26 Higher and Super Calculus of Zeta Function etc

26.1 Higher and Super Calculus of Riemann Zeta Function

26.1.1 Higher and Super Integral of Riemann Zeta Function

Formula 26.1.1h (Higher Integral)

When \( \zeta(z) \) is Riemann zeta function, \( \zeta^{(n)}(z) \) is the lineal \( n \)th order primitive, \( H_n = \sum_{k=1}^{n} \frac{1}{k} \) is a harmonic number (where \( H_0 = 0 \)), the following expression holds on whole complex plane.

\[
\zeta^{(n)}(z) = \frac{(z-1)^{n-1}}{(n-1)!} \{ \log(z-1) - H_{n-1} \} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r+n}}{(r+n)!} 
\]

\( n = 1, 2, 3, \ldots \) 

(1.1h)

Where, \( \gamma_r \) is Stieltjes constant defined by the following expression.

\[
\gamma_r = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} \frac{(\log k)^r}{k} - \frac{(\log n)^{r+1}}{r+1} \right\}
\]

**Proof**

It is known that the Riemann zeta function \( \zeta(z) \) is expanded to Laurent series around 1 as follows.

\[
\zeta(z) = \frac{1}{z-1} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^r}{r!}
\]

Then, integrating the both sides with respect to \( z \) without considering the constant of the integration,

\[
\zeta^{(1)}(z) = \log(z-1) + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r+1}}{(r+1)!}
\]

Integrating this once more without considering the constant of the integration,

\[
\zeta^{(2)}(z) = \frac{(z-1)^2}{2!} \left\{ \log(z-1) - \frac{1}{1+ \frac{1}{2}} \right\} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r+2}}{(r+2)!}
\]

Integrating this once more without considering the constant of the integration,

\[
\zeta^{(3)}(z) = \frac{(z-1)^3}{3!} \left\{ \log(z-1) - \left(1+ \frac{1}{2} + \cdots + \frac{1}{n-1} \right) \right\} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r+3}}{(r+3)!}
\]

Hereafter, by induction,

\[
\zeta^{(n)}(z) = \frac{(z-1)^{n-1}}{(n-1)!} \left\{ \log(z-1) - \left(1+ \frac{1}{2} + \cdots + \frac{1}{n-1} \right) \right\} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r+n}}{(r+n)!}
\]

Rewriting the harmonic number as \( H_{n-1} \), we obtain the desired expression.

**Example** The 1st order integral

When the real part and the imaginary part of \( \zeta^{(1)}(x + iy) \) are illustrated, it is as follows. The left figure is the real part and the right figure is the imaginary part.

\[
H_n := \text{HarmonicNumber}[n] \quad \gamma_s := \text{StieltjesGamma}[s]
\]
Note

If the left side of (1.1h) is represented by integral symbols, it is as follows.

\[
\int \cdots \int \zeta(z) \, dz = \frac{(z-1)^{n-1}}{(n-1)!} \left\{ \log(z-1) - H_{n-1} \right\} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r+n}}{(r+n)!}
\]

Here, the lower limits of the integral are as follows.

\[ a_1 = 1.669008 \cdots, \quad a_2 = 2.641300 \cdots, \quad a_3 = 3.610288 \cdots, \quad \cdots \]

That is, this is a higher integral with variable lower limits. And this is a lineal higher integral.

cf.

The higher integral with variable lower limits \( a_1 = a_2 = \cdots = a_n = 0 \) is as follows.

\[
\int \cdots \int \zeta(z) \, dz^n = \frac{(z-1)^{n-1}}{(n-1)!} \left\{ \log(z-1) - H_{n-1} \right\} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r+n}}{(r+n)!} - \frac{i\pi(z-1)^{n-1}}{(n-1)!} + \sum_{r=1}^{n-1} \frac{(-1)^r H_r}{r!} \frac{z^{n-1-r}}{(n-1-r)!} - \sum_{r=0}^{n} \sum_{s=1}^{r} \frac{(-1)^s \gamma_r z^{n-s}}{(r+s)! (n-s)!}
\]

As seen from the existence of constant-of-integration polynomials, this is a collateral higher integral.

