10 Convergence Acceleration & Summation Method by Double Series of Functions

Abstract

A series of functions can be the iterated series (row sum or column sum) of a certain double series of functions. On the other hand, a diagonal series of functions is made from the double series of functions. The double series of functions is not unique to one series of functions. So, the diagonal series of functions is also not unique. Euler-Knopp transformation is one of such a diagonal series of functions.

The diagonal series of functions converges faster than the iterated series of functions in general. Then, when both are equal in a certain domain, we can accelerate convergence of the original series of functions by using the diagonal series of functions

When either is convergent among the iterated series of functions or the diagonal series of functions, and the other is divergent, the divergent series of functions have to be interpreted as being convergent by a summation method.

When the convergence ranges of the iterated series of functions and the diagonal series of functions differ and a common domain exists, analytic continuation may arise.

10.1 Double Series of Functions & Summation Method

The theorems about a double series are prepared for the beginning.

Lemma 10.1.1

When a_{mn} $(m,n=0,1,2,\cdots)$ are complex numbers, the following four double series exist.

$$\sum_{m,n=0}^{\infty} a_{mn} \;,\; \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{mn}\right) \;,\; \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{nm}\right) \;,\; \sum_{m=0}^{\infty}\sum_{n=0}^{m} a_{m-n\,n}$$

Then, If any one among these is absolutely convergent, a certain number S exists and the following holds.

$$\sum_{m,n=0}^{\infty} a_{mn} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{mn}\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{nm}\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} a_{m-n\,n}\right) = S$$

Proof

See Eissa D. Habil " Double Sequences and Double Series "

This lemma is extensible to a double series of functions.

Lemma 10.1.2

When $a_{mn}(z)$ $(m,n=0,1,2,\cdots)$ are complex functions in a domain D , the following four double series of functions exist.

$$\sum_{m,n=0}^{\infty} a_{mn}(z) \;,\; \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{mn}(z)\right) \;,\; \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{nm}(z)\right) \;,\; \sum_{m=0}^{\infty}\sum_{n=0}^{m} a_{m-n\,n}(z)$$

Then, If any one among these is absolutely convergent in the domain D , a certain function $S(z)$ exists in the D and the following holds.

$$\sum_{m,n=0}^{\infty} a_{mn}(z) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{mn}(z)\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{nm}(z)\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} a_{m-n\,n}(z)\right) = S(z)$$

Using this lemma, we can prove the following thoerem.

Theorem 10.1.3

Let $b(s)$ be a real function, D_1, D_2 are small domains, $f_1(z), f_2(z)$ are series of functions defined as follows respectively.

$$b(s) = \sum_{r=s}^{\infty} b_{rs} = 1 \qquad s=0,1,2,\cdots \tag{b}$$

$$f_1(z) = \sum_{s=0}^{\infty} a_s(z) \qquad z \in D_1 \tag{3.1}$$

$$f_2(z) = \sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z) \tag{3.2}$$

(1) When $f_1(z)$ is absolutely convergent,

i if $f_2(z)$ is convergent in the D_1,

$$\sum_{s=0}^{\infty} a_s(z) = \sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z) \qquad z \in D_1 \tag{3.1$_1$}$$

ii if $f_2(z)$ is divergent in the D_1,

$$\sum_{s=0}^{\infty} a_s(z) = "\sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z)" \qquad z \in D_1 \tag{3.1$_2$}$$

where, "S(z)" means that S(z) is interpreted as convergent.

(2) When $f_2(z)$ is absolutely convergent in the D_1,

i if $f_1(z)$ is convergent,

$$\sum_{s=0}^{\infty} a_s(z) = \sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z) \qquad z \in D_1 \tag{3.2$_1$}$$

ii if $f_1(z)$ is divergent,

$$"\sum_{s=0}^{\infty} a_s(z)" = \sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z) \qquad z \in D_1 \tag{3.2$_2$}$$

where, "S(z)" means that S(z) is interpreted as convergent.

(3) When $f_2(z)$ is absolutely convergent in the $D_2\ (\neq D_1)$,

$$"\sum_{s=0}^{\infty} a_s(z)" = \sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z) \qquad z \in D_2 \tag{3.3}$$

where, "S(z)" means that S(z) is interpreted as convergent.

Proof

We consider the following double series of functions.

$$\begin{aligned}\sum_{s=0,\, r=s}^{\infty} b_{rs} a_s(z) = \ & b_{00} a_0(z) + b_{10} a_0(z) + b_{20} a_0(z) + b_{30} a_0(z) + \cdots \\ & + b_{11} a_1(z) + b_{21} a_1(z) + b_{31} a_1(z) + b_{41} a_1(z) + \cdots \\ & + b_{22} a_2(z) + b_{32} a_2(z) + b_{42} a_2(z) + b_{52} a_2(z) + \cdots \\ & \vdots \end{aligned}$$

Calculating the iterated series of functions,

$$\sum_{s=0}^{\infty}\sum_{r=s}^{\infty} b_{rs}\,a_s(z) = \left(b_{00}+b_{10}+b_{20}+b_{30}+\cdots\right)a_0(z)$$
$$+\left(b_{11}+b_{21}+b_{31}+b_{41}+\cdots\right)a_1(z)$$
$$+\left(b_{22}+b_{32}+b_{42}+b_{52}+\cdots\right)a_2(z)$$
$$\vdots$$

Substituting (b) for this,

$$\sum_{s=0}^{\infty}\sum_{r=s}^{\infty} b_{rs}\,a_s(z) = a_0(z)+a_1(z)+a_2(z)+\cdots = \sum_{s=0}^{\infty} a_s(z) \tag{3.1'}$$

On the other hand, calculating the diagonal series of functions,

$$b_{00}\,a_0(z)$$
$$+\,b_{10}\,a_0(z)+b_{11}\,a_1(z)$$
$$+\,b_{20}\,a_0(z)+b_{21}\,a_1(z)+b_{22}\,a_2(z)$$
$$+\,b_{30}\,a_0(z)+b_{31}\,a_1(z)+b_{32}\,a_2(z)+b_{33}\,a_3(z)$$
$$\vdots$$
$$= \sum_{r=0}^{\infty}\sum_{s=0}^{r} b_{rs}\,a_s(z)$$

(1) When $f_1(z)$ is absolutely convergent, the left side of (3.1') is also absolutely convergent. Then according to Lemma 10.1.2 , (3.1') and (3.2') become equal in the D_1 . At this time, if $f_2(z)$ is convergent in the D_1 , (3.1$_1$) holds immediatly. If $f_2(z)$ is divergent, it have to be interpreted as convergent. If not so, it is contrary to Lemma 10.1.2. Thus (3.1$_2$) holds.

(2) When $f_2(z)$ is absolutely convergent in the D_1 , according to Lemma 10.1.2 , (3.1') and (3.2') become equal in the D_1 . At this time, if $f_1(z)$ is convergent, (3.2$_1$) holds immediatly. If $f_1(z)$ is divergent, it have to be interpreted as convergent. If not so, it is contrary to Lemma 10.1.2. Thus (3.2$_2$) holds.

(3) When $f_2(z)$ is absolutely convergent in the $D_2\ (\neq D_1)$, according to Lemma 10.1.2 , (3.1') and (3.2') become equal in the D_2 . At this time, regardless of whether $f_1(z)$ is convergent, it have to be interpreted as convergent. If not so, it is contrary to Lemma 10.1.2. Thus (3.3) holds. Q.E.D.

Remark

This theorem often forces a summation method for the divergent series. Although there are Cesaro summation, Abel summation, etc. in the summation method, what is required here seems to be matrix summation method (Silverman–Toeplitz) .

