Analytical Proof of the Riemann Hypothesis

Abstract

(1) The problem of Zeros of the Riemann zeta function is reduced to the system of transcendental equations consisting of 4 equations with 2 real variables, by functional equation.

(2) On the critical line, certain 2 equations are identically 0, and the remaining 2 equations have simultaneous solutions.

(3) Except on the critical line, the two equations do not have simultaneous solutions in the critical strip. This can be proved analytically by mediating the primitive functions of these expressions.

(4) As a result of (3), the system of transcendental equations of (1) have no solution in the critical strip except on the critical line. Thus, the Riemann Hypothesis holds true.

1 Introduction

Riemann Zeta Function

Riemann Zeta Function $\zeta(z)$ is defined by the following Dirichlet series.

$$\zeta(z) = \sum_{r=1}^{\infty} e^{-z\log r} = \frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\cdots \qquad Re(z)>1 \qquad (1.\zeta)$$

This function is analytically continued to $Re(z)<1$, and has trivial zeros $z=-2n\ (n=1,2,3,\cdots)$ and **non-trivial zeros** $z=1/2\pm b_n\ (n=1,2,3,\cdots)$. So, it is the Riemann hypothesis that there will be no non-trivial zeros other than these. In addition, it is known that non-trivial zeros exist only in the **critical strip** $\mathbf{0<Re(z)<1}$. Also, the center line $\mathbf{Re(z)=1/2}$ is called the **critical line**.

Dirichlet Eta Function

Dirichlet Eta Function $\eta(z)$ is defined by the following Dirichlet series.

$$\eta(z) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-z\log r} = \frac{1}{1^z}-\frac{1}{2^z}+\frac{1}{3^z}-\frac{1}{4^z}+-\cdots \qquad Re(z)>0 \qquad (1.\eta)$$

It is known that this series converges conditionally for $0<Re(z)\le 1$.

This function is analytically continued to $Re(z)\le 0$, and has the following relation to $\zeta(z)$.

$$\zeta(z) = \frac{1}{1-2^{1-z}}\eta(z) \qquad z\neq 1$$

Therefore $\zeta(z)$ and $\eta(z)$ share trivial and non-trivial zeros. In addition, $\eta(z)$ has **$\eta(z)$-specific zeros** $\mathbf{z=1\pm 2n\pi/\log 2\ (n=1,2,3,\cdots)}$. These are the zeros of $1-2^{1-z}=0$.

Dirichlet Series to use

The right-hand sides of $(1.\zeta)$ and $(1.\eta)$ are called Dirichlet series. $(1.\zeta)$, which is the definition of $\zeta(z)$, is not suitable for analysis in the critical strip. This is because even if the Euler transformation or the like is applied, it will only be an asymptotic expansion. On the other hand, $(1.\eta)$, which is the definition of $\eta(z)$, can be used as it is in the critical strip. So, in this paper, we use $(1.\eta)$ to analyze the zeros of the Riemann zeta function $\zeta(z)$.

2 Zeros of $\eta(z)$ and System of Equations

In this chapter, we consider the problem of zeros of the Dirichlet Eta function $\eta(z)$ from the point of view of the system of equations.

Lemma 2.1

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy\ ,\ x,y\in R)$, $\eta(z)=0$ in $0<x<1$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \eta(z) = \sum_{r=1}^{\infty}(-1)^{r-1}e^{-z\log r} = 0 & (2.1_+) \\ \eta(1-z) = \sum_{r=1}^{\infty}(-1)^{r-1}e^{-(1-z)\log r} = 0 & (2.1_-) \end{cases}$$

Proof

The following functional equation holds for the Dirichlet Eta function $\eta(z)$.

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left(1-2^{z}\right)\eta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\left(1-2^{1-z}\right)\eta(1-z) \qquad 0<Re(z)<1$$

Gamma function and powers of π have no zeros, and $1-2^{z}\ ,1-2^{1-z}$ have no zeros in $0<Re(z)<1$
Therefore, at the zeros of $\eta(z)$,

$$\eta(z)=\eta(1-z)=0 \qquad 0<Re(z)<1$$

Representing $\eta(z)\ ,\eta(1-z)$ by Dirichlet series respectively, we obtain the desired expressions.

Note1

Since there are 2 equations for 1 complex variable in the lemma, this system of equations is an overdetermined system. Such a system of equations generally has no solution. What forces this overdetermined system is the functional equation clearly.