**Formula 26.1.1s ( Super Integral )**

When \( p \) is a complex number, \( \zeta(z) \) is Riemann zeta function, \( \zeta^{(p)}(z) \) is the lineal \( p \)-th order primitive, \( \Gamma(p) \) is gamma function, \( \psi(p) \) is digamma function and \( \gamma_r \) is Stieltjes constant, the following expression holds on whole complex plane.

\[
\zeta^{(p)}(z) = \frac{\log(z-1) - \psi(p) - \gamma_0}{\Gamma(p)} (z-1)^{p-1} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r+p}}{\Gamma(1+r+p)}
\] (1.1s)

**Proof**

From (1.1h),
\[
\zeta^{(n)}(z) = \frac{(z-1)^{n-1}}{(n-1)!} \{ \log (z-1) - H_{n-1} \} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r+n}}{(r+n)!} \quad n=1, 2, 3, \ldots
\]

At first,
\[
(n-1)! = \Gamma(n), \quad H_{n-1} = \psi(n) + \gamma_0, \quad (r+n)! = \Gamma(1+r+n)
\]

Using these,
\[
\zeta^{(n)}(z) = \frac{(z-1)^{n-1}}{\Gamma(n)} \{ \log (z-1) - \psi(n) + \gamma_0 \} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r+n}}{\Gamma(1+r+n)}
\]

Then, this expression also holds at \( n = 0 \). Because,
\[
\frac{\log (z-1) - \gamma_0}{\Gamma(0)} \rightarrow \frac{\log (z-1) - \gamma_0}{\infty} = 0 \quad , \quad \frac{-\psi(0)}{\Gamma(0)} = 1
\]

So, replacing the natural number \( n \) with a complex number \( p \), we obtain the desired expression.

**Example The 0.1th order integral**

When \( z = x + iy \), \( \zeta(z) \) and \( \zeta^{(0.1)}(z) \) are illustrated as follows. The left figure is a real part and the right figure is an imaginary part. In both figures, the orange is \( \zeta(z) \) and the blue is \( \zeta^{(0.1)}(z) \). Since \( p \) is near \( 0 \), both curved surfaces look double.

![Real and Imaginary Parts of 0.1th Order Integral](image)

**26.1.2 Higher and Super Derivative of Riemann Zeta Function**

**Formula 26.1.2h (Higher Derivative)**

When \( \zeta(z) \) is Riemann zeta function, \( \zeta^{(n)}(z) \) is the lineal \( n \)-th order derivative and \( \gamma_r \) is Stieltjes constant, the following expression holds on whole complex plane.

\[
\zeta^{(n)}(z) = \frac{(-1)^{-n} n!}{(z-1)^{n+1}} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r-n}}{\Gamma(1+r-n)} \quad (1.2h)
\]

**Proof**

It is known that the Riemann zeta function \( \zeta(z) \) is expanded to Laurent series around 1 as follows.

![Example of Higher and Super Derivative](image)
\[ \zeta(z) = \frac{1}{z-1} + \sum_{r=0}^{\infty} (-1)^r \frac{\gamma_r}{r!} \left(\frac{z-1}{r} \right) \]

Differentiating the both sides \( n \) times with respect to \( z \),

\[ \zeta^{(n)}(z) = \left(\frac{1}{z-1} \right)^{(n)} + \sum_{r=0}^{\infty} (-1)^r \frac{\gamma_r}{r!} \left\{ \left(\frac{z-1}{r} \right)^{(n)} \right\} \]

According to Formula 9.2.1 in "09 Higher Derivative",

\[ (x^\alpha)^{(n)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)} x^{\alpha-n} \quad (\alpha \geq 0) \]

\[ = (-1)^{-n} \frac{\Gamma(-\alpha+n)}{\Gamma(-\alpha)} x^{\alpha-n} \quad (\alpha < 0) \]

Applying this,

\[ \left\{ \left(\frac{z-1}{r} \right)^{(n)} \right\} = \frac{\Gamma(1+r)}{\Gamma(1+r-n)} \left(\frac{z-1}{r} \right)^{r-n} = \frac{r!}{\Gamma(1+r-n)} \left(\frac{z-1}{r} \right)^{-n-n} \]

\[ \left(\frac{1}{z-1} \right)^{(n)} = (-1)^{-n} \frac{\Gamma(1+n)}{\Gamma(1)} \left(\frac{z-1}{r} \right)^{-1-n} = (-1)^{-n} \frac{n!}{(z-1)^{n+1}} \]

Substituting these for the above,

\[ \zeta^{(n)}(z) = \frac{(-1)^{-n}}{(z-1)^{n+1}} + \sum_{r=0}^{\infty} (-1)^r \frac{\gamma_r}{r!} \frac{(z-1)^{-r-n}}{\Gamma(1+r-n)} \quad (1.2h) \]

**Example \( \zeta^{(1)}(0.3) \), \( \zeta^{(2)}(-1.1+2.3i) \)

When these are calculated by formula manipulation soft Mathematica, it is as follows. We can see that this formula is numerically right.

