25 Higher and Super Calculus of Zeta Function etc

25.1 Higher and Super Calculus of Riemann Zeta Function

25.1.1 Higher and Super Integral of Riemann Zeta Function

Formula 25.1.1h (Higher Integral)

When $\zeta(z)$ is Riemann zeta function, $\zeta^{<n>}(z)$ is the lineal n -th order primitive, H_n ($=\sum_{k=1}^{n} 1/k$) is a harmonic number (where $H_0 = 0$), the following expression holds on whole complex plane.

$$\zeta^{\langle n\rangle}(z) = \frac{(z-1)^{n-1}}{(n-1)!}\left\{\log(z-1) - H_{n-1}\right\} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r+n}}{(r+n)!} \qquad n=1,2,3,\cdots$$

(1.1h)

Where, γ_r is Stieltjes constant defined by the following expression.

$$\gamma_r = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{(\log k)^r}{k} - \frac{(\log n)^{r+1}}{r+1}\right\}$$

Proof

It is known that the Riemann zeta function $\zeta(z)$ is expanded to Laurent series around 1 as follows.

$$\zeta(z) = \frac{1}{z-1} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^r}{r!}$$

Then, integrating the both sides with respect to z without considering the constant of the integration,

$$\zeta^{<1>}(z) = \log(z-1) + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r+1}}{(r+1)!}$$

Integrating this once more without considering the constant of the integration,

$$\zeta^{<2>}(z) = \frac{(z-1)^1}{1!}\left\{\log(z-1) - 1\right\} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r+2}}{(r+2)!}$$

Integrating this once more without considering the constant of the integration,

$$\zeta^{<3>}(z) = \frac{(z-1)^2}{2!}\left\{\log(z-1) - \left(1+\frac{1}{2}\right)\right\} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r+3}}{(r+3)!}$$

Hereafter, by induction,

$$\zeta^{\langle n\rangle}(z) = \frac{(z-1)^{n-1}}{(n-1)!}\left\{\log(z-1) - \left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)\right\} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r+n}}{(r+n)!}$$

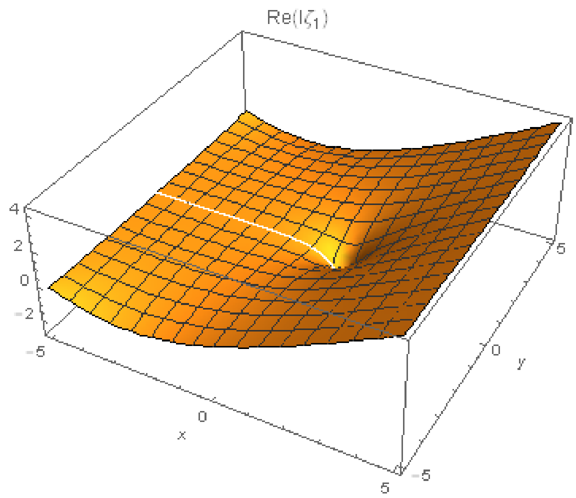
Rewriting the harmonic number as H_{n-1}, we obtain the desired expression.

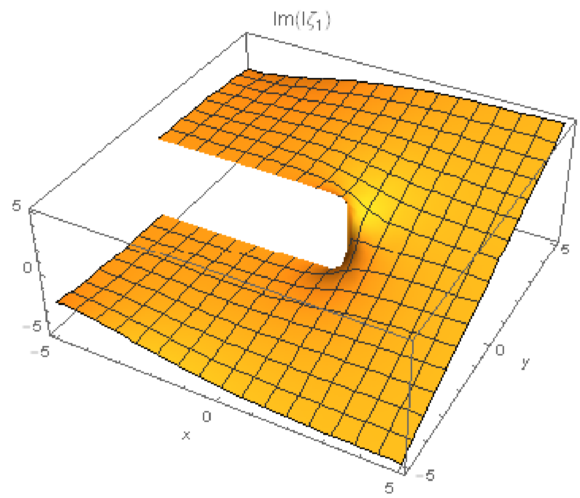
Example The 1st order integral

When the real part and the imaginary part of $\zeta^{<1>}(x+iy)$ are illustrated, it is as follows. The left figure is the real part and the right figure is the imaginary part.

```
H_n_ := HarmonicNumber[n]          γ_s_ := StieltjesGamma[s]
```

$$I\zeta_n[z\_, m\_] := \frac{(z-1)^{n-1}}{(n-1)!}\left(\text{Log}[z-1] - H_{n-1}\right) + \sum_{r=0}^{m} (-1)^r \gamma_r \frac{(z-1)^{r+n}}{(r+n)!}$$





Note

If the left side of (1.1h) is represented by integral symbols, it is as follows.

$$\int_{a_n}^{z}\cdots\int_{a_1}^{z}\zeta(z)dz = \frac{(z-1)^{n-1}}{(n-1)!}\left\{log(z-1) - H_{n-1}\right\} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r+n}}{(r+n)!}$$

Here, the lower limits of the integral are as follows.

$$a_1 = 1.669008\cdots,\ a_2 = 2.641300\cdots,\ a_3 = 3.610288\cdots,\ \cdots$$

That is, this is a higher integral with variable lower limits. And this is a lineal higher integral .

cf.

The higher integral with variable lower limits $a_1 = a_2 = \cdots = a_n = 0$ is as folows.

$$\int_0^{z}\cdots\int_0^{z}\zeta(z)dz^n = \frac{(z-1)^{n-1}}{(n-1)!}\left\{log(z-1) - H_{n-1}\right\} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r+n}}{(r+n)!}$$

$$- i\pi\frac{(z-1)^{n-1}}{(n-1)!} + \sum_{r=1}^{n-1}\frac{(-1)^r H_r z^{n-1-r}}{r!(n-1-r)!} - \sum_{r=0}^{\infty}\sum_{s=1}^{n}\frac{(-1)^s \gamma_r z^{n-s}}{(r+s)!(n-s)!}$$

As seen from the existence of constant-of-integration polynomials, this is a collateral higher integral.

Formula 25.1.1s (Super Integral)

When p is a complex number, $\zeta(z)$ is Riemann zeta function, $\zeta^{<p>}(z)$ is the lineal p -th order primitive, $\Gamma(p)$ is gamma function, $\psi(p)$ is digamma function and γ_r is Stieltjes constant, the following expression holds on whole complex plane.

$$\zeta^{\langle p\rangle}(z) = \frac{log(z-1) - \psi(p) - \gamma_0}{\Gamma(p)}(z-1)^{p-1} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r+p}}{\Gamma(1+r+p)} \qquad (1.1s)$$

Proof

From (1.1h) ,

$$\zeta^{\langle n \rangle}(z) = \frac{(z-1)^{n-1}}{(n-1)!}\{\log(z-1) - H_{n-1}\} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r+n}}{(r+n)!} \qquad n=1,2,3,\cdots$$

At first,

$$(n-1)! = \Gamma(n) \quad , \quad H_{n-1} = \psi(n) + \gamma_0 \quad , \quad (r+n)! = \Gamma(1+r+n)$$

Using these,

$$\zeta^{\langle n \rangle}(z) = \frac{(z-1)^{n-1}}{\Gamma(n)}\{\log(z-1) - \psi(n) + \gamma_0\} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r+n}}{\Gamma(1+r+n)}$$

