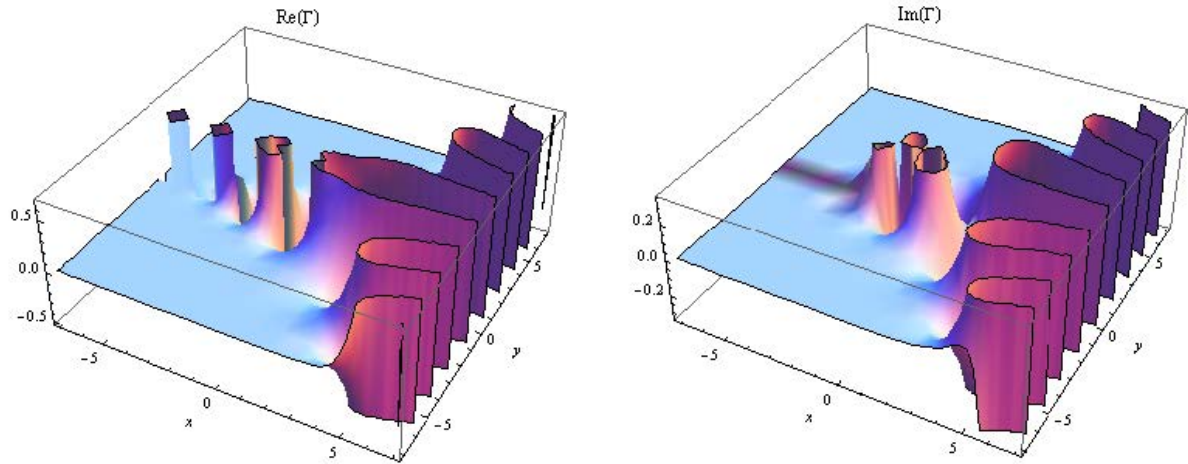
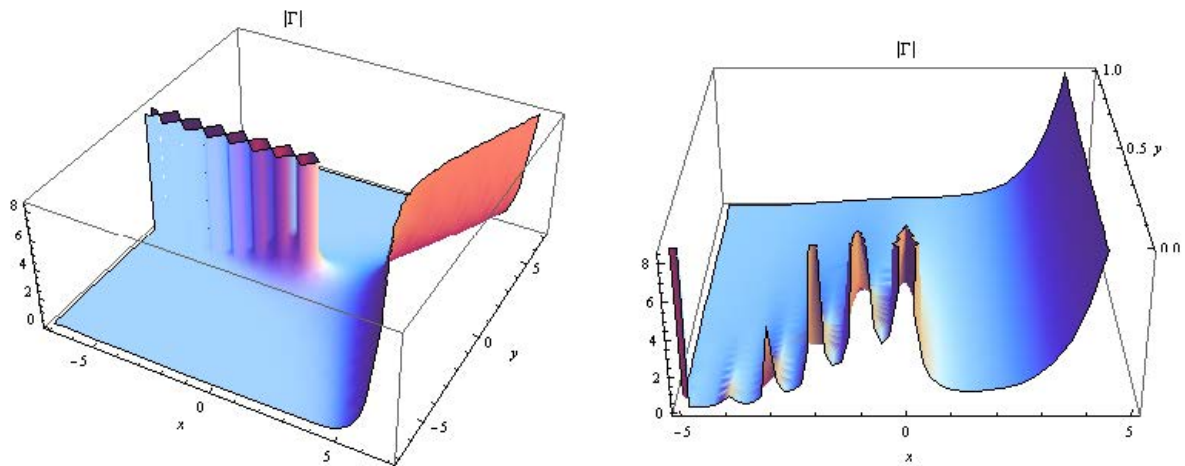


## 09 Absolute Value of Gamma Function

About the gamma function in the complex plane, both the real part and the imaginary part have very complicated shapes, as shown in the following figure.



However, when the attention is paid to the absolute value, especially in the positive half planes, it seems to have a simple shape. In this chapter, we examine this analytically.



## 9.1 Some infinite products

### Formula 9.1.1

$$\prod_{r=1}^{\infty} \left(1 + \frac{y}{r+x}\right) \left(1 - \frac{y}{r-x}\right) = \frac{\Gamma(1+x)\Gamma(1-x)}{\Gamma(1+x+y)\Gamma(1-x-y)} \quad (1.1)$$

$$\prod_{r=1}^{\infty} \left(1 + \frac{y}{r-x}\right) \left(1 - \frac{y}{r+x}\right) = \frac{\Gamma(1+x)\Gamma(1-x)}{\Gamma(1+x-y)\Gamma(1-x+y)} \quad (1.2)$$

$$\prod_{r=1}^{\infty} \left(1 + \frac{y}{r+x}\right) \left(1 - \frac{y}{r+x}\right) = \frac{\Gamma^2(1+x)}{\Gamma(1+x+y)\Gamma(1+x-y)} \quad (1.3)$$

$$\prod_{r=1}^{\infty} \left(1 + \frac{y}{r-x}\right) \left(1 - \frac{y}{r-x}\right) = \frac{\Gamma^2(1-x)}{\Gamma(1-x+y)\Gamma(1-x-y)} \quad (1.4)$$

**Proof**

According to "岩波 数学公式 II" p86,

$$\prod_{r=1}^{\infty} \left( 1 + \frac{y}{r\pi-x} \right) \left( 1 - \frac{y}{r\pi+x} \right) = \frac{\sin(x-y)}{\sin x} / \left( 1 - \frac{y}{x} \right)$$

Replacing  $x, y$  with  $\pi x, \pi y$  respectively,

$$\prod_{r=1}^{\infty} \left( 1 + \frac{y}{r-x} \right) \left( 1 - \frac{y}{r+x} \right) = \frac{\sin(\pi x - \pi y)}{\sin \pi x} / \left( 1 - \frac{y}{x} \right)$$

Replacing  $y$  with  $-y$ ,

$$\prod_{r=1}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r-x} \right) = \frac{\sin(\pi x + \pi y)}{\sin \pi x} / \left( 1 + \frac{y}{x} \right)$$

Multiplying both equations,

$$\begin{aligned} \left\{ \prod_{r=1}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r-x} \right) \right\} \left\{ \prod_{r=1}^{\infty} \left( 1 + \frac{y}{r-x} \right) \left( 1 - \frac{y}{r+x} \right) \right\} \\ = \frac{\sin(\pi x - \pi y) \sin(\pi x + \pi y)}{\sin^2 \pi x} / \left( 1 - \frac{y^2}{x^2} \right) \end{aligned} \quad (1.0)$$

