15 Higher and Super Calculus of Elliptic Integral

15.1 Double series expansion of Elliptic Integral

15.1.1 Double series expansion of Elliptic Integral of the 1st kind

Formula 15.1.1

The following expressions hold for $|k| \leqq 1$, $|x| \leqq 1$.

$$F(x,k) = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r}{2r+1}\binom{-1/2}{r-s}\binom{-1/2}{s} k^{2s}x^{2r+1} \tag{1.1}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{1}{2r+1}\frac{(2r-2s-1)!!}{(2r-2s)!!}\frac{(2s-1)!!}{(2s)!!} k^{2s}x^{2r+1} \tag{1.1'}$$

Proof

Since $|k| \leqq 1$, $|x| \leqq 1$, from Generalized Binomial Theorem (See **3.2**) .

$$(1-x^2)^{-\frac{1}{2}} = \sum_{r=0}^{\infty}(-1)^r\binom{-1/2}{r}x^{2r}$$

$$(1-k^2x^2)^{-\frac{1}{2}} = \sum_{r=0}^{\infty}(-1)^r\binom{-1/2}{r}k^{2r}x^{2r}$$

Multiplying each other, it is as follows.

$$(1-x^2)^{-\frac{1}{2}} = \binom{-1/2}{0}x^0 - \binom{-1/2}{1}x^2 + \binom{-1/2}{2}x^4 - \binom{-1/2}{3}x^6 +-$$

$$\times \quad (1-k^2x^2)^{-\frac{1}{2}} = \binom{-1/2}{0}k^0x^0 - \binom{-1/2}{1}k^2x^2 + \binom{-1/2}{2}k^4x^4 - \binom{-1/2}{3}k^6x^6 +-$$

$$= \binom{-1/2}{0}\binom{-1/2}{0}k^0x^0$$

$$- \left\{\binom{-1/2}{0}\binom{-1/2}{1}k^0 + \binom{-1/2}{1}\binom{-1/2}{0}k^2\right\}x^2$$

$$+ \left\{\binom{-1/2}{0}\binom{-1/2}{2}k^0 + \binom{-1/2}{1}\binom{-1/2}{1}k^2 + \binom{-1/2}{2}\binom{-1/2}{0}k^4\right\}x^4$$

$$\vdots$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^r\binom{-1/2}{r-s}\binom{-1/2}{s}k^{2s}x^{2r}$$

That is

$$\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} = \sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^r\binom{-1/2}{r-s}\binom{-1/2}{s}k^{2s}x^{2r}$$

Then, integrating both sides of this with respect to x from 0 to x, we obtain (1.1).

Next, from the definition of General Binomial Coefficient (See **3.2**) ,

$$\binom{-1/2}{s} = \frac{\Gamma(-1/2+1)}{\Gamma(-1/2-s+1)\Gamma(s+1)} = \frac{\Gamma(1/2)}{\Gamma(1/2-s)\Gamma(s+1)}$$

On the other hand, from the properties of the Gamma Function (See **1.1.6**) ,

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}(2n-1)!! \quad , \quad \Gamma\left(\frac{1}{2}-n\right) = (-1)^n \frac{2^n\sqrt{\pi}}{(2n-1)!!}$$

Then, for non-negative integer s,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad , \quad \Gamma\left(\frac{1}{2}-s\right) = (-1)^s \frac{2^s\sqrt{\pi}}{(2s-1)!!}$$

Using these,

$$\binom{-1/2}{s} = (-1)^s \frac{\sqrt{\pi}\,(2s-1)!!}{2^s s!\,\sqrt{\pi}} = (-1)^s \frac{(2s-1)!!}{2^s s!}$$

i.e.

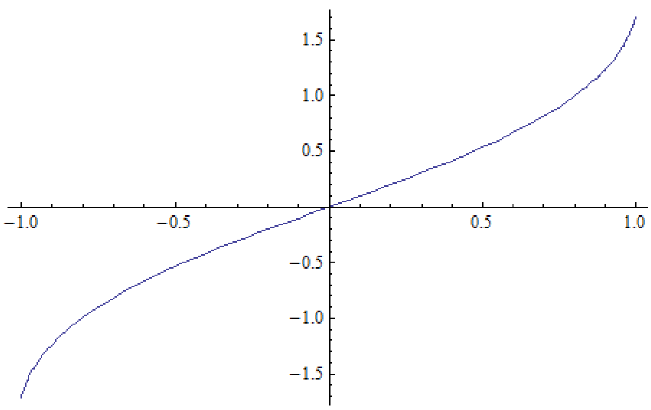
$$\binom{-1/2}{s} = (-1)^s \frac{(2s-1)!!}{(2s)!!} \quad , \quad \binom{-1/2}{r-s} = (-1)^{r-s} \frac{(2r-2s-1)!!}{(2r-2s)!!}$$

Hence, substituting these for (1.1) , we obtain (1.1') .