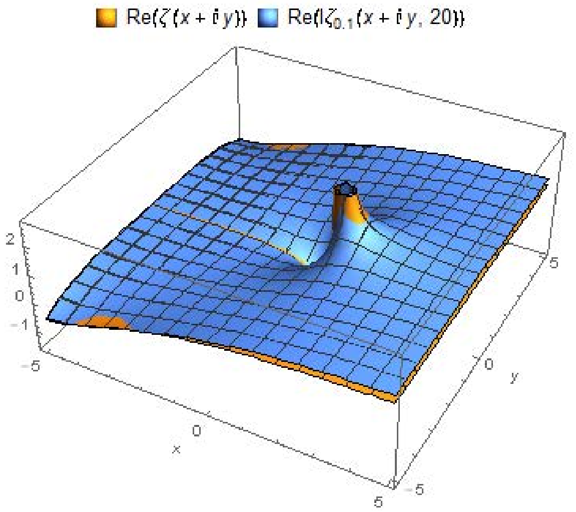
Then, this expression also holds at $n=0$. Because,

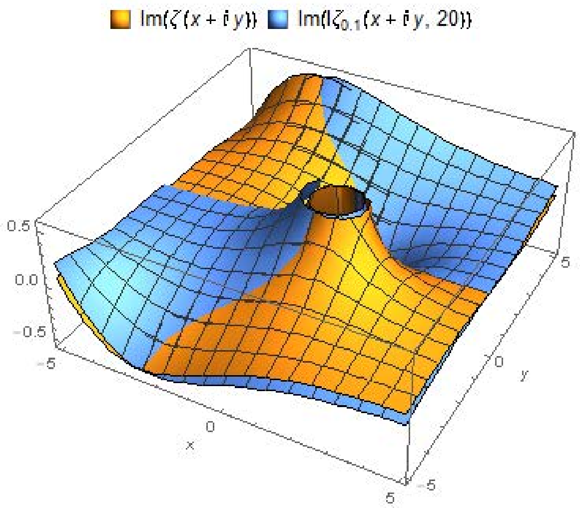
$$\frac{\log(z-1) - \gamma_0}{\Gamma(0)} = \frac{\log(z-1) - \gamma_0}{\infty} = 0 \quad , \quad \frac{-\psi(0)}{\Gamma(0)} = 1$$

So, replacing the natural number n with a complex number p, we obtain the desired expression.

Example The 0.1th order integral

When $z = x+iy$, $\zeta(z)$ and $\zeta^{<0.1>}(z)$ are illustrated as follows. The left figure is a real part and the right figure is an imaginary part. In both figures, the orange is $\zeta(z)$ and the blue is $\zeta^{<0.1>}(z)$. Since p is neer 0, both curved surfaces look double.





25.1.2 Higher and Super Derivative of Riemann Zeta Function

Formula 25.1.2h (Higher Derivative)

When $\zeta(z)$ is Riemann zeta function, $\zeta^{(n)}(z)$ is the lineal n-th order derivative and γ_r is Stieltjes constant, the following expression holds on whole complex plane.

$$\zeta^{(n)}(z) = \frac{(-1)^{-n}n!}{(z-1)^{n+1}} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r-n}}{\Gamma(1+r-n)} \tag{1.2h}$$

Proof

It is known that the Riemann zeta function $\zeta(z)$ is expanded to Laurent series around 1 as follows.

$$\zeta(z) = \frac{1}{z-1} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^r}{r!}$$

Differentiating the both sides n times with respect to z ,

$$\zeta^{(n)}(z) = \left(\frac{1}{z-1}\right)^{(n)} + \sum_{r=0}^{\infty}(-1)^r \frac{\gamma_r}{r!}\left\{(z-1)^r\right\}^{(n)}$$

According to Formula 9.2.1 in " **09 Higher Derivative** ",

$$\left(x^{\alpha}\right)^{(n)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)}x^{\alpha-n} \qquad (\alpha \geqq 0)$$

$$= (-1)^{-n}\frac{\Gamma(-\alpha+n)}{\Gamma(-\alpha)}x^{\alpha-n} \qquad (\alpha < 0)$$

Applying this,

$$\left\{(z-1)^r\right\}^{(n)} = \frac{\Gamma(1+r)}{\Gamma(1+r-n)}(z-1)^{r-n} = \frac{r!}{\Gamma(1+r-n)}(z-1)^{r-n}$$

$$\left(\frac{1}{z-1}\right)^{(n)} = (-1)^{-n}\frac{\Gamma(1+n)}{\Gamma(1)}(z-1)^{-1-n} = (-1)^{-n}\frac{n!}{(z-1)^{n+1}}$$

Substituting these for the above,

$$\zeta^{(n)}(z) = \frac{(-1)^{-n}n!}{(z-1)^{n+1}} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r-n}}{\Gamma(1+r-n)} \qquad (1.2h)$$

Example $\zeta^{(1)}(0.3)$, $\zeta^{(2)}(-1.1+2.3i)$

When these are calculated by formula manipulation soft ***Mathematica***, it is as follows. We can see that this formula is numerically right.

```
γr_ := StieltjesGamma[r]
Dζn_[z_, m_] := (-1)^-n n! / (z - 1)^(n+1) + Sum[(-1)^r γr (z - 1)^(r-n) / Gamma[1 + r - n], {r, 0, m}]

N[{Zeta'[0.3] , Dζ1[0.3, 20]}]
    {-1.96186 , -1.96186}

N[{Zeta''[-1.1 + 2.3 i] , Dζ2[-1.1 + 2.3 i, 20]}]
{0.0446687 - 0.0601275 i , 0.0446687 - 0.0601275 i}
```

Formula 25.1.2s (Super Derivative)

When p is a complex number, $\zeta(z)$ is Riemann zeta function, $\zeta^{(p)}(z)$ is the lineal p-th order derivative, $\Gamma(p)$ is gamma function, $\psi(p)$ is digamma function and γ_r is Stieltjes constant, the following expression holds on whole complex plane.

$$\zeta^{(p)}(z) = \frac{\log(z-1) - \psi(-p) - \gamma_0}{\Gamma(-p)}(z-1)^{-p-1} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r-p}}{\Gamma(1+r-p)} \qquad (1.2s)$$

Proof

From Formula 25.1.2h ,

$$\zeta^{(n)}(z) = \frac{(-1)^{-n}n!}{(z-1)^{n+1}} + \sum_{r=0}^{\infty}(-1)^r\gamma_r\frac{(z-1)^{r-n}}{(r-n)!} \tag{1.2h}$$

According to Formula 1.3.1 in " **01 Gamma Function & Digamma Function** " ,

$$(-1)^{-n}n! = -\frac{\psi(-n)}{\Gamma(-n)} \quad , \quad n=0,1,2,3,\cdots$$

Using this, (1.2h) is rewritten as follows.

$$\zeta^{(n)}(z) = \frac{log(z-1)-\psi(-n)-\gamma_0}{\Gamma(-n)}(z-1)^{-n-1} + \sum_{r=0}^{\infty}(-1)^r\gamma_r\frac{(z-1)^{r-n}}{\Gamma(1+r-p)}$$

Because,

$$\frac{log(z-1)-\gamma_0}{\Gamma(-n)} = \frac{log(z-1)-\gamma_0}{\pm\infty} = 0 \quad for\ n=0,1,2,3,\cdots$$

So, replacing the natural number n with a complex number p , we obtain the desired expression.

$$\zeta^{(p)}(z) = \frac{log(z-1)-\psi(-p)-\gamma_0}{\Gamma(-p)}(z-1)^{-p-1} + \sum_{r=0}^{\infty}(-1)^r\gamma_r\frac{(z-1)^{r-p}}{\Gamma(1+r-p)} \tag{1.2s}$$