Note2

(1) When $x=1/2$, the overdetermined property disappears. Because,

$$\begin{cases} \eta(1/2+iy) = \sum_{r=1}^{\infty}(-1)^{r-1}e^{-(1/2+iy)\log r} = 0 & (2.1_+) \\ \eta(1/2-iy) = \sum_{r=1}^{\infty}(-1)^{r-1}e^{-(1/2-iy)\log r} = 0 & (2.1_-) \end{cases}$$

i.e.

$$\begin{cases} \eta(1/2+iy) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\{\cos(y\log r) - i\sin(y\log r)\} = 0 & (2.1_+) \\ \eta(1/2-iy) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\{\cos(y\log r) + i\sin(y\log r)\} = 0 & (2.1_-) \end{cases}$$

At zero point $(1/2\ ,y)$,

$$-\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\sin(y\log r) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\sin(y\log r) = 0$$

So, (2.1_+) and (2.1_-) become substantially the same equation.

(2) When $x \neq 1/2$, This system of equations is an overdetermined system.

Even though (2.1_+) and (2.1_+) are different equations, they must share one complex number. The Riemann hypothesis says that such a thing will not happen.

Replacing z with $1/2+z$ in Lemma 2.1 , we obtain the following equivalent lemma.

Lemma 2.1'

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy\ ,\ x,y \in R)$, $\eta(1/2 \pm z)=0$ in $-1/2<x<1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \eta\left(\dfrac{1}{2}+z\right) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} e^{-z\log r} = 0 & (2.1'_+) \\ \eta\left(\dfrac{1}{2}-z\right) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} e^{z\log r} = 0 & (2.1'_-) \end{cases}$$

Note

(1) The known non-trivial zeros are moved parallel onto the **new critical line** $Re(z) = 0$

(2) When $-1/2<x<1/2$, these series conditionally converge .

(3) When $x=0$, the overdetermined property disappears.

(4) When $x \neq 0$, if there are zeros, the one set consists of the following four.

$$a \pm ib\ ,\quad -a \pm ib \qquad (\ -1/2 < a < 1/2\ )$$

Hyperbolic Function Series

Lemma 2.1' is equivalent to the following

Lemma 2.2

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy\ ,\ x,y \in R)$, $\eta(1/2 \pm z)=0$ in $-1/2<x<1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \eta_c(z) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \cosh(z\log r) = 0 & (2.2c) \\ \eta_s(z) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(z\log r) = 0 & (2.2s) \end{cases}$$

Proof

From (2.1'+) , (2.1'-) ,

$$\frac{1}{2}\left\{\eta\left(\frac{1}{2}-z\right)+\eta\left(\frac{1}{2}+z\right)\right\} = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \frac{e^{z\log r}+e^{-z\log r}}{2}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \cosh(z\log r) = 0$$

$$\frac{1}{2}\left\{\eta\left(\frac{1}{2}-z\right)-\eta\left(\frac{1}{2}+z\right)\right\} = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\frac{e^{z\log r}-e^{-z\log r}}{2}$$

$$= \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\sinh\{z\log r\} = 0$$

Describing these as $\eta_c(z), \eta_s(z)$ respectively, we obtain the desired expressions.
Conversely, by adding or subtracting these, (2.1'+) , (2.1'-) are obtained.

Hyperbolic Function Series (real part, imaginary part)

Theorem 2.3

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z = x+iy\,,\ x,y \in R)$, $\eta(1/2 \pm z) = 0$ in $-1/2 < x < 1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} u_c(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\cosh(x\log r)\cos(y\log r) = 0 \\ v_c(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\sinh(x\log r)\sin(y\log r) = 0 \\ u_s(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\sinh(x\log r)\cos(y\log r) = 0 \\ v_s(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\cosh(x\log r)\sin(y\log r) = 0 \end{cases}$$

Proof

$$\cosh(x+iy) = \cosh x\cos y + i\sinh x\sin y$$
$$\sinh(x+iy) = \sinh x\cos y + i\cosh x\sin y$$

Replacing x with $x\log r$ and y with $y\log r$ respectively,

$$\cosh(z\log r) = \cosh(x\log r)\cos(y\log r) + i\sinh(x\log r)\sin(y\log r)$$
$$\sinh(z\log r) = \sinh(x\log r)\cos(y\log r) + i\cosh(x\log r)\sin(y\log r)$$

Substituting these for (2.2c) , (2.2s) respectively,

$$\eta_c(z) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\cosh(z\log r)$$

$$= \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\{\cosh(x\log r)\cos(y\log r) + i\sinh(x\log r)\sin(y\log r)\}$$

$$\eta_s(z) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\sinh(z\log r)$$

$$= \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\{\sinh(x\log r)\cos(y\log r) + i\cosh(x\log r)\sin(y\log r)\}$$

Describing the real and imaginary parts as $u_c(x,y)$, $v_c(x,y)$, $u_s(x,y)$, $v_s(x,y)$ respectively, we obtain the desired expressions.