Example 1 : Double series expansion of $F\left(x, \frac{1}{\sqrt{2}}\right)$

When arbitrary point x= 0.7 is given to this elliptic integral and its double series, both are compared and the right side is illustrated, it is as follows.

```
m = 30;
Fl[x_, k_] := ∫_0^x 1/Sqrt[(1 - t^2) (1 - k^2 t^2)] dt
Fr[x_, k_] := Sum_{r=0}^{m} Sum_{s=0}^{r} (-1)^r/(2 r + 1) Binomial[-1/2, r - s] Binomial[-1/2, s] k^(2 s) x^(2 r+1)
N[Fl[0.7, 1/Sqrt[2]]]          N[Fr[0.7, 1/Sqrt[2]]]
0.814489                       0.814489
```



15.1.2 Double series expansion of Elliptic Integral of the 2nd kind

Formula 15.1.2

The following expressions hold for $|k| \leqq 1$, $|x| \leqq 1$.

$$E(x,k) = \int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r}{2r+1}\binom{-1/2}{r-s}\binom{1/2}{s} k^{2s}x^{2r+1} \tag{1.2}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{1}{(2r+1)(1-2s)} \frac{(2r-2s-1)!!}{(2r-2s)!!} \frac{(2s-1)!!}{(2s)!!} k^{2s}x^{2r+1} \tag{1.2'}$$

Proof

Since $|k| \leqq 1$, $|x| \leqq 1$, from Generalized Binomial Theorem .

$$\left(1-x^2\right)^{-\frac{1}{2}} = \sum_{r=0}^{\infty}(-1)^r\binom{-1/2}{r}x^{2r}$$

$$\left(1-k^2x^2\right)^{\frac{1}{2}} = \sum_{r=0}^{\infty}(-1)^r\binom{1/2}{r}k^{2r}x^{2r}$$

Multiplying each other and integrating both sides of the result with respect to x from 0 to x, we obtain (1.2). This is only what reversed the sign of 1/2 in the 2nd binomial coefficient in (1.1).

Next, from the definition of General Binomial Coefficient ,

$$\binom{1/2}{s} = \frac{\Gamma(3/2)}{\Gamma(3/2-s)\Gamma(s+1)} = \frac{\Gamma(3/2)}{(1/2-s)\Gamma(1/2-s)\Gamma(s+1)}$$

On the other hand, from the properties of the Gamma Function (See **1.1.6**) , the following expressions hold for non-negative integer s,

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \quad , \quad \Gamma\left(\frac{1}{2}-s\right) = (-1)^s\frac{2^s\sqrt{\pi}}{(2s-1)!!}$$

Using these ,

$$\binom{1/2}{s} = (-1)^s\frac{\sqrt{\pi}}{(1-2s)s!}\frac{(2s-1)!!}{2^s\sqrt{\pi}} = \frac{(-1)^s}{(1-2s)}\frac{(2s-1)!!}{(2s)!!}$$

Substituting this and previous $\binom{-1/2}{r-s} = (-1)^{r-s}\frac{(2r-2s-1)!!}{(2r-2s)!!}$ for (1.2) , we obtain (1.2') .

Example 2 : Double series expansion of $E\left(x,\sqrt{\frac{2}{3}}\right)$

When arbitrary point x= 0.8 is given to this elliptic integral and its double series and both are compared, it is as follows.

```
m = 30;
El[x_, k_] := ∫_0^x Sqrt[(1 - k^2 t^2)/(1 - t^2)] dt
Er[x_, k_] := Sum_{r=0}^{m} Sum_{s=0}^{r} (-1)^r/(2 r + 1) Binomial[-1/2, r - s] Binomial[1/2, s] k^(2 s) x^(2 r+1)
```

```
N[El[0.8, Sqrt[2/3]]]
0.84667
```

```
N[Er[0.8, Sqrt[2/3]]]
0.84667
```

15.1.3 Triple series expansion of Elliptic Integral of the 3rd kind

Formula 15.1.3

The following expressions hold for $|c| \leqq 1$, $|k| \leqq 1$, $|x| \leqq 1$.

$$\Pi(x,c,k) = \int_0^x \frac{dx}{(1+cx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} \frac{(-1)^r}{2r+1} c^{r-s} \binom{-1/2}{s-t} \binom{-1/2}{t} k^{2t} x^{2r+1} \tag{1.3}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} \frac{(-c)^{r-s}}{2r+1} \frac{(2s-2t-1)!!}{(2s-2t)!!} \frac{(2t-1)!!}{(2t)!!} k^{2t} x^{2r+1} \tag{1.3'}$$

Proof

Since $|c| \leqq 1$, $|k| \leqq 1$, $|x| \leqq 1$, from Generalized Binomial Theorem .

$$\frac{1}{1+cx^2} = \sum_{r=0}^{\infty} (-1)^r c^r x^{2r}$$

$$(1-x^2)^{-\frac{1}{2}} = \sum_{r=0}^{\infty} (-1)^r \binom{-1/2}{r} x^{2r}$$

$$(1-k^2x^2)^{-\frac{1}{2}} = \sum_{r=0}^{\infty} (-1)^r \binom{-1/2}{r} k^{2r} x^{2r}$$

Multiplying each other and integrating both sides of the result with respect to x from 0 to x, we obtain (1.3) .
And replacing the binomial coefficients with the double factorial in (1.3) , we obtain (1.3') .

Example 3 : Triple series expansion of $\Pi\left(x, \frac{1}{3}, \frac{1}{\sqrt{2}}\right)$

When arbitrary point x= 0.9 is given to this elliptic integral and its triple series and both are compared, it is as follows.

```
m = 35;
Pl[x_, c_, k_] := ∫_0^x 1/((1 + c t^2) Sqrt[(1 - t^2) (1 - k^2 t^2)]) dt
Pr[x_, c_, k_] := Sum_{r=0}^{m} Sum_{s=0}^{r} Sum_{t=0}^{s} (-1)^r/(2 r + 1) c^(r-s) Binomial[-1/2, s - t] Binomial[-1/2, t] k^(2 t) x^(2 r+1)
```

```
N[Pl[0.9, 1/3, 1/Sqrt[2]]]
1.11438
```

```
N[Pr[0.9, 1/3, 1/Sqrt[2]]]
1.11438
```

15.2 Arc length of an ellipse

The ellipse which has foci on x-axis is shown by the following formula.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (0<b<a) \tag{1}$$

Comparing this with $\cos^2\theta + \sin^2\theta = 1$,

$$\frac{x^2}{a^2} = \cos^2\theta \quad , \quad \frac{y^2}{b^2} = \sin^2\theta$$

i.e.

$$x = a\cos\theta \quad , \quad y = b\sin\theta \quad , \quad 0 \leqq \theta \leqq 2\pi$$

This is illustrated as shown in Fig.A. By this expression, since the arc length is calculated by the counter-rclockwise rotation from x-axis, the calculation is complicated.