Note

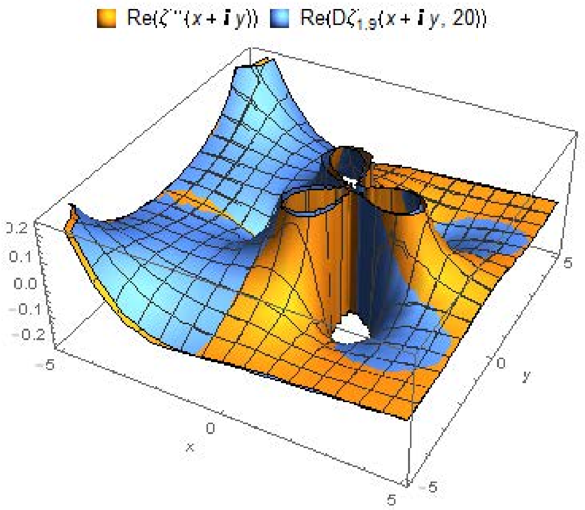
If the sign of p is inverted in (1.2s), it becomes as follows.

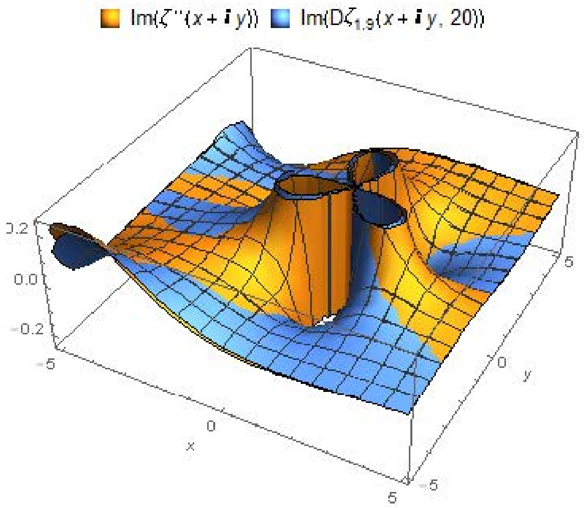
$$\zeta^{(-p)}(z) = \frac{log(z-1)-\psi(p)-\gamma_0}{\Gamma(p)}(z-1)^{p-1} + \sum_{r=0}^{\infty}(-1)^r\gamma_r\frac{(z-1)^{r+p}}{\Gamma(1+r+p)}$$

This results in Formula 25.1.1s . That is, the lineal super calculus of Riemann zeta function is seamless.

Example The 1.9th order derivative

When $z=x+iy$, $\zeta^{(2)}(z)$ and $\zeta^{(1.9)}(z)$ are illustrated as follows. The left figure is a real part and the right figure is an imaginary part. In both figures, the orange is $\zeta^{(2)}(z)$ and the blue is $\zeta^{(1.9)}(z)$. Since p is near 2 , both curved surfaces look double.





25.2 Higher and Super Calculus of Dirichlet Lambda Function

25.2.1 Higher and Super Integral of Dirichlet Lambda Function

Formula 25.2.1h (Higher Integral)

When $\lambda(z)$ is Dirichlet lambda function, $\lambda^{<n>}(z)$ is the lineal n -th order primitive, H_n $\left(=\sum_{k=1}^{n} 1/k\right)$ is a harmonic number (where $H_0=0$) , the following expression holds on whole complex plane.

$$\lambda^{<n>}(z) = \frac{(z-1)^{n-1}}{2(n-1)!}\{log(z-1)-H_{n-1}\}$$
$$+\frac{1}{2}\sum_{r=0}^{\infty}(-1)^r\left\{\gamma_r+\frac{log^{r+1}2}{r+1}-\sum_{s=0}^{r-1}\binom{r}{s}\gamma_s(log\,2)^{r-s}\right\}\frac{(z-1)^{r+n}}{(r+n)!} \qquad n=1,2,3,\cdots \tag{2.1h}$$

Where, γ_r is Stieltjes constant defined by the following expression.

$$\gamma_r = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{(log\,k)^r}{k}-\frac{(log\,n)^{r+1}}{r+1}\right\}$$

Proof

According to Formula 3.1.3 in " **03 Complementary Series of Dirichlet Series** " (Dirichlet Series), Dirichlet lambda function is expanded to Laurent series around 1 as follows.

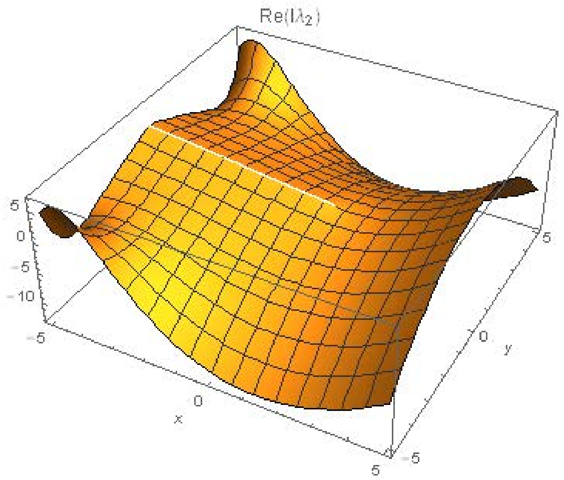
$$\lambda(z) = \frac{1}{2(z-1)}+\frac{1}{2}\sum_{r=0}^{\infty}(-1)^r\left\{\gamma_r+\frac{log^{r+1}2}{r+1}-\sum_{s=0}^{r-1}\binom{r}{s}\gamma_s(log\,2)^{r-s}\right\}\frac{(z-1)^r}{r!}$$

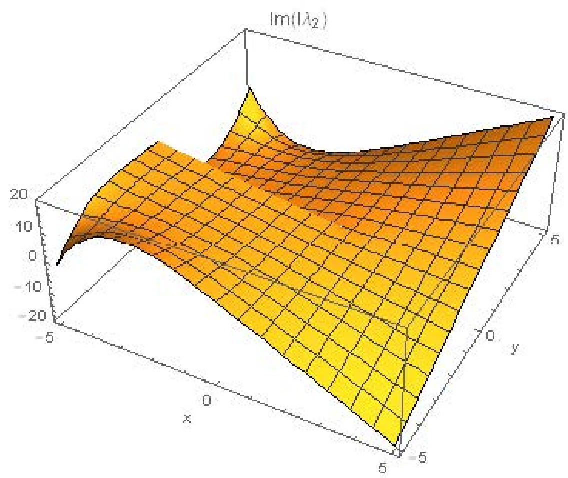
Where, Σ in { } is absent for $r=0$.

Hereafter, in a way similar to the proof of Formula 25.1.1h , the desired expression is obtained.

Example The 2nd order integral

When the real part and the imaginary part of $\lambda^{<1>}(x+iy)$ are illustrated, it is as follows. The left figure is the real part and the right figure is the imaginary part.





Note

This is also lineal higher integral with variable lower limits. The first few of the integral lower limits are as follows.

$$a_1 = 1.509052\cdots,\ a_2 = 2.203125\cdots,\ a_3 = 2.891846\cdots,\ \cdots$$

Formula 25.2.1s (Super Integral)

When p is a complex number, $\lambda(z)$ is Dirichlet lambda function, $\lambda^{<p>}(z)$ is the lineal p -th order primitive, $\Gamma(p)$ is gamma function, $\psi(p)$ is digamma function and γ_r is Stieltjes constant, the following expression holds on whole complex plane.