Overdetermined System

Since there are 4 equations for 2 real variable in Theorem 2.3 , this system of equations is an overdetermined system. Such a system of equations generally has no solution.

Zeros on the Critical Line

However, such a system of equations may exceptionally has solution. That is the case when $x=0$. Note that $x=0$ is the critical line of function $\eta(1/2+z)$. Substituting $x=0$ for the equations in Theorem 2.3

$$\begin{cases} u_c(0,y) = 1\sum_{r=1}^{\infty}\dfrac{(-1)^{r-1}}{\sqrt{r}}\cos(y\log r) = 0 \\ v_c(0,y) = 0\sum_{r=1}^{\infty}\dfrac{(-1)^{r-1}}{\sqrt{r}}\sin(y\log r) = 0 \\ u_s(0,y) = 0\sum_{r=1}^{\infty}\dfrac{(-1)^{r-1}}{\sqrt{r}}\cos(y\log r) = 0 \\ v_s(0,y) = 1\sum_{r=1}^{\infty}\dfrac{(-1)^{r-1}}{\sqrt{r}}\sin(y\log r) = 0 \end{cases}$$

Since $v_c(0,y)$, $u_s(0,y)$ are equal to non-existent, the overdetermined property disappears. As the result,

$$0 = u_c(0,y) - i\,v_s(0,y) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\{\cos(y\log r) - i\sin(y\log r)\}$$
$$= \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\{\cos(y\log r) + i\sin(y\log r)\}$$

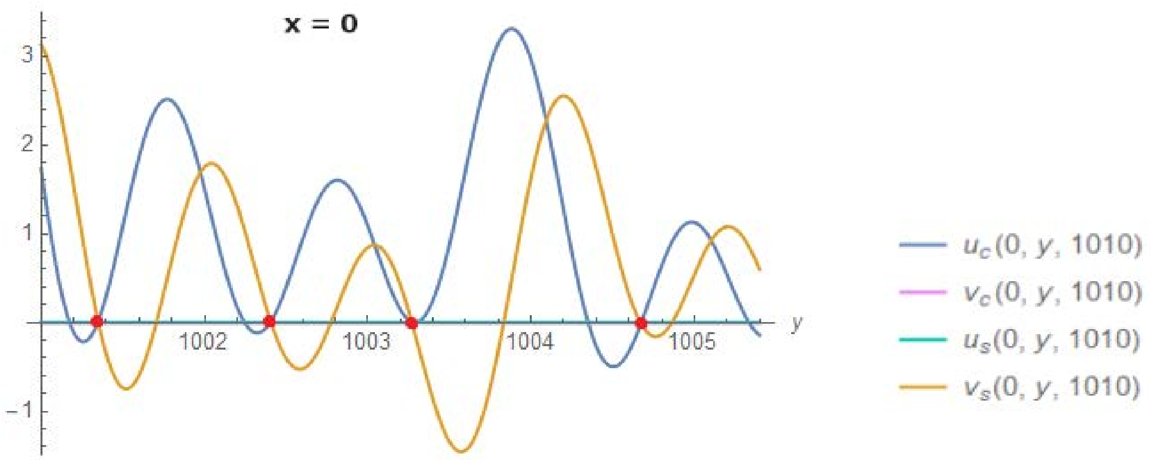
i.e.

$$0 = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}e^{-y\log r} = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}e^{y\log r}$$

That is, this results in the case of $x=0$ in Lemma 2.1' .

It is known that non-trivial zeros of the Riemann zeta function do not exist up to very large y values in the critical strip ($-1/2 < x < 1/2$ in this paper) . So, in the following examples, $y=1001\sim1005.4$ is used.