Therefore we think about only the 1st quadrant, and let us replace θ with $\pi/2-\theta$.

i.e.

$$x = a\sin\theta \quad , \quad y = b\cos\theta \quad , \quad 0 \leqq \theta \leqq \pi/2 \tag{1'}$$

Then, we can calculate the arc length clockwise from y-axis as shown in Fig.B. Because the calculation is easy, we adopt expression (1').

Fig.A

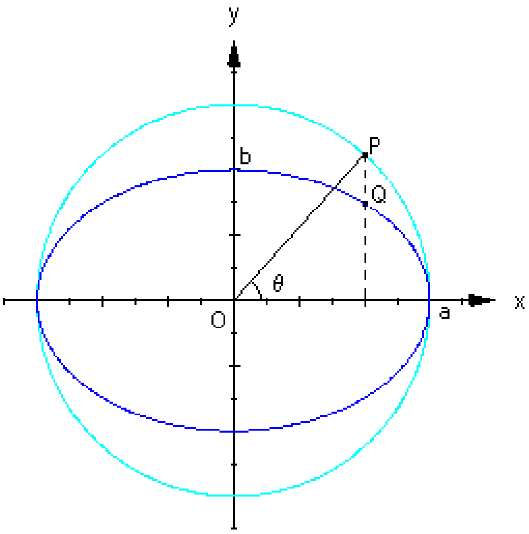
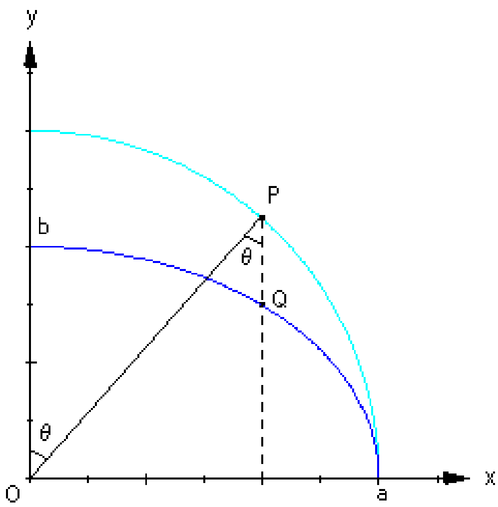


Fig.B



When the parametric equation of a curve on a plane is $x=f(\theta)$, $y=g(\theta)$, the length l is given by the following expression.

$$l = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \qquad \alpha \leqq \theta \leqq \beta$$

From (1') ,

$$\frac{dx}{d\theta} = a\cos\theta \quad , \quad \frac{dy}{d\theta} = -b\sin\theta$$

Then

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2 - (a^2-b^2)\sin^2\theta$$

$$= a^2(1-k^2\sin^2\theta) \qquad \left(k \equiv \sqrt{(a^2-b^2)/a^2} \ : \text{ Eccentricity}\right)$$

Therefore, the length l of $\widehat{bQ}$ is as follows.

$$l = a\int_0^{\theta}\sqrt{1-k^2\sin^2\theta}\,d\theta \tag{2.0}$$

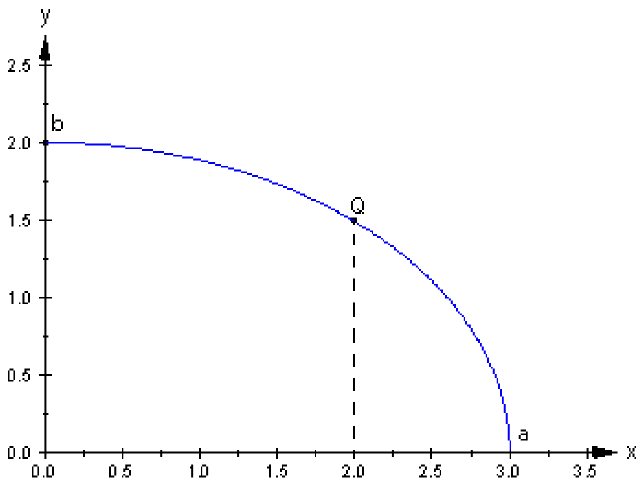
Furthermore, let $t = \sin\theta \ (= x/a)$. Then,

$$l = a\int_0^{\frac{x}{a}}\sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt = aE\left(\frac{x}{a},k\right) \qquad \left(k=\sqrt{\frac{a^2-b^2}{a^2}}\right) \tag{2.1}$$

Since the right side is a Elliptic Integral of the 2nd kind, using (1.2) in previous section , we obtain

$$l = a\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-1)^r}{2r+1}\binom{-1/2}{r-s}\binom{1/2}{s}k^{2s}\left(\frac{x}{a}\right)^{2r+1} \qquad \left(k=\sqrt{\frac{a^2-b^2}{a^2}}\right) \tag{2.1'}$$

Example : The length of $\widehat{bQ}$ in the following figure



From $a=3$, $b=2$, $x_q=2$,

$$k = \sqrt{\frac{3^2-2^2}{3^2}} = \frac{\sqrt{5}}{3} \quad , \qquad \frac{x_q}{a} = \frac{2}{3}$$

Then the length l of $\widehat{bQ}$ is as follows from (2.1') .

$$l = 3\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-1)^r}{2r+1}\binom{-1/2}{r-s}\binom{1/2}{s}\left(\frac{\sqrt{5}}{3}\right)^{2s}\left(\frac{2}{3}\right)^{2r+1}$$

When this is calculated, it is as follows.