\[ \gamma_{\infty} := \text{StieltjesGamma}[z] \]

\[ D_{\infty}[z, m_{\infty}] := \frac{(-1)^{-n}}{(z-1)^{n+1}} + \sum_{r=0}^{n} (-1)^r \frac{\gamma_r}{r!} \frac{(z-1)^{-r-n}}{\Gamma(1+r-n)} \]

\[ \text{N}\{\text{Zeta}'[0.3], D_{\infty}[0.3, 20]\}] \]

\[ \{-1.96186, -1.96186\} \]

\[ \text{N}\{\text{Zeta}'[-1.1+2.3i], D_{\infty}[-1.1+2.3i, 20]\}] \]

\[ \{0.0446687 - 0.0601275 i, 0.0446687 - 0.0601275 i\} \]

**Formula 26.1.2s ( Super Derivative )**

When \( p \) is a complex number, \( \zeta(z) \) is Riemann zeta function, \( \zeta^{(p)}(z) \) is the lineal \( p \)-th order derivative, \( \Gamma(p) \) is gamma function, \( \psi(p) \) is digamma function and \( \gamma_r \) is Stieltjes constant, the following expression holds on whole complex plane.

\[ \zeta^{(p)}(z) = \frac{\log(z-1) - \psi(-p) - \gamma_0}{\Gamma(-p)} (z-1)^{-p-1} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{-r-p}}{\Gamma(1+r-p)} \quad (1.2s) \]
Proof

From Formula 26.1.2h,

\[
\zeta^{(n)}(z) = \frac{(-1)^{-n} n!}{(z-1)^{n+1}} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r-n}}{(r-n)!}
\]

(1.2h)

According to Formula 1.3.1 in "01 Gamma Function & Digamma Function",

\[
(-1)^{-n} n! = -\frac{\psi(-n)}{\Gamma(-n)}, \quad n=0, 1, 2, 3, \ldots
\]

Using this, (1.2h) is rewritten as follows.

\[
\zeta^{(n)}(z) = \frac{\log(z-1) - \psi(-n) - \gamma_0}{\Gamma(-n)} (z-1)^{-n-1} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r-n}}{\Gamma(1+r-p)}
\]

Because,

\[
\frac{\log(z-1) - \gamma_0}{\Gamma(-n)} = \frac{\log(z-1) - \gamma_0}{\pm \infty} = 0 \quad \text{for } n=0, 1, 2, 3, \ldots
\]

So, replacing the natural number \( n \) with a complex number \( p \), we obtain the desired expression.

\[
\zeta^{(p)}(z) = \frac{\log(z-1) - \psi(-p) - \gamma_0}{\Gamma(-p)} (z-1)^{-p-1} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r-p}}{\Gamma(1+r-p)}
\]

(1.2s)

Note

If the sign of \( p \) is inverted in (1.2s), it becomes as follows.

\[
\zeta^{(-p)}(z) = \frac{\log(z-1) - \psi(p) - \gamma_0}{\Gamma(p)} (z-1)^{p-1} + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r+p}}{\Gamma(1+r+p)}
\]

This results in Formula 26.1.1s. That is, the lineal super calculus of Riemann zeta function is seamless.

Example

The 1.9th order derivative

When \( z = x + iy \), \( \zeta^{(2)}(z) \) and \( \zeta^{(1,1.9)}(z) \) are illustrated as follows. The left figure is a real part and the right figure is an imaginary part. In both figures, the orange is \( \zeta^{(2)}(z) \) and the blue is \( \zeta^{(1,1.9)}(z) \). Since \( p \) is near 2, both curved surfaces look double.
26.2 Higher and Super Calculus of Dirichlet Lambda Function

26.2.1 Higher and Super Integral of Dirichlet Lambda Function

Formula 26.2.1h (Higher Integral)

When \( \lambda (z) \) is Dirichlet lambda function, \( \lambda^{<n>}(z) \) is the lineal \( n \)-th order primitive, \( H_n\) (\( = \sum_{k=1}^{n} 1/k \)) is a harmonic number (where \( H_0 = 0 \)), the following expression holds on whole complex plane.

\[
\lambda^{<n>}(z) = \frac{(z-1)^{n-1}}{2(n-1)!} \left\{ \log (z-1) - H_{n-1} \right\} + \frac{1}{2} \sum_{r=0}^{\infty} (-1)^r \left\{ \gamma_r + \frac{\log r+1}{r+1} - \sum_{s=0}^{r-1} \binom{r}{s} \gamma_s (\log 2)^{r-s} \right\} \frac{(z-1)^{r+n}}{(r+n)!} \quad n=1, 2, 3, \ldots
\]

Where, \( \gamma_r \) is Stieltjes constant defined by the following expression.

\[
\gamma_r = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} \frac{(\log k)^r}{k} - \frac{(\log n)^{r+1}}{r+1} \right\}
\]

Proof

According to Formula 3.1.3 in "03 Complementary Series of Dirichlet Series" (Dirichlet Series), Dirichlet lambda function is expanded to Laurent series around 1 as follows.

\[
\lambda (z) = \frac{1}{2(z-1)} + \frac{1}{2} \sum_{r=0}^{\infty} (-1)^r \left\{ \gamma_r + \frac{\log r+1}{r+1} - \sum_{s=0}^{r-1} \binom{r}{s} \gamma_s (\log 2)^{r-s} \right\} \frac{(z-1)^r}{r!}
\]

Where, \( \Sigma \) in \{ \} is absent for \( r=0 \).