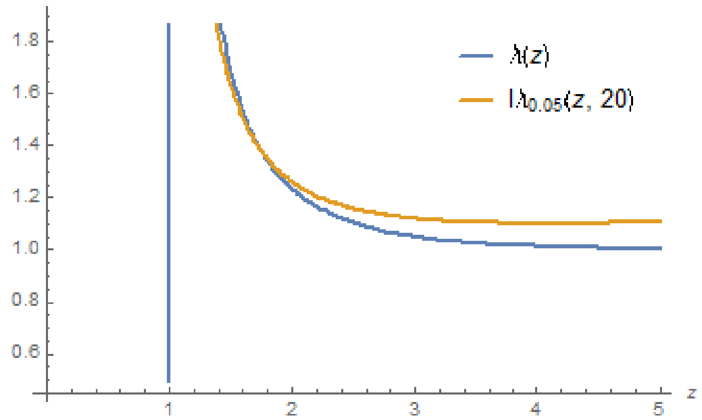
$$\lambda^{<p>}(z) = \frac{\log(z-1) - \psi(p) - \gamma_0}{2\Gamma(p)}(z-1)^{p-1}$$
$$+ \frac{1}{2}\sum_{r=0}^{\infty}(-1)^r\left\{\gamma_r + \frac{\log^{r+1}2}{r+1} - \sum_{s=0}^{r-1}\binom{r}{s}\gamma_s(\log 2)^{r-s}\right\}\frac{(z-1)^{r+p}}{\Gamma(1+r+p)} \qquad (2.1s)$$

Proof

In a way similar to the proof of Formula 25.1.1s , the desired expression is obtained.

Example The 0.05th order integral

$\lambda(z)$ and $\lambda^{<0.05>}(z)$ are illustrated as follows.



25.2.2 Higher and Super Derivative of Dirichlet Lambda Function

Formula 25.2.2h (Higher Derivative)

When $\lambda(z)$ is Dirichlet lambda function, $\lambda^{(n)}(z)$ is the lineal n -th order derivative and γ_r is Stieltjes constant, the following expression holds on whole complex plane.

$$\lambda^{(n)}(z) = \frac{(-1)^{-n}n!}{2(z-1)^{n+1}}$$
$$+ \frac{1}{2}\sum_{r=0}^{\infty}(-1)^r\left\{\gamma_r + \frac{\log^{r+1}2}{r+1} - \sum_{s=0}^{r-1}\binom{r}{s}\gamma_s(\log 2)^{r-s}\right\}\frac{(z-1)^{r-n}}{\Gamma(1+r-n)} \qquad n=0,1,2,\cdots$$

(2.2h)

Proof

As seen in the proof of Formula 25.2.1h , the Dirichlet lambda function $\lambda(z)$ is expanded to Laurent series around 1 as follows.

$$\lambda(z) = \frac{1}{2(z-1)} + \frac{1}{2}\sum_{r=0}^{\infty}(-1)^r\left\{\gamma_r + \frac{\log^{r+1}2}{r+1} - \sum_{s=0}^{r-1}\binom{r}{s}\gamma_s(\log 2)^{r-s}\right\}\frac{(z-1)^r}{r!}$$

Where, Σ in { } is absent for $r=0$.

Hereafter, in a way similar to the proof of Formula 25.1.2h , the desired expression is obtained.

Example $\lambda^{(2)}(-2.1)$, $\lambda^{(3)}(-0.8-1.9i)$

When these are calculated by formula manipulation soft ***Mathematica***, it is as follows. We can see that this formula is numerically right.

```
γr_ := StieltjesGamma[r]
Dλn_[z_, m_] := (-1)^-n n! / (2 (z - 1)^(n+1))
  + 1/2 Sum[(-1)^r (γr + Log[2]^(r+1)/(r + 1) - Sum[Binomial[r, s] γs Log[2]^(r-s), {s, 0, r-1}]) (z - 1)^(r-n)/Gamma[1 + r - n], {r, 0, m}]
N[{DirichletLambda''[-2.1] , Dλ2[-2.1, 20]}]
           {0.0404744 , 0.0404744}
N[{DirichletLambda'''[-0.8 - 1.9 i] , Dλ3[-0.8 - 1.9 i, 20]}]
           {0.0682879 - 0.0999407 i , 0.0682879 - 0.0999407 i}
```

Formula 25.2.2s (Super Derivative)

When p is a complex number, $\lambda(z)$ is Dirichlet lambda function, $\lambda^{(p)}(z)$ is the lineal p-th order derivative, $\Gamma(p)$ is gamma function, $\psi(p)$ is digamma function and γ_r is Stieltjes constant, the following expression holds on whole complex plane.

$$\lambda^{(p)}(z) = \frac{\log(z-1)-\psi(-p)-\gamma_0}{2\Gamma(-p)}(z-1)^{-p-1}$$
$$+\frac{1}{2}\sum_{r=0}^{\infty}(-1)^r\left\{\gamma_r + \frac{\log^{r+1}2}{r+1} - \sum_{s=0}^{r-1}\binom{r}{s}\gamma_s(\log 2)^{r-s}\right\}\frac{(z-1)^{r-p}}{\Gamma(1+r-p)} \tag{2.2s}$$

Proof

In a way similar to the proof of Formula 25.1.2s , the desired expression is obtained.

Note

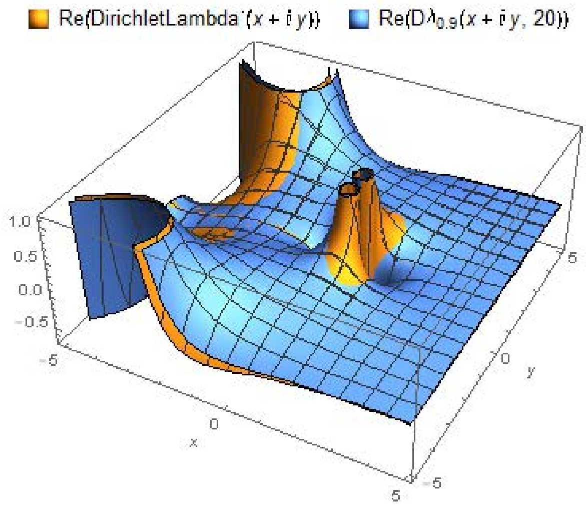
If the sign of p is inverted in (2.2s), it becomes as follows.

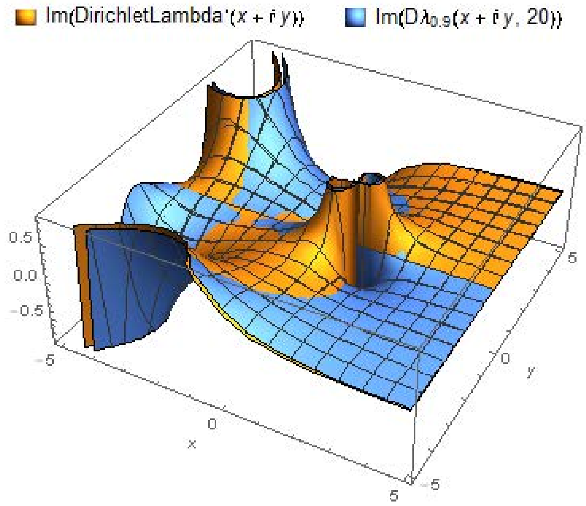
$$\lambda^{(-p)}(z) = \frac{\log(z-1)-\psi(p)-\gamma_0}{2\Gamma(p)}(z-1)^{p-1}$$
$$+\frac{1}{2}\sum_{r=0}^{\infty}(-1)^r\left\{\gamma_r + \frac{\log^{r+1}2}{r+1} - \sum_{s=0}^{r-1}\binom{r}{s}\gamma_s(\log 2)^{r-s}\right\}\frac{(z-1)^{r+p}}{\Gamma(1+r+p)}$$

This results in Formula 25.2.1s . That is, the lineal super calculus of Dirichlet lambda function is seamless.