Here,

$$\sin(\pi x + \pi y) = \frac{\pi(x+y)}{\Gamma(1+x+y) \Gamma(1-x-y)}$$

$$\sin(\pi x - \pi y) = \frac{\pi(x-y)}{\Gamma(1+x-y) \Gamma(1-x+y)}, \quad \sin \pi x = \frac{\pi x}{\Gamma(1+x) \Gamma(1-x)}$$

From these ,

$$\frac{\sin(\pi x + \pi y) \sin(\pi x - \pi y)}{\sin^2 \pi x} = \frac{\frac{x+y}{\Gamma(1+x+y) \Gamma(1-x-y)}}{\frac{x}{\Gamma(1+x) \Gamma(1-x)}} \cdot \frac{\frac{x-y}{\Gamma(1+x-y) \Gamma(1-x+y)}}{\frac{x}{\Gamma(1+x) \Gamma(1-x)}}$$

i.e.

$$\begin{aligned} \frac{\sin(\pi x + \pi y) \sin(\pi x - \pi y)}{\sin^2 \pi x} &= \frac{\Gamma(1+x) \Gamma(1-x)}{\Gamma(1+x+y) \Gamma(1-x-y)} \cdot \frac{\Gamma(1+x) \Gamma(1-x)}{\Gamma(1+x-y) \Gamma(1-x+y)} \\ &\quad \times \left( 1 - \frac{y^2}{x^2} \right) \end{aligned}$$

Substituting these for (1.0) ,

$$\begin{aligned} \left\{ \prod_{r=1}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r-x} \right) \right\} \left\{ \prod_{r=1}^{\infty} \left( 1 + \frac{y}{r-x} \right) \left( 1 - \frac{y}{r+x} \right) \right\} \\ = \frac{\Gamma(1+x) \Gamma(1-x)}{\Gamma(1+x+y) \Gamma(1-x-y)} \cdot \frac{\Gamma(1+x) \Gamma(1-x)}{\Gamma(1+x-y) \Gamma(1-x+y)} \end{aligned} \quad (1.0')$$

From this,

$$\prod_{r=1}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r-x} \right) = \frac{\Gamma(1+x) \Gamma(1-x)}{\Gamma(1+x+y) \Gamma(1-x-y)} \quad (1.1)$$

$$\prod_{r=1}^{\infty} \left( 1 + \frac{y}{r-x} \right) \left( 1 - \frac{y}{r+x} \right) = \frac{\Gamma(1+x) \Gamma(1-x)}{\Gamma(1+x-y) \Gamma(1-x+y)} \quad (1.2)$$

Next, we rearrange (1.0) as follows.

$$\begin{aligned}
 & \left\{ \prod_{r=1}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r+x} \right) \right\} \left\{ \prod_{r=1}^{\infty} \left( 1 + \frac{y}{r-x} \right) \left( 1 - \frac{y}{r-x} \right) \right\} \\
 &= \frac{\Gamma(1+x)\Gamma(1+x)}{\Gamma(1+x+y)\Gamma(1+x-y)} \cdot \frac{\Gamma(1-x)\Gamma(1-x)}{\Gamma(1-x+y)\Gamma(1-x-y)} \\
 &= \frac{\Gamma^2(1+x)}{\Gamma(1+x+y)\Gamma(1+x-y)} \cdot \frac{\Gamma^2(1-x)}{\Gamma(1-x+y)\Gamma(1-x-y)} \quad (1.0'')
 \end{aligned}$$

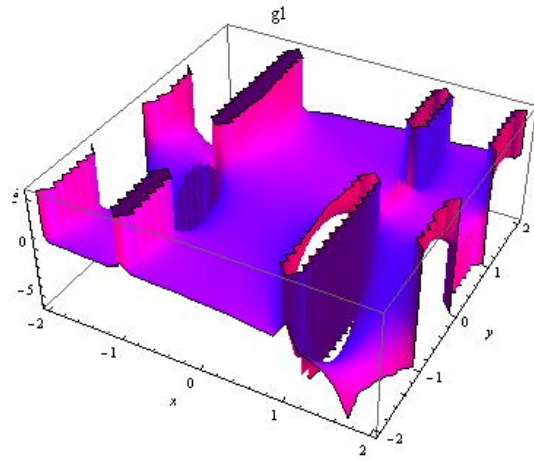
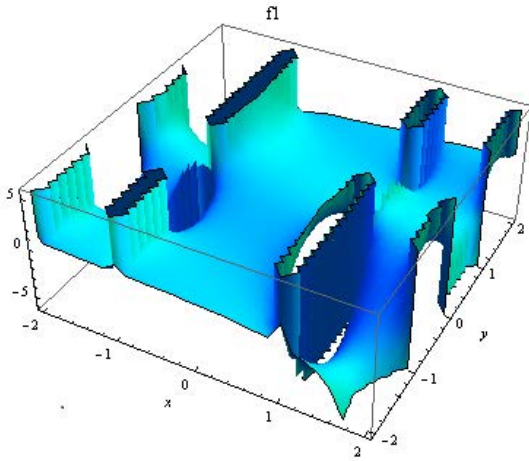
From this,

$$\prod_{r=1}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r+x} \right) = \frac{\Gamma^2(1+x)}{\Gamma(1+x+y)\Gamma(1+x-y)} \quad (1.3)$$

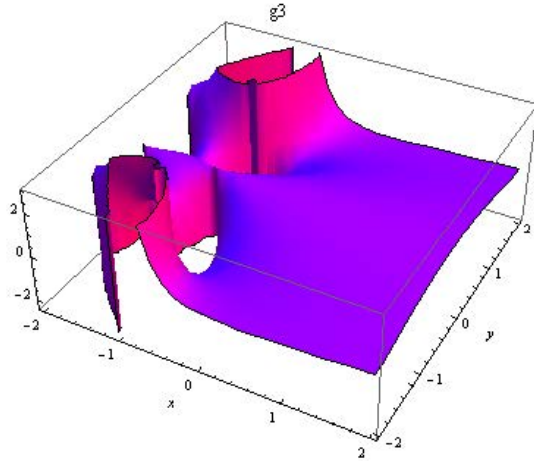
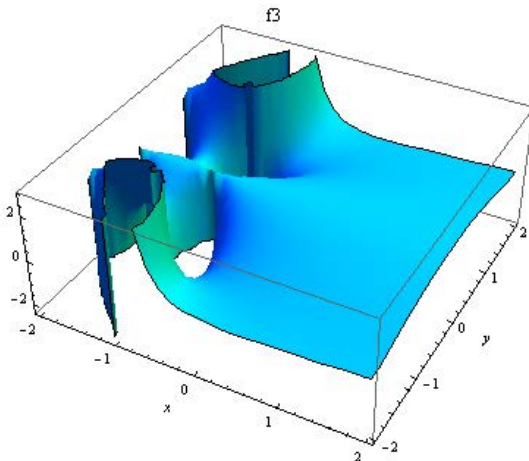
$$\prod_{r=1}^{\infty} \left( 1 + \frac{y}{r-x} \right) \left( 1 - \frac{y}{r-x} \right) = \frac{\Gamma^2(1-x)}{\Gamma(1-x+y)\Gamma(1-x-y)} \quad (1.4)$$

