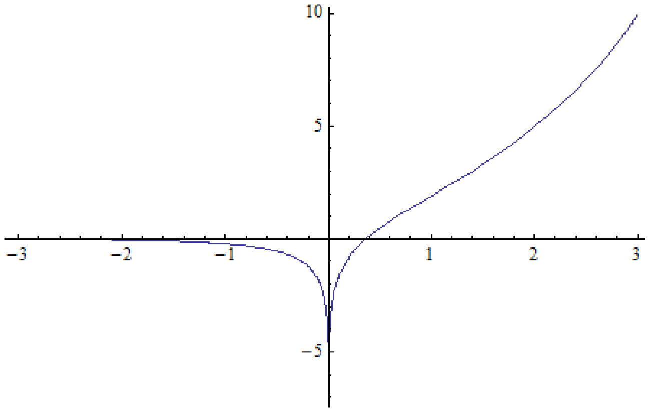
14 Higher and Super Calculus of Logarithmic Integral etc.

14.1 Higher Integral of Exponential Integral

Exponential Integral is defined as follows.

$$Ei(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt \tag{1.0}$$



Integrating both sides of (1.0) with respect to x repeatedly by ONLINE INTEGRATOR (Wolfram Mathematica) and arranging the results, we obtain the following higher indefinite integrals.

$$\int Ei(x)dx = \frac{1}{1!}\{xEi(x) - e^x 0!\}$$

$$\int\int Ei(x)dx^2 = \frac{1}{2!}\{x^2 Ei(x) - e^x(0!x+1!)\}$$

$$\int\int\int Ei(x)dx^3 = \frac{1}{3!}\{x^3 Ei(x) - e^x(0!x^2+1!x+2!)\}$$

$$\vdots$$

$$\int\cdots\int Ei(x)dx^n = \frac{1}{n!}\left\{x^n Ei(x) - e^x \sum_{r=0}^{n-1} r!\, x^{n-1-r}\right\}$$

Although these right sides are the lineal primitive functions of Ei(x), since both zeros of Ei(x) and e^x are $-\infty$, zeros of the right sides are all $-\infty$. Therfore, the lineal higher primitive function of Ei(x) can be expressed by the higher integral with a fixed lower limit $-\infty$.

Formula 14.1.1

When Exponential Integral is $Ei(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt$, the following expressions hold.

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} Ei(x)dx^n = \frac{1}{n!}\left\{x^n Ei(x) - e^x \sum_{r=0}^{n-1} r!\, x^{n-1-r}\right\} \tag{1.n}$$

Example : 3rd order integral of Ei (x)

```
f1[x_] := ∫_{-∞}^{x} ( ∫_{-∞}^{x} ( ∫_{-∞}^{x} ExpIntegralEi[x] dx ) dx ) dx
```

```
fr[n_, x_] := 1/n! (x^n ExpIntegralEi[x] - e^x Sum[r! x^(n-1-r), {r, 0, n-1}])
```

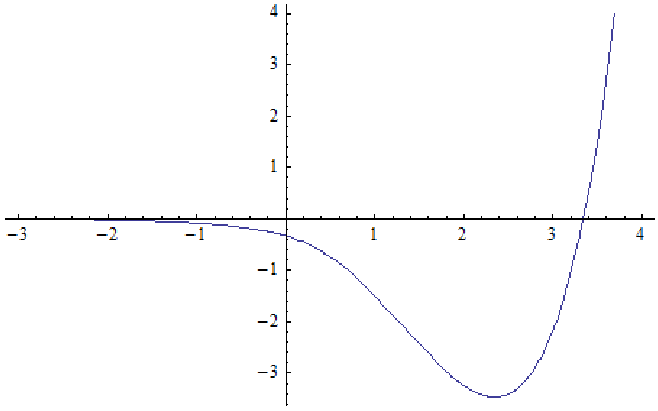
$$fr[n\_, x\_] := \frac{1}{n!}\left(x^n\,\mathrm{ExpIntegralEi}[x] - e^x \sum_{r=0}^{n-1} r!\, x^{n-1-r}\right)$$

fl[x]

$$\frac{1}{6}\left(-e^x\left(2 + x + x^2\right) + x^3\,\mathrm{ExpIntegralEi}[x]\right)$$

fr[3, x]

$$\frac{1}{6}\left(-e^x\left(2 + x + x^2\right) + x^3\,\mathrm{ExpIntegralEi}[x]\right)$$



14.2 Higher Integral of Cosine Integral

Cosine Integral is defined as follows.

$$Ci(x)=\int_{\infty}^{x}\frac{\cos t}{t}dt$$

Integrating both sides of this with respect to x repeatedly by ONLINE INTEGRATOR and arranging the results, we obtain the following higher indefinite integrals.