Example The 0.9th order derivative

When $z = x + iy$, $\lambda^{(1)}(z)$ and $\lambda^{(0.9)}(z)$ are illustrated as follows. The left figure is a real part and the right figure is an imaginary part. In both figures, the orange is $\lambda^{(1)}(z)$ and the blue is $\lambda^{(0.9)}(z)$. Since p is near 1 , both curved surfaces look double.





25.3 Higher and Super Calculus of Dirichlet Eta Function

25.3.1 Higher and Super Integral of Dirichlet Eta Function

Formula 25.3.1h (Higher Integral)

When $\eta(z)$ is Dirichlet eta function, $\eta^{<n>}(z)$ is the lineal n -th order primitive,

(1) The following expression holds for z $s.t.$ $Re(z) > 0$.

$$\eta^{\langle n \rangle}(z) = \frac{z^n}{n!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^{-n} r}{r^z} \qquad n=0,1,2,\cdots \tag{3.1h}$$

(2) The following expression holds on whole complex plane.

$$\eta^{\langle n \rangle}(z) = \frac{z^n}{n!} + (-1)^n \sum_{k=2}^{\infty} \sum_{r=2}^{k} \frac{(-1)^{r-1}}{2^{k+1}} \binom{k}{r} \frac{\log^{-n} r}{r^z} \qquad n=0,1,2,\cdots \tag{3.1h'}$$

Proof

At $Re(z) > 0$, Dirichlet eta function $\eta(z)$ is expressed with the following series which is called DIrichlet eta series.

$$\eta(z) = 1 + \sum_{r=2}^{\infty} (-1)^{r-1} e^{-z\log r} = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} +- \cdots$$

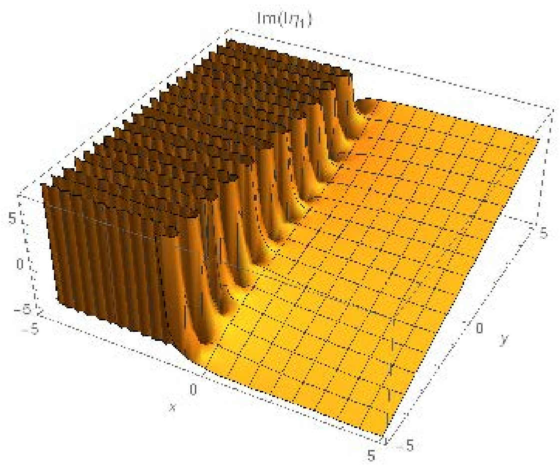
So, integrating the both sides n times with respect to z without considering the constant of the integration,

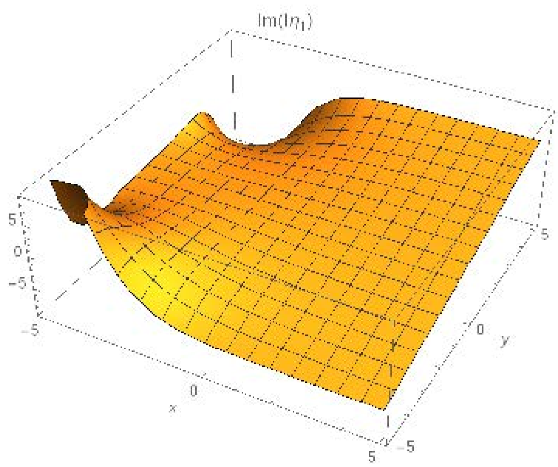
$$\eta^{\langle n \rangle}(z) = \frac{z^n}{n!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^{-n} r}{r^z} \qquad n=0,1,2,\cdots \tag{3.1h}$$

Applying Euler transformation to this second term, we obtain (3.1h') . By this transformation, (3.1h) is analytically continued from $Re(z) > 0$ to the whole complex plane. In addition, about Euler transformation, see " **10 Convergence Acceleration & Summation Method by Double Series of Functions** " (A la carte).

Example The 1st order integral

The imaginary part of $\eta^{<1>}(x+iy)$ are illustrated as follows. The left is (3.1h) and the right is (3.1h') . In the left figure, the line of convergence is visible at $x=0$.





Note

If the left side of (3.1h) is represented by integral symbols, it is as follows.

$$\int_{a_n}^{z}\cdots\int_{a_1}^{z}\eta(z)\,dz = \frac{z^n}{n\,!} + (-1)^n\sum_{r=2}^{\infty}(-1)^{r-1}\frac{\log^{-n} r}{r^z}$$

Here, the lower limits of the integral are as follows.

$$a_1 = -1.809613\cdots + i\,1.766080\cdots,\ a_2 = 1.216967\cdots,$$

$$a_3 = 1.337211\cdots + i\,1.289222\cdots,\ a_4 = 2.163768\cdots,\ \cdots$$

That is, this is a higher integral with variable lower limits. And this is a lineal higher integral .

Formula 25.3.1s (Super Integral)

When p is a complex number, $\eta(z)$ is Dirichlet eta function, $\eta^{<p>}(z)$ is the lineal p-th order primitive and $\Gamma(p)$ is gamma function,

(1) The following expression holds for z s.t. $Re(z) > 0$.

$$\eta^{\langle p\rangle}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i}\sum_{r=2}^{\infty}(-1)^{r-1}\frac{\log^{-p} r}{r^z} \tag{3.1s}$$

(2) The following expression holds on whole complex plane.

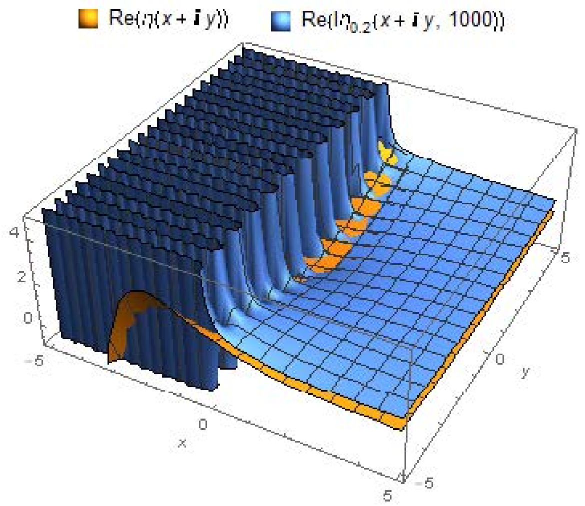
$$\eta^{\langle p\rangle}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i}\sum_{k=2}^{\infty}\sum_{r=2}^{k}\frac{(-1)^{r-1}}{2^{k+1}}\binom{k}{r}\frac{\log^{-p} r}{r^z} \tag{3.1s'}$$

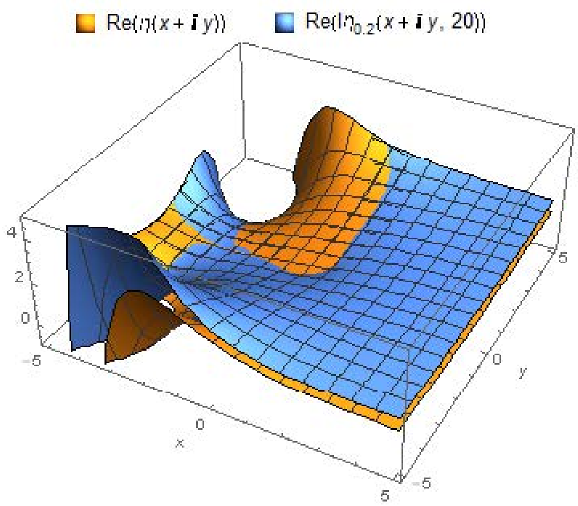
Proof

In Formula 25.3.1h , replacing $n!$ with $\Gamma(1+n)$, replacing $(-1)^{-n}$ with $e^{-n\pi i}$ and replacing the natural number n with a complex number p , we obtain the desired expressions.