Hereafter, in a way similar to the proof of Formula 26.1.1h, the desired expression is obtained.

Example The 2nd order integral

When the real part and the imaginary part of \( \lambda^{<1>}(x+iy) \) are illustrated, it is as follows. The left figure is the real part and the right figure is the imaginary part.

Note

This is also lineal higher integral with variable lower limits. The first few of the integral lower limits are as follows.
\[ a_1 = 1.509052\ldots, \ a_2 = 2.203125\ldots, \ a_3 = 2.891846\ldots, \ \ldots \]

**Formula 26.2.1s (Super Integral)**

When \( p \) is a complex number, \( \lambda(z) \) is Dirichlet lambda function, \( \lambda^{p\gamma}(z) \) is the lineal \( p \)-th order primitive, \( \Pi(p) \) is gamma function, \( \psi(p) \) is digamma function and \( \gamma_r \) is Stieltjes constant, the following expression holds on whole complex plane.

\[
\lambda^{p\gamma}(z) = \frac{\log(z-1) - \psi(p) - \gamma_0}{2\Pi(p)}(z-1)^{p-1} + \frac{1}{2} \sum_{r=0}^{\infty} (-1)^r \left( \gamma_r + \frac{\log^{r+1} 2}{r+1} - \sum_{s=0}^{r-1} \binom{r}{s} \gamma_s \left( \log 2 \right)^{r-s} \right) \frac{(z-1)^{r+p}}{\Pi(1+r+p)}
\]  

(2.1s)

**Proof**

In a way similar to the proof of Formula 26.1.1s, the desired expression is obtained.

**Example** The 0.05th order integral

\( \lambda(z) \) and \( \lambda^{<0.05>}(z) \) are illustrated as follows.

26.2.2 Higher and Super Derivative of Dirichlet Lambda Function

**Formula 26.2.2h (Higher Derivative)**

When \( \lambda(z) \) is Dirichlet lambda function, \( \lambda^{(n)}(z) \) is the lineal \( n \)-th order derivative and \( \gamma_r \) is Stieltjes constant, the following expression holds on whole complex plane.

\[
\lambda^{(n)}(z) = \frac{(-1)^n n!}{2(z-1)^{n+1}} + \frac{1}{2} \sum_{r=0}^{\infty} (-1)^r \left( \gamma_r + \frac{\log^{r+1} 2}{r+1} - \sum_{s=0}^{r-1} \binom{r}{s} \gamma_s \left( \log 2 \right)^{r-s} \right) \frac{(z-1)^{r-n}}{\Pi(1+r-n)} \quad n=0, 1, 2, \ldots
\]  

(2.2h)

**Proof**

As seen in the proof of Formula 26.2.1h, the Dirichlet lambda function \( \lambda(z) \) is expanded to Laurent series.
around 1 as follows.

\[
\lambda(z) = \frac{1}{2(z-1)} + \frac{1}{2} \sum_{r=0}^{\infty} (-1)^r \left( \kappa_r + \frac{\log^{r+1} 2}{r+1} - \sum_{s=0}^{r-1} \binom{r}{s} \gamma_s \log(2)^{r-s} \right) \frac{(z-1)^r}{r!}
\]

Where, \(\Sigma\) in \(\{\}\) is absent for \(r=0\).

Hereafter, in a way similar to the proof of Formula 26.1.2h, the desired expression is obtained.

**Example \(\lambda^{(2)}(-2.1), \lambda^{(3)}(-0.8-1.9i)\)**

When these are calculated by formula manipulation soft *Mathematica*, it is as follows. We can see that this formula is numerically right.

\[
\begin{align*}
\gamma_s := \text{StieltjesGamma}[s] \\
D\lambda_2(z_2, \gamma) := \frac{(z_2 - 1)^{-\gamma}}{2} \\
\times 1 \frac{1}{\Gamma(p)} \sum_{r=0}^{\infty} (-1)^r \left( \psi(r+1) - \sum_{s=0}^{r-1} \binom{r}{s} \gamma_s \log(2)^{r-s} \right) \frac{(z_2 - 1)^{-\gamma}}{\Gamma(1+r-p)}
\end{align*}
\]

\[
\begin{align*}
N[\{\text{DirichletLambda}'[-2.1], \text{D}\lambda_2[-2.1, 20]\}] \\
\{0.0404744, 0.0404744\}
\end{align*}
\]

\[
\begin{align*}
N[\{\text{DirichletLambda}'[-0.8-1.9i], \text{D}\lambda_3[-0.8-1.9i, 20]\}] \\
\{0.0682879 - 0.0999407i, 0.0682879 - 0.0999407i\}
\end{align*}
\]