10.2 Accelerator

The following equations were obtained in the previous section.

$$\sum_{s=0}^{\infty} a_s(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} b_{rs}\, a_s(z) \tag{1.1}$$

$$b(s) = \sum_{r=s}^{\infty} b_{rs} = 1 \qquad for\ s=0,1,2,\cdots \tag{b}$$

When (b) converges faster than the left side of (1.1), by transforming this like the right side of (1.1), we can accelerate the convergence of the left side. In this case, we can call the series (b) **accelerator** .

Of course, the accelerator is desired fast-convergence more. Although this is considered variously, I recommend the following.

Formula 10.2.1 (Knopp's accelerator)

$$b(s) = \sum_{r=s}^{\infty} \frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s} = 1 \qquad for \quad \begin{array}{l} s = 0,1,2,\cdots \\ q > 0 \end{array}$$

Proof

$$\begin{aligned} b(0) &= \frac{q^0{}_0C_0}{(q+1)^1} + \frac{q^1{}_1C_0}{(q+1)^2} + \frac{q^2{}_2C_0}{(q+1)^3} + \frac{q^3{}_3C_0}{(q+1)^4} + \cdots \\ &= \frac{q^0}{(q+1)^1} + \frac{q^1}{(q+1)^2} + \frac{q^2}{(q+1)^3} + \frac{q^3}{(q+1)^4} + \cdots \\ &= \frac{1}{q+1}\left\{1+\left(\frac{q}{q+1}\right)^1 + \left(\frac{q}{q+1}\right)^2 + \left(\frac{q}{q+1}\right)^3 + \cdots\right\} \\ &= \frac{1}{q+1}\,\frac{q+1}{1} = 1 \end{aligned}$$

Next, since ${}_nC_1 = \sum_{k=0}^{n-1} {}_kC_0 \quad n=1,2,3,\cdots$,

$$\begin{aligned} b(1) &= \frac{q^0{}_1C_1}{(q+1)^2} + \frac{q^1{}_2C_1}{(q+1)^3} + \frac{q^2{}_3C_1}{(q+1)^4} + \frac{q^3{}_4C_1}{(q+1)^5} + \cdots \\ &= \frac{q^0{}_0C_0}{(q+1)^2} + \frac{q^1{}_1C_0}{(q+1)^3} + \frac{q^2{}_2C_0}{(q+1)^4} + \frac{q^3{}_3C_0}{(q+1)^5} + \cdots \\ &\qquad + \frac{q^1{}_0C_0}{(q+1)^3} + \frac{q^2{}_1C_0}{(q+1)^4} + \frac{q^3{}_2C_0}{(q+1)^5} + \frac{q^4{}_3C_0}{(q+1)^6} + \cdots \\ &\qquad\qquad + \frac{q^2{}_0C_0}{(q+1)^4} + \frac{q^3{}_1C_0}{(q+1)^5} + \frac{q^4{}_2C_0}{(q+1)^6} + \frac{q^5{}_3C_0}{(q+1)^7} + \cdots \\ &\qquad\qquad\qquad \vdots \end{aligned}$$

Here,

$$\frac{q^0{}_0C_0}{(q+1)^2}+\frac{q^1{}_1C_0}{(q+1)^3}+\frac{q^2{}_2C_0}{(q+1)^4}+\frac{q^3{}_3C_0}{(q+1)^5}+\cdots = \frac{q^0}{(q+1)^1}b(0)$$

$$\frac{q^1{}_0C_0}{(q+1)^3}+\frac{q^2{}_1C_0}{(q+1)^4}+\frac{q^3{}_2C_0}{(q+1)^5}+\frac{q^4{}_3C_0}{(q+1)^6}+\cdots = \frac{q^1}{(q+1)^2}b(0)$$

$$\frac{q^2{}_0C_0}{(q+1)^4}+\frac{q^3{}_1C_0}{(q+1)^5}+\frac{q^4{}_2C_0}{(q+1)^6}+\frac{q^5{}_3C_0}{(q+1)^7}+\cdots = \frac{q^2}{(q+1)^3}b(0)$$

$$\vdots$$

Substituting these for the above,

$$b(1) = \left\{\frac{q^0}{(q+1)^1}+\frac{q^1}{(q+1)^2}+\frac{q^2}{(q+1)^3}+\frac{q^3}{(q+1)^4}+\cdots\right\}b(0)$$

$$= \frac{1}{q+1}\left\{1+\left(\frac{q}{q+1}\right)^1+\left(\frac{q}{q+1}\right)^2+\left(\frac{q}{q+1}\right)^3+\cdots\right\}b(0)$$

$$= \frac{1}{q+1}\frac{q+1}{1}b(0) = 1$$

Next, since ${}_nC_2 = \sum_{k=1}^{n-1}{}_kC_1 \quad n=2,3,4,\cdots$,

$$b(2) = \frac{q^0{}_2C_2}{(q+1)^3}+\frac{q^1{}_3C_2}{(q+1)^4}+\frac{q^2{}_4C_2}{(q+1)^5}+\frac{q^3{}_5C_2}{(q+1)^6}+\cdots$$

$$= \frac{q^0{}_1C_1}{(q+1)^3}+\frac{q^1{}_2C_1}{(q+1)^4}+\frac{q^2{}_3C_1}{(q+1)^5}+\frac{q^3{}_4C_1}{(q+1)^6}+\cdots$$

$$+\frac{q^1{}_1C_1}{(q+1)^4}+\frac{q^2{}_2C_1}{(q+1)^5}+\frac{q^3{}_3C_1}{(q+1)^6}+\frac{q^4{}_4C_1}{(q+1)^7}+\cdots$$

$$+\frac{q^2{}_1C_1}{(q+1)^5}+\frac{q^3{}_2C_1}{(q+1)^6}+\frac{q^4{}_3C_1}{(q+1)^7}+\frac{q^5{}_4C_1}{(q+1)^8}+\cdots$$

$$\vdots$$

Here,

$$\frac{q^0{}_1C_1}{(q+1)^3}+\frac{q^1{}_2C_1}{(q+1)^4}+\frac{q^2{}_3C_1}{(q+1)^5}+\frac{q^3{}_4C_1}{(q+1)^6}+\cdots = \frac{q^0}{(q+1)^1}b(1)$$

$$\frac{q^1{}_1C_1}{(q+1)^4}+\frac{q^2{}_2C_1}{(q+1)^5}+\frac{q^3{}_3C_1}{(q+1)^6}+\frac{q^4{}_4C_1}{(q+1)^7}+\cdots = \frac{q^1}{(q+1)^2}b(1)$$

$$\frac{q^2{}_1C_1}{(q+1)^5}+\frac{q^3{}_2C_1}{(q+1)^6}+\frac{q^4{}_3C_1}{(q+1)^7}+\frac{q^5{}_4C_1}{(q+1)^8}+\cdots = \frac{q^2}{(q+1)^3}b(1)$$

$$\vdots$$

Substituting these for the above,

$$b(2) = \left\{ \frac{q^0}{(q+1)^1} + \frac{q^1}{(q+1)^2} + \frac{q^2}{(q+1)^3} + \frac{q^3}{(q+1)^4} + \cdots \right\} b(1)$$
$$= \frac{1}{q+1} \left\{ 1 + \left(\frac{q}{q+1} \right)^1 + \left(\frac{q}{q+1} \right)^2 + \left(\frac{q}{q+1} \right)^3 + \cdots \right\} b(1)$$
$$= \frac{1}{q+1} \frac{q+1}{1} b(1) = 1$$

Hereafter, by induction, we obtain the desired expression.

