

7 Completed Riemann Zeta

7.1 Even Function & Odd Function

Definition 7.1.0

Let $f(z)$ be a function in the domain D .

- (1) When $f(z) = f(-z)$, $f(z)$ is an **even function**.
- (2) When $f(z) = -f(-z)$, $f(z)$ is an **odd function**.

Note

According to this definition, we can immediately say the following.

- (1) When $f(z)$ is an even functions, if z_1 is a zero of $f(z)$, $-z_1$ is also zero of $f(z)$.
- (2) When $f(z)$ is an odd functions, if z_1 is a zero of $f(z)$, $-z_1$ is also zero of $f(z)$.

Theorem 7.1.1

Let $f_1(z), f_2(z)$ are functions in the domain D .

- (1) If $f_1(z), f_2(z)$ are even functions, $f_1(z) \pm f_2(z)$ is also an even function.
- (2) If $f_1(z), f_2(z)$ are odd functions, $f_1(z) \pm f_2(z)$ is also an odd function.

Proof

Whent $f_1(z), f_2(z)$ are even functions, since $f_k(z) = f_k(-z) \quad k=1, 2$,

$$f_1(z) \pm f_2(z) = f_1(-z) \pm f_2(-z)$$

Whent $f_1(z), f_2(z)$ are odd functions, since $f_k(z) = -f_k(-z) \quad k=1, 2$,

$$f_1(z) \pm f_2(z) = -f_1(-z) \pm \{-f_2(-z)\} = -\{f_1(-z) \pm f_2(-z)\}$$

Theorem 7.1.2

Let $f_1(z), f_2(z)$ are functions in the domain D .

- (1) If $f_1(z), f_2(z)$ are even functions, $f_1(z) f_2(z)$ is an even function.
- (2) If $f_1(z), f_2(z)$ are odd functions, $f_1(z) f_2(z)$ is an even function.
- (3) If $f_1(z)$ is an odd function and $f_2(z)$ is an even function, $f_1(z) f_2(z)$ is an odd function.

Proof

Whent $f_1(z), f_2(z)$ are odd functions, since $f_k(z) = -f_k(-z) \quad k=1, 2$,

$$f_1(z) f_2(z) = -f_1(-z) \{-f_2(-z)\} = f_1(-z) f_2(-z)$$

Thus, (2) is proved.

(1), (3) are also proved in a similar way.

When $f(z)$ is separated into a real part and an imaginary part, the following theorems hold.

Theorem 7.1.3

Let $f(z)$ be a complex function in the domain D .

(1) If $f(z)$ is an even function, both the real part and the imaginary part are even functions.

(2) If $f(z)$ is an odd function, both the real part and the imaginary part are odd functions.

Proof

Let R be a real number space in a domain D and let

$$f(z) = u(z) + iv(z), \quad u, v \in R$$

Replacing z with $-z$,

$$f(-z) = u(-z) + iv(-z)$$

From these,

if $f(z) = f(-z)$ then $u(z) = u(-z)$, $v(z) = v(-z)$.

if $f(z) = -f(-z)$ then $u(z) = -u(-z)$, $v(z) = -v(-z)$.

Theorem 7.1.4

Let $f(z)$ be a complex function in the domain D . Then,

if $f(z)$ is an even function or an odd function, $|f(z)|^2$ is an even function.

Proof

Let R be a real number space in a domain D and let

$$f(z) = u(z) + iv(z), \quad u, v \in R$$

Then,

$$|f(z)|^2 = u^2(z) + v^2(z)$$

if $f(z)$ is an even function, both $u(z)$ and $v(z)$ are even functions from Theorem 7.1.3 (1) and

if $f(z)$ is an odd function, both $u(z)$ and $v(z)$ are odd functions from Theorem 7.1.3 (2).

Then, in the case of which, both $u^2(z)$ and $v^2(z)$ become even functions from Theorem 7.1.2.

Furthermore, $u^2(z) + v^2(z)$ becomes an even function from Theorem 7.1.1 (1).

Regarding the derivative of $f(z)$ and the Maclaurin series, the well-known following theorems hold.

Theorem 7.1.5

Let a complex function $f(z)$ be holomorphic in the whole domain D .

(1) If $f(z)$ is an even function, the 1st order derivative $f^{(1)}(z)$ is an odd function.

(2) If $f(z)$ is an odd function, the 1st order derivative $f^{(1)}(z)$ is an even function.

Proof

If a function $f(z)$ is an even function, $f(z) = f(-z)$.

Differentiating both sides with respect to z ,

$$f^{(1)}(z) = f^{(1)}(-z) \cdot (-z)^{(1)} = -f^{(1)}(-z)$$

Therefore, $f^{(1)}(z)$ is an odd function.