**Formula 26.2.2s (Super Derivative)**

When \(p\) is a complex number, \(\lambda(z)\) is Dirichlet lambda function, \(\lambda^{(p)}(z)\) is the lineal \(p\)-th order derivative, \(\Gamma(p)\) is gamma function, \(\psi(p)\) is digamma function and \(\gamma_r\) is Stieltjes constant, the following expression holds on whole complex plane.

\[
\lambda^{(p)}(z) = \frac{\log(z-1) - \psi(-p) - \gamma_0}{2\Gamma(-p)} (z-1)^{-p-1} + \frac{1}{2} \sum_{r=0}^{\infty} (-1)^r \left( \kappa_r + \frac{\log^{r+1} 2}{r+1} - \sum_{s=0}^{r-1} \binom{r}{s} \gamma_s \log(2)^{r-s} \right) \frac{(z-1)^{r-p}}{\Gamma(1+r-p)}
\] (2.2s)

**Proof**

In a way similar to the proof of Formula 26.1.2s, the desired expression is obtained.

**Note**

If the sign of \(p\) is inverted in (2.2s), it becomes as follows.

\[
\lambda^{(-p)}(z) = \frac{\log(z-1) - \psi(p) - \gamma_0}{2\Gamma(p)} (z-1)^{p-1} + \frac{1}{2} \sum_{r=0}^{\infty} (-1)^r \left( \kappa_r + \frac{\log^{r+1} 2}{r+1} - \sum_{s=0}^{r-1} \binom{r}{s} \gamma_s \log(2)^{r-s} \right) \frac{(z-1)^{r+p}}{\Gamma(1+r+p)}
\]

This results in Formula 26.2.1s. That is, the lineal super calculus of Dirichlet lambda function is seamless.
Example  The 0.9th order derivative

When $z = x + iy$, $\lambda^{(1)}(z)$ and $\lambda^{(0,9)}(z)$ are illustrated as follows. The left figure is a real part and the right figure is an imaginary part. In both figures, the orange is $\lambda^{(1)}(z)$ and the blue is $\lambda^{(0,9)}(z)$. Since $p$ is near 1, both curved surfaces look double.
26.3 Higher and Super Calculus of Dirichlet Eta Function

26.3.1 Higher and Super Integral of Dirichlet Eta Function

Formula 26.3.1h (Higher Integral)

When \( \eta^{(n)}(z) \) is Dirichlet eta function, \( \eta^{(n)}(z) \) is the lineal \( n \)-th order primitive,

(1) The following expression holds for \( z \) s.t. \( \Re(z) > 0 \).

\[
\eta^{(n)}(z) = \frac{z^n}{n!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^r n}{r^z} \quad n = 0, 1, 2, \ldots \tag{3.1h}
\]

(2) The following expression holds on whole complex plane.

\[
\eta^{(n)}(z) = \frac{z^n}{n!} + (-1)^n \sum_{k=2}^{\infty} \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{2^{k+1}} \binom{k}{r} \frac{\log^r n}{r^z} \quad n = 0, 1, 2, \ldots \tag{3.1h'}
\]

Proof

At \( \Re(z) > 0 \), Dirichlet eta function \( \eta(z) \) is expressed with the following series which is called Dirichlet eta series.

\[
\eta(z) = 1 + \sum_{r=2}^{\infty} (-1)^{r-1} e^{-z \log r} = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \ldots
\]

So, integrating the both sides \( n \) times with respect to \( z \) without considering the constant of the integration,

\[
\eta^{(n)}(z) = \frac{z^n}{n!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^r n}{r^z} \quad n = 0, 1, 2, \ldots \tag{3.1h}
\]

Applying Euler transformation to this second term, we obtain \( 3.1h' \). By this transformation, \( 3.1h \) is analytically continued from \( \Re(z) > 0 \) to the whole complex plane. In addition, about Euler transformation, see "[10 Convergence Acceleration & Summation Method by Double Series of Functions](#)" (A la carte).

Example The 1st order integral

The imaginary part of \( \eta^{<1>}(x + iy) \) are illustrated as follows. The left is \( 3.1h \) and the right is \( 3.1h' \).