**Example 1**  $\prod_{r=1}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r-x} \right) = \frac{\Gamma(1+x)\Gamma(1-x)}{\Gamma(1+x+y)\Gamma(1-x-y)}$

If we draw the left side with cyan and the right side with magenta, it is as follows. (Same in the following.)



**Example 2**  $\prod_{r=1}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r+x} \right) = \frac{\Gamma^2(1+x)}{\Gamma(1+x+y)\Gamma(1+x-y)}$



**Formula 9.1.1'**

$$\prod_{r=0}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r-x} \right) = \frac{\Gamma(x)\Gamma(-x)}{\Gamma(x+y)\Gamma(-x-y)} \quad (1.1')$$

$$\prod_{r=0}^{\infty} \left( 1 + \frac{y}{r-x} \right) \left( 1 - \frac{y}{r+x} \right) = \frac{\Gamma(x)\Gamma(-x)}{\Gamma(x-y)\Gamma(-x+y)} \quad (1.2')$$

$$\prod_{r=0}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r+x} \right) = \frac{\Gamma^2(x)}{\Gamma(x+y)\Gamma(x-y)} \quad (1.3')$$

$$\prod_{r=0}^{\infty} \left( 1 + \frac{y}{r-x} \right) \left( 1 - \frac{y}{r-x} \right) = \frac{\Gamma^2(-x)}{\Gamma(-x+y)\Gamma(-x-y)} \quad (1.4')$$

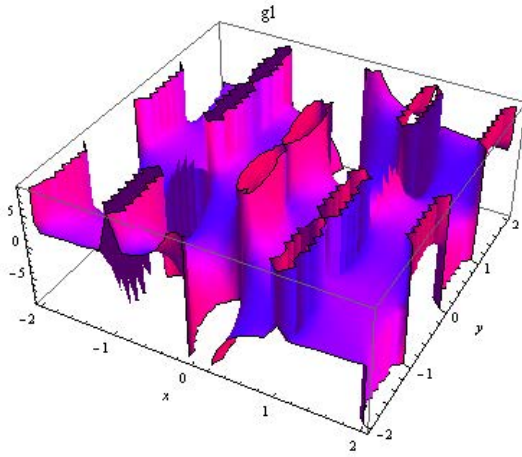
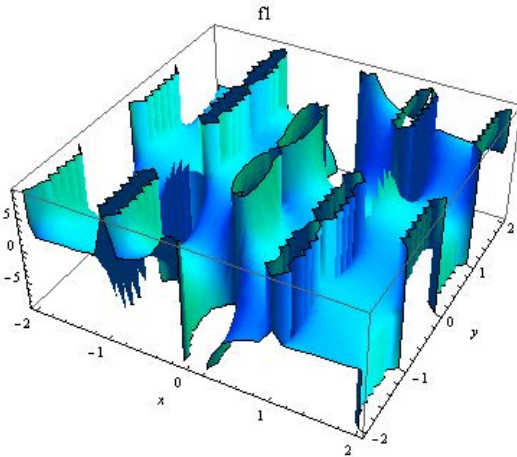
**Proof**

Multiplying by  $(1 + y/x)^2$  both sides of (1.1), we obtain (1.1').

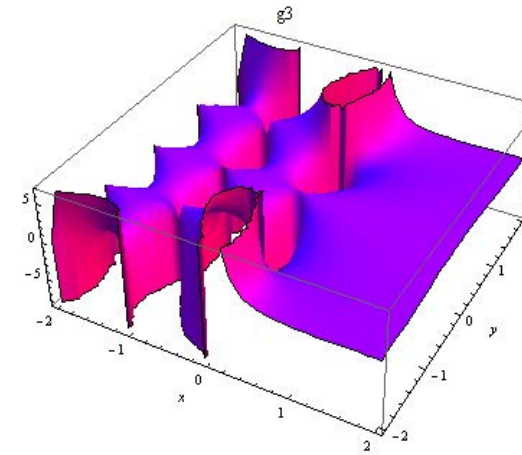
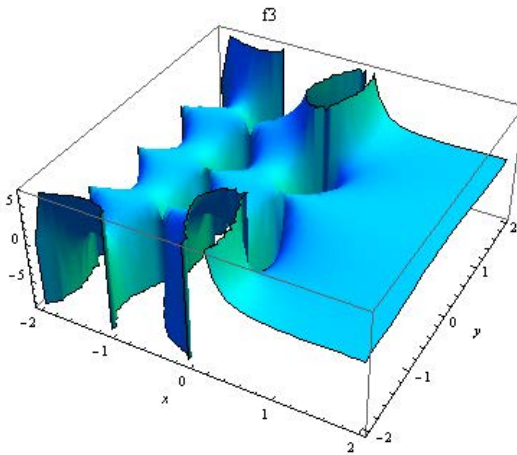
Multiplying by  $(1 - y/x)^2$  both sides of (1.2), we obtain (1.2').

Multiplying by  $1 - y^2/x^2$  both sides of (1.3) and (1.4), we obtain (1.3') and (1.4').