$$\int Ci(x)dx = \frac{1}{1!}\left\{x^1 Ci(x) - 0!\sin x\right\}$$

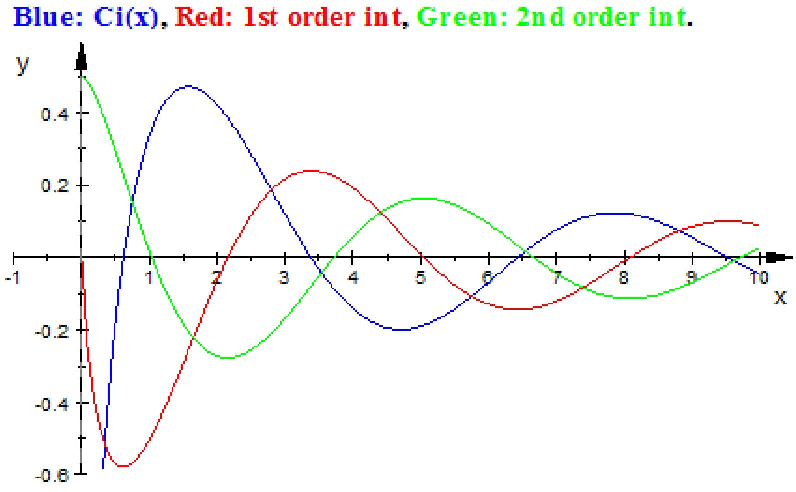
$$\iint Ci(x)dx^2 = \frac{1}{2!}\left\{x^2 Ci(x) - 0!x\sin x + 1!\cos x\right\}$$

$$\iiint Ci(x)dx^3 = \frac{1}{3!}\left\{x^3 Ci(x) - \left(0!x^2-2!\right)\sin x + 1!x\cos x\right\}$$

$$\int\cdots\int Ci(x)dx^4 = \frac{1}{4!}\left\{x^4 Ci(x) - \left(0!x^3-2!x\right)\sin x + \left(1!x^2-3!\right)\cos x\right\}$$

$$\vdots$$

$$\int\cdots\int Ci(x)dx^n = \frac{1}{n!}\left\{Ci(x)x^n - \sin x\sum_{r=0}^{\lfloor(n-1)/2\rfloor}(-1)^r(2r)!\,x^{n-1-2r} + \cos x\sum_{r=0}^{\lfloor(n-2)/2\rfloor}(-1)^r(2r+1)!\,x^{n-2-2r}\right\}$$



Although these right sides are the lineal primitive functions of Ci(x), these zeros are all ∞. (See the above figure.) Therfore, the lineal higher primitive function of Ci(x) can be expressed by the higher integral with a fixed lower limit ∞.

Formula 14.2.1

When $\downarrow$ is floor function and $Ci(x)=\int_{\infty}^{x}\frac{\cos t}{t}dt$,is Cosine Integral, the following expressions hold.

$$\int_{\infty}^{x} Ci(x)dx = \frac{1}{1!}\left\{x^1 Ci(x) - 0!\sin x\right\}$$

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} Ci(x)dx^n = \frac{1}{n!}\left\{Ci(x)x^n - \sin x\sum_{r=0}^{\lfloor(n-1)/2\rfloor}(-1)^r(2r)!\,x^{n-1-2r}\right.$$

$$+ \cos x \sum_{r=0}^{(n-2)/2\downarrow} (-1)^r (2r+1)!\, x^{n-2-2r} \Big\} \qquad n \geqq 2$$

Example : 4th order integral of Ci (x)

If both sides are illustrated, it is as follows. Since both sides overlap exactly, the left side (blue) is not visible.

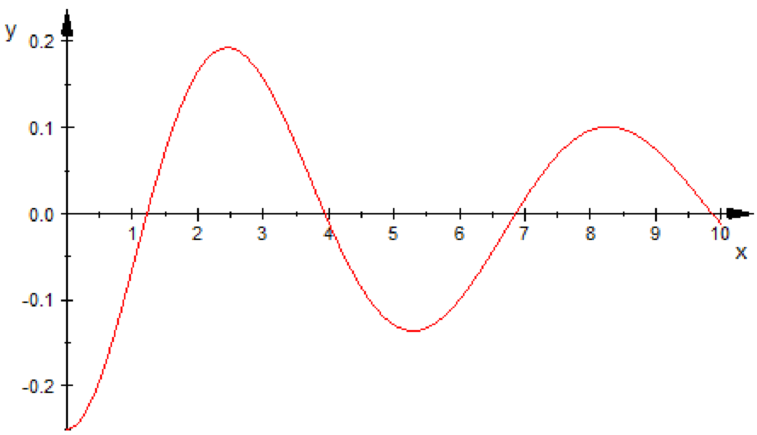
Left: 4 times integral

- $f1 := x \rightarrow \int_{\infty}^{x} \int_{\infty}^{t4} \int_{\infty}^{t3} \int_{\infty}^{t2} \mathrm{Ci}(t1)\, dt1\, dt2\, dt3\, dt4$