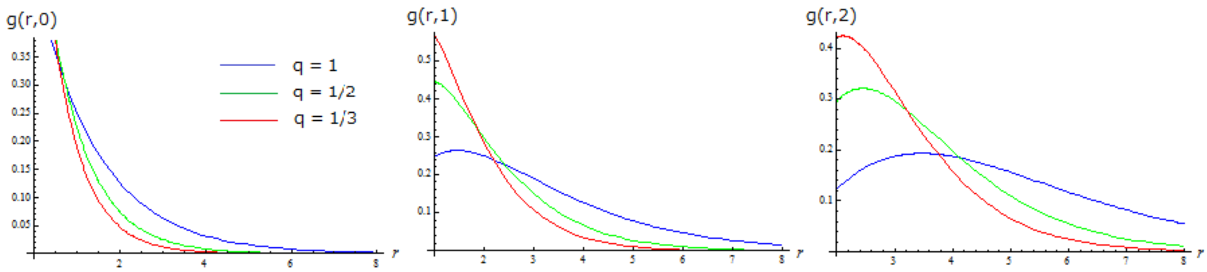
Properties

If $q=1$ is adopted in Formula 10.2.1 , it gives the **Euler's Transformation** . However, the superiority of Knopp's accelerator is that the larger acceleration effect is obtained by adopting the smaller q . We show it below.

Let the term of $b(s)$ be

$$g(r,s,q) = \frac{q^{r-s}}{(q+1)^{r+1}} \binom{r}{s}$$

This is illustrated as follows. $s=0,1,2$ from the left to the right. In each figure, $q=1,1/2,1/3$ is blue, green and red respectively.



In each figure, g converges to 0 more quickly as q decreases. This is an acceleration factor. On the other hand, the initial value of g is deviated more from 0 as q decreases. This is a slowdown factor. Although it depends on the series to be accelerated, in many cases, it seems that $q=1\sim 1/3$ are appropriate.

c.f.

The accelerator which I discovered in August, 2014 was as follows.

$$b_1(s) = \sum_{r=s}^{\infty} \frac{q^{s+1}}{(q+1)^{r+1}} \binom{r}{s} = 1 \quad \text{for} \quad \begin{array}{l} s = 0,1,2,\cdots \\ q > 0 \end{array}$$

This is very similar to Formula10.2.1, In fact, if q is replaced with $1/q$, this expression reduces to Formula10.2.1
I did not notice this and mistook that both were another. And the mistake became a starting point of this paper.

Note

I also found out the following accelerators besides the above.

$$b_2(s) = \sum_{r=s}^{\infty} e^{-1} \frac{s!}{r!} \binom{r}{s} = 1 \quad \text{for } s = 0,1,2,\cdots$$

$$b_3(s) = \sum_{r=s}^{\infty} \frac{e^{s+1} - e^{s}}{e^{r+1}} = 1 \quad for \;\; s = 0, 1, 2, \cdots$$

Regrettably, these accelerations are quite smaller than Formula 10.2.1 . However, these existence leaves possibility of the existence of better accelerator.

10.3 Knopp Transformation & Double Series of Functions

Transformation of a series using Knopp's accelerator is called **Knopp Transformation**. First, we prepar a lemma about Knopp transformation.

Lemma 10.3.1

Let q be a positive number, $f_1(z)=\sum_{s=0}^{\infty} a_s(z)$ be a series of functions in a small domain D_1 and the Knopp transformation be as follows.

$$f_2(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}a_s(z)$$

Then, if $f_1(z)$ is bounded, $f_2(z)$ is absolutely convergent,

Proof

According to d'Alembert's ratio test,

$$\rho(r,z,q) = \frac{\dfrac{1}{(q+1)^{r+2}}\left|\sum_{s=1}^{r+1} q^{r+1-s}\binom{r+1}{s}a_s(z)\right|}{\dfrac{1}{(q+1)^{r+1}}\left|\sum_{s=1}^{r} q^{r-s}\binom{r}{s}a_s(z)\right|}$$

$$= \frac{q}{q+1}\left|\frac{\sum_{s=1}^{r+1} q^{r-s}\binom{r+1}{s}a_s(z)}{\sum_{s=1}^{r} q^{r-s}\binom{r}{s}a_s(z)}\right|$$

Here,

$$\binom{r+1}{s} = \frac{r+1}{r+1-s}\binom{r}{s}$$

Using this,

$$\rho(r,z,q) = \frac{q}{q+1}\left|\frac{\sum_{s=1}^{r+1}\frac{r+1}{r+1-s}q^{r-s}\binom{r}{s}a_s(z)}{\sum_{s=1}^{r} q^{r-s}\binom{r}{s}a_s(z)}\right|$$

The numerator in the $|\ \ |$ is

$$\sum_{s=1}^{r+1}\frac{r+1}{r+1-s}q^{r-s}\binom{r}{s}a_s(z) = \sum_{s=1}^{r}\frac{r+1}{r+1-s}q^{r-s}\binom{r}{s}a_s(z)$$
$$+\frac{r+1}{r+1-(r+1)}q^{r-(r+1)}\binom{r}{r+1}a_{r+1}(z)$$

For non-negative number r ,

$$\binom{r}{r+1} = 0$$

Therefore,

$$\rho(r,z,q) = \frac{q}{q+1}\left| \frac{\displaystyle\sum_{s=1}^{r} \frac{1}{1-\dfrac{s}{r+1}} q^{r-s}\binom{r}{s} a_s(z)}{\displaystyle\sum_{s=1}^{r} q^{r-s}\binom{r}{s} a_s(z)} \right|$$

Since $r \to \infty \Rightarrow \dfrac{s}{r+1} \to 0$ and $a_s(z)$ is bounded,

$$\lim_{r\to\infty} \rho(r,z,q) = \frac{q}{q+1} < 1 \qquad for \;\; q>0$$

Therefore, $f_2(z)$ is absolutely convergent for $a_s(z)$ $s=0,1,2,\cdots$ defined in the small domain D_1.

Theorem 10.3.2 (Knopp Transformation)

Let q be a positive number, D_1, D_2 are small domains, $f_1(z), f_2(z)$ are series of functions defined as follows respectively.

$$f_1(z) = \sum_{s=0}^{\infty} a_s(z) \qquad\qquad z \in D_1 \tag{2.1}$$

$$f_2(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s} a_s(z) \tag{2.2}$$

(1) When $f_1(z)$ is convergent,

$$\sum_{s=0}^{\infty} a_s(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s} a_s(z) \qquad z \in D_1 \tag{3.1}$$

(2) When $f_2(z)$ is absolutely convergent in the D_1, if $f_1(z)$ is divergent,

$$"\sum_{s=0}^{\infty} a_s(z)" = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s} a_s(z) \qquad z \in D_1 \tag{3.2}$$

where, "S(z)" means that S(z) is interpreted as convergent.

(3) When $f_2(z)$ is absolutely convergent in the D_2 $(\neq D_1)$,

$$"\sum_{s=0}^{\infty} a_s(z)" = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s} a_s(z) \qquad z \in D_2 \tag{3.3}$$

where, "S(z)" means that S(z) is interpreted as convergent.

Proof

The following expression holds from **Formula 10.2.1 .**

$$b(s) = \sum_{r=s}^{\infty} \frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s} = 1 \qquad for \quad \begin{array}{l} s = 0,1,2,\cdots \\ q > 0 \end{array}$$

Then this $b(s)$ satisfies the requirements for **Theorem 10.1.3** .