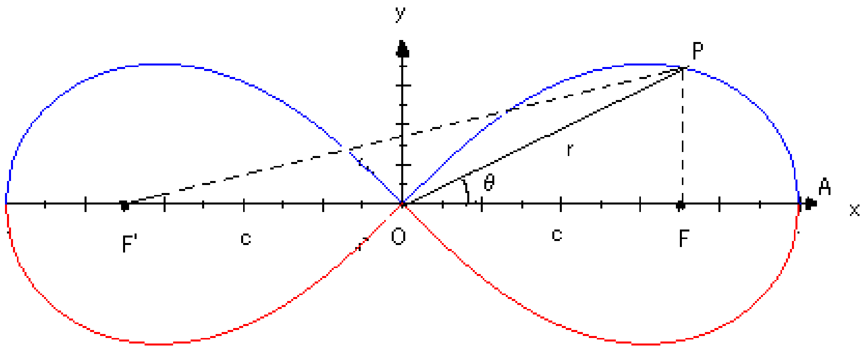
```
a = 3;  b = 2;  m = 20;     k = Sqrt[(a^2 - b^2)/a^2] ;

l[x_] := a Sum[Sum[(-1)^r/(2 r + 1) Binomial[-1/2, r - s] Binomial[1/2, s] k^(2 s) (x/a)^(2 r + 1), {s, 0, r}], {r, 0, m}]

N[l[2]]
2.08808
```

15.3 Arc length of a lemniscate

15.3.0 Definition of a lemniscate

The locus of the point for which the product of the distance from some fixed points on a plane is a constant is called Lemniscate of a wide sense. Especially, when the number of the fixed points is 2, and the distance is $FF'=2c$, and the constant product is c^2, it is called Lemniscate in a narrow sense.

Fig.1 Whole figure



The equation is drawn from $FP^2 \cdot F'P^2 = c^4$ and is as follows.

Orthogonal coordinates : $\left(x^2+y^2\right)^2 - 2c^2\left(x^2-y^2\right) = 0$

Polar coordinates : $r^2 - 2c^2 \cos 2\theta = 0$

Although these equations are used with this, the following formula replaced by $2c^2 = a^2$ is often used.
In this chapter, these are used hereafter.

Orthogonal coordinates : $\left(x^2+y^2\right)^2 - a^2\left(x^2-y^2\right) = 0$ (3.0o)

Polar coordinates : $r^2 - a^2 \cos 2\theta = 0 \qquad (a>0)$ (3.0p)

Explicit function (Orthogonal coordinates)

From (3.0o),

$$y^2 = \frac{-\left(2x^2+a^2\right) \pm a\sqrt{8x^2+a^2}}{2}$$

Since $a > 0$, $\sqrt{8x^2+a^2} > 0$, Employing +, we obtain

$$y = \pm\sqrt{\frac{-\left(2x^2+a^2\right) + a\sqrt{8x^2+a^2}}{2}} \tag{3.0h}$$

These zeros are obtained from $\left(x^2-a^2\right)x^2 = 0$ which is drawn from (3.0h) = 0. They are as follows.

$$x_0 = \pm a \ , \ =\pm 0$$

That is, the x-coordinate of A points is a in Fig.1. The expression of $+$ of (3.h) is an upper half (blue) and the expression of $-$ is a lower half (red) in Fig.1.

Relation between orthogonal coordinates and polar coordinates

Substituting $r = a\sqrt{\cos 2\theta}$ for $x = r\cos\theta$, $y = r\sin\theta$, we obtain the following equations.

$$x = a\sqrt{\cos 2\theta}\cos\theta \quad , \quad y = a\sqrt{\cos 2\theta}\sin\theta \tag{3.xy}$$

Next, substituting the following expressions for (3.xy),

$$\cos\theta = \sqrt{1-\sin^2\theta}\ , \qquad \cos 2\theta = \sqrt{1-2\sin^2\theta}$$

we can extract $\sin\theta$ as follows.

$$\sin\theta = \frac{\sqrt{3a^2 - a\sqrt{a^2+8x^2}}}{2a} = \frac{\sqrt{a^2 - a\sqrt{a^2-8y^2}}}{2a} \tag{3.0'}$$

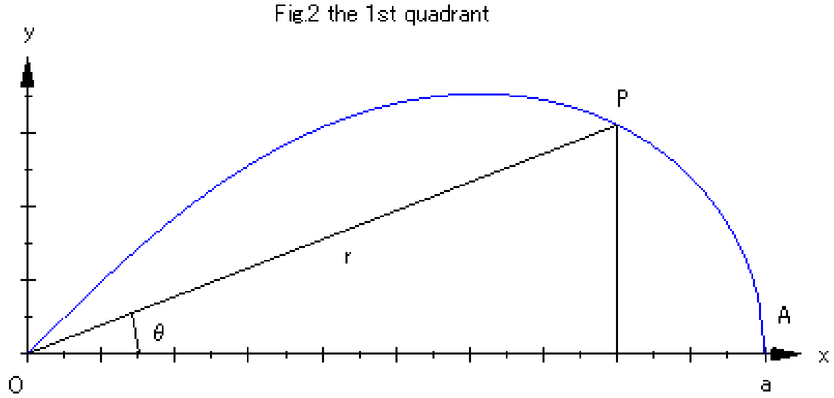
Furthermore, from this, we obtain θ .

$$\theta = \sin^{-1}\frac{\sqrt{3a^2 - a\sqrt{a^2+8x^2}}}{2a} = \sin^{-1}\frac{\sqrt{a^2 - a\sqrt{a^2-8y^2}}}{2a} \tag{3.0''}$$

Arc length from the right end point (polar coordinates)

In Fig.1, the figure of each quadrant is symmetrical with a point, and also symmetrical with a line. So, we pay our attention only to the 1st quadrant.