Right: Polynomial

- $n:=4:$
- $fr := x \rightarrow \frac{1}{n!} \cdot \left(\mathrm{Ci}(x) \cdot x^n - \sin(x) \cdot \left(\sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^r \cdot (2 \cdot r)! \cdot x^{n-1-2 \cdot r} \right) + \cos(x) \cdot \left(\sum_{r=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} (-1)^r \cdot (2 \cdot r+1)! \cdot x^{n-2-2 \cdot r} \right) \right)$

Left: 4 times integral , Right: Polynomial



14.3 Collateral Higher Integral of Sine Integral

Sine Integral is defined as follows.

$$Si(x)=\int_0^x \frac{\sin t}{t}dt$$

Integrating both sides of this with respect to x repeatedly by ONLINE INTEGRATOR and arranging the results, we obtain the following higher indefinite integrals.

$$\int Si(x)dx = \frac{1}{1!}\left\{x^1 Si(x) + 0!\cos x\right\}$$

$$\int\int Si(x)dx^2 = \frac{1}{2!}\left\{x^2 Si(x) + 0!x\cos x + 1!\sin x\right\}$$

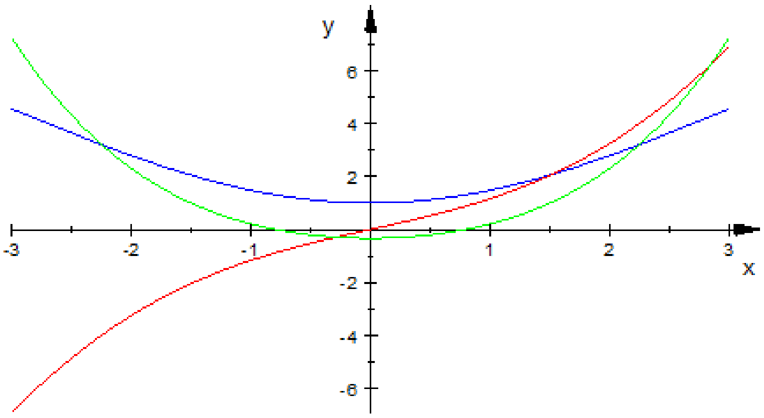
$$\int\int\int Si(x)dx^3 = \frac{1}{3!}\left\{x^3 Si(x) + \left(0!x^2-2!\right)\cos x + 1!\ x\sin x\right\}$$

$$\int\cdots\int Si(x)dx^4 = \frac{1}{4!}\left\{x^4 Si(x) + \left(0!x^3-2!x\right)\cos x + \left(1!x^2-3!\right)\sin x\right\}$$

$$\vdots$$

$$\int\cdots\int Si(x)dx^n = \frac{1}{n!}\left\{Si(x)x^n + \cos x\sum_{r=0}^{(n-1)/2\downarrow}(-1)^r(2r)!\,x^{n-1-2r} + \sin x\sum_{r=0}^{(n-2)/2\downarrow}(-1)^r(2r+1)!\,x^{n-2-2r}\right\}$$

Blue: Ci(x), Red: 1st order int, Green: 2nd order int.



Although these right sides are the lineal primitive functions of Si(x), those zeros are all 0 at the time of even order and are not 0 at the time of odd order. That is, the lineal higher primitive function of Si(x) can not be expressed by the higher integral with a fixed lower limit.(See the above figure.) Therfore, the higher integra of Si(x) with a fixed lower limit 0 is not lineal but collateral. However, the idea which makes 0 a common lower limit is natural. It is because the Si(x) itself is defined by the integral with a lower limit 0.

Collateral Higher Integral of Sine Integral

Collateral Higher Integrals of Si(x) are obtained by compensating the above lineal higher primitive functions with Constant-of-integration Polynomials .

Formula 14.3.1

When $\downarrow$ is floor function and $Si(x)=\int_0^x \frac{\sin t}{t}dt$,is Sine Integral, the following expressions hold.

$$\int_0^x Si(x)dx = \frac{1}{1!}\left\{x^1 Si(x) + 0!\cos x\right\} - \frac{x^0}{1\cdot 0!}$$

$$\int_0^x\int_0^x Si(x)dx^2 = \frac{1}{2!}\left\{x^2 Si(x) + 0!\,x\cos x + 1!\sin x\right\} - \frac{x^1}{1\cdot 1!}$$

$$\int_0^x\int_0^x\int_0^x Si(x)dx^3 = \frac{1}{3!}\left\{x^3 Si(x) + \left(0!\,x^2 - 2!\right)\cos x + 1!\,x\sin x\right\} - \frac{x^2}{1\cdot 2!} + \frac{x^0}{3\cdot 0!}$$

$$\int_0^x\cdots\int_0^x Si(x)dx^4 = \frac{1}{4!}\left\{x^4 Si(x) + \left(0!\,x^3 - 2!\,x\right)\cos x + \left(1!\,x^2 - 3!\right)\sin x\right\} - \frac{x^3}{1\cdot 3!} + \frac{x^1}{3\cdot 1!}$$