Example The 0.2th order integral

When $z = x+iy$, the real parts of $\eta(z)$ and $\eta^{(0.2)}(z)$ are illustrated as follows. The left figure is (3.1s) and the right figure is (3.1s'). In both figures, the orange is $\eta(z)$ and the blue is $\eta^{(0.2)}(z)$. Since p is near 0 , both curved surfaces look double. In the left figure, the line of convergence is visible at $x=0$.





25.3.2 Higher and Super Derivative of Dirichlet Eta Function

Formula 25.3.2h (Higher Integral)

When $\eta(z)$ is Dirichlet eta function, $\eta^{(n)}(z)$ is the lineal n -th order drivative and $\Gamma(n)$ is gamma function,

(1) The following expression holds for z $s.t.$ $Re(z) > 0$.

$$\eta^{(n)}(z) = \frac{z^{-n}}{\Gamma(1-n)} + (-1)^{-n} \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\log^n r}{r^z} \qquad n=0,1,2,\cdots \tag{3.2h}$$

(2) The following expression holds on whole complex plane.

$$\eta^{(n)}(z) = \frac{z^{-n}}{\Gamma(1-n)} + (-1)^{-n} \sum_{k=2}^{\infty} \sum_{r=2}^{k} \frac{(-1)^{r-1}}{2^{k+1}} \binom{k}{r} \frac{\log^n r}{r^z} \qquad n=0,1,2,\cdots \tag{3.2h'}$$

Proof

As seen in the proof of Formula 25.3.1h ,

$$\eta(z) = 1 + \sum_{r=2}^{\infty} (-1)^{r-1} e^{-z\log r} = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} +- \cdots$$

So, assuming $1 = z^0/0!$ and differentiating the both sides n times with respect to z ,

$$\eta^{(n)}(z) = \frac{z^{-n}}{(-n)!} + (-1)^{-n} \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\log^n r}{r^z} \qquad n=0,1,2,\cdots$$

Replacing $(-n)!$ with $\Gamma(1-n)$, we obtain (3.2h) . In addition,

$$\frac{z^{-n}}{\Gamma(1-n)} = \begin{cases} 1 & for\ n=0 \\ 0 & for\ n=1,2,3,\cdots \end{cases}$$

And applying Euler transformation to this second term, we obtain (3.2h') . By this transformation, (3.2h) is analytically continued from $Re(z) > 0$ to the whole complex plane.

Example $\eta^{(1)}(0.2)$, $\eta^{(3)}(0.5+14.1i)$

If these are calculated according to (3,2h') by formula manipulation soft ***Mathematica***, it is as follows. We can see that this formula is numerically right.

$$D\eta_n[z\_, m\_] := \frac{z^{-n}}{\text{Gamma}[1-n]} + (-1)^{-n} \sum_{k=2}^{m} \sum_{r=2}^{k} \frac{(-1)^{r-1}}{2^{k+1}} \text{Binomial}[k, r] \frac{(\text{Log}[r])^n}{r^z}$$

```
N[{DirichletEta'[0.2], Dη1[0.2, 23]}]
          {0.21318 , 0.21318}

N[{DirichletEta''[0.5 + 14.1 i], Dη2[0.5 + 14.1 i, 45]}]
     {-3.05246 - 0.0494394 i , -3.05246 - 0.0494394 i}
```

Formula 25.3.2s (Super Derivative)

When p is a complex number, $\eta(z)$ is Dirichlet eta function, $\eta^{(p)}(z)$ is the lineal p -th order derivative and $\Gamma(p)$ is gamma function,

(1) The following expression holds for z s.t. $Re(z) > 0$.

$$\eta^{(p)}(z) = \frac{z^{-p}}{\Gamma(1-p)} + e^{-p\pi i}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\log^p r}{r^z} \tag{3.2s}$$

(2) The following expression holds on whole complex plane.

$$\eta^{(p)}(z) = \frac{z^{-p}}{\Gamma(1-p)} + e^{-p\pi i}\sum_{k=2}^{\infty}\sum_{r=2}^{k}\frac{(-1)^{r-1}}{2^{k+1}}\binom{k}{r}\frac{\log^p r}{r^z} \tag{3.2s'}$$

Proof

In Formula 25.3.2h , replacing $(-1)^{-n}$ with $e^{-n\pi i}$ and replacing the natural number n with a complex number p , we obtain the desired expressions.

Note

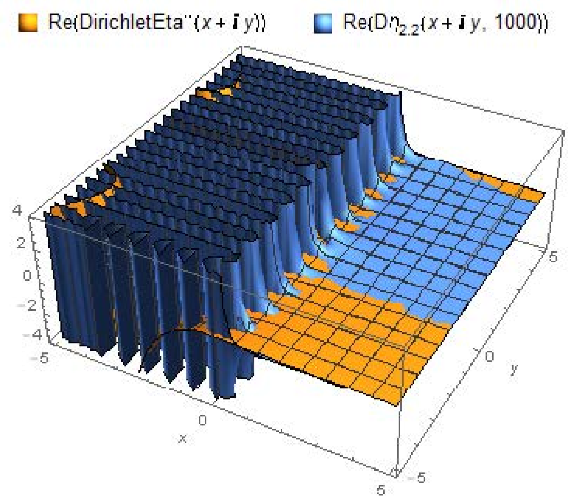
If the sign of p is inverted in (3.2s), it becomes as follows.

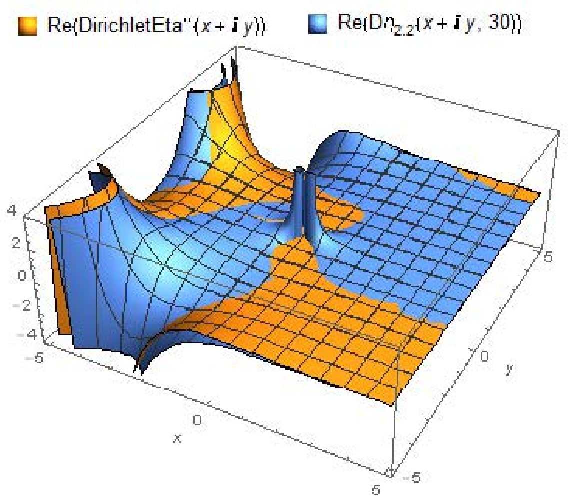
$$\eta^{(-p)}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\log^{-p} r}{r^z}$$

This results in Formula 25.3.1s . That is, the lineal super calculus of Dirichlet eta function is seamless.