**Example 1'** 
$$\prod_{r=0}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r-x} \right) = \frac{\Gamma(x)\Gamma(-x)}{\Gamma(x+y)\Gamma(-x-y)}$$



**Example 2'** 
$$\prod_{r=0}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r+x} \right) = \frac{\Gamma^2(x)}{\Gamma(x+y)\Gamma(x-y)}$$



## 9.2 Square of the Absolute Value

### Formula 9.2.1

When  $x, y$  are real numbers and  $\Gamma(x+iy)$  is the gamma function in the complex plane,

$$|\Gamma(x+iy)|^2 = \Gamma^2(x) \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} \quad (1.1)$$

i.e.

$$|\Gamma(x+iy)| = |\Gamma(x)| \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1/2} \quad (1.1')$$

### Proof

Formula 9.1.1' (1.3') was as follows.

$$\prod_{r=0}^{\infty} \left( 1 + \frac{y}{r+x} \right) \left( 1 - \frac{y}{r+x} \right) = \frac{\Gamma^2(x)}{\Gamma(x+y)\Gamma(x-y)}$$

Replacing  $y$  with  $iy$ ,

$$\prod_{r=0}^{\infty} \left\{ 1 + \frac{y^2}{(r+x)^2} \right\} = \frac{\Gamma^2(x)}{\Gamma(x+iy)\Gamma(x-iy)}$$

Here, let  $z = x+iy$ ,  $\bar{z} = x-iy$ . Then, since the gamma function  $\Gamma(z)$  is known to have complex conjugate property  $\Gamma(\bar{z}) = \overline{\Gamma(z)}$ ,

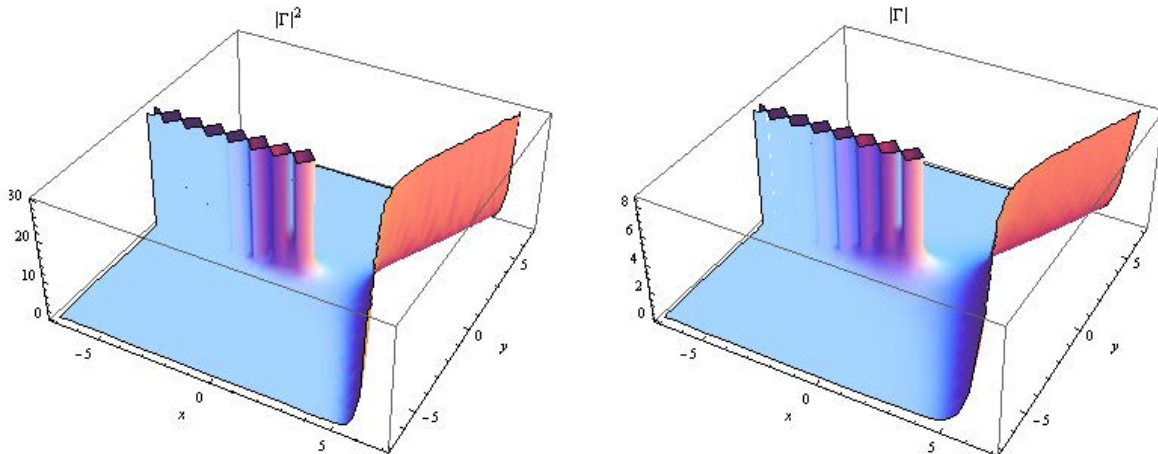
$$\Gamma(x+iy)\Gamma(x-iy) = \Gamma(z)\Gamma(\bar{z}) = \Gamma(z)\overline{\Gamma(z)} = |\Gamma(z)|^2 = |\Gamma(x+iy)|^2$$

Therefore,

$$\prod_{r=0}^{\infty} \left\{ 1 + \frac{y^2}{(r+x)^2} \right\} = \frac{\Gamma^2(x)}{|\Gamma(x+iy)|^2}$$

From this, we obtain (1.1). And taking the square root of this, we obtain (1.1').

(1.1), (1.1') are drawn as follows. The right figure is the same as the figure of  $|\Gamma|$  on the previous page.



Both figures are different only in the scale of the vertical axis, and are the same things essentially. Therefore, we discuss for  $|\Gamma|$  exclusively in the following sections.

**Note 1**

From Formula 9.2.1 , we can see immediately that the absolute value of  $|I|$  can not be 0, therefore  $|I|$  does not have a zero point.

**Note 2**

Formula 9.2.1 is already known. (See Abramowitz and Stegun: Handbook of Mathematical Functions p256.)

### 9.3 Partial Derivative with respect to y

#### Formula 9.3.1

When  $\Gamma(z)$  ( $z = x + iy$ ) is the gamma function in the complex plane  $C$ ,

$$\frac{\partial |\Gamma(z)|^2}{\partial y} = -2\Gamma^2(x) \left[ \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} \right] \sum_{s=0}^{\infty} \frac{y}{(s+x)^2 + y^2}$$

#### Calculation

Formula 9.2.1 was as follows.

$$|\Gamma(z)|^2 = \Gamma^2(x) \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1}$$

Differentiating this with respect to  $y$ ,

$$\frac{\partial |\Gamma(z)|^2}{\partial y} = \Gamma^2(x) \frac{\partial}{\partial y} \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1}$$

Let  $\left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} = f_r$ . Then the right side is

$$\frac{\partial}{\partial y} \prod_{r=0}^{\infty} f_r = f_0^{(1)} f_1 f_2 f_3 \cdots + f_0 f_1^{(1)} f_2 f_3 \cdots + f_0 f_1 f_2^{(1)} f_3 \cdots + \cdots$$

Here,

$$f_r^{(1)} = \frac{\partial}{\partial y} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} = - \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-2} \cdot \frac{2y}{(r+x)^2} = - \frac{2y}{(r+x)^2} f_r^2$$

Then,

$$\begin{aligned} \frac{\partial}{\partial y} \prod_{r=0}^{\infty} f_r &= - \frac{2y}{(0+x)^2} f_0^2 f_1 f_2 f_3 \cdots - \frac{2y}{(1+x)^2} f_0 f_1^2 f_2 f_3 \cdots - \frac{2y}{(2+x)^2} f_0 f_1 f_2^2 f_3 \cdots - \cdots \\ &= - \frac{2y f_0}{(0+x)^2} f_0 f_1 f_2 f_3 \cdots - \frac{2y f_1}{(1+x)^2} f_0 f_1 f_2 f_3 \cdots - \frac{2y f_2}{(2+x)^2} f_0 f_1 f_2 f_3 \cdots + \cdots \\ &= - \sum_{s=0}^{\infty} \frac{2y f_s}{(s+x)^2} \cdot \prod_{r=0}^{\infty} f_r \end{aligned}$$

i.e.

$$\frac{\partial}{\partial y} \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} = - \sum_{s=0}^{\infty} \frac{2y}{(s+x)^2} \left\{ 1 + \left( \frac{y}{s+x} \right)^2 \right\}^{-1} \cdot \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1}$$

Here,

$$\frac{y}{(s+x)^2} \left\{ 1 + \left( \frac{y}{s+x} \right)^2 \right\}^{-1} = \frac{y}{(s+x)^2 + y^2}$$