Fig.2 the 1st quadrant



Then, the length l of $\overset{\frown}{AP}$ in Fig.2 is obtained as follows.

Differentiating (3.xy) with respect to θ ,

$$\frac{dx}{d\theta} = -a\sin\theta\sqrt{\cos 2\theta} - \frac{a\cos\theta\ \sin 2\theta}{\sqrt{\cos 2\theta}}$$

$$\frac{dy}{d\theta} = a\cos\theta\sqrt{\cos 2\theta} - \frac{a\sin\theta\ \sin 2\theta}{\sqrt{\cos 2\theta}}$$

From these,

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{a^2}{\cos 2\theta}$$

Hence,

$$l = a\int_0^\theta \frac{d\theta}{\sqrt{\cos 2\theta}} = a\int_0^\theta \frac{d\theta}{\sqrt{1-2\sin^2\theta}} \qquad \left(0 \leqq \theta \leqq \frac{\pi}{4}\right) \tag{3.0}$$

15.3.1 Elliptic integral expression of arc length (Part 1)

Let $t = \sqrt{2}\sin\theta$ in (3.0) , then

$$dt = \sqrt{2}\cos\theta\, d\theta\ , \qquad \sin\theta = \frac{t}{\sqrt{2}}\ , \quad \cos\theta = \sqrt{1-\frac{t^2}{2}}$$

From these

$$\frac{d\theta}{\sqrt{1-2\sin^2\theta}} = \frac{1}{\sqrt{1-2\cdot\frac{t^2}{2}}}\frac{dt}{\sqrt{2}\cos\theta} = \frac{1}{\sqrt{2}}\frac{dt}{\sqrt{(1-t^2)\left(1-\frac{1}{2}t^2\right)}}$$

Then, the length l of $\overset{\frown}{AP}$ in Fig.2 is obtained as follows.

$$l = \frac{a}{\sqrt{2}}\int_0^u \frac{dt}{\sqrt{(1-t^2)\left\{1-\left(\frac{1}{\sqrt{2}}\right)^2 t^2\right\}}} = \frac{a}{\sqrt{2}}F\left(u,\frac{1}{\sqrt{2}}\right) \tag{3.1}$$

$$u = \sqrt{2}\sin\theta = \sqrt{\frac{3}{2}-\frac{\sqrt{a^2+8x^2}}{2a}} \tag{3.1u}$$

Where, (3.1u) is obtained from $t=\sqrt{2}\sin\theta$ and (3.0') .

Since the right side of (3.1) is the elliptic Integral of the 1st kind, using (1.1) in previous section, we obtain

$$l = \frac{a}{\sqrt{2}}\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-1)^r}{2r+1}\binom{-1/2}{r-s}\binom{-1/2}{s}\left(\frac{1}{\sqrt{2}}\right)^{2s}u^{2r+1} \tag{3.1'}$$

Example1: Arc length from x= a to x= 0.8 at the time of a= 1 (Part 1)

```
a = 1;  m = 100;   u[x_] := Sqrt[3/2 - Sqrt[a^2 + 8 x^2]/(2 a)]

l[x_] := a/Sqrt[2] Sum[Sum[(-1)^r/(2 r + 1) Binomial[-1/2, r - s]
          × Binomial[-1/2, s] (1/Sqrt[2])^(2 s) u[x]^(2 r+1), {s, 0, r}], {r, 0, m}]

N[l[0.8]]
0.390075
```

Note

When $K(k)\ \{=F(1,k)\}$ is a complete elliptic integral of the 1st kind, the length $L/4$ of $\overset{\frown}{AO}$ in Fig.2 is given by the following expression.

$$\frac{L}{4} = \frac{a}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right) = \frac{a}{\sqrt{2}}\times 1.85407467730137\cdots \tag{3.1q}$$

Furthermore, the length $\bar{l}$ of $\overset{\frown}{OP}$ in Fig.2 is drawn from (3.1) and (3.1q) as follows.

$$\bar{l} = \frac{a}{\sqrt{2}}\left\{K\left(\frac{1}{\sqrt{2}}\right) - F\left(t,\frac{1}{\sqrt{2}}\right)\right\} \qquad (0\leqq t\leqq 1) \tag{3.$\bar{1}$}$$

15.3.2 Elliptic integral expression of arc length (Part 2)

Let $t = \tan\theta$ in (3.0) , then $d\theta = \dfrac{du}{1+t^2}$ from $\theta = \tan^{-1}t$.

On the other hand,

$$\frac{1}{1-2\sin^2\theta} = \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta-\sin^2\theta} = \frac{1+\tan^2\theta}{1-\tan^2\theta} = \frac{1+t^2}{1-t^2}$$

Then, the length l of $\overset{\frown}{AP}$ in Fig.2 is

$$l = a\int_0^u \frac{\sqrt{1+t^2}}{\sqrt{1-t^2}}\frac{dt}{1+t^2} = a\int_0^u \frac{dt}{\sqrt{1-t^2}\sqrt{1+t^2}} .$$

That is,

$$l = a\int_0^u \frac{dt}{\sqrt{1-t^2}\sqrt{1-i^2t^2}} = aF(t,i) \tag{3.2}$$

$$u = \tan\left(\sin^{-1}\frac{\sqrt{3a^2-a\sqrt{a^2+8x^2}}}{2a}\right) \tag{3.2u}$$

Where, (3.2u) is obtained from $u=\tan\theta$ and (3.0") .