Example The 2.2th order derivative

When $z = x+iy$, the real parts of $\eta^{(2)}(z)$ and $\eta^{(2.2)}(z)$ are illustrated as follows. The left figure is (3.2s) and the right figure is (3.2s'). In both figures, the orange is $\eta^{(2)}(z)$ and the blue is $\eta^{(2.2)}(z)$. In the left figure, the line of convergence is visible at $x=0$.





25.4 Higher and Super Calculus of Dirichlet Beta Function

25.4.1 Higher and Super Integral of Dirichlet Beta Function

Formula 25.4.1h (Higher Integral)

When $\beta(z)$ is Dirichlet beta function, $\beta^{<n>}(z)$ is the lineal n -th order primitive,

(1) The following expression holds for z $s.t.$ $Re(z) > 0$.

$$\beta^{\langle n\rangle}(z) = \frac{z^n}{n!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^{-n}(2r-1)}{(2r-1)^z} \qquad n=0,1,2,\cdots \tag{4.1h}$$

(2) The following expression holds on whole complex plane.

$$\beta^{<n>}(z) = \frac{z^n}{n!} + (-1)^{-n} \sum_{k=2}^{\infty} \sum_{r=2}^{k} \frac{(-1)^{r-1}}{2^{k+1}} \binom{k}{r} \frac{\log^{-n}(2r-1)}{(2r-1)^z} \qquad n=0,1,2,\cdots \tag{4.1h'}$$

Proof

At $Re(z) > 0$, Dirichlet beta function $\beta(z)$ is expressed with the following series which is called DIrichlet beta series.

$$\beta(z) = 1 + \sum_{r=2}^{\infty} (-1)^{r-1} e^{-z\log(2r-1)} = 1 - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} +- \cdots$$

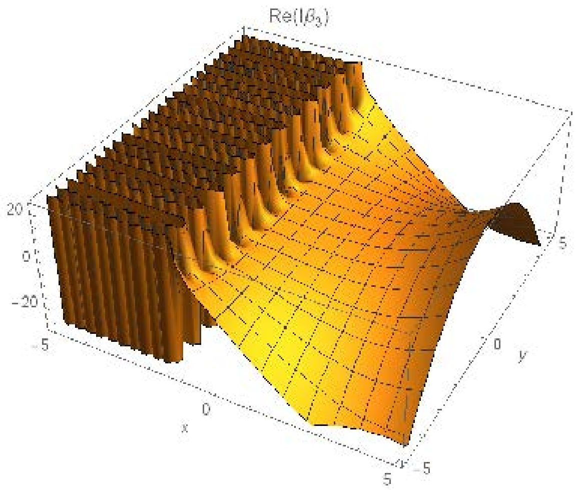
So, integrating the both sides n times with respect to z without considering the constant of the integration,

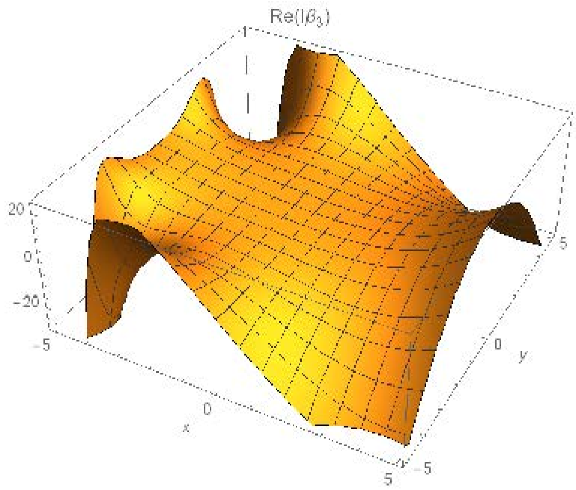
$$\beta^{\langle n\rangle}(z) = \frac{z^n}{n!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^{-n}(2r-1)}{(2r-1)^z} \qquad n=0,1,2,\cdots \tag{4.1h}$$

Applying Euler transformation to this second term, we obtain (4.1h') . By this transformation, (4.1h) is analytically continued from $Re(z) > 0$ to the whole complex plane.

Example The 3rd order integral

The imaginary part of $\beta^{<3>}(x+iy)$ are illustrated as follows. The left is (4.1h) and the right is (4.1h') . In the left figure, the line of convergence is visible at $x=0$.





Note

If the left side of (4.1h) is represented by integral symbols, it is as follows.

$$\int_{a_n}^{z}\cdots\int_{a_1}^{z}\beta(z)dz = \frac{z^n}{n!} + (-1)^n\sum_{r=2}^{\infty}(-1)^{r-1}\frac{\log^{-n}(2r-1)}{(2r-1)^z}$$

Here, the lower limits of the integral are as follows.

$a_1 = -1.027077\cdots + i\,0.978760\cdots$, $a_2 = 0.754520\cdots$,

$a_3 = 0.831711\cdots + i\,0.807718\cdots$, $a_4 = 1.357089\cdots$, $\cdots$

That is, this is a higher integral with variable lower limits. And this is a lineal higher integral .

Formula 25.4.1s (Super Integral)

When p is a complex number, $\beta(z)$ is Dirichlet beta function, $\beta^{<p>}(z)$ is the lineal p-th order primitive and $\Gamma(p)$ is gamma function,

(1) The following expression holds for z s.t. $Re(z) > 0$.

$$\beta^{<p>}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i}\sum_{r=2}^{\infty}(-1)^{r-1}\frac{\log^{-p}(2r-1)}{(2r-1)^z} \tag{4.1s}$$

(2) The following expression holds on whole complex plane.

$$\beta^{<p>}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i}\sum_{k=2}^{\infty}\sum_{r=2}^{k}\frac{(-1)^{r-1}}{2^{k+1}}\binom{k}{r}\frac{\log^{-p}(2r-1)}{(2r-1)^z} \tag{4.1s'}$$

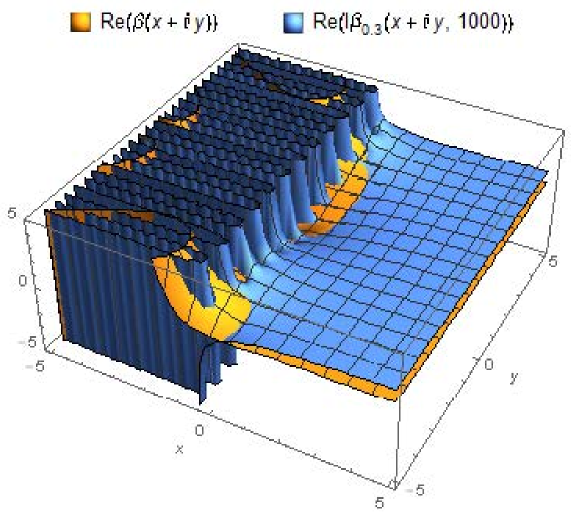
Proof

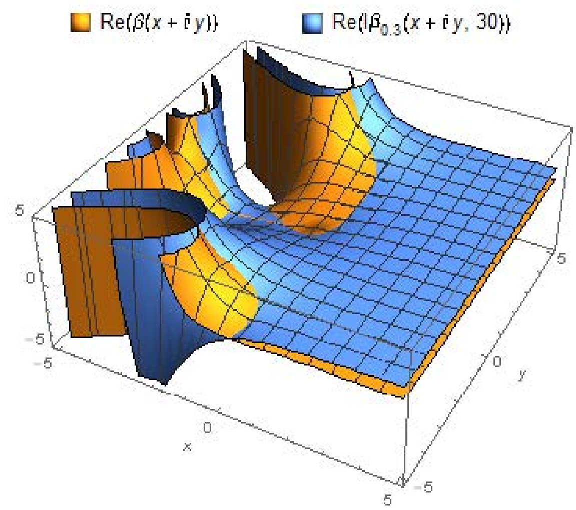
In Formula 25.4.1h , replacing $n!$ with $\Gamma(1+n)$, replacing $(-1)^{-n}$ with $e^{-n\pi i}$ and replacing the natural number n with a complex number p , we obtain the desired expressions.