If a function $f(z)$ is an odd function, $f(z) = -f(-z)$.

Differentiating both sides with respect to z ,

$$f^{(1)}(z) = -f^{(1)}(-z) \cdot (-z)^{(1)} = f^{(1)}(-z)$$

Therefore, $f^{(1)}(z)$ is an even function.

Theorem 7.1.6

Let a complex function $f(z)$ be holomorphic in the whole domain D .

(1) $f(z)$ is an even function if and only if the Maclaurin series includes only even powers.

(2) $f(z)$ is an odd function if and only if the Maclaurin series includes only odd powers.

Proof

By assumption, $f(z)$ is expanded to Maclaurin series as follows.

$$f(z) = \frac{f^{(0)}(0)}{0!}z^0 + \frac{f^{(1)}(0)}{1!}z^1 + \frac{f^{(2)}(0)}{2!}z^2 + \frac{f^{(3)}(0)}{3!}z^3 + \dots$$

(1) According to Theorem 7.1.5, if $f(z)$ is an even function, $f^{(2n-1)}(z)$ $n=1, 2, 3, \dots$ are odd functions.

$$\text{So, } f^{(2n+1)}(0) = -f^{(2n+1)}(-0) \quad n=1, 2, 3, \dots$$

$$\text{From this, we obtain } f^{(2n+1)}(0) = 0 \quad n=1, 2, 3, \dots$$

Conversely, if the Maclaurin series includes only even powers, each terms are all even functions.

Then, according to Theorem 7.1.1, the Maclaurin series becomes an even function.

(2) According to Theorem 7.1.5, if $f(z)$ is an odd function, $f^{(2n)}(z)$ $n=1, 2, 3, \dots$ are even functions.

$$\text{So, } f^{(2n)}(0) = -f^{(2n)}(-0) \quad n=1, 2, 3, \dots$$

$$\text{From this, we obtain } f^{(2n)}(0) = 0 \quad n=1, 2, 3, \dots$$

Conversely, if the Maclaurin series includes only odd powers, each terms are all odd functions.

Then, according to Theorem 7.1.1, the Maclaurin series becomes an odd function.

7.2 Complex Conjugate Property

Definition 7.2.0

When the function $f(z)$ defined in the domain D satisfies

$$f(\bar{z}) = \overline{f(z)} \quad z \in D$$

We say that the $f(z)$ has **complex conjugate property**.

Example1 $f(z) = e^{z-1}$

$$\because f(z) = f(x, y) = e^{x-1+iy} = e^{x-1}(\cos y + i \sin y)$$

$$f(\bar{z}) = f(x, -y) = e^{x-1-iy} = e^{x-1}(\cos y - i \sin y) = \overline{f(z)}$$

Example2 $f(z) = (z-1)^{\pm n}$

Convert $z = x-1+iy$ to spherical coordinates (r, θ) as follows.

$$r = \sqrt{(x-1)^2 + y^2}, \quad \cos \theta = \frac{x-1}{\sqrt{(x-1)^2 + y^2}}, \quad \sin \theta = \frac{y}{\sqrt{(x-1)^2 + y^2}}.$$

Then,

$$f(z) = \{r(\cos \theta + i \sin \theta)\}^{\pm n} = r^{\pm n} \{\cos(n\theta) \pm i \sin(n\theta)\}$$

$$f(\bar{z}) = \{r(\cos \theta - i \sin \theta)\}^{\pm n} = r^{\pm n} \{\cos(n\theta) \mp i \sin(n\theta)\} = \overline{f(z)}$$

cf.

$$f(z) = e^{z-i} \quad \text{and} \quad f(z) = (z-i)^{\pm n} \quad \text{do not have complex conjugate property.}$$

The following formula and theorem are easily proved.

Formula 7.2.1

When $f_1(z), f_2(z)$ are functions with the complex conjugate property in the domain D ,

$$(1) f_1(\bar{z}) \pm f_2(\bar{z}) = \overline{f_1(z) \pm f_2(z)}$$

$$(2) f_1(\bar{z}) f_2(\bar{z}) = \overline{f_1(z) f_2(z)}$$

Proof

$$f_1(z) = u_1 + i v_1, \quad f_1(\bar{z}) = u_1 - i v_1$$

$$f_2(z) = u_2 + i v_2, \quad f_2(\bar{z}) = u_2 - i v_2$$

From these,

$$\begin{aligned} f_1(\bar{z}) \pm f_2(\bar{z}) &= u_1 - i v_1 \pm (u_2 - i v_2) = u_1 \pm u_2 - i(v_1 \pm v_2) \\ &= \overline{u_1 \pm u_2 + i(v_1 \pm v_2)} \\ &= \overline{f_1(z) \pm f_2(z)} \end{aligned}$$