Using this,

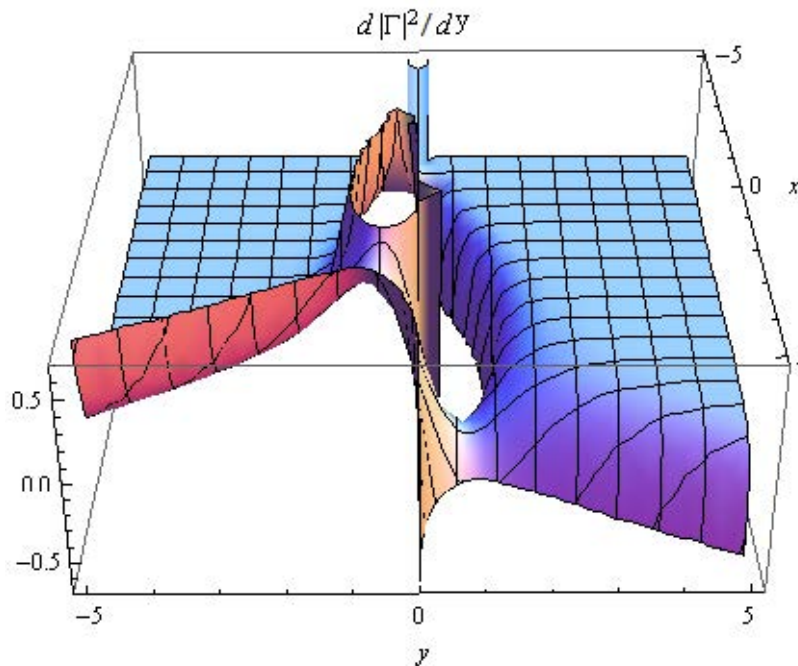
$$\frac{\partial}{\partial y} \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} = -2 \sum_{s=0}^{\infty} \frac{y}{(s+x)^2 + y^2} \cdot \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1}$$

Thus,

$$\frac{\partial |\Gamma(z)|^2}{\partial y} = -2\Gamma^2(x) \left[ \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} \right] \sum_{s=0}^{\infty} \frac{y}{(s+x)^2 + y^2}$$

Q.E.D.

Formula 9.3.1 is drawn as follows.



### Sign of the partial derivative with respect to $y$

According to this figure, the partial derivative of  $|\Gamma(z)|^2$  with respect to  $y$  seems to be negative at  $y > 0$  and to be positive at  $y < 0$ . Actually, this is so. Below, we prove this analytically.

Since  $x, y$  are real numbers, the followings hold for  $r, s = 0, 1, 2, \dots$ .

$$\Gamma^2(x) > 0, \quad \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} > 0, \quad \frac{1}{(s+x)^2} > 0$$

therefore, the sign of this partial derivative depends only on the sign of  $y$ . That is,

$$\text{When } y > 0, \quad \frac{\partial |\Gamma(z)|^2}{\partial y} < 0$$

$$\text{When } y < 0, \quad \frac{\partial |\Gamma(z)|^2}{\partial y} > 0$$

Thus, the following holds.



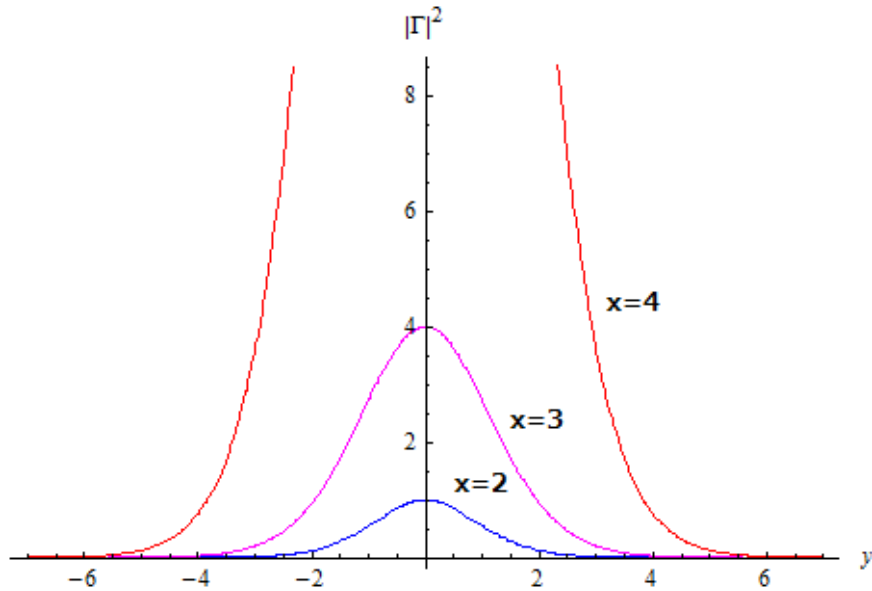
**Theorem 9.3.2**

When  $\Gamma(z)$  ( $z = x + iy$ ) is the gamma function in the complex plane  $C$ ,

**For any  $x$ , if  $y > 0$  then  $|\Gamma(z)|^2$  is monotonic decreasing with respect to  $y$ ,**

**if  $y < 0$  then  $|\Gamma(z)|^2$  is monotonic increasing with respect to  $y$ .**

For example, the figures of  $|\Gamma(z)|^2$  which was cut at  $x=2, 3, 4$  are as follows.



## 9.4 Partial Derivative with respect to x

### Lemma 9.4.1

Let  $x, y$  are real numbers,  $\psi(x)$  be the digamma function, and define  $h(x, y)$  as follows.

$$h(x, y) = \psi(x) + \sum_{s=0}^{\infty} \frac{1}{s+x} \frac{y^2}{(s+x)^2 + y^2} \quad (1.1)$$

Then, the followings hold.

$$\frac{\partial h(x, y)}{\partial y} > 0 \quad \text{for } y > 0, x \geq 0$$

$$\frac{\partial h(x, y)}{\partial y} < 0 \quad \text{for } y < 0, x \geq 0$$

### Proof

If we differentiate  $h(x, y)$  with respect to  $y$ , it is as follows.

$$\frac{\partial h(x, y)}{\partial y} = \sum_{s=0}^{\infty} \frac{2}{(s+x)y} \frac{y^2}{(s+x)^2 + y^2} \left\{ 1 - \frac{y^2}{(s+x)^2 + y^2} \right\}$$