Although the right side of (3.2) is slightly irregular ($k=\sqrt{-1}$), since it is the elliptic Integral of the 1st kind, using (1.1) in previous section, we obtain the following expression.

$$l = a\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-1)^r}{2r+1}\binom{-1/2}{r-s}\binom{-1/2}{s} i^{2s}u^{2r+1} \tag{3.2'}$$

Example2: **Arc length from x= a to x= 0.8 at the time of a= 1 (Part 2)**

```
a = 1;  m = 20;       u[x_] := Tan[ArcSin[Sqrt[3 a^2 - a Sqrt[a^2 + 8 x^2]] / (2 a)]]

l[x_] := a Sum[Sum[(-1)^r/(2 r + 1) Binomial[-1/2, r - s] Binomial[-1/2, s] i^(2 s) u[x]^(2 r + 1), {s, 0, r}], {r, 0, m}]

N[l[0.8]]
0.390075
```

15.3.3 Single series expression of arc length

In the previous two examples, elliptic integral was calculated by the double series. On the other hand, (3.2) is calculable by single series. According to the Generalized Binomial Theorem,

$$\frac{1}{\sqrt{1-x^4}} = \sum_{r=0}^{\infty}(-1)^r\binom{-1/2}{r}x^{4r}$$

From this,

$$l = a\int_0^u \frac{dt}{\sqrt{1-t^4}} = a\sum_{r=0}^{\infty}\frac{(-1)^r}{4r+1}\binom{-1/2}{r}u^{4r+1} \tag{3.3}$$

or

$$l = a\int_0^u \frac{dt}{\sqrt{1-t^4}} = a\sum_{r=0}^{\infty}\frac{(2r-1)!!}{(2r)!!}\frac{u^{4r+1}}{4r+1} \tag{3.3'}$$

The upper limit of the integration is the same as (3.2u), as follows.

$$u = \tan\left(\sin^{-1}\frac{\sqrt{3a^2-a\sqrt{a^2+8x^2}}}{2a}\right)$$

Example3: Arc length from x= a to x= 0.8 at the time of a= 1 (Part 3)

Using (3.3'), we try to calculate Example 2 by hand. The significant figure is to three numbers below a decimal point. The upper limit of (3.3') at the time of a= 1 and x= 0.8 is same as Example2. Then,

$$u = \tan\left(\sin^{-1} \frac{\sqrt{3\cdot 1^2 - 1\sqrt{1^2+8(0.8)^2}}}{2\cdot 1} \right) = 0.389$$

Substituting this for (3.3') and calculating the first 5 terms, we obtain the following.

$$s = \frac{(-1)!!}{0!!}\frac{0.389^1}{1} + \frac{0!!}{1!!}\frac{0.389^5}{5} + \frac{3!!}{4!!}\frac{0.389^9}{9} + \frac{5!!}{6!!}\frac{0.389^{13}}{13} + \frac{7!!}{8!!}\frac{0.389^{17}}{17}$$
$$= 0.390$$

Note

After all, there are the following relations between the three integrals.

$$\int_0^\theta \frac{d\theta}{\sqrt{1-2\sin^2\theta}} = \frac{1}{\sqrt{2}}\int_0^t \frac{dt}{\sqrt{(1-t^2)\left(1-\frac{1}{2}t^2\right)}} = \int_0^u \frac{du}{\sqrt{1-u^4}}$$

$$t = \sqrt{2}\sin\theta = \sqrt{2}\sin\left(\tan^{-1}u\right) = \frac{\sqrt{2}\,u}{\sqrt{1+u^2}}$$

$$u = \tan\theta = \tan\left(\sin^{-1}\frac{t}{\sqrt{2}}\right) = \frac{t}{\sqrt{2-t^2}}$$

15.4 Termwise Higher Calculus of Elliptic Integral

15.4.1 Termwise Higher Integral of Elliptic Integral

Formula 15.4.1

When $|x| \leqq 1$, $|k| \leqq 1$, $|c| \leqq 1$

$$F(x,k) = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad , \quad E(x,k) = \int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$$

$$\Pi(x,c,k) = \int_0^x \frac{dx}{(1+cx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

the following expressions hold for natural number n.

$$\int_0^x \cdots \int_0^x F(x,k)\,dx^n = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r(2r)!}{(2r+n+1)!}\binom{-1/2}{r-s}\binom{-1/2}{s} k^{2s} x^{2r+n+1} \tag{4.1}$$

$$\int_0^x \cdots \int_0^x E(x,k)\,dx^n = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r(2r)!}{(2r+n+1)!}\binom{-1/2}{r-s}\binom{1/2}{s} k^{2s} x^{2r+n+1} \tag{4.2}$$

$$\int_0^x \cdots \int_0^x \Pi(x,c,k)\,dx^n = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} \frac{(-1)^r(2r)!}{(2r+n+1)!} c^{r-s}\binom{-1/2}{s-t}\binom{-1/2}{t} k^{2t} x^{2r+n+1} \tag{4.3}$$

Proof

From Formula 15.1.1 ,

$$F(x,k) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r}{2r+1}\binom{-1/2}{r-s}\binom{-1/2}{s} k^{2s} x^{2r+1}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r(2r)!}{(2r+1)!}\binom{-1/2}{r-s}\binom{-1/2}{s} k^{2s} x^{2r+1}$$

Integrating both sides of this with respect to x from 0 to x n times, we obtain (4.1) . (4.2) and (4.3) are also obtained from the Formula 15.1.2 and 15.1.3 in a similar way.