Next,

$$f_1(z) f_2(z) = u_1 u_2 - v_1 v_2 + i(u_1 v_2 + u_2 v_1)$$

$$\begin{aligned}\overline{f_1(z)f_2(z)} &= u_1u_2 - v_1v_2 - i(u_1v_2 + u_2v_1) \\ f_1(\bar{z})f_2(\bar{z}) &= u_1u_2 - v_1v_2 - i(u_1v_2 + u_2v_1) \\ \therefore f_1(\bar{z})f_2(\bar{z}) &= \overline{f_1(z)f_2(z)}\end{aligned}$$

Theorem 7.2.2

Assume that the function $f(z)$ defined in the domain D is expanded to the series as follows.

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

At this time, if a, c_n ($n=0, \pm 1, \pm 2, \dots$) are real numbers, $f(z)$ has complex conjugate property .

Proof

As seen in Example2 above, $(z-1)^{\pm n}$ ($n=0, \pm 1, \pm 2, \dots$) has complex conjugate property . So if a is a real number, $(z-a)^{\pm n}$ ($n=0, \pm 1, \pm 2, \dots$) has also complex conjugate property . Since c_n ($n=0, \pm 1, \pm 2, \dots$) are real numbers, from Formula 7.2.1 (1), $\sum_{n=-\infty}^{\infty} c_n (z-a)^n$ has also complex conjugate property .

Note 1

From this theorem we can see that each of the following functions has complex conjugate property .

$$\pi^z = e^{z \log \pi} = \sum_{n=0}^{\infty} \frac{\log^n \pi}{n!} z^n$$

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (z-1)^n \quad \gamma_s : \text{Stieltjes constant}$$

$$\Gamma(z) = 1 + \sum_{n=1}^{\infty} \frac{c_n(a)}{n!} (z-a)^n \quad a > 0$$

Where, $c_n(a) = \Gamma(a) \sum_{k=1}^n B_{n,k}(\psi_0(a), \psi_1(a), \dots, \psi_{n-1}(a))$ $n=1, 2, 3, \dots$

$\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \dots)$ are Bell polynomials.

Note 2

When $f(z)$ is a function defined in the domain D and has the complex conjugate property, if z_1 is a zero of $f(z)$, \bar{z}_1 is also zero of $f(z)$. Because, when $f(z) = u(z) + i v(z)$, if z_1 is a zero of $f(z)$,

$$\begin{aligned}0 &= f(z_1) = u(z_1) + i v(z_1) \quad (i.e. \ u(z_1) = v(z_1) = 0) \\ \therefore &= u(z_1) - i v(z_1) = \overline{f(z_1)} = f(\bar{z}_1)\end{aligned}$$

Even Function & Odd Function with the complex conjugate property

When $f(z)$ is an even function or an odd function and has the complex conjugate property, it becomes complicated as follows.

Theorem 7.2.3

When $f(x, y) = u(x, y) + iv(x, y)$ is a function with the complex conjugate property in the domain D ,

(1) if $f(x, y)$ is an even function,

$$u(x, y) = u(x, -y) = u(-x, y) = u(-x, -y) \quad (2.1u)$$

$$v(x, y) = -v(x, -y) = -v(-x, y) = v(-x, -y) \quad (2.1v)$$

(2) if $f(x, y)$ is an odd function,

$$u(x, y) = u(x, -y) = -u(-x, y) = -u(-x, -y) \quad (2.2u)$$

$$v(x, y) = -v(x, -y) = v(-x, y) = -v(-x, -y) \quad (2.2v)$$

Proof

Since $f(x, y)$ has the complex conjugate property,

$$u(x, -y) + iv(x, -y) = u(x, y) - iv(x, y)$$

$$u(-x, -y) + iv(-x, -y) = u(-x, y) - iv(-x, y)$$

From these,

$$u(x, y) = u(x, -y) \quad , \quad u(-x, y) = u(-x, -y) \quad (2.u)$$

$$v(x, y) = -v(x, -y) \quad , \quad -v(-x, y) = v(-x, -y) \quad (2.v)$$

(1) If $f(x, y)$ is an even function, from Theorem 7.1.3 (1),

$$u(x, y) = u(-x, -y) \quad , \quad v(x, y) = v(-x, -y)$$

From (2.u), (2.v) and these, (2.1u), (2.1v) are obtained.