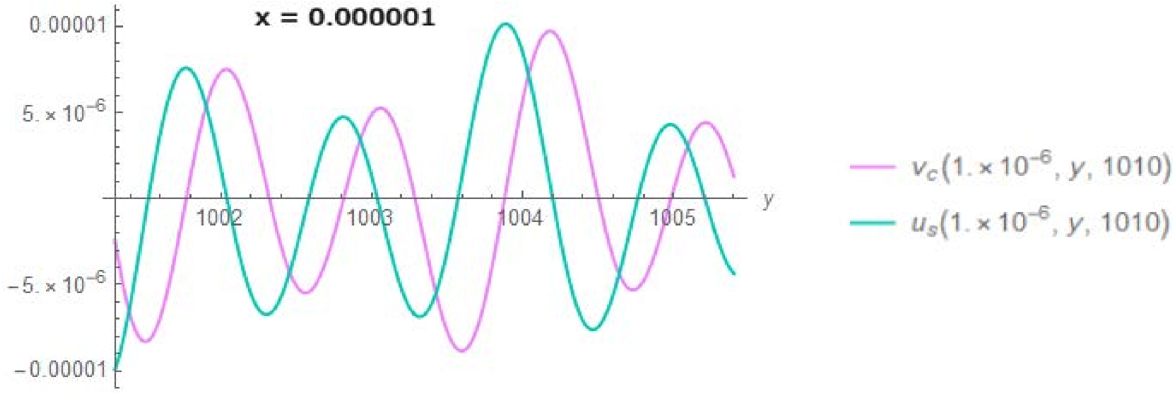
When $x=0$, $u_c \sim v_s$ are drawn as follows. Blue is u_c and orange is v_s .



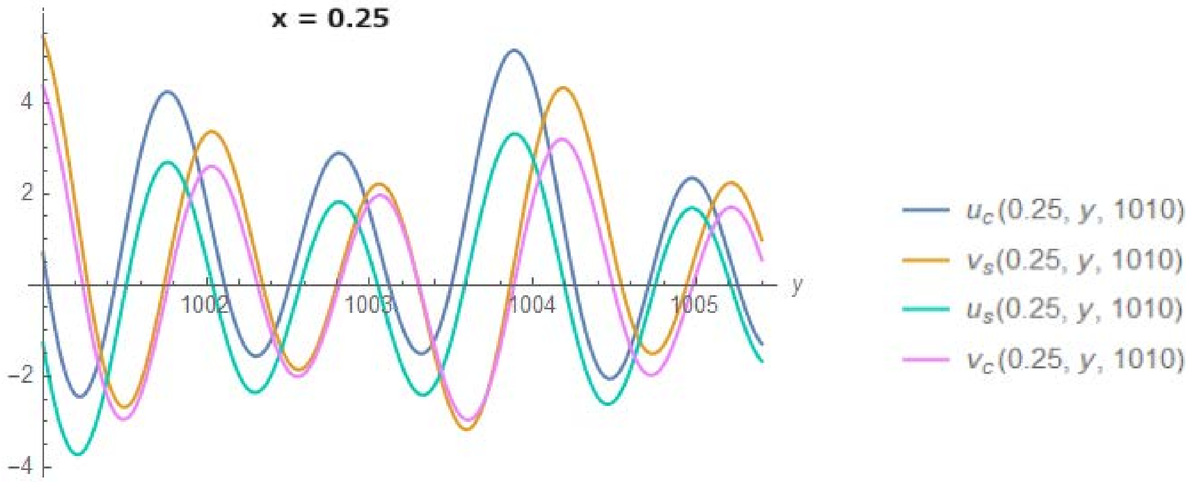
The points (red) where these intersect on the y -axis are the zeros of $\eta(1/2\pm z)$. Magenta is v_c and cyan is u_s . They overlap on the y -axis. Of course, these 2 straight lines also pass through the red points.

Zeros outside the Critical Line

If x deviates even slightly from 0 , v_c , u_s cease to be straight lines. For example, when $x = 0.000001$,