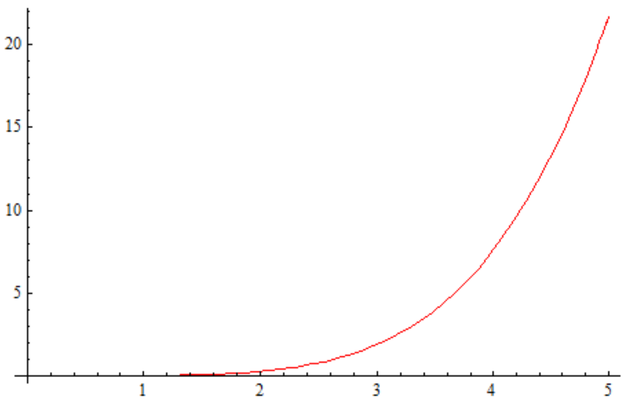
$$\vdots$$

$$\int_0^x\cdots\int_0^x Si(x)dx^n = \frac{1}{n!}\left\{Si(x)x^n + \cos x\sum_{r=0}^{(n-1)/2\downarrow}(-1)^r(2r)!\,x^{n-1-2r} + \sin x\sum_{r=0}^{(n-2)/2\downarrow}(-1)^r(2r+1)!\,x^{n-2-2r}\right\} - \sum_{r=0}^{(n-1)/2\downarrow}(-1)^r\frac{x^{n-1-2r}}{(2r+1)(n-1-2r)!}$$

Example : Collateral the 4th order integral of Si (x)

If both sides are illustrated, it is as follows. Since both sides overlap exactly, the left side (blue) is not visible.

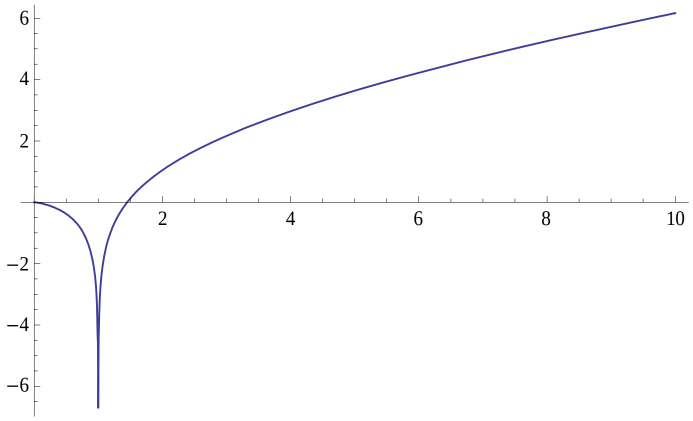
```
n = 4;
fl[x_] := 1/Gamma[n] ∫_0^x (x - t)^(n-1) SinIntegral[t] dt
fr[x_] := 1/n! ( SinIntegral[x] x^n + Cos[x] Sum_{r=0}^{Floor[(n-1)/2]} (-1)^r (2 r)! x^(n-1-2 r)
                 + Sin[x] Sum_{r=0}^{Floor[(n-2)/2]} (-1)^r (2 r + 1)! x^(n-2-2 r) )
              - Sum_{r=0}^{Floor[(n-1)/2]} (-1)^r x^(n-1-2 r) / ((2 r + 1) (n - 1 - 2 r)!)
```



14.4 Higher Integral of Logarithmic Integral

Logarithmic Integral is defined as follows.

$$li(x)=\int_0^x \frac{1}{\log t}dt \tag{1.0}$$



First, we prepare two Lemmas.

Lemma 14.4.1

When Exponential Integral is $Ei(x)=\int_{-\infty}^x \frac{e^t}{t}dt$, the following expressions hold.

$$\int Ei(2\log x)dx = xEi(2\log x) - Ei(3\log x)$$

$$\int Ei(3\log x)dx = xEi(3\log x) - Ei(4\log x)$$

$$\vdots$$

$$\int Ei(n\log x)dx = xEi(n\log x) - Ei\{(n+1)\log x\} \tag{1.n}$$

Proof

Let $2\log x = t$. Then $x = e^{\frac{t}{2}}$, $dx = \frac{x}{2}dt = \frac{1}{2}e^{\frac{t}{2}}dt$. Hence

$$\int Ei(2\log x)dx = \frac{1}{2}\int Ei(t)e^{\frac{t}{2}}dt$$

Calculating the integral of the right side by ONLINE INTEGRATOR, we obtain

$$\int Ei(t)e^{\frac{t}{2}}dt = 2Ei(t)e^{\frac{t}{2}} - 2Ei\left(\frac{3}{2}t\right)$$

Using this,

$$\int Ei(2\log x)dx = Ei(t)e^{\frac{t}{2}} - Ei\left(\frac{3}{2}t\right) = xEi(2\log x) - Ei(3\log x)$$

Next, let $3\log x = t$. Then we obtain the following expresssion by the same calculation.

$$\int Ei(3\log x)dx = \int Ei(t)\frac{1}{3}e^{\frac{t}{3}}dt = \frac{1}{3}\int Ei(t)e^{\frac{t}{3}}dt = Ei(t)e^{\frac{t}{3}} - Ei\left(\frac{4}{3}t\right)$$

$$= xEi(3\log x) - Ei(4\log x)$$

Hereafter, by induction, we obtain the desired expresson.