When $f_1(z)$ is convergent, $f_2(z)$ is absolutely convergent in D_1 according to Lemma 10.3.1 . Then (1) holds by **Theorem 10.1.3 (1) i** . (2) and (3) follow immediatly from **Theorem 10.1.3**.

Parameter in Knopp Transformation

Knopp transformation has a parameter q . And if it is a positive number, anything is all right. In fact, it may be a complex number $s.t.\ Re(q)>0$. Therefore, there is no raison d'etre of the parameter q in infinity. This is just the reason why Knopp transformation is not famous as Euler transformation.

However, we cannot calculate an infinite series. We have to cut an infinite series with somewhere. Then, the parameter q begins to play a big role suddenly. That is,

If q increases, the approximate range spreads, and the convergence speed descends.

If q decreases, the approximate range reduces, and the convergence speed ascends.

In this way. the parameter in Knopp transformation is very useful in numerical computation. It will be seen in the following sections. Although Knopp transformation is combined with Euler transformation and called Euler-Knopp transformation, the significance of its having a parameter should be evaluated more.

10.4 Acceleration of Power Series

Since the convergence of Powe Series is generally quick, the acceleration effect by Knopp transformation is not so much expectable. However, the remarkable acceleration effect and the asymptotic effect are seen at the circumference of the convergence circle.

10.4.1 Acceleration of Mercator Series

$$f(z) = log(z+1) \tag{1.0}$$

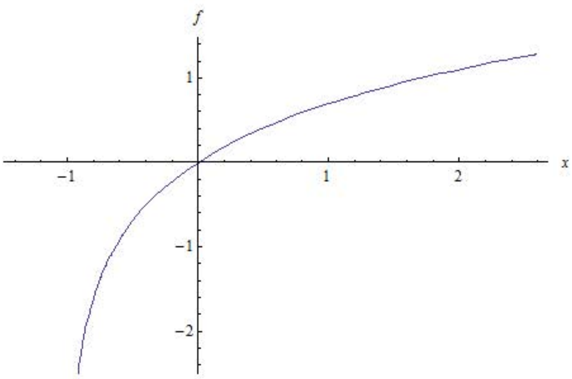
The Maclaurin series is called **Mercator Series** and it is as follows.

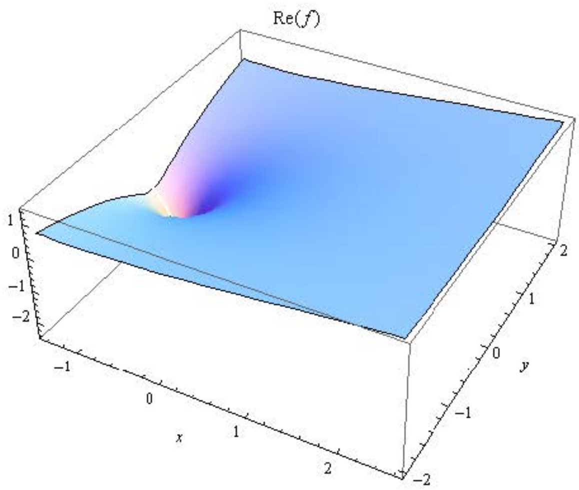
$$f_1(z) = \sum_{s=1}^{\infty} (-1)^{s-1} \frac{z^s}{s} \qquad |z| \le 1 \tag{1.1}$$

Applying Knopp Transformation to this,

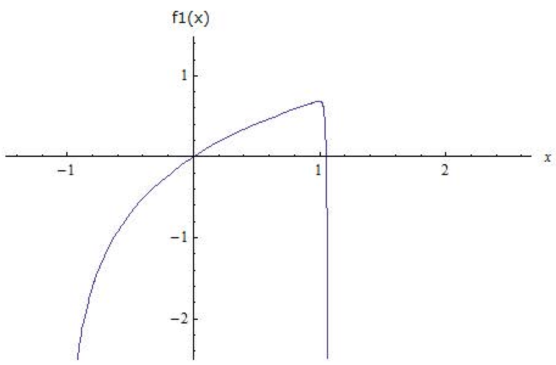
$$f_2(z,q) = \sum_{r=1}^{\infty} \sum_{s=1}^{r} \frac{q^{r-s}}{(q+1)^{r+1}} \binom{r}{s} (-1)^{s-1} \frac{z^s}{s} \qquad |z| \le 1 \tag{1.2}$$

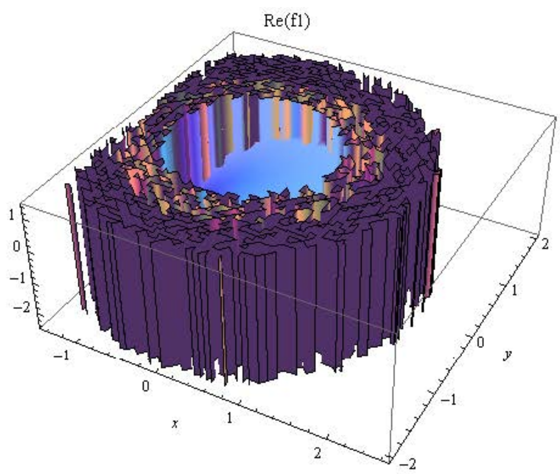
First, log function $f(z)$ is illustrated. The left is 2-D figure and the right is 3-D figure of the real number part.





Next, the Maclaurin series $f_1(z)$ is illustrated. The left is 2-D and the right is 3-D of the real number part.





From both figures, we can see that the convergence radius of $f_1(z)$ is 1. The series $f_1(z)$ is absolutely convergent in this circle. Then, the following expression holds from **Theorem 10.3.2 (1)** .

$$\sum_{s=1}^{\infty} (-1)^{s-1} \frac{z^s}{s} = \sum_{r=1}^{\infty} \sum_{s=1}^{r} \frac{q^{r-s}}{(q+1)^{r+1}} \binom{r}{s} (-1)^{s-1} \frac{z^s}{s} \qquad |z| \le 1 \tag{1.3}$$

Aceleration Effect

It is expected that the right side of (1.3) will accelerate the left side. In order to investigate this, we describe the partial sums of (1,1) and (1.2) as follows.

$$f_1(z,m) = \sum_{s=1}^{m} (-1)^{s-1} \frac{z^s}{s} \qquad |z| \le 1 \tag{1.1'}$$

$$f_2(z,q,m) = \sum_{r=1}^{m} \sum_{s=1}^{r} \frac{q^{r-s}}{(q+1)^{r+1}} \binom{r}{s} (-1)^{s-1} \frac{z^s}{s} \tag{1.2'}$$

Then, giving suitable values to z,q and calculating m required for the significant 6 digits, we obtain the following table.

m required for the significant 6 digits

z	$f(z)$	$f_1(z,m)$	$f_2(z,q,m)$		Acceleration Effect
		m	m	q	
-0.9	-2.30258	93	96	1/30	-
			97	1/25	-
			98	1/20	-
0.1	0.0953101	6	5	1/20	0
			7	1/10	-
			9	1/5	-
1	0.693147	about 10^6	15	1/2	+++
			24	1	+++
			41	2	+++

Asymptotic Effect

We cannot obtain the value in $|z|>1$ by (1.2) . It is because (1.2) diverges for such a z . However, using (1.2') , we can obtain the value in $|z|>1$ with very high precision. Indeed, giving suitable values to z,q (where, $|z|>1$) and calculating m required for the significant 6 digits, we obtain the following table.