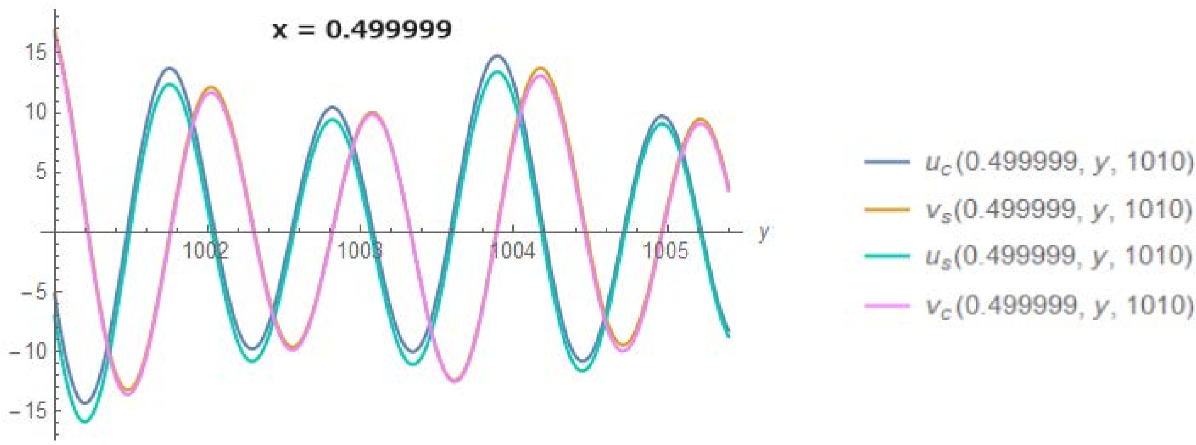


As the result, the property of overdetermination is restored. For example, when $x = 0.25$, $u_c \sim v_s$ are



The amplitudes of v_c , u_s are expanding, and the 4 curves are unlikely to intersect at one point on the y -axis.

When $x = 0.499999$ (near the boundary of the critical strip) , $u_c \sim v_s$ are drawn as follows.



The peaks and valleys of u_c (blue) and u_s (cyan) are almost the same, and the functions themselves of v_c (magenta) and v_s (yellow) are almost the same. These results are due to the fact that the difference between $cosh(x\log r)$ and $sinh(x\log r)$ decreases as x increases.

Note

When $x \geq 0.5$, in the interval where y is very large, it becomes as follows.

$$u_c(x,y) \approx u_s(x,y) \quad , \quad v_s(x,y) \approx v_c(x,y)$$

Although such a drawing is not possible using the series in Theorems 2.3 , it is possible by using the $\eta(z)$ calculation routine of the formula manipulation software ***Mathematica*** according to the following equations.

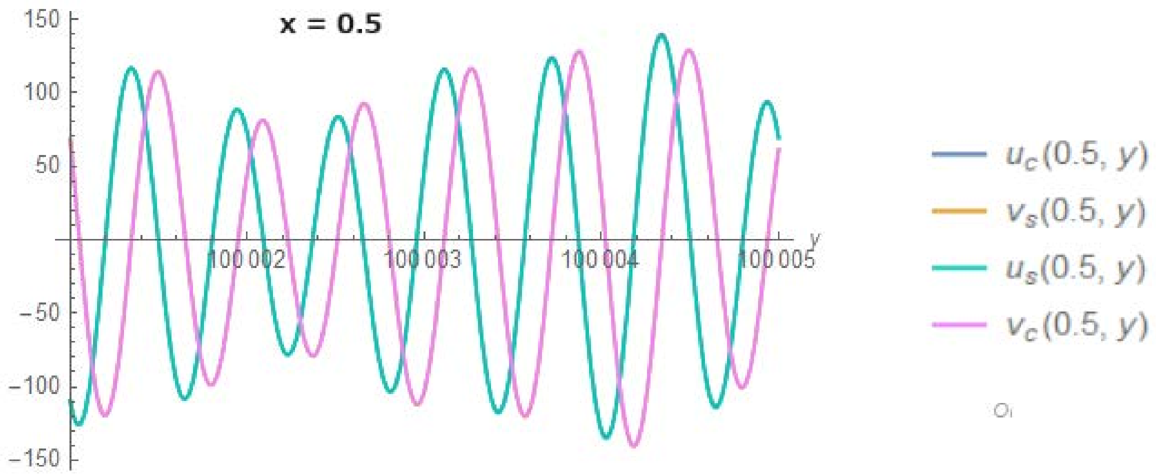
$$u_c(x,y) = \frac{1}{2}\left[Re\left\{ \eta\left(\frac{1}{2} - x - iy \right) \right\} + Re\left\{ \eta\left(\frac{1}{2} + x + iy \right) \right\} \right]$$

$$v_c(x,y) = \frac{1}{2}\left[Im\left\{ \eta\left(\frac{1}{2} - x - iy \right) \right\} + Im\left\{ \eta\left(\frac{1}{2} + x + iy \right) \right\} \right]$$

$$u_s(x,y) = \frac{1}{2}\left[Re\left\{ \eta\left(\frac{1}{2} - x - iy \right) \right\} - Re\left\{ \eta\left(\frac{1}{2} + x + iy \right) \right\} \right]$$

$$v_s(x,y) = \frac{1}{2}\left[Im\left\{ \eta\left(\frac{1}{2} - x - iy \right) \right\} - Im\left\{ \eta\left(\frac{1}{2} + x + iy \right) \right\} \right]$$

When $x=0.5$, $y=100001 \sim 100005$, $u_c \sim v_s$ are drawn as follows.



u_c and u_s overlap exactly, and v_s and v_c also overlap exactly. As the result, only u_s (cyan) and v_c (magenta) are visible.

