5.1y}$$

These cutting figure at $x=1/2$ & $x=1$ are as follows. The left is a cut figure at $x=1/2$ and the left is a cut figure at $x=1$. The red points are zeros on the critical line and the purple points are η specific zeros.





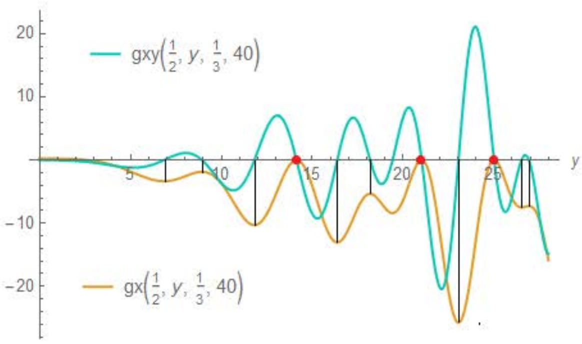
As seen in Theorem 4.7.1 , g_x shares zeros with g on the y -axis, and they are local maximum. And $g_x = g_y = 0$ from Theorem 4.6.1 .Thus, the red and purple dots are their zeros. And these shared zeros of g_x and g_y are always the uphill zeros of g_y. (i.e. $g_{yy}(a,b) > 0$)

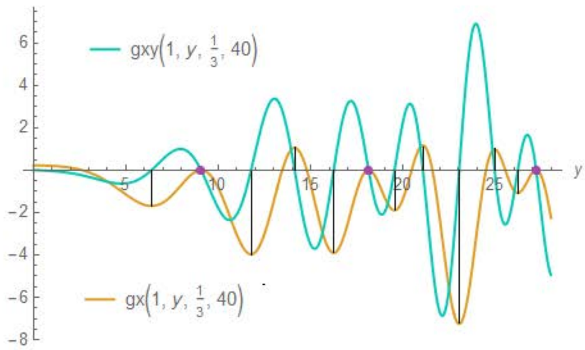
4.7.4 $g(x,y)$ and $g_{xy}(x,y)$

$$g_x(x,y) = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \tag{5.1x}$$

$$g_{xy}(x,y) = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2 r}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) \qquad \{= g_{yx}(x,y)\} \tag{5.xy}$$

These cutting figure at $x=1/2$ & $x=1$ are as follows. The left is a cut figure at $x=1/2$ and the left is a cut figure at $x=1$. The red points are zeros on the critical line and the purple points are η specific zeros.





g_{xy} is a derivative of g_x with respect to y. So, the extremums of g_x correspond to the zeros of g_{xy}.

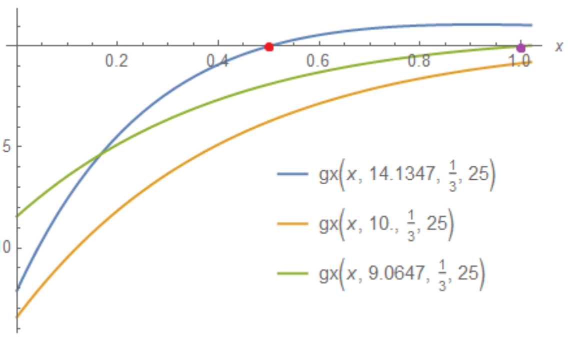
The black vertical lines show that. Further, the local maximum points of g_x coincide with the downhill zeros of g_{xy}.

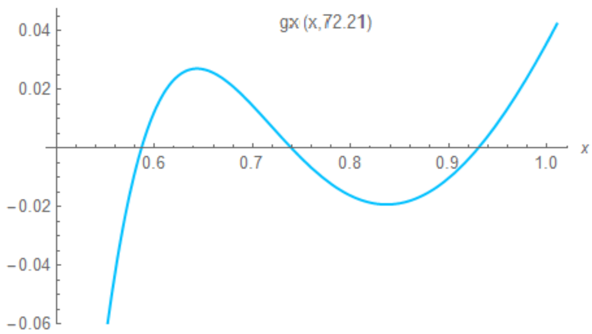
4.7.4 2D figures of $g_x(x,y)$

Of the first-order partial derivatives, the one of particular interest is g_x. So let us observe this in a little more detail.

(1) Cutting figure at y

The left is a cut figure at $y = 14.1347\cdots$, 10 , $9.0647\cdots$ of $g_x(x,y)$, and the right is a cut figure at $y = 72.21$. The red points are zeros on the critical line and the purple points are η specific zeros.

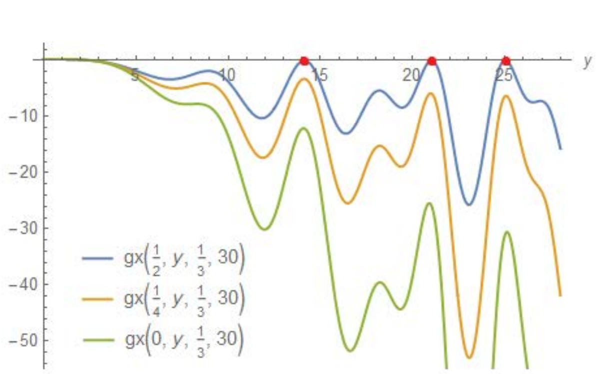


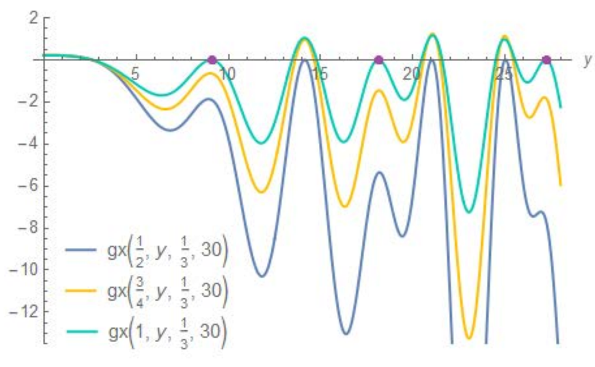


Observing both figures, it seems that $g_x(x,y) < 0$ at $x < 1/2$.

(2) Cutting figure at x

The left is a cut figure at $x = 0$, $1/4$, $1/2$ of $g_x(x,y)$, and the right is a cut figure at $x = 1/2$, $3/4$, 1 . The red points are zeros on the critical line and the purple points are η specific zeros.





In $x \le 1/2$, $y \ge 3$ (left figure), $|g_x(x,y)|$ is inversely proportional to x , minimum at $x = 1/2$ and maximum at $x = 0$. That is, $g_x(x,y)$ has no zero in $x \le 1/2$, $y \ge 3$.

If the left figures of (1) and (2) are expressed analytically, the following hypothesis is obtained, which is equivalent to the Riemann hypothesis.

Hypothesis 4.7.5

When $\eta(x,y)$ is the Dirichlet eta function on the complex plane, the following inequality holds.

$$g_x(x,y) = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) < 0 \qquad for \quad \begin{array}{l} 0 < x < 1/2 \\ y \ge 3 \end{array} \tag{7.5}$$

Note

This hypothesis excludes the possibility of $g_x(x,y) \geq 0$ in the critical strip. Unlike Hypothesis 4.3.3 , not only " $=$ " but " $>$ " must be excluded. It may be more difficult than the proof of Hypothesis 4.3.3 .

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