Note

Since $\log x \to -\infty$ at the time $x \to +0$, $x=0$ is clearly a zero of these functions. Then, (1.n) can be written as follows.

$$\int_0^x Ei(n\log x)dx = xEi(n\log x) - Ei\{(n+1)\log x\} \tag{1.n'}$$

Lemma 14.4.2

When Exponential Integral is $Ei(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt$, the following expressions hold.

$$\int x^n Ei(\log x)dx = \frac{1}{n+1}x^{n+1}Ei(\log x) - \frac{1}{n+1}Ei\{(n+2)\log x\} \tag{2.n}$$

Calculation

Calculating by ONLINE INTEGRATOR , we obtain (2.n) immediatly.

Note

Since $\log x \to -\infty$ at the time $x \to +0$, $x=0$ is crealy a zero of these functions. Then, (2.n) can be written as follows.

$$\int_0^x x^n Ei(\log x)dx = \frac{1}{n+1}x^{n+1}Ei(\log x) - \frac{1}{n+1}Ei\{(n+2)\log x\} \tag{2.n'}$$

Formula 14.4.3

When

$$li(x)=\int_0^x \frac{1}{\log t}dt \quad , \quad Ei(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt$$

the following expression holds for $x \geqq 0$.

$$\int_0^x \cdots \int_0^x li(x)dx^n = \frac{1}{n!}\sum_{r=0}^{n}(-1)^r {}_nC_r x^{n-r} Ei\{(r+1)\log x\} \tag{3.n}$$

Proof

Let $t = \log x$. Then $[0, x] \to [-\infty, t]$, $dx = e^t dt$. Hence

$$\int_0^x \frac{1}{\log x}dx = \int_{-\infty}^{t}\frac{e^t}{t}dt = [Ei(t)]_{-\infty}^{t} = Ei(\log x) = li(x)$$

Next, let

$$\int_0^x\int_0^x \frac{1}{\log x}dx^2 = \int_0^x li(x)dx$$

Calculating the integral of the right side by ONLINE INTEGRATOR, we obtain

$$\int li(x)dx = xli(x) - Ei(2\log x)$$

Since the zero of this right side is x=0 obviously,

$$\int_0^x li(x)dx = xEi(\log x) - Ei(2\log x)$$

Next, integrating both sides of this with respect to x and applying Lemma 14.4.1 , 14.4.2 to the result, we obtain the following

$$\int_0^x\int_0^x li(x)dx^2 = \int_0^x xEi(\log x)dx - \int_0^x Ei(2\log x)dx$$

$$= \frac{1}{2}x^2Ei(\log x) - \frac{1}{2}Ei(3\log x) - \{xEi(2\log x) - Ei(3\log x)\}$$

$$= \frac{1}{2}\{x^2Ei(\log x) - 2xEi(2\log x) + Ei(3\log x)\}$$

Next, integrating both sides of this with respect to x and applying Lemma 14.4.1 , 14.4.2 to the result, we obtain the following

$$\int_0^x\int_0^x\int_0^x li(x)dx^3 = \frac{1}{2}\int_0^x x^2Ei(\log x)dx - \int_0^x xEi(2\log x)dx + \frac{1}{2}\int_0^x Ei(3\log x)dx$$

$$= \frac{1}{3!}x^3Ei(\log x) - \frac{1}{3!}Ei(4\log x) - \frac{1}{2}x^2Ei(2\log x) + \frac{1}{2}Ei(4\log x)$$

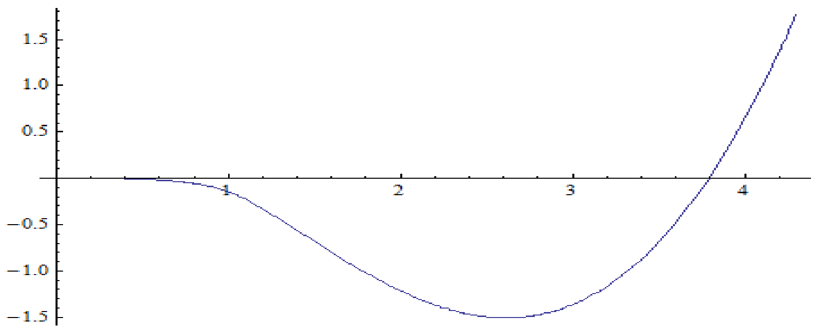
$$+ \frac{1}{2}xEi(3\log x) - \frac{1}{2}Ei(4\log x)$$

$$= \frac{1}{3!}\{x^3Ei(\log x) - 3x^2Ei(2\log x) + 3xEi(3\log x) - Ei(4\log x)\}$$

Hereafter, by induction, we obtain the desired expresson .