From this result, it is seems that (1.2) is an asymptotic series in $|z|>1$.

m required for the significant 6 digits

z	$f(z)$	$f_1(z)$	$f_2(z,q,m)$	
			m	q
2	1.09861	$\pm\infty$	20	1
			28	3/2
			35	2
$3+i$	$1.41660+0.244978\,i$	$\pm\infty$	40	4/3
			26	3/2
			33	2

10.4.2 Acceleration of Madhava Series

Madhava series (Gregory–Leibniz series) is as follows.

$$\frac{\pi}{4} = \sum_{s=0}^{\infty} \frac{(-1)^s}{2s+1} \tag{2.1}$$

And applying Knopp Transformation to the right side,

$$\frac{\pi}{4} = \sum_{r=0}^{\infty} \sum_{s=0}^{r} \frac{q^{r-s}}{(q+1)^{r+1}} \binom{r}{s} \frac{(-1)^s}{2s+1} \tag{2.2}$$

Especially when $q=1$, (2.2) is as follows.

$$\begin{aligned}\frac{\pi}{4} &= \frac{1}{2^1}\frac{1}{1} + \frac{1}{2^2}\left(\frac{1}{1}-\frac{1}{3}\right) + \frac{1}{2^3}\left(\frac{1}{1}-\frac{2}{3}+\frac{1}{5}\right) + \frac{1}{2^4}\left(\frac{1}{1}-\frac{3}{3}+\frac{3}{5}-\frac{1}{7}\right) + \cdots \\ &= \frac{1}{2^1}\frac{1}{1} + \frac{1}{2^2}\frac{2}{1\cdot 3} + \frac{1}{2^3}\frac{8}{1\cdot 3\cdot 5} + \frac{1}{2^4}\frac{48}{1\cdot 3\cdot 5\cdot 7} + \cdots \\ &= \frac{1}{2^1}\frac{1}{1} + \frac{1}{2^2}\frac{2}{1\cdot 3} + \frac{1}{2^3}\frac{2\cdot 4}{1\cdot 3\cdot 5} + \frac{1}{2^4}\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7} + \cdots\end{aligned}$$

i.e.

$$\frac{\pi}{4} = \frac{1}{2^1}\frac{0!!}{1!!} + \frac{1}{2^2}\frac{2!!}{3!!} + \frac{1}{2^3}\frac{4!!}{5!!} + \frac{1}{2^4}\frac{6!!}{7!!} + \cdots = \sum_{r=0}^{\infty} \frac{1}{2^{r+1}}\frac{(2r)!!}{(2r+1)!!}$$

This formula was discovered by Euler. Thus, the circular constant π is

$$\pi = 4\sum_{s=0}^{\infty} \frac{(-1)^s}{2s+1} \tag{2.1'}$$

$$= 2\sum_{r=0}^{\infty} \frac{1}{2^r}\frac{(2r)!!}{(2r+1)!!} \tag{2.2'}$$

In order to investigate the acceleration effect of (2.2') , we describe the partial sums of these as follows.

$$f_1(m) = 4\sum_{s=0}^{m} \frac{(-1)^s}{2s+1}$$

$$f_2(m) = 2\sum_{r=0}^{m} \frac{1}{2^r}\frac{(2r)!!}{(2r+1)!!}$$

And calculating m required for the significant 1,001 digits of π , we obtain the following table.

	$f_1(m)$	$f_2(m)$
m	about $10^{1000}/2$	3,330

From these calculations, it is guessed that m required for the significant $k+1$ digits is given by $\frac{10}{3}k$.

10.5 Acceleration of Fourier Series

Since the convergence of Fourier Series is generally slow, the large acceleration effect is obtained by Knopp transformation. Furthermore, the strong asymptotic effect is also seen on the both sides of the convergence interval. Here, Fourier series of $-log(2sin(z/2))$ is taken up as an example.

$$f(z) = -log\left(2sin\frac{z}{2}\right) \tag{1.0}$$

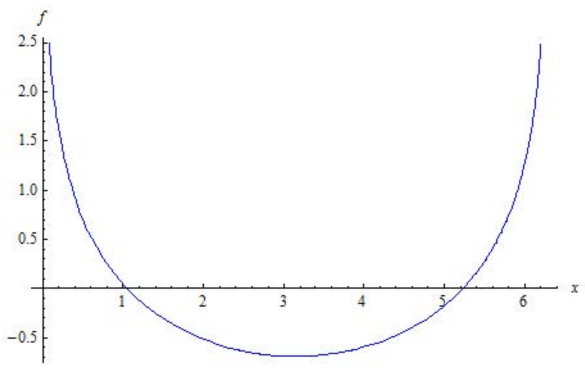
The Fourier series is

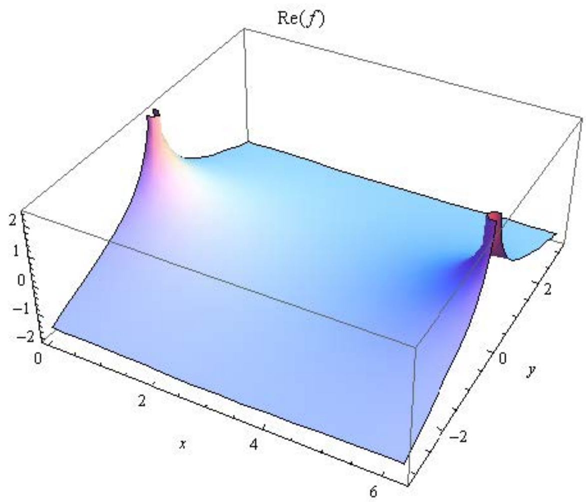
$$f_1(z) = \sum_{s=1}^{\infty}\frac{cos(sz)}{s} \qquad \begin{array}{c} 0<Re(z)<2\pi \\ Im(z) = 0 \end{array} \tag{1.1}$$

Applying Knopp Transformation to this,

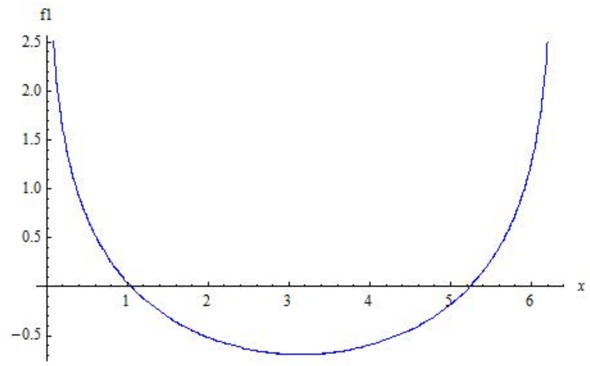
$$f_2(z,q) = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}\frac{cos(sz)}{s} \tag{1.2}$$

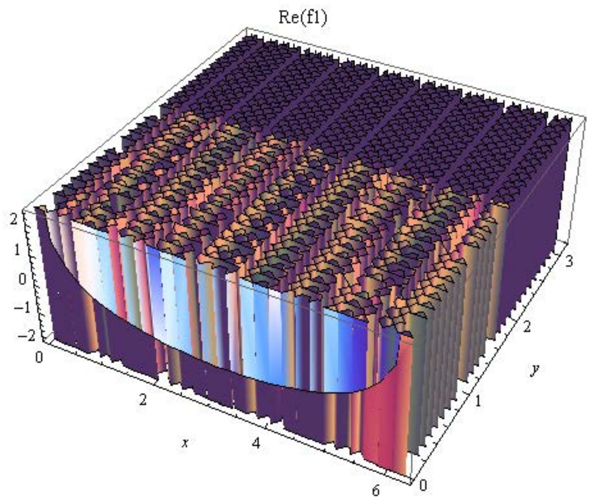
First, function $f(z)$ is illustrated. The left is 2-D figure and the right is 3-D figure of the real number part.