3 Lemma about a System of Equations

Theorem 2.3 is equivalent to that the following 6 pairs have a common solution. Each pair is one of the necessary conditions for $\eta(1/2+z)$ to have zeros.

$$\begin{cases} u_c = 0 \\ v_c = 0 \end{cases}, \quad \begin{cases} u_c = 0 \\ u_s = 0 \end{cases}, \quad \begin{cases} u_c = 0 \\ v_s = 0 \end{cases}, \quad \begin{cases} v_c = 0 \\ u_s = 0 \end{cases}, \quad \begin{cases} v_c = 0 \\ v_s = 0 \end{cases}, \quad \begin{cases} u_s = 0 \\ v_s = 0 \end{cases}$$

Therefore, to prove the Riemann hypothesis, it is sufficient to show that any one of these pairs does not have a solution such as $x \neq 0$.

The most interesting of these is $v_c = 0$ and $u_s = 0$ pair. The reason is as follows.

(1) When $x = 0$, $v_c = u_s = 0$ *for any* y.

(2) v_c and u_s series share a coefficient part $\sinh(x \log r)/\sqrt{r}$ that has a large effect on the amplitude.

(3) v_c and u_s series have a first term $(r = 1)$ of 0.

In particular, from (3), we can change the first terms of the v_c and u_s series from $r = 1$ to $r = 2$. That is

$$v_c(x,y) = \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x \log r) \sin(y \log r) \tag{3.1c}$$

$$u_s(x,y) = \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x \log r) \cos(y \log r) \tag{3.1s}$$

As the result, we can prove the following lemma for both expressions.

Lemma 3.1

When y is a real number, x is a real number s.t. $-1/2 < x < 1/2$, the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} v_c(x,y) = \displaystyle\sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x \log r) \sin(y \log r) = 0 & (3.1c) \\ u_s(x,y) = \displaystyle\sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x \log r) \cos(y \log r) = 0 & (3.1s) \end{cases}$$

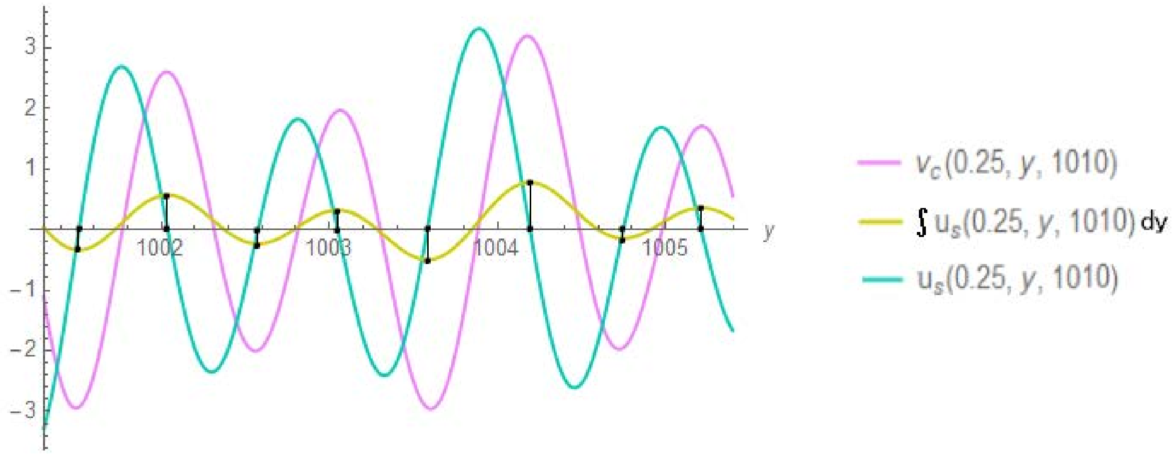
Proof

1. Since the first term of the series (3.1s) is $r = 2$, term-wise integration is possible for both x and y.

So, integrating this term by term from 0 to y with respect to y,

$$\int u_s(x,y)\,dy = \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}\,\log r} \sinh(x \log r) \sin(y \log r) \tag{3.1sy}$$

When $x = 0.25$, $y = 1001.3 \sim 1005.4$, the 2D figures of (3.1c), (3.1s) and (3.1sy) are drawn on the next page. Magenta is $v_c(0.25, y)$, yellow is $\int u_s(0.25, y)\,dy$, and cyan is $u_s(0.25, y)$.