Example The 0.3th order integral

When $z = x+iy$, the real parts of $\beta(z)$ and $\beta^{(0.3)}(z)$ are illustrated as follows. The left figure is (4.1s) and the right figure is (4.1s'). In both figures, the orange is $\beta(z)$ and the blue is $\beta^{(0.3)}(z)$.

In the left figure, the line of convergence is visible at $x=0$.





25.4.2 Higher and Super Derivative of Dirichlet Beta Function

Formula 25.4.2h (Higher Integral)

When $\beta(z)$ is Dirichlet beta function, $\beta^{(n)}(z)$ is the lineal n -th order drivative and $\Gamma(n)$ is gamma function,

(1) The following expression holds for z $s.t.$ $Re(z) > 0$.

$$\beta^{(n)}(z) = \frac{z^{-n}}{\Gamma(1-n)} + (-1)^{-n}\sum_{r=2}^{\infty}(-1)^{r-1}\frac{\log^n(2r-1)}{(2r-1)^z} \qquad n=0,1,2,\cdots \tag{4.2h}$$

(2) The following expression holds on whole complex plane.

$$\beta^{(n)}(z) = \frac{z^{-n}}{\Gamma(1-n)} + (-1)^{-n}\sum_{k=2}^{\infty}\sum_{r=2}^{k}\frac{(-1)^{r-1}}{2^{k+1}}\binom{k}{r}\frac{\log^n(2r-1)}{(2r-1)^z} \qquad n=0,1,2,\cdots$$

(4.2h')

Proof

As seen in the proof of Formula 25.4.1h ,

$$\beta(z) = 1 + \sum_{r=2}^{\infty}(-1)^{r-1}e^{-z\log(2r-1)} = 1 - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} +- \cdots$$

Hereafter, in a way similar to the proof of Formula 25.3.2h , (4.2h) is obtained.
And applying Euler transformation to this second term, we obtain (4.2h') . By this transformation, (4.2h) is analytically continued from $Re(z) > 0$ to the whole complex plane.

Example $\beta^{(2)}(0.9+8.7i)$, $\beta^{(3)}(1.3)$

If these are calculated according to (4,2h) by formula manipulation soft ***Mathematica***, it is as follows. Though the convergence is slow, we can see that this formula is numerically right.

$$D\beta_{n\_}[z\_, m\_] := \frac{z^{-n}}{\text{Gamma}[1-n]} + (-1)^n \sum_{s=2}^{m} (-1)^{s-1} \frac{(\text{Log}[2s-1])^n}{(2s-1)^z}$$

```
N[{DirichletBeta''[0.9 + 8.7 i] , Dβ2[0.9 + 8.7 i, 3000000]}]
        { 1.26788 - 0.982596 i , 1.26795 - 0.982658 i }

N[{DirichletBeta'''[1.3] , Dβ3[1.3, 2000000]}]
        { 0.0876124 , 0.087617 }
```

Formula 25.4.2s (Super Derivative)

When p is a complex number, $\beta(z)$ is Dirichlet beta function, $\beta^{(p)}(z)$ is the lineal p -th order derivative and $\Gamma(p)$ is gamma function,

(1) The following expression holds for z $s.t.$ $Re(z) > 0$.

$$\beta^{(p)}(z) = \frac{z^{-p}}{\Gamma(1-p)} + e^{-p\pi i}\sum_{r=2}^{\infty}(-1)^{r-1}\frac{\log^p(2r-1)}{(2r-1)^z} \tag{4.2s}$$

(2) The following expression holds on whole complex plane.

$$\beta^{(p)}(z) = \frac{z^{-p}}{\Gamma(1-p)} + e^{-p\pi i} \sum_{k=2}^{\infty} \sum_{r=2}^{k} \frac{(-1)^{r-1}}{2^{k+1}} \binom{k}{r} \frac{\log^{p}(2r-1)}{(2r-1)^{z}} \tag{4.2s'}$$

Proof

In Formula 25.4.2h , replacing $(-1)^{-n}$ with $e^{-n\pi i}$ and replacing the natural number n with a complex number p , we obtain the desired expressions.

Note

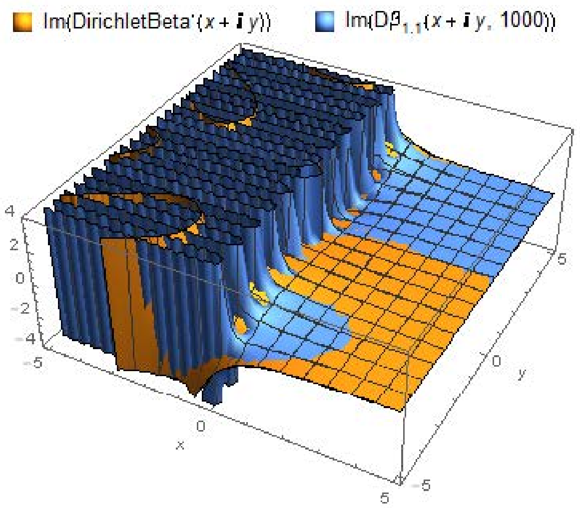
If the sign of p is inverted in (4.2s), it becomes as follows.

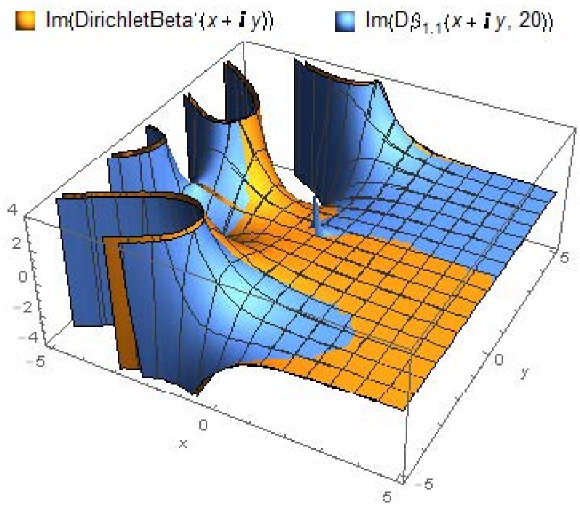
$$\beta^{(-p)}(z) = \frac{z^{p}}{\Gamma(1+p)} + e^{p\pi i} \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^{-p}(2r-1)}{(2r-1)^{z}}$$

This results in Formula 25.4.1s . That is, the lineal super calculus of Dirichlet beta function is seamless.

Example The 1.1th order derivative

When $z = x+iy$, the imaginary parts of $\beta^{(1)}(z)$ and $\beta^{(1.1)}(z)$ are illustrated as follows. The left figure is (4.2s) and the right figure is (4.2s'). In both figures, the orange is $\beta^{(1)}$ and the blue is $\beta^{(1.1)}(z)$. In the left figure, the line of convergence is visible at $x=0$.





2018.01.13

Kano Kono

Alien's Mathematics