When  $x \geq 0, y > 0$ , the followings hold for  $s = 1, 2, 3, \dots$

$$\frac{2}{(s+x)y} > 0$$

$$\frac{y^2}{(s+x)^2 + y^2} > 0, \quad 1 - \frac{y^2}{(s+x)^2 + y^2} > 0$$

Therefore,

$$\frac{\partial h(x, y)}{\partial y} > 0 \quad \text{for } y > 0, x \geq 0$$

And this partial derivative is odd function with respect to  $y$ . That is,

$$\frac{\partial h(x, -y)}{\partial y} = - \frac{\partial h(x, y)}{\partial y}$$

Therefore,

$$\frac{\partial h(x, y)}{\partial y} < 0 \quad \text{for } y < 0, x \geq 0$$

### Formula 9.4.2

When  $\Gamma(z)$  ( $z = x + iy$ ) is the gamma function in the complex plane  $C$  and  $\psi(x)$  is the digamma function,

$$\frac{\partial |\Gamma(z)|^2}{\partial x} = 2\Gamma^2(x) \left[ \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} \right] \cdot \left[ \psi(x) + \sum_{s=0}^{\infty} \frac{1}{s+x} \frac{y^2}{(s+x)^2 + y^2} \right]$$

## Calculation

Formula 9.2.1 was as follows.

$$|\Gamma(z)|^2 = \Gamma^2(x) \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1}$$

Differentiating this with respect to  $x$ ,

$$\frac{\partial |\Gamma(z)|^2}{\partial x} = \{\Gamma^2(x)\}^{(1)} \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} + \Gamma^2(x) \frac{\partial}{\partial x} \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1}$$

When  $\psi(x)$  denotes the digamma function, the 1st term of the right side is

$$\{\Gamma^2(x)\}^{(1)} = 2\Gamma(x) \cdot \Gamma^{(1)}(x) = 2\Gamma(x) \cdot \Gamma(x) \psi(x) = 2\Gamma^2(x) \psi(x)$$

Using this,

$$\{\Gamma^2(x)\}^{(1)} \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} = 2\Gamma^2(x) \psi(x) \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1}$$

When  $\left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} = f_r$ , the 2nd term of the right side is

$$\frac{\partial}{\partial x} \prod_{r=0}^{\infty} f_r = f_0^{(1)} f_1 f_2 f_3 \cdots + f_0 f_1^{(1)} f_2 f_3 \cdots + f_0 f_1 f_2^{(1)} f_3 \cdots + \cdots$$

Here,

$$f_r^{(1)} = \frac{\partial}{\partial x} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} = \frac{2y^2}{(r+x)^3} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-2} = \frac{2y^2}{(r+x)^3} f_r^2$$

Then,

$$\begin{aligned} \frac{\partial}{\partial x} \prod_{r=0}^{\infty} f_r &= \frac{2y^2}{(0+x)^3} f_0^2 f_1 f_2 f_3 \cdots + \frac{2y^2}{(1+x)^3} f_0 f_1^2 f_2 f_3 \cdots + \frac{2y^2}{(2+x)^3} f_0 f_1 f_2^2 f_3 \cdots + \cdots \\ &= \frac{2y^2 f_0}{(0+x)^3} f_0 f_1 f_2 f_3 \cdots + \frac{2y^2 f_1}{(1+x)^3} f_0 f_1 f_2 f_3 \cdots + \frac{2y^2 f_2}{(2+x)^3} f_0 f_1 f_2 f_3 \cdots + \cdots \\ &= \sum_{s=0}^{\infty} \frac{2y^2 f_s}{(s+x)^3} \cdot \prod_{r=0}^{\infty} f_r \end{aligned}$$

i.e.

$$\frac{\partial}{\partial x} \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} = \sum_{s=0}^{\infty} \frac{2y^2}{(s+x)^3} \left\{ 1 + \left( \frac{y}{s+x} \right)^2 \right\}^{-1} \cdot \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1}$$

Here,

$$\frac{y^2}{(s+x)^2} \left\{ 1 + \left( \frac{y}{s+x} \right)^2 \right\}^{-1} = \frac{y^2}{(s+x)^2 + y^2}$$

Then,

$$\frac{\partial}{\partial x} \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} = 2 \sum_{s=0}^{\infty} \frac{1}{s+x} \frac{y^2}{(s+x)^2 + y^2} \cdot \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1}$$

Substituting these for the above,

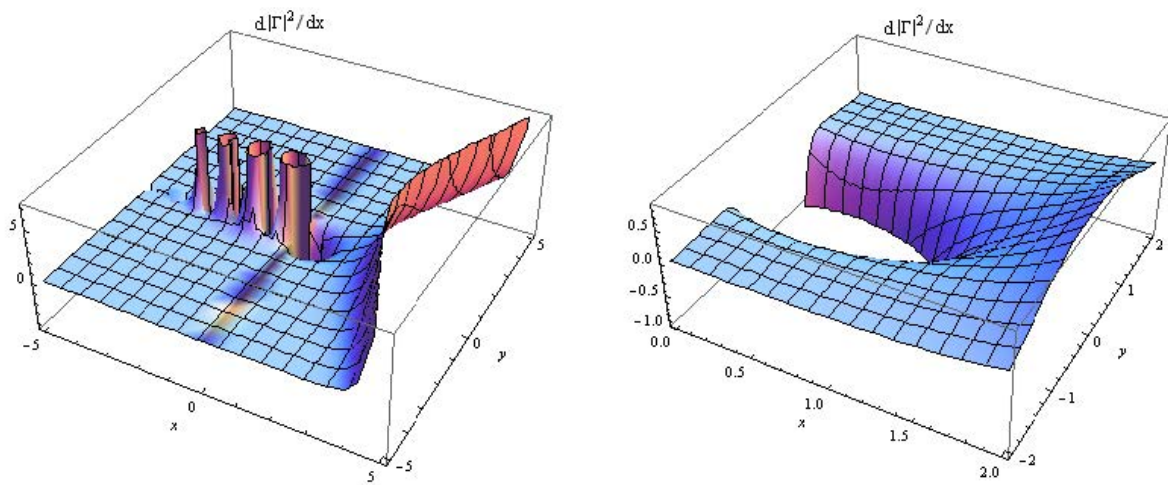
$$\frac{\partial |\Gamma(z)|^2}{\partial x} = 2\Gamma^2(x) \psi(x) \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} + 2\Gamma^2(x) \sum_{s=0}^{\infty} \frac{1}{s+x} \frac{y^2}{(s+x)^2 + y^2} \cdot \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1}$$

i.e.