The y coordinates of the peaks and valleys of $v_c(0.25, y)$ (magenta) and $\int u_s(0.25, y)\,dy$ (yellow) almost match. For example, the first valleys of $v_c(0.25, y)$ and $\int u_s(0.25, y)\,dy$ are $y = 1001.49$ and $y = 1001.51$ respectively, and the difference between them is as small as 0.02. This is due to the fact that both terms are $\sin(y \log r)$.

The y coordinates of the peaks and valleys of $\int u_s(0.25, y)\,dy$ (yellow) and the zeros of $u_s(0.25, y)$ (cyan) exactly match. This is natural since the latter is the derivative of the former with respect to y.

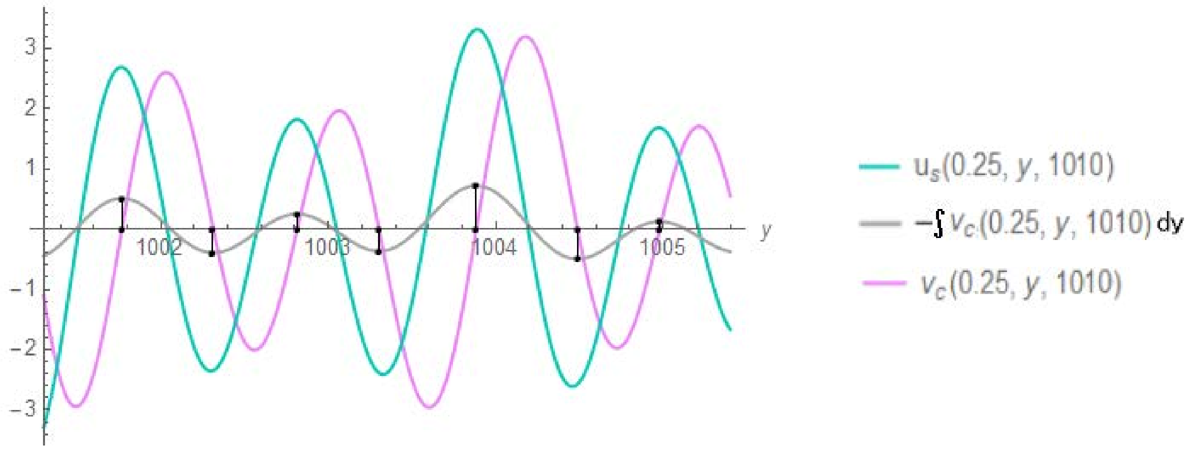
So, the y coordinates of the peaks and valleys of $v_c(0.25, y)$ (magenta) and the zeros of $u_s(0.25, y)$ (cyan) almost match. This also holds true for any $-1/2 < x < 1/2$, $x \neq 0$.

2. Since the first term of the series (3.1c) is $r = 2$, term-wise integration is possible for both x and y.

So, integrating this term by term from 0 to y with respect to y,

$$\int v_c(x,y)\,dy = -\sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}\,\log r} \sinh(x \log r)\cos(y \log r) \tag{3.1cy}$$

When $x = 0.25$, $y = 1001.3 \sim 1005.4$, the 2D figures of (3.1c), (3.1s) and (3.1cy) are drawn as follows. Magenta is $v_c(0.25, y)$, gray is $-\int v_c(0.25, y)\,dy$, and cyan is $u_s(0.25, y)$.



The y coordinates of the peaks and valleys of $u_s(0.25, y)$ (cyan) and $-\int v_c(0.25, y)\,dy$ (gray) almost match. For example, the last peaks of $u_s(0.25, y)$ and $-\int v_c(0.25, y)\,dy$ are $y = 1004.97$ and $y = 1004.98$ respectively, and the difference between them is as small as 0.01. This is due to the fact that both terms are $\cos(y \log r)$.

The y coordinates of the peaks and valleys of $-\int v_c(0.25, y)\,dy$ (gray) and the zeros of $v_c(0.25, y)$ (magenta) exactly match. This is natural since the latter is the derivative of the former with respect to y.

So, the y coordinates of the peaks and valleys of $u_s(0.25, y)$ (cyan) and the zeros of $v_c(0.25, y)$ (magenta) almost match. This also holds true for any $-1/2 < x < 1/2\ , x \neq 0$.