In the left figure, the line of convergence is visible at \( x = 0 \).
Note
If the left side of (3.1h) is represented by integral symbols, it is as follows.
\[
\int_{a_n}^{z} \int_{a_1}^{z} \eta(z) \, dz = \frac{z^n}{n!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^r r}{r^2}
\]
Here, the lower limits of the integral are as follows.
\[
a_1 = -1.809613\ldots + i 1.766080\ldots, \quad a_2 = 1.216967\ldots,
\]
\[
a_3 = 1.337211\ldots + i 1.289222\ldots, \quad a_4 = 2.163768\ldots, \ldots
\]
That is, this is a higher integral with variable lower limits. And this is a lineal higher integral.

Formula 26.3.1s (Super Integral)
When \( p \) is a complex number, \( \eta(z) \) is Dirichlet eta function, \( \eta^{<p>}(z) \) is the lineal \( p \)-th order primitive and \( \Gamma(p) \) is gamma function,

(1) The following expression holds for \( z \) s.t. \( \text{Re}(z) > 0 \).
\[
\eta^{<p>}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i} \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^r r}{r^2}
\]

(2) The following expression holds on whole complex plane.
\[
\eta^{<p>}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i} \sum_{k=2}^{\infty} \sum_{r=2}^{k} (-1)^{r-1} \frac{1}{2^{k+1}} \left( \begin{array}{c} k \\ r \end{array} \right) \frac{\log^r r}{r^2}
\]

Proof
In Formula 26.3.1h, replacing \( n! \) with \( \Gamma(1+n) \), replacing \( (-1)^n \) with \( e^{-n\pi i} \) and replacing the natural number \( n \) with a complex number \( p \), we obtain the desired expressions.

Example The 0.2th order integral
When \( z = x + iy \), the real parts of \( \eta(z) \) and \( \eta^{<0.2>}(z) \) are illustrated as follows. The left figure is (3.1s) and the right figure is (3.1s'). In both figures, the orange is \( \eta(z) \) and the blue is \( \eta^{<0.2>}(z) \). Since \( p \) is near 0, both curved surfaces look double. In the left figure, the line of convergence is visible at \( x = 0 \).
26.3.2 Higher and Super Derivative of Dirichlet Eta Function

Formula 26.3.2h (Higher Derivative)

When $\eta(z)$ is Dirichlet eta function, $\eta^{(n)}(z)$ is the lineal $n$-th order derivative and $\Gamma(n)$ is gamma function,

(1) The following expression holds for $z \ s.t. \ Re(z) > 0$.

$$\eta^{(n)}(z) = \frac{z^{-n}}{\Gamma(1-n)} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^r r}{r^z}$$  \hspace{1cm} n=0, 1, 2, \ldots \ (3.2h)

(2) The following expression holds on whole complex plane.

$$\eta^{(n)}(z) = \frac{z^{-n}}{\Gamma(1-n)} + (-1)^n \sum_{k=2}^{\infty} \sum_{r=2}^{k} (-1)^{r-1} \left( \frac{k}{2k+1} \right) \frac{\log^r r}{r^z}$$  \hspace{1cm} n=0, 1, 2, \ldots \ (3.2h')

Proof

As seen in the proof of Formula 26.3.1h.

So, assuming $1 = z^0 / 0!$ and differentiating the both sides $n$ times with respect to $z$,

$$\eta^{(n)}(z) = \frac{z^{-n}}{(-n)!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^r r}{r^z}$$  \hspace{1cm} n=0, 1, 2, \ldots

Replacing $(-n)!$ with $\Gamma(1-n)$, we obtain (3.2h). In addition,

$$\frac{z^{-n}}{\Gamma(1-n)} = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n=1, 2, 3, \ldots \end{cases}$$

And applying Euler transformation to this second term, we obtain (3.2h'). By this transformation, (3.2h) is analytically continued from $Re(z) > 0$ to the whole complex plane.

Example $\eta(0.2), \eta^{(3)}(0.5+14.1i)$

If these are calculated according to (3.2h') by formula manipulation soft Mathematica, it is as follows. We can see that this formula is numerically right.

$$D[z, m] := \frac{z^{-n}}{\Gamma[1-n]} \ast (-1)^n \sum_{k=2}^{m} \sum_{r=2}^{k} \frac{(-1)^{r-1}}{2^{k+1}} \text{Binomial}[k, r] \frac{\text{Log}[z]^n}{r^z}$$

$$N[\{\text{DirichletEta}'[0.2], D[0.2, 23]\}]$$

$$\{0.21318, 0.21318\}$$

$$N[\{\text{DirichletEta}'[0.5+14.1i], D[0.5+14.1i, 45]\}]$$

$$\{-3.05246 - 0.0494394i, -3.05246 - 0.0494394i\}$$

Formula 26.3.2s (Super Derivative)

When $p$ is a complex number, $\eta(z)$ is Dirichlet eta function, $\eta^{(p)}(z)$ is the lineal $p$-th order derivative
and $\Gamma(p)$ is gamma function,