Example : 2nd order integral of $li(x)$

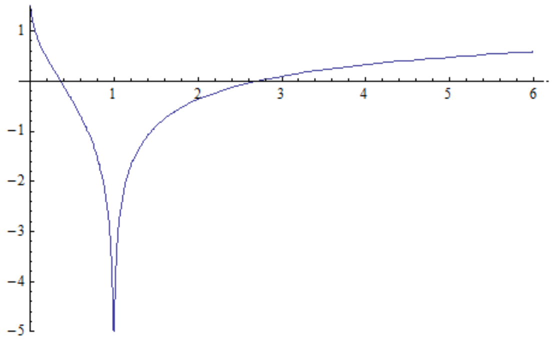
```
f1[x_] := ∫_0^x (∫_0^x LogIntegral[x] dx) dx
fr[n_, x_] = 1/n! Σ_{r=0}^{n} (-1)^r Binomial[n, r] x^(n-r) ExpIntegralEi[(r + 1) Log[x]];
f1[x]
1/2 (-2 x ExpIntegralEi[2 Log[x]] + ExpIntegralEi[3 Log[x]] + x² LogIntegral[x])
FullSimplify[fr[2, x]]
1/2 (-2 x ExpIntegralEi[2 Log[x]] + ExpIntegralEi[3 Log[x]] + x² LogIntegral[x])
```



14.5 Higher Integral of Double Logarithmic Function

Double Logarithmic Function is defined as follows.

$$f(x) = \log|\log x| \tag{1.0}$$



Integrating both sides of this with respect to x repeatedly and arranging the results, we obtain the following higher indefinite integrals. Where, $li(x)\{=Ei(\log x)\}$ is Logarithmic Integral mentioned in the previous.

$$\int \log|\log x|\,dx = \frac{1}{1!}\{x\log|\log x| - li(x)\}$$

$$\int\int \log|\log x|\,dx^2 = \frac{1}{2!}\{x^2\log|\log x| - 2x\,li(x) + Ei(2\log x)\}$$

$$\int\int\int \log|\log x|\,dx^3 = \frac{1}{3!}\{x^3\log|\log x| - 3x^2 li(x) + 3xEi(2\log x) - Ei(3\log x)\}$$

$$\vdots$$

$$\int\cdots\int \log|\log x|\,dx^n = \frac{1}{n!}\left\{x^n\log|\log x| + \sum_{r=1}^{n}(-1)^r {}_nC_r\, x^{n-r} Ei(r\log x)\right\}$$

Although these right sides are the lineal primitive functions of $\log|\log x|$, since both zeros of $x^n\log|\log x|$ and $Ei(n\log x)$ are 0, zeros of the right sides are all 0. Therfore, the lineal higher primitive function of $\log|\log x|$ can be expressed by the higher integral with a fixed lower limit 0.

Formula 14.5.1

When $Ei(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt$, the following expressions hold for $x \geqq 0$.

$$\int_0^x\cdots\int_0^x \log|\log x|\,dx^n = \frac{1}{n!}\left\{x^n\log|\log x| + \sum_{r=1}^{n}(-1)^r {}_nC_r\, x^{n-r} Ei(r\log x)\right\} \tag{1.n}$$

Example : 2nd order integral of $\log|\log x|$

When the one arbitrary point $x=1.6$ is given) , the values of the both sides are as follows.

$$fl[x\_] := \int_0^x \int_0^u \mathrm{Log[Abs[Log[t]]]}\, dt\, du$$

$$fr[n\_, x\_] := \frac{1}{n!}\left(x^n\,\mathrm{Log[Abs[Log[x]]]} + \sum_{r=1}^{n}(-1)^r\,\mathrm{Binomial}[n, r]\, x^{n-r}\,\mathrm{ExpIntegralEi}[r\,\mathrm{Log}[x]]\right)$$

```
N[fl[1.6]]          N[fr[2, 1.6]]
-0.666445           -0.666445
```

14.6 Super Calculus of Logarithmic Integral

Among the higher integrals mentioned in previous sections, Higher Integral of Logarithmic Integral is extensible even to Super Calculus. It is because this higher integral is expressed with binomial coefficients.

14.6.1 Super Integral of Logarithmic Integral

Formula 14.6.1

When

$$li(x)=\int_0^x \frac{1}{\log t}dt \quad , \qquad Ei(x)=\int_{-\infty}^x \frac{e^t}{t}dt$$

the following expression holds for $p \geqq 0$ and $x \geqq 0$.

$$\int_0^x \sim \int_0^x li(x)\,dx^p = \frac{1}{\Gamma(1+p)} \sum_{r=0}^{\infty} (-1)^r \binom{p}{r} x^{p-r} Ei\{(r+1)\log x\}$$

Proof

First, replace $n!\,,\,{}_nC_r$ with $\Gamma(1+n)\,,\,\binom{n}{r}$ respectively in Formula 14.4.3 . Next, analytically continuing the index of the integration operator to $[0,p]$ from $[1,n]$, we obtain the desired formula.