Note

The following expression using the double factorial is also possible. However, it is complicated and does not have a merit.

$$\int_0^x \cdots \int_0^x F(x,k)\,dx^n = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(2r)!}{(2r+n+1)!}\frac{(2r-2s-1)!!}{(2r-2s)!!}\frac{(2s-1)!!}{(2s)!!} k^{2s} x^{2r+n+1}$$

Example : $\int_0^x \cdots \int_0^x E\left(x, \sqrt{\frac{2}{3}}\right) dx^3$

In the left side, the integrand of elliptic integral of the 2nd kind is integrated 3+1 times by Riemann-Liouville integral. In the right side, this higher integral is calculated by the double series (4.2). Moreover, one arbitrary point x= 0.8 is given to the both sides, and the values of the both sides are compared. Both are corresponding.

```
n = 3;   m = 25;
```

$$\text{El}[x\_, k\_] := \frac{1}{\text{Gamma}[n+1]} \int_0^x (x-t)^{n+1-1} \sqrt{\frac{1-k^2 t^2}{1-t^2}}\, dt$$

```
Er[x_, k_] := Sum[Sum[(-1)^r (2 r)! / (2 r + n + 1)! Binomial[-1/2, r - s] Binomial[1/2, s] k^(2 s) x^(2 r+n+1), {s, 0, r}], {r, 0, m}]
```

$$\mathrm{Er}[x\_,k\_] := \sum_{r=0}^{m}\sum_{s=0}^{r}\frac{(-1)^r(2r)!}{(2r+n+1)!}\mathrm{Binomial}\left[-\frac{1}{2},r-s\right]\mathrm{Binomial}\left[\frac{1}{2},s\right]k^{2s}x^{2r+n+1}$$

$\mathrm{N}\left[\mathrm{El}\left[0.8,\sqrt{\frac{2}{3}}\right]\right]$ $\mathrm{N}\left[\mathrm{Er}\left[0.8,\sqrt{\frac{2}{3}}\right]\right]$

0.0172074   0.0172074

15.4.2 Termwise Higher Derivative of Elliptic Integral

Formula 15.4.2

When $|x| \leqq 1$, $|k| \leqq 1$, $|c| \leqq 1$, $x\uparrow\ (=\lceil x \rceil)$

$$F(x,k) = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad , \quad E(x,k) = \int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$$

$$\Pi(x,c,k) = \int_0^x \frac{dx}{(1+cx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

the following expressions hold for natural number n.

$$\frac{d^n}{dx^n}F(x,k) = \sum_{r=\frac{n-1}{2}\uparrow}^{\infty}\sum_{s=0}^{r}\frac{(-1)^r(2r)!}{(2r-n+1)!}\binom{-1/2}{r-s}\binom{-1/2}{s}k^{2s}x^{2r-n+1} \tag{4.4}$$

$$\frac{d^n}{dx^n}E(x,k) = \sum_{r=\frac{n-1}{2}\uparrow}^{\infty}\sum_{s=0}^{r}\frac{(-1)^r(2r)!}{(2r-n+1)!}\binom{-1/2}{r-s}\binom{1/2}{s}k^{2s}x^{2r-n+1} \tag{4.5}$$

$$\frac{d^n}{dx^n}\Pi(x,c,k) = \sum_{r=\frac{n-1}{2}\uparrow}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^r(2r)!}{(2r-n+1)!}c^{r-s}\binom{-1/2}{s-t}\binom{-1/2}{t}k^{2t}x^{2r-n+1}$$

(4.6)

Proof

From Formula 15.1.1 ,

$$F(x,k) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-1)^r(2r)!}{(2r+1)!}\binom{-1/2}{r-s}\binom{-1/2}{s}k^{2s}x^{2r+1}$$

Differentiating both sides of this with respect to x n times, we obtain (4.4). The 1st term r_0 of outside $\sum$ is decided as the power $2r-n+1$ of x is nonnegative. And there is a following relation between the 1st term r_0 and the order n of derivative.

n	0	1	2	3	4	5	...
r_0	0	0	1	1	2	2	...

Such a relation can be expressed as follows using a ceiling function $x\uparrow\ (=\lceil x \rceil)$.

$$r_0 = \frac{n-1}{2}\uparrow \qquad n \geqq 0$$

(4.5) and (4.6) are also obtained from the Formula 15.1.2 and 15.1.3 in a similar way.

Example : $\dfrac{d^3}{dx^3}\Pi\left(x, \dfrac{1}{3}, \dfrac{1}{\sqrt{2}}\right)$

In the left side, the integrand of the elliptic integral of the 3rd kind is differentiated 3-1 times directly. In the right side, this higher derivative is calculated by the double series (4.6). Moreover, one arbitrary point x= -0.9 is given to the both sides, and the values of the both sides are compared. Both are corresponding.

```
n = 3; m = 80;
```

$$\text{Pl}[x\_, c\_, k\_] = \partial_x\left(\partial_x \frac{1}{(1 + c\,x^2)\sqrt{(1 - x^2)\,(1 - k^2\,x^2)}}\right);$$

$$\text{Pr}[x\_, c\_, k\_] := \sum_{r=\text{Ceiling}\left[\frac{n-1}{2}\right]}^{m} \sum_{s=0}^{r}\sum_{t=0}^{s} \frac{(-1)^r\,(2\,r)!}{(2\,r - n + 1)!}\, c^{r-s}\,\text{Binomial}\left[-\frac{1}{2}, s - t\right] \times \text{Binomial}\left[-\frac{1}{2}, t\right] k^{2\,t}\, x^{2\,r-n+1}$$

$\text{Pl}\left[-0.9, \frac{1}{3}, \frac{1}{\sqrt{2}}\right]$

180.39

$\text{N}\left[\text{Pr}\left[-0.9, \frac{1}{3}, \frac{1}{\sqrt{2}}\right]\right]$