1. The following expression holds for $z \ s.t. \ Re(z) > 0$.

$$\eta(p)(z) = \frac{z^{-p}}{\Gamma(1-p)} + e^{-p\pi i} \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log p}{r^z} \tag{3.2s}$$

2. The following expression holds on whole complex plane.

$$\eta(p)(z) = \frac{z^{-p}}{\Gamma(1-p)} + e^{-p\pi i} \sum_{k=2}^{\infty} \sum_{r=2}^{k} (-1)^{r-1} \frac{\log p}{r^z} \tag{3.2s'}$$

Proof

In [Formula 26.3.2h], replacing $(-1)^n$ with $e^{-n\pi i}$ and replacing the natural number $n$ with a complex number $p$, we obtain the desired expressions.

Note

If the sign of $p$ is inverted in (3.2s), it becomes as follows.

$$\eta(-p)(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i} \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log p}{r^z}$$

This results in [Formula 26.3.1s]. That is, the lineal super calculus of Dirichlet eta function is seamless.

Example The 2.2th order derivative

When $z = x + iy$, the real parts of $\eta^{(2)}(z)$ and $\eta^{(2,2)}(z)$ are illustrated as follows. The left figure is (3.2s) and the right figure is (3.2s'). In both figures, the orange is $\eta^{(2)}(z)$ and the blue is $\eta^{(2,2)}(z)$.

In the left figure, the line of convergence is visible at $x = 0$.
26.4 Higher and Super Calculus of Dirichlet Beta Function

26.4.1 Higher and Super Integral of Dirichlet Beta Function

Formula 26.4.1h (Higher Integral)

When $\beta(z)$ is Dirichlet beta function, $\beta^{<n>}(z)$ is the lineal $n$-th order primitive,

(1) The following expression holds for $z$ s.t. $Re(z) > 0$.

$$\beta^{<n>}(z) = \frac{z^n}{n!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^n(2r-1)}{(2r-1)^{z}}$$

(2) The following expression holds on whole complex plane.

$$\beta^{<n>}(z) = \frac{z^n}{n!} + (-1)^n \sum_{k=2}^{\infty} \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{2^{k+1}} \frac{k}{r} \frac{\log^n(2r-1)}{(2r-1)^z}$$

Proof

At $Re(z) > 0$, Dirichlet beta function $\beta(z)$ is expressed with the following series which is called Dirichlet beta series.

$$\beta(z) = 1 + \sum_{r=2}^{\infty} (-1)^{r-1} e^{-z\log(2r-1)} = 1 - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + ...$$

So, integrating the both sides $n$ times with respect to $z$ without considering the constant of the integration,

$$\beta^{<n>}(z) = \frac{z^n}{n!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^n(2r-1)}{(2r-1)^z}$$

Applying Euler transformation to this second term, we obtain (4.1h'). By this transformation, (4.1h) is analytically continued from $Re(z) > 0$ to the whole complex plane.

Example The 3rd order integral

The imaginary part of $\beta^{<3>}(x+iy)$ are illustrated as follows. The left is (4.1h) and the right is (4.1h').

In the left figure, the line of convergence is visible at $x=0$.
Note

If the left side of (4.1h) is represented by integral symbols, it is as follows.

\[
\int_{a_n}^{z} \int_{a_1}^{z} \beta(z) \, dz = \frac{z^n}{n!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^n (2r-1)}{(2r-1)^z}
\]

Here, the lower limits of the integral are as follows.

\[a_1 = -1.027077 \cdots + i 0.978760 \cdots, \quad a_2 = 0.754520 \cdots, \quad a_3 = 0.831711 \cdots + i 0.807718 \cdots, \quad a_4 = 1.357089 \cdots, \quad \cdots\]

That is, this is a higher integral with variable lower limits. And this is a lineal higher integral.

Formula 26.4.1s (Super Integral)

When \(p\) is a complex number, \(\beta(z)\) is Dirichlet beta function, \(\beta^{(p)}(z)\) is the lineal \(p\)-th order primitive and \(\Gamma(p)\) is gamma function,

1. The following expression holds for \(z\) s.t. \(\text{Re}(z) > 0\).

\[
\beta^{(p)}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i} \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^p (2r-1)}{(2r-1)^z}
\]  (4.1s)

2. The following expression holds on whole complex plane.

\[
\beta^{(p)}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i} \sum_{k=2}^{\infty} \sum_{r=2}^{k} (-1)^{r-1} \frac{k^{r-1}}{2^{k+1}} \left(\frac{k}{r}\right) \frac{\log^p (2r-1)}{(2r-1)^z}
\]  (4.1s')

Proof

In Formula 26.4.1h, replacing \(n!\) with \(\Gamma(1+n)\), replacing \((-1)^n\) with \(e^{-n\pi i}\) and replacing the natural number \(n\) with a complex number \(p\), we obtain the desired expressions.