3. As the result of 1 and 2, for any $-1/2 < x < 1/2\ , x \neq 0$, the zeros of $v_c(x, y)$ and the zeros of $u_s(x, y)$ exist alternately on the y-axis. That is, $v_c(x, y)$ and $u_s(x, y)$ do not have common zeros in $-1/2 < x < 1/2\ , x \neq 0$.

Q.E.D.

4 Proof of the Riemann Hypothesis

In this chapter, we will prove the Riemann hypothesis by organizing and summarizing the above.

Theorem 4.1 (Riemann Hypothesis)

Let $\zeta(z)$ be the function defined by the following Dirichlet series.

$$\zeta(z) = \sum_{r=1}^{\infty} e^{-z\log r} = \frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\cdots \qquad Re(z) > 1 \qquad (1.\zeta)$$

This function has no non-trivial zeros except on the critical line $Re(z)=1/2$.

Proof

Dirichlet Eta Function $\eta(z)$ is defined by the following Dirichlet series.

$$\eta(z) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-z\log r} = \frac{1}{1^z}-\frac{1}{2^z}+\frac{1}{3^z}-\frac{1}{4^z}+-\cdots \qquad Re(z) > 0 \qquad (1.\eta)$$

This function is analytically continued to $Re(z) \le 0$, and has the following relation to $\zeta(z)$.

$$\zeta(z) = \frac{1}{1-2^{1-z}}\eta(z) \qquad z\neq 1$$

Therefore, the non-trivial zeros of $\zeta(z)$ and $\eta(z)$ coincide in the critical strip $0<Re(z)<1$.

First, by functional equation, the solution for $\eta(z)=0$ is consistent with the solution of the following system of equations. (Lemma 2.1)

$$\begin{cases} \eta(z) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-z\log r} = 0 \\ \eta(1-z) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-(1-z)\log r} = 0 \end{cases} \qquad 0 < Re(z) < 1$$

Second, by translation, the solution for $\eta(1/2+z)=0$ is consistent with the solution of the following system of equations. (Lemma 2.1 ')

$$\begin{cases} \eta\left(\frac{1}{2}+z\right) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} e^{-z\log r} = 0 \\ \eta\left(\frac{1}{2}-z\right) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} e^{z\log r} = 0 \end{cases} \qquad -\frac{1}{2} < Re(z) < \frac{1}{2}$$

Third, by addition and subtraction, the solution for $\eta(1/2+z)=0$ is consistent with the solution of the following system of equations. (Lemma 2.2)

$$\begin{cases} \eta_c(z) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \cosh(z\log r) = 0 \\ \eta_s(z) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(z\log r) = 0 \end{cases} \qquad -\frac{1}{2} < Re(z) < \frac{1}{2}$$

Last, expressing these by real and imaginary parts, we obtain the following theorem.

Theorem 2.3 (reprint)

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy,\ x,y\in R)$, $\eta(1/2\pm z)=0$ in $-1/2<x<1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} u_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \cosh(x\log r)\cos(y\log r) = 0 \\ v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x\log r)\sin(y\log r) = 0 \\ u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x\log r)\cos(y\log r) = 0 \\ v_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \cosh(x\log r)\sin(y\log r) = 0 \end{cases}$$

According to this theorem, if a system of equations consisting of any two of these equations does not have a solution in the critical strip except on the critical line, the Riemann hypothesis holds.

So, if we focus on the pair $v_c(x,y) = u_s(x,y) = 0$, both of these first terms $(r=1)$ are 0. Therefore, we can change the index of the first term from $r=1$ to $r=2$.

Then, $v_c(x,y)$ and $u_s(x,y)$ can be integrated term by term from 0 to y with respect to y.

In the previous chapter, Lemma 3.1 was proven using this fact.

Lemma 3.1 (reprint)

When y is a real number, x is a real number s.t. $-1/2 < x < 1/2$, the following system of equations has no solution such as $x \neq 0$.

$$\begin{cases} v_c(x,y) = \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x\log r)\sin(y\log r) = 0 & (3.1c) \\ u_s(x,y) = \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x\log r)\cos(y\log r) = 0 & (3.1s) \end{cases}$$

Thus, according to Theorem 2.3, $\eta(1/2+z)$ has no zeros other than $x=0$ in $-1/2 < x < 1/2$.

That is, Dirichlet eta function $\eta(z)$ has no zeros other than $x=1/2$ in $0 < x < 1$.

Therefore, Riemann zeta function $\zeta(z)$ also has no zeros other than $x=1/2$ in $0 < x < 1$.

Q.E.D.

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Kano Kono

Hiroshima, Japan

Alien's Mathematics