Next, the Fourier series $f_1(z)$ is illustrated. The left is 2-D and the right is 3-D of the real number part.





2-D is exactly the same as the figure above, but, 3-D is very different. This figure shows that the Fourier series $f_1(z)$ is expanded only in the real number interval. This Fourier series $f_1(z)$ converges conditionally in the interval of (1.1) . Then, the following expression holds from **Theorem 10.3.2 (1)** .

$$\sum_{s=1}^{\infty}\frac{cos(sz)}{s} = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}\frac{cos(sz)}{s} \qquad \begin{array}{l}0<Re(z)<2\pi \\ Im(z)=0\end{array} \tag{1.3}$$

Aceleration Efect

It is expected that the right side of (1.3) will accelerate the left side. In order to investigate this, we describe the partial sums of (1,1) and (1.2) as follows.

$$f_1(z,m) = \sum_{s=1}^{m}\frac{cos(sz)}{s} \qquad \begin{array}{l}0<Re(z)<2\pi \\ Im(z)=0\end{array} \tag{1.1'}$$

$$f_2(z,q,m) = \sum_{r=1}^{m}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}\frac{cos(sz)}{s} \tag{1.2'}$$

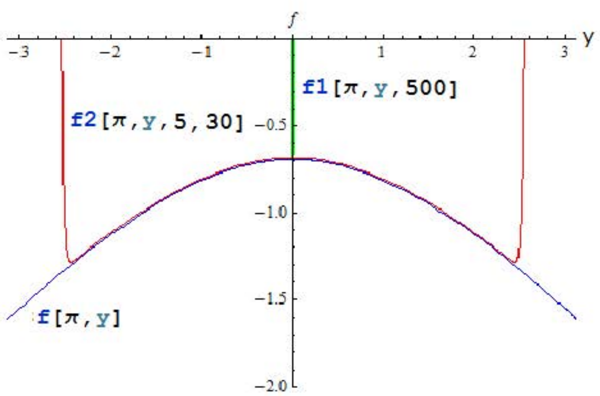
Then, giving suitable values to z,q and calculating m required for the significant 6 digits, we obtain the following table.

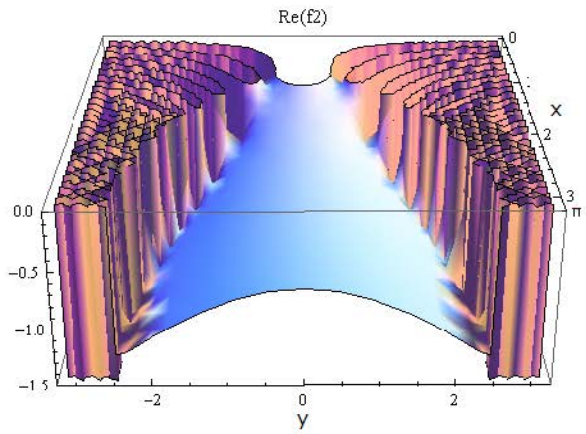
m required for the significant 6 digits

z	$f(z)$	$f_1(z,m)$	$f_2(z,q,m)$		Acceleration Effect
		m	m	q	
1	0.0420195	about 10^7	102	1/2	++
			89	1	++
			101	2	++
π	-0.693147	about 10^6	15	1/2	+++
			22	1	+++
			41	2	+++
5	-0.179771	about 10^6	50	1/2	++
			45	1	++
			51	2	++

Asymptotic Effect

We cannot obtain the value in the complex plane by (1.2) . It is because (1.2) diverges for complex number z . However, using (1.2') , we can obtain the value in the complex plane. To show this, if we put $z = x+iy$ and illustrate $f(x,y), f_1(x,y,m), f_2(x,y,q,m)$, it is as follows.





The right side is 3-D figure and the real number part of $f_2(x,y,5,30)$ is illustrated for $0 < x \le \pi$.

On the left side, by using y as an imaginary axis, $f(\pi), f_1(\pi,y,500), f_2(\pi,y,5,30)$ are drawn with blue, green and red respectively. The left figure shows that $f_2(\pi,y,5,30)$ is the asymptotic expansion of $f(z)$. Of course, if $m \to \infty$, $f_2(\pi,y,5,m)$ (red) moves to the center and overlaps with $f_1(\pi,y,m)$ (green) .

Although it is asymptotic expansion, $f_2(z,p,m)$ can obtain a very precise approximate value. Indeed, giving suitable values to z,q (where, z is a complex number) and calculating m required for the significant 6 digits, we obtain the following table.

m required for the significant 6 digits

z	$f(z)$	$f_2(z,q,m)$	
		m	q
$3+i$	$-0.811290-0.0327592\,i$	67	1
		41	2
		57	3

10.6 Acceleration of Dirichlet Series

If Knopp Transformation is applied to Dirichlet Series, large acceleration effect is seen near the convergence axis. Furthermore, analytic continuation arises beyond the convergence axis. Here, Dirichlet Eta Series is taken up as an example.

Let $\zeta(z)$ be Riemann Zeta Function and

$$\eta(z) = \begin{cases} \left(1-2^{1-z}\right)\zeta(z) & z \neq 1 \\ \log 2 & z = 1 \end{cases} \tag{1.0}$$

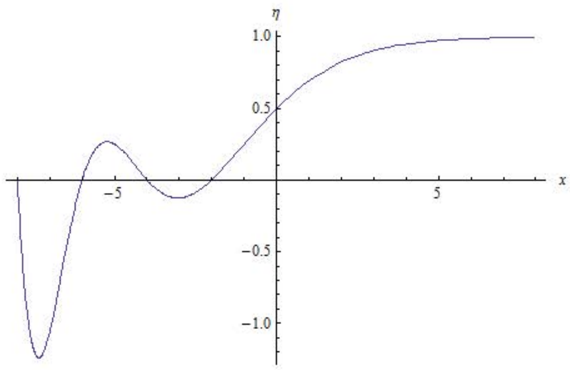
A series of this is called Dirichlet Series and is as follows.

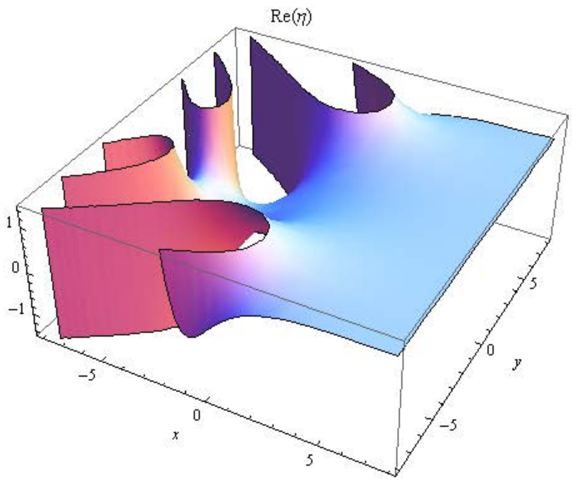
$$f_1(z) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s^z} \qquad Re(z) \geq 0 \tag{1.1}$$