Example : 3/2th order integral of $li(x)$

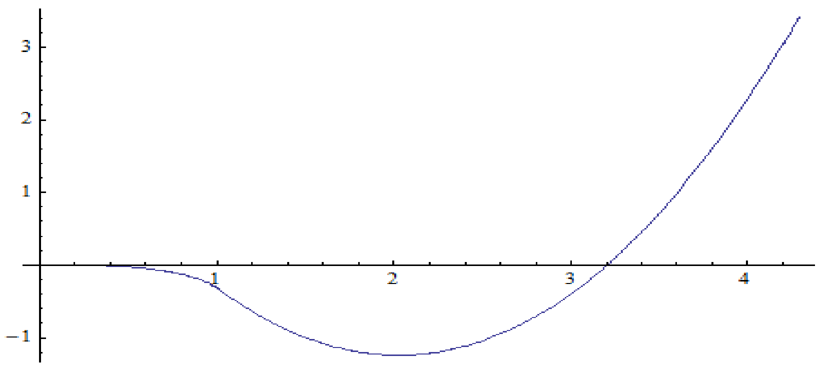
We calculated the function values on arbitrary one point $x=4$ according to the formula and Riemann-Liouville integral. As the result, two values were almost corresponding.
Compared with the figure of the 2nd order integral in **14.4** , we can find that this curvature is loose.

```
p = 3 / 2;  m = 100;
```

$$\text{fl}[x\_] := \frac{1}{\text{Gamma}[p]} \int_0^x (x-t)^{p-1}\, \text{LogIntegral}[t]\, dt$$

$$\text{fr}[x\_] := \frac{1}{\text{Gamma}[1+p]} \sum_{r=0}^{m} (-1)^r\, \text{Binomial}[p, r]\, x^{p-r}\, \text{ExpIntegralEi}[(r+1)\, \text{Log}[x]]$$

```
N[fl[4]]        N[fr[4]]
2.27357         2.27343
```



14.6.2 Super Derivative of Logarithmic Integral

Formula 14.6.2

When

$$li(x)=\int_0^x \frac{1}{\log t}dt \quad , \qquad Ei(x)=\int_{-\infty}^x \frac{e^t}{t}dt$$

the following expression holds for $p>0$, $p\neq 1,2,3,\cdots$ and $x\geqq 0$.

$$\{li(x)\}^{(p)} = \frac{1}{\Gamma(1-p)}\sum_{r=0}^{\infty}(-1)^r\binom{-p}{r}x^{-p-r}Ei\{(r+1)\log x\}$$

Proof

In fact, Formula 14.6.1 holds for $p\neq -1,-2,-3,\cdots$, $x\geqq 0$. Therefore, in Formula 14.6.1 , replacing the integration operator $<p>$ with the differentiation operator (p) $\{=<-p>\}$, we obtain the desired expression.

Example : 1/2th order derivative of $li(x)$

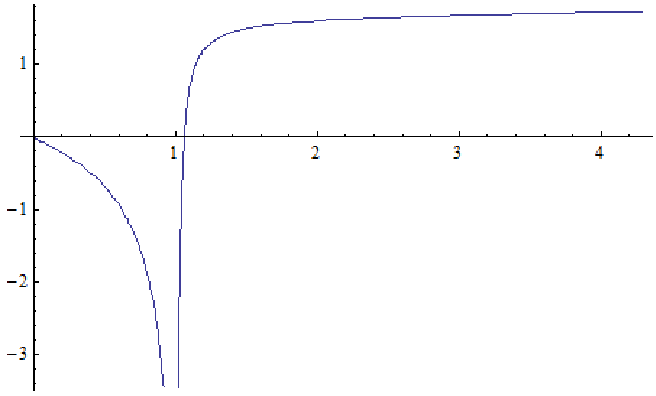
We calculated the the super differential coefficients on arbitrary one point $x=2$ according to the formula and Riemann-Liouville differintegral. As the result, two values were almost corresponding.

```
p = 1 / 2;  h = 10^-6;  m = 7000;

f[x_] := 1/Gamma[1 - p] ∫_0^x (x - t)^(1-p-1) LogIntegral[t] dt        fl = (f[2 + h] - f[2]) / h ;

fr[x_] := 1/Gamma[1 - p] Σ_{r=0}^{m} (-1)^r Binomial[-p, r] x^(-p-r) ExpIntegralEi[(r + 1) Log[x]]

N[fl]                                  N[fr[2]]
1.76931  - 0.000171431  i              1.76409
```



14.7 Super Calculus of Double Logarithmic Function

Among the higher integrals mentioned in previous sections, Higher Integral of Double Logarithmic Function is extensible even to Super Calculus. It is because this higher integral is expressed with binomial coefficients.