180.39

15.5 Termwise Super Calculus of Elliptic Integral

15.5.1 Termwise Super Integral of Elliptic Integral

Formula 15.5.1

When $|x| \leqq 1$, $|k| \leqq 1$, $|c| \leqq 1$

$$F(x,k) = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad , \quad E(x,k) = \int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$$

$$\Pi(x,c,k) = \int_0^x \frac{dx}{(1+cx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

the following expressions hold for $p \geqq 0$.

$$\int_0^x \sim \int_0^x F(x,k)\,dx^p = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r(2r)!}{\Gamma(2r+p+2)} \binom{-1/2}{r-s}\binom{-1/2}{s} k^{2s} x^{2r+p+1} \tag{5.1}$$

$$\int_0^x \sim \int_0^x E(x,k)\,dx^p = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r(2r)!}{\Gamma(2r+p+2)} \binom{-1/2}{r-s}\binom{1/2}{s} k^{2s} x^{2r+p+1} \tag{5.2}$$

$$\int_0^x \sim \int_0^x \Pi(x,c,k)\,dx^p = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} \frac{(-1)^r(2r)!}{\Gamma(2r+p+2)} c^{r-s} \binom{-1/2}{s-t}\binom{-1/2}{t} k^{2t} x^{2r+p+1} \tag{5.3}$$

Proof

Analytically continuing the index of the integration operator in Formula 15.4.1 to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Example : $\int_0^x \sim \int_0^x F\left(x, \frac{1}{\sqrt{2}}\right) dx^{\frac{3}{2}}$

In the left side, the integrand of elliptic integral of the 1st kind is integrated 3/2+1 times by Riemann-Liouville integral. In the right side, this higher integral is calculated by the double series (5.1). Moreover, one arbitrary point x= 0.7 is given to the both sides, and the values of the both sides are compared. Both are corresponding,

```
p = 3 / 2;  m = 20;
```

$$\text{Fl}[x\_, k\_] := \frac{1}{\text{Gamma}[p+1]} \int_0^x (x-t)^{p+1-1} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\, dt$$

$$\text{Fr}[x\_, k\_] := \sum_{r=0}^{m}\sum_{s=0}^{r} \frac{(-1)^r (2r)!}{\text{Gamma}[2r+p+2]} \text{Binomial}\left[-\frac{1}{2}, r-s\right] \text{Binomial}\left[-\frac{1}{2}, s\right] k^{2s} x^{2r+p+1}$$

$\text{N}\left[\text{Fl}\left[0.7, \frac{1}{\sqrt{2}}\right]\right]$ $\quad$ $\text{N}\left[\text{Fr}\left[0.7, \frac{1}{\sqrt{2}}\right]\right]$

0.130055 $\quad$ 0.130055

15.5.2 Termwise Super Derivative of Elliptic Integral

Formula 15.5.2

When $|x| \leqq 1$, $|k| \leqq 1$, $|c| \leqq 1$, $x\uparrow \; (=\lceil x \rceil)$

$$F(x,k)=\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad , \quad E(x,k)=\int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$$

$$\Pi(x,c,k)=\int_0^x \frac{dx}{(1+cx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

the following expressions hold for $p \geqq 0$.

$$\frac{d^p}{dx^p}F(x,k)=\sum_{r=\frac{p-1}{2}\uparrow}^{\infty}\sum_{s=0}^{r}\frac{(-1)^r(2r)!}{\Gamma(2r-p+2)}\binom{-1/2}{r-s}\binom{-1/2}{s}k^{2s}x^{2r-p+1} \tag{5.4}$$

$$\frac{d^p}{dx^p}E(x,k)=\sum_{r=\frac{p-1}{2}\uparrow}^{\infty}\sum_{s=0}^{r}\frac{(-1)^r(2r)!}{\Gamma(2r-p+2)}\binom{-1/2}{r-s}\binom{1/2}{s}k^{2s}x^{2r-p+1} \tag{5.5}$$

$$\frac{d^p}{dx^p}\Pi(x,c,k)=\sum_{r=\frac{p-1}{2}\uparrow}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^r(2r)!}{\Gamma(2r-n+2)}c^{r-s}\binom{-1/2}{s-t}\binom{-1/2}{t}k^{2t}x^{2r-p+1} \tag{5.6}$$

Proof

Analytically continuing the index of the differentiation operator in Formula 15.4.2 to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Example : $\dfrac{d^{3/2}}{dx^{3/2}}E\left(x,\sqrt{\dfrac{2}{3}}\right)$

In the left side, the 2nd order derivative of elliptic integral of the 2nd kind is integrated 1/2 times by Riemann-Liouville integral. In the right side, this super derivative is calculated by the double series (5.5). Moreover, one arbitrary point x= 0.6 is given to the both sides, and the values of the both sides are compared. Both sides are corresponding,

$$p = 3/2;\ m = 15;\quad d2[t\_, k\_] = \partial_t \sqrt{\frac{1-k^2t^2}{1-t^2}};$$

$$El[x\_, k\_] := \frac{1}{Gamma[2-p]}\int_0^x (x-t)^{2-p-1}\, d2[t, k]\, dt$$

$$Er[x\_, k\_] := \sum_{r=Ceiling[\frac{p-1}{2}]}^{m}\sum_{s=0}^{r}\frac{(-1)^r (2r)!}{Gamma[2r-p+2]} Binomial\left[-\frac{1}{2}, r-s\right] \times Binomial\left[\frac{1}{2}, s\right] k^{2s} x^{2r-p+1}$$

$N\left[El\left[0.6, \sqrt{\frac{2}{3}}\right]\right]$

0.203623

$N\left[Er\left[0.6, \sqrt{\frac{2}{3}}\right]\right]$

0.203623

2007.11.07

K. Kono

Alien's Mathematics