Applying Knopp Transformation to this,

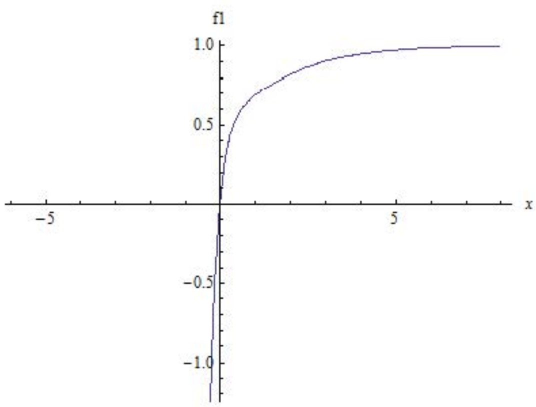
$$f_2(z,q) = \sum_{r=1}^{\infty} \sum_{s=1}^{r} \frac{q^{r-s}}{(q+1)^{r+1}} \binom{r}{s} \frac{(-1)^{s-1}}{s^z} \tag{1.2}$$

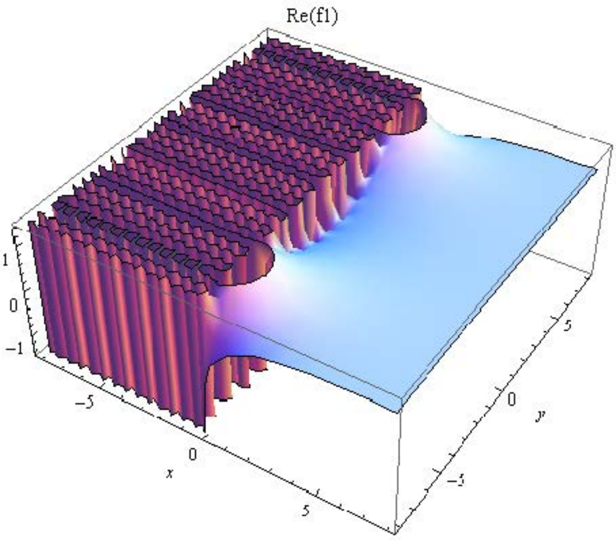
First, Dirichlet Eta function $\eta(z)$ is illustrated. The left is 2-D and the right is 3-D of the real number part.





Next, Dirichlet Eta series $f_1(z)$ is illustrated. The left is 2-D and the right is 3-D of the real number part.





These figures show that $x=0$ is the convergence axis of this Dirichlet Eta series.

This Dirichlet Eta series $f_1(z)$ converges conditionally in the half complex plane $Re(z) \geq 0$. Then, the following expression holds from **Theorem 10.3.2 (1)** .

$$\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^z} = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}\frac{(-1)^{s-1}}{s^z} \qquad Re(z)\ge 0 \tag{1.3}$$

Aceleration Efect

It is expected that the right side of (1.3) will accelerate the left side. In order to investigate this, we describe the partial sums of (1,1) and (1.2) as follows.

$$f_1(z,m) = \sum_{s=1}^{m}\frac{(-1)^{s-1}}{s^z} \qquad Re(z)\ge 0 \tag{1.1'}$$

$$f_2(z,q,m) = \sum_{r=1}^{m}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}\frac{(-1)^{s-1}}{s^z} \tag{1.2'}$$

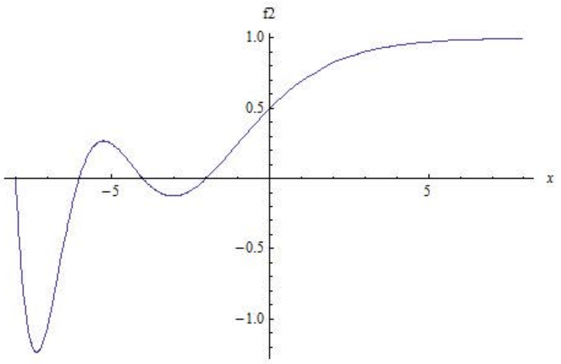
Then, giving suitable values to z,q and calculating m required for the significant 6 digits, we obtain the following table.

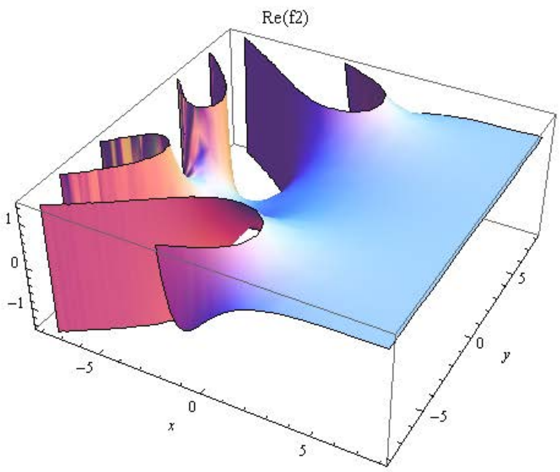
m required for the significant 6 digits

z	$\eta(z)$	$f_1(z,m)$	$f_2(z,q,m)$		Acceleration Effect
		m	m	q	
5	0.972119	18	9	1/6	+
			15	6/13	0
			24	1	-
1	0.693147	about 10^6	15	1/2	+++
			24	1	+++
			41	2	+++
$\frac{1}{2}$	0.604898	about 10^{12}	13	1/2	+++
			21	1	+++
			36	2	+++

Analytic Continuation

When a series converges, the Knopp transformation converges absolutely in the convergence area of the series. However, in the case of this example, (1.2) converges absolutely in the whole complex plane. Indeed, if (1.2) is illustrated, it is as follows.





This figure is exactly the same as the figure of Dirichlet Eta Function $\eta(z)$. A convergence axis is not found anywhere. This figure shows that the following expression holds from **Theorem 10.3.2 (3)** .

$$"\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^z}" = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}\frac{(-1)^{s-1}}{s^z} \qquad z \in C \tag{1.4}$$

In $Re(z)<0$, the right side is not asymptotic series but convergent series. This must be because $f_1(z)$ and $f_1(1-z)$ share a domain in $0 \le Re(z) \le 1$ and are connected by a functional equation. Actually, the value in $Re(z)<0$ is also calculable as follows by (1.2') .

***m* required for the significant 6 digits**

z	$\eta(z)$	$"f_1(z)"$	$f_2(z,q,m)$	
			m	q
-1	0.250000	$"\sum_{s=1}^{\infty}(-1)^{s-1}s^1"$	15	1/2
			1	1
			18	2
-3	-0.125000	$"\sum_{s=1}^{\infty}(-1)^{s-1}s^3"$	22	1/2
			3	1
			19	2
$-3+i$	$-0.268443+0.0300057i$	$"\sum_{s=1}^{\infty}(-1)^{s-1}s^{3+i}"$	25	1/2
			21	1
			36	2

Therefore, $f_1(z)$ have to be interpreted as follows.

$$"1^1-2^1+3^1-4^1+-\cdots" = \frac{1}{4}$$

$$"1^3-2^3+3^3-4^3+-\cdots" = -\frac{1}{8}$$

Definition expression of various function

Hence, the right side of (1.4) can be used as a definition expression of Dirichlet Eta Function $\eta(p)$.

$$\eta(p) = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}\frac{(-1)^{s-1}}{s^p} \qquad q>0$$

Therefore, Riemann Zeta Function $\zeta(p)$ can be defined as follows.

$$\zeta(p) = \frac{1}{1-2^{1-p}}\sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}\frac{(-1)^{s-1}}{s^p} \qquad \begin{array}{l} p \ne 1 \\ q>0 \end{array}$$

Moreover, Tangent Number T_p and Bernouilli Number B_p can be defined respectively in the complex plane as follows. (See " 06 Global definition of Riemann Zeta, and generalization of related coefficients")

$$T_p = \begin{cases} 0 & p = 0 \\ 2^{p+1} \sum_{r=1}^{\infty} \sum_{s=1}^{r} \frac{q^{s+1}}{(q+1)^{r+1}} \binom{r}{s} (-1)^{s-1} s^p & p \neq 0 \\ & q > 0 \end{cases}$$

$$B_p = \begin{cases} -\frac{1}{2} & p = 1 \\ \frac{p}{2^p - 1} \sum_{r=1}^{\infty} \sum_{s=1}^{r} \frac{q^{s+1}}{(q+1)^{r+1}} \binom{r}{s} (-1)^{s-1} s^{p-1} & p \neq 1 \\ & q > 0 \end{cases}$$

10.7 Application to Divergent Series

Knopp transformation is applicable also to a divergent series. Of course, it is not applicable to any divergent series. For example, if this is applied to a divergent nonnegative term series, nothing happens. The divergent series which can be candidate are an oscillating series and an alternating series. Below, wel show the 2 cases.