14.7.1 Super Integral of Double Logarithmic Function

Formula 14.7.1

When $Ei(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt$, the following expressions hold for $p>0\,,\,x\geqq 0$.

$$\int_0^x\overset{\sim}{\cdots}\int_0^x log\,|log\,x\,|\,dx^p = \frac{1}{\Gamma(1+p)}\left\{x^p log\,|logx| + \sum_{r=1}^{\infty}(-1)^r\binom{p}{r}x^{p-r}Ei(r\,log\,x)\right\}$$

Proof

First, replace $n!\,,\,{}_nC_r$ with $\Gamma(1+n)\,,\,\binom{n}{r}$ respectively in Formula 14.5.1 . Next, analytically continuing the index of the integration operator to $[0,p]$ from $[1,n]$, we obtain the desired formula.

Example : 5/3th order integral of $log|logx|$

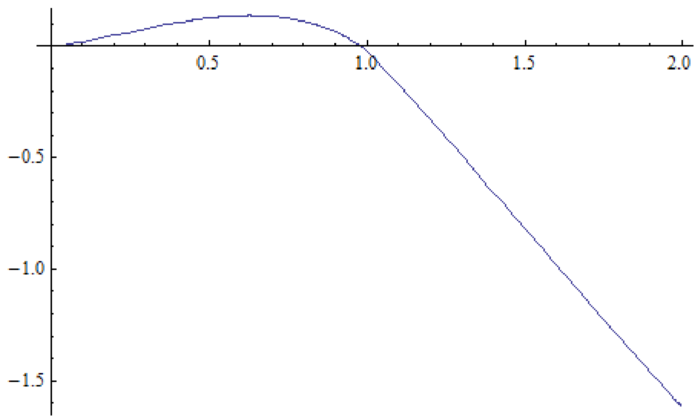
We calculated the function values on arbitrary one point $x=1.5$ according to the formula and Riemann-Liouville integral. As the result, two values were almost corresponding.

```
p = 5 / 3;  m = 150;
```

$$fl[x\_] := \frac{1}{Gamma[p]}\int_0^x (x-t)^{p-1}\,Log[Abs[Log[t]]]\,dt$$

$$fr[x\_] := \frac{1}{Gamma[1+p]}\left(x^p\,Log[Abs[Log[x]]] + \sum_{r=1}^{m}(-1)^r\,Binomial[p,\,r]\,x^{p-r}\,ExpIntegralEi[r\,Log[x]]\right)$$

N[fl[1.5]]	N[fr[1.5]]
-0.818661	-0.818662



14.7.2 Super Derivative of Double Logarithmic Function

Formula 14.7.2

When $Ei(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt$, the following expression holds for $p>0\,,\,p\neq 1,2,3,\cdots\,,\;x\geqq 0$, the following expression holds.

$$\left(\log|\log x|\right)^{(p)} = \frac{1}{x^p\Gamma(1-p)}\left\{\log|\log x| + \sum_{r=1}^{\infty}(-1)^r\binom{-p}{r}\frac{Ei(r\log x)}{x^r}\right\}$$

Proof

In fact, Formula 14.7.1 holds for $p \neq -1,-2,-3,\cdots$, $x \geqq 0$. Therefore, in Formula 14.7.1 , replacing the integration operator $<p>$ with the differentiation operator (p) $\{= <-p>\}$, we obtain the desired expressions.

Example : 0.3th order derivative of $\log|\log x|$

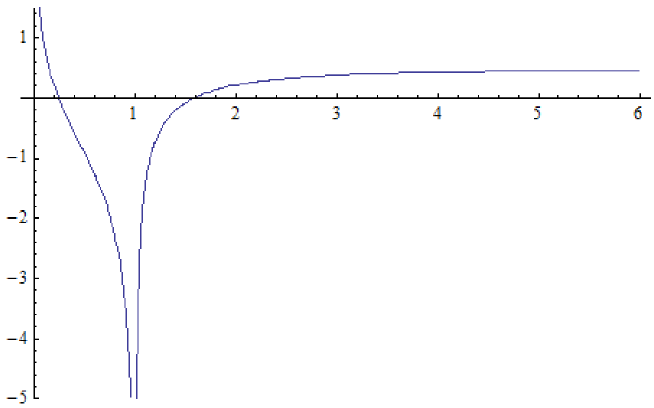
We calculated the the super differential coefficients on arbitrary one point $x=0.5$ according to the formula and Riemann-Liouville differintegral. As the result, two values were almost corresponding.
Since the number of the order of the differentiation is small, it resembles the figure of $\log|\log x|$ well.

```
p = 0.3;  h = 10^-6;  m = 30000;
```

$$f[x\_] := \frac{1}{Gamma[1-p]}\int_0^x (x-t)^{1-p-1} Log[Abs[Log[t]]]\, dt \qquad f1 = \frac{f[0.5+h]-f[0.5]}{h};$$

$$fr[x\_] := \frac{1}{x^p\, Gamma[1-p]}\left(Log[Abs[Log[x]]] + \sum_{r=1}^{m}(-1)^r Binomial[-p, r]\frac{ExpIntegralEi[r\, Log[x]]}{x^r}\right)$$

```
N[f1]          N[fr[0.5]]
-0.93656       -0.936078
```



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Alien's Mathematics