(2) If $f(x, y)$ is an odd function, from Theorem 7.1.3 (2),

$$u(x, y) = -u(-x, -y) \quad , \quad v(x, y) = -v(-x, -y)$$

From (2.u), (2.v) and these, (2.2u), (2.2v) are obtained.

Example The 4th-degree function

The 4th-degree function $f(z) = z^4 + 1$ is given as an example of an even function. When (2.2u) and (2.2v) are calculated at $x = \pm 1/\sqrt{2}$, $y = \pm 1/\sqrt{2}$, it is as follows.

$$\begin{aligned} f[z_] &:= z^4 + 1 && \text{ComplexExpand}[f[x + i y]] \\ &&& 1 + x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3) \\ u[x_ , y_] &:= 1 + x^4 - 6x^2y^2 + y^4 && v[x_ , y_] := 4x^3y - 4xy^3 \\ \left\{ u\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], u\left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right], -u\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], -u\left[-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right] \right\} \\ \{0, 0, 0, 0\} \\ \left\{ v\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], -v\left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right], v\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], -v\left[-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right] \right\} \\ \{0, 0, 0, 0\} \end{aligned}$$

Corollary 7.2.3

Let $f(x,y) = u(x,y) + iv(x,y)$ be a function with the complex conjugate property in the domain D .

Then, the followings hold for any real number $x, y \in D$.

- (1) When $f(x,y)$ is an even function, $v(x,0) = 0$, $v(0,y) = 0$.
- (2) When $f(x,y)$ is an odd function, $u(0,y) = 0$, $v(x,0) = 0$.

Proof

(1) When $f(x,y)$ is an even function, from (2.1v),

$$v(x,y) = -v(x,-y), \quad v(x,y) = -v(-x,y)$$

Putting $y=0, x=0$ respectively,

$$v(x,0) = -v(x,0), \quad v(0,y) = -v(0,y)$$

From these, $v(x,0) = 0, v(0,y) = 0$

(2) When $f(x,y)$ is an odd function, from (2.2u) and (2.2v),

$$u(x,y) = -u(-x,y), \quad v(x,y) = -v(x,-y)$$

Putting $x=0, y=0$ respectively,

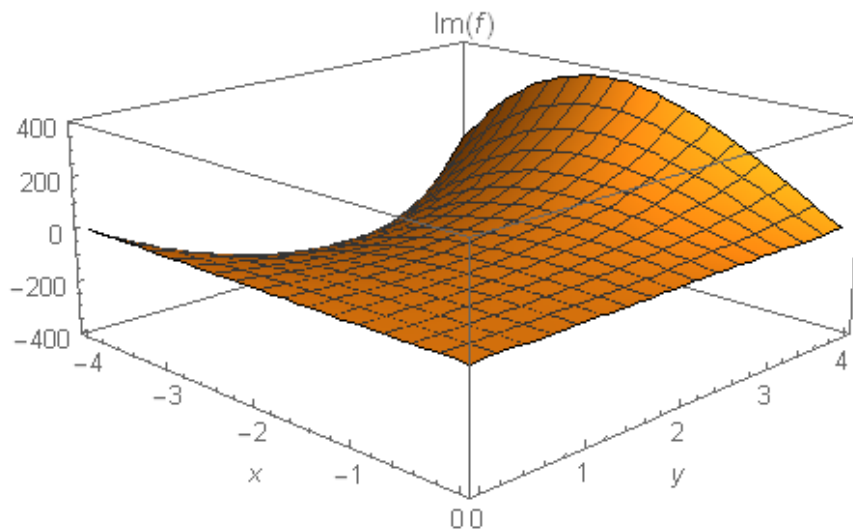
$$u(0,y) = -u(0,y), \quad v(x,0) = -v(x,0)$$

From these, $u(0,y) = 0, v(x,0) = 0$

Example 1 Imaginary part of the 4th-degree function

When $f(z) = z^4 + 1$, the sectional view taken along $x = 0, y = 0$ of the imaginary part is drawn as follows.

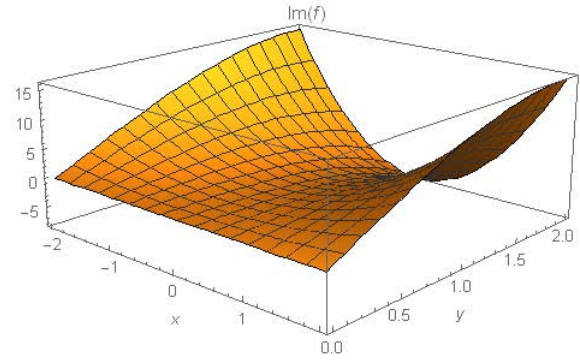