10.7.1 Application to Oscillating Series

$$f(z) = \frac{1}{2}cot\frac{z}{2} \tag{1.0}$$

The Fourier series (?) is

$$f_1(z) = \sum_{s=1}^{\infty} sin(sz) \qquad \begin{matrix} 0 < Re(z) < 2\pi \\ Im(z) = 0 \end{matrix} \tag{1.1}$$

Applying Knopp Transformation to this,

$$f_2(z,q) = \sum_{r=1}^{\infty}\sum_{s=1}^{r} \frac{q^{r-s}}{(q+1)^{r+1}} \binom{r}{s} sin(sz) \tag{1.2}$$

In addition, (1.0) and (1.1) were obtained by differentiating with respect to z both sides of the following expression in **10.5** .

$$-log\left(2sin\frac{z}{2}\right) = \sum_{s=1}^{\infty}\frac{cos(sz)}{s} \qquad \begin{matrix} 0 < Re(z) < 2\pi \\ Im(z) = 0 \end{matrix}$$

Summation Method

Fourier series (1.1) oscillates.Then, the value is not decided therefore is divergent. Naturally, $f(z) \neq f_1(z)$.

Nevertheless, this Konpp transformation (1.2) converges absolutely for any $q>0$ in the interval of (1.1) .

Then, the following expression holds from **Theorem 10.3.2 (2)** .

$$"\sum_{s=1}^{\infty} sin(sz)" = \sum_{r=1}^{\infty}\sum_{s=1}^{r} \frac{q^{r-s}}{(q+1)^{r+1}} \binom{r}{s} sin(sz) \qquad \begin{matrix} 0 < Re(z) < 2\pi \\ Im(z) = 0 \end{matrix}$$

In order to investigate this, we describe the partial sum of (1.2) as follows.

$$f_2(z,q,m) = \sum_{r=1}^{m}\sum_{s=1}^{r} \frac{q^{r-s}}{(q+1)^{r+1}} \binom{r}{s} sin(sz) \tag{1.2'}$$

Then, giving suitable values to z and calculating the optimal m,q for obtaining the significant 6 digits, we obtain the following table.

Optimal m,q for obtaining the significant 6 digits

z	$f(z)$	$"f_1(z)"$	$f_2(z,q,m)$	
			m	q
$\frac{\pi}{4}$	1.20710	$"\sum_{s=1}^{\infty} sin\frac{s\pi}{4}"$	155	1
$\frac{\pi}{3}$	0.866025	$"\sum_{s=1}^{\infty} sin\frac{s\pi}{3}"$	98	1

z	$f(z)$	$"f_1(z)"$	$f_2(z,q,m)$	
			m	q
$\frac{\pi}{2}$	0.500000	$"\sum_{s=1}^{\infty} \sin\frac{s\pi}{2}"$	42	1
1	0.915243	$"\sum_{s=1}^{\infty} \sin s"$	121	1

Since $"f_1(z)" = f_2(z,q) = f(z)$, these mean as follows.

$$"\frac{1}{\sqrt{2}}+1+\frac{1}{\sqrt{2}}+0-\frac{1}{\sqrt{2}}-1-\frac{1}{\sqrt{2}}-0 \;++++---- \;\cdots" = \frac{1}{2}cot\frac{\pi}{8} = \frac{1+\sqrt{2}}{2}$$

$$"\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+0-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}-0 \;+++--- \;\cdots" = \frac{1}{2}cot\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$"1+0-1-0 \;++-- \;\cdots" = \frac{1}{2}cot\frac{\pi}{4} = \frac{1}{2}$$

$$"sin\,1+sin\,2+sin\,3+sin\,4+\cdots" = \frac{1}{2}cot\frac{1}{2}$$

That is, Fourier series (?) $f_1(z)$ have to be interpreted in this way by a certain summation method.

Asymptotic Effect

As well as **10.5** , (1.2') is asymptotic expansion of (1.0) in the complex plane. And it can obtain a very precise approximate value. Indeed, giving a suitable complex number to z and calculating the optimal m,q for obtaining the significant 6 digits, we obtain the following table.

Optimal m,q for obtaining the significant 6 digits

z	$f(z)$	$f_2(z,q,m)$	
		m	q
$2+i$	$0.232055-0.299914\,i$	50	3

10.7.2 Application to Divergent Alternating Series

Let us consider the following series.

$$f_1(n) = \sum_{s=\frac{n-1}{2}\uparrow}^{\infty} (-1)^s \frac{(2s)!}{(2s+1-n)!} \qquad n=1,2,3,\cdots$$

When $n=3,4$,

$$f_1(3) = \sum_{s=1}^{\infty} (-1)^s \frac{(2s)!}{(2s-2)!} = -1\cdot 2 + 3\cdot 4 - 5\cdot 6 + 7\cdot 8 -+ \cdots$$

$$f_1(4) = \sum_{s=2}^{\infty} (-1)^s \frac{(2s)!}{(2s-3)!} = 2\cdot 3\cdot 4 - 4\cdot 5\cdot 6 + 6\cdot 7\cdot 8 - 8\cdot 9\cdot 10 +- \cdots$$

These are clearly divergent series.

Nevertheless, these Konpp transformations converge absolutely for any $q>0$. And they become as follows.

$$f_2(3,q) = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}(-1)^s\frac{(2s)!}{(2s-2)!} = \frac{1}{2}$$

$$f_2(4,q) = \sum_{r=2}^{\infty}\sum_{s=2}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}(-1)^s\frac{(2s)!}{(2s-3)!} = 0$$

Since the domain of these is all naural numbers, these have to be interpreted as follows from **Theorem 10.3.2 (2)** .

$$"1\cdot 2 - 3\cdot 4 + 5\cdot 6 - 7\cdot 8 +- \cdots" = -\frac{1}{2}$$

$$"2\cdot 3\cdot 4 - 4\cdot 5\cdot 6 + 6\cdot 7\cdot 8 - 8\cdot 9\cdot 10 +- \cdots" = 0$$

In fact, these are obtained also by putting $x=1$ in the following formula, ignoring the convergence condition. (See " 11 Termwise Higher Derivative (Inv-Trigonometric, Inv-Hyperbolic) ")

$$\left(\tan^{-1}x\right)^{(n)} = \sum_{k=\left\lceil\frac{n-1}{2}\right\rceil}^{\infty}(-1)^k\frac{(2k)!}{(2k+1-n)!}x^{2k+1-n} \qquad |x|<1$$

$$\left(\tan^{-1}x\right)^{(n)} = (-1)^n\frac{(n-1)!}{\left(x^2+1\right)^n}\sum_{r=1}^{\lceil n/2\rceil}(-1)^r{}_nC_{n+1-2r}\,x^{n+1-2r}$$

This example gives legitimacy to the convergence condition is ignored.

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Kano. Kono

Alien's Mathematics