Example The 0.3th order integral

When \(z = x + iy\), the real parts of \(\beta(z)\) and \(\beta^{(0.3)}(z)\) are illustrated as follows. The left figure is (4.1s) and the right figure is (4.1s'). In both figures, the orange is \(\beta(z)\) and the blue is \(\beta^{(0.3)}(z)\).

In the left figure, the line of convergence is visible at \(x = 0\).
26.4.2 Higher and Super Derivative of Dirichlet Beta Function

Formula 26.4.2h (Higher Derivative)

When $\beta(z)$ is Dirichlet beta function, $\beta^{(n)}(z)$ is the lineal $n$-th order derivative and $\Gamma(n)$ is gamma function,

1. The following expression holds for $z$ s.t. $\Re(z) > 0$.

$$\beta^{(n)}(z) = \frac{z^{-n}}{\Gamma(1-n)} + (-1)^n \sum_{r=2}^{\infty} (-1)^r \frac{\log^n(2r-1)}{(2r-1)^z} \quad n=0, 1, 2, \ldots$$  \hspace{1cm} (4.2h)

2. The following expression holds on whole complex plane.

$$\beta^{(n)}(z) = \frac{z^{-n}}{\Gamma(1-n)} + (-1)^n \sum_{k=2}^{\infty} \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{2^{k+1}} \frac{\log^n(2r-1)}{(2r-1)^z} \quad n=0, 1, 2, \ldots$$  \hspace{1cm} (4.2h')

Proof

As seen in the proof of Formula 26.4.1h,

$$\beta(z) = 1 + \sum_{r=2}^{\infty} (-1)^{r-1} (1 - \frac{z}{r}) e^{-z \log(2r-1)} = 1 - \frac{1}{3z} + \frac{1}{5z} - \frac{1}{7z} + \ldots$$

Hereafter, in a way similar to the proof of Formula 26.3.2h, (4.2h) is obtained.

And applying Euler transformation to this second term, we obtain (4.2h'). By this transformation, (4.2h) is analytically continued from $\Re(z) > 0$ to the whole complex plane.

Example $\beta^{(2)}(0.9 + 8.7i)$, $\beta^{(3)}(1.3)$

If these are calculated according to (4.2h) by formula manipulation soft Mathematica, it is as follows. Though the convergence is slow, we can see that this formula is numerically right.

$$\text{N}[\{\text{DirichletBeta}'[0.9 + 8.7\text{i}], \%[0.9 + 8.7\text{i}, 3\,000\,000]\}], \{1.26788 - 0.982596\text{i}, 1.26795 - 0.982658\text{i}\}$$

$$\text{N}[\{\text{DirichletBeta}''[1.3], \%[1.3, 2\,000\,000]\}], \{0.0876124, 0.087617\}$$

Formula 26.4.2s (Super Derivative)

When $p$ is a complex number, $\beta(z)$ is Dirichlet beta function, $\beta^{(p)}(z)$ is the lineal $p$-th order derivative and $\Gamma(p)$ is gamma function

1. The following expression holds for $z$ s.t. $\Re(z) > 0$.

$$\beta^{(p)}(z) = \frac{z^{-p}}{\Gamma(1-p)} + e^{-p\pi z} \sum_{r=2}^{\infty} (-1)^r \frac{\log^p(2r-1)}{(2r-1)^z}$$  \hspace{1cm} (4.2s)
The following expression holds on whole complex plane.

$$\beta^p(z) = \frac{z^{-p}}{\Gamma(1-p)} + e^{-p\pi i} \sum_{r=2}^{\infty} \left( \frac{-1}{2^{r-1}r} \right) \log^p (2r-1)$$

(4.2s')

**Proof**

In Formula 26.4.2h, replacing $(-1)^n$ with $e^{-n\pi i}$ and replacing the natural number $n$ with a complex number $p$, we obtain the desired expressions.

**Note**

If the sign of $p$ is inverted in [4.2s], it becomes as follows.

$$\beta^{-p}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i} \sum_{r=2}^{\infty} \left( \frac{1}{2^{r-1}r} \right) \log^{-p} (2r-1)$$

This results in Formula 26.4.1s. That is, the lineal super calculus of Dirichlet beta function is seamless.

**Example The 1.1th order derivative**

When $z = x + iy$, the imaginary parts of $\beta^{(1)}(z)$ and $\beta^{(1,1)}(z)$ are illustrated as follows. The left figure is (4.2s) and the right figure is (4.2s'). In both figures, the orange is $\beta^{(1)}$ and the blue is $\beta^{(1,1)}$. In the left figure, the line of convergence is visible at $x = 0$.

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