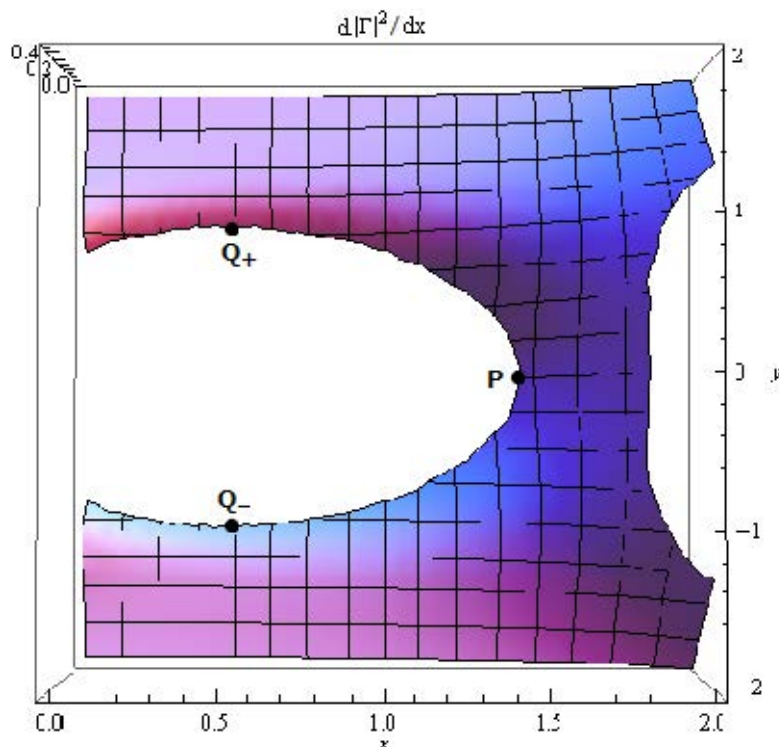
$$\frac{\partial |\Gamma(z)|^2}{\partial x} = 2\Gamma^2(x) \left[ \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} \right] \cdot \left[ \psi(x) + \sum_{s=0}^{\infty} \frac{1}{s+x} \frac{y^2}{(s+x)^2 + y^2} \right]$$

Q.E.D.

Formula 9.4.2 is drawn as follows. The right figure is an enlarged picture of the neighborhood of the origin.



And, the following figure is one that is viewed from above by removing the negative portion of the right figure.



### Sign of the partial derivative with respect to $x$

According to this figure, the partial derivative of  $|I(z)|^2$  with respect to  $x$  seems to be positive on the right of  $P$  ( $x \doteq 1.5$ ), and to be positive above or below points  $Q_+$ ,  $Q_-$  ( $y \doteq \pm 1$ ). Actually, this is so. Below, we prove these analytically.

Formula 9.4.2 can be written separately as follows:

$$\frac{\partial |I(z)|^2}{\partial x} = 2\Gamma^2(x) \left[ \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} \right] \times h(x, y)$$

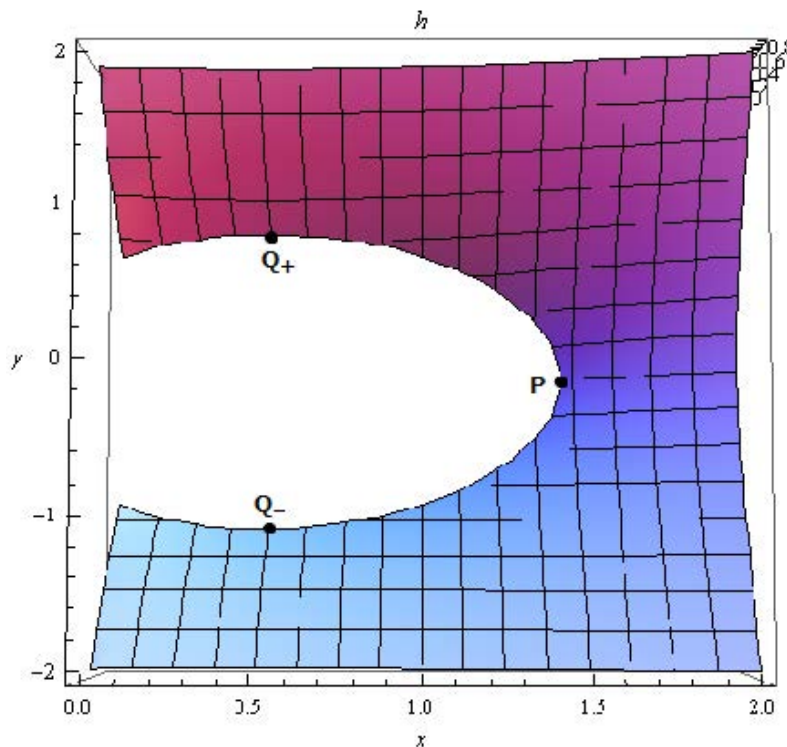
$$h(x, y) = \psi(x) + \sum_{s=0}^{\infty} \frac{1}{s+x} \frac{y^2}{(s+x)^2 + y^2} \quad (1.1)$$

The sign of [ ] is

$$\prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{y}{r+x} \right)^2 \right\}^{-1} > 0 \quad \text{for any } x, y$$

Therefore, the sign of this partial derivative depends on the sign of  $h(x, y)$ . That is, we may examine the sign of  $h(x, y)$  instead of  $\partial |I(z)|^2 / \partial x$ .

Then, if  $h(x, y)$  is illustrated, it is as follows. This is just like the above figure and the positions of  $P$ ,  $Q_+$ ,  $Q_-$  are entirely consistent with the above figure.



### Right side of the point $P$

As the result of the numerical calculation, coordinates of the zero  $P$  farthest from the origin is  $y = 0$ ,  $x = 1.461632144968\dots$ . This corresponds to the zero point of the 1st term  $\psi(x)$  of  $h(x, y)$ . That is

$$\psi(x) > 0 \quad \text{for } x > 1.461632144968\dots$$

And

$$\sum_{s=0}^{\infty} \frac{1}{s+x} \frac{y^2}{(s+x)^2 + y^2} \geq 0 \quad \text{for any } x, y$$

Then,

$$h(x, y) = \psi(x) + \sum_{s=0}^{\infty} \frac{1}{s+x} \frac{y^2}{(s+x)^2 + y^2} > 0 \quad \text{for } x > 1.461632144968 \dots$$

Zeros and the sign of  $\partial |\Gamma(z)|^2 / \partial x$  are consistent with those of  $h(x, y)$ . Therefore,

$$\frac{\partial |\Gamma(z)|^2}{\partial x} > 0 \quad \text{for } x > 1.461632144968 \dots \text{ and any } y$$

Thus, the following holds.

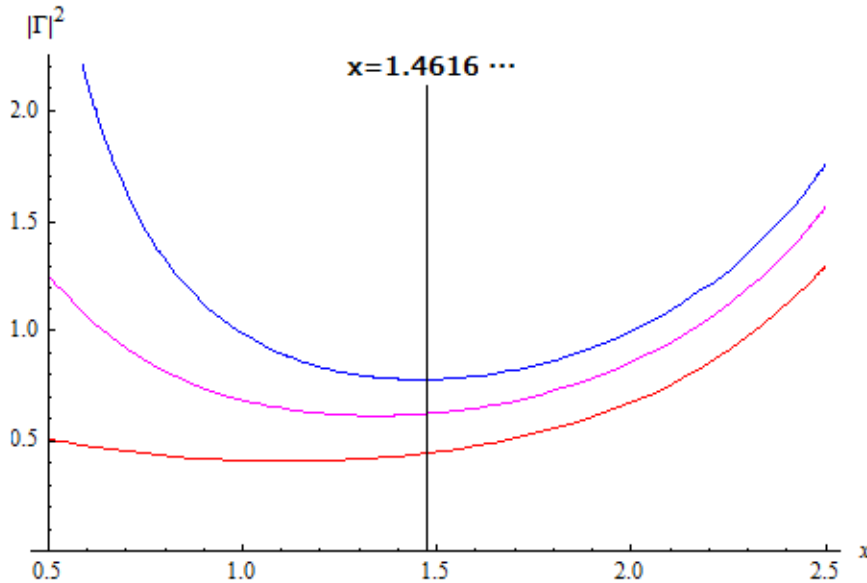
### Theorem 9.4.3

When  $\Gamma(z)$  ( $z = x + iy$ ) is the gamma function in the complex plane  $C$ ,

**For any  $y$ , if  $x > 1.461632144968 \dots$  then  $|\Gamma(z)|^2$  is monotonic increasing with respect to  $x$ .**

For example, the figures of  $|\Gamma(z)|^2$  which was cut at  $y = 0.1, 0.5, 0.8$  are as follows.

In any case, the curve is monotonic increasing at  $x > 1.461632144968 \dots$ .



### Above the point $Q_+$ and Below the point $Q_-$

As the result of the numerical calculation, coordinates of the furthest two zeros  $Q_+, Q_-$  along the  $y$ -axis are  $x = 1/2, y = \pm 1.047662675461731 \dots$ . According to **Lemma 9.4.1**,

$$\frac{\partial h(x, y)}{\partial y} > 0 \quad \text{for } y > 0, x \geq 0$$

Then, when  $y > 0, x \geq 0$ ,  $h(x, y)$  is monotonic increasing with respect to  $y$ . Therefore,

$$h(x, y) > 0 \quad \text{for } y > 1.047662675461731 \dots, x \geq 0$$

And  $h(x, y)$  is even function with respect to  $y$ , that is,  $h(x, y) = h(x, -y)$ . Therefore,

$$h(x, y) > 0 \quad \text{for } y < -1.047662675461731\cdots, x \geq 0$$

Zeros and the sign of  $\partial |\Gamma(z)|^2 / \partial x$  are consistent with those of  $h(x, y)$ . Therefore,

$$\frac{\partial |\Gamma(z)|^2}{\partial x} > 0 \quad \text{for } |y| > 1.047662675461731\cdots, x \geq 0$$

Thus, the following holds.

### Theorem 9.4.3'

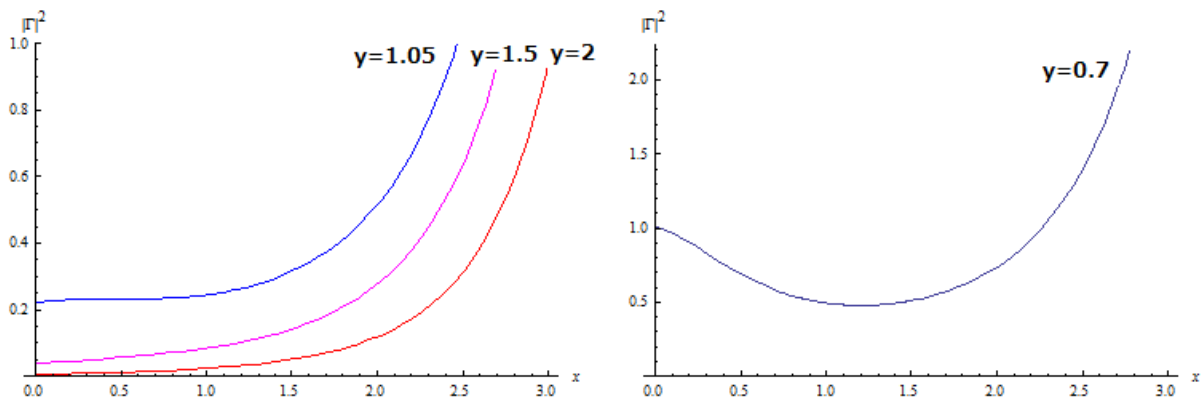
When  $\Gamma(z)$  ( $z = x + iy$ ) is the gamma function in the complex plane  $C$ ,

**If  $x \geq 0$  &  $|y| > 1.047662675461731\cdots$ ,  $|\Gamma(z)|^2$  is monotonic increasing with respect to  $x$ .**

For example, the figures of  $|\Gamma(z)|^2$  which was cut at  $y = 1.05, 1.5, 2$  are shown in a left figure..

Every curves are monotonic increasing with respect to  $x$ . On the contrary, the right figure shows that

$|\Gamma(z)|^2$  is not monotonic function with respect to  $x$  when  $y = 0.7$ .



### Note 1

As the result of the numerical calculation, the condition  $x \geq 0$  seems to be unnecessary. That is, for any  $x$ , the following seems to hold.

$$h(x, y) > 0 \quad \text{for } y > 1.047662675461731\cdots$$

Regrettably, I could not prove this at last.

### Note 2

It is very interesting that the  $x$ -coordinates of the furthest two zeros  $Q_+$ ,  $Q_-$  along the  $y$ -axis is  $1/2$ .

Because, the gamma function is a key element of the functional equation of the Riemann Zeta.

2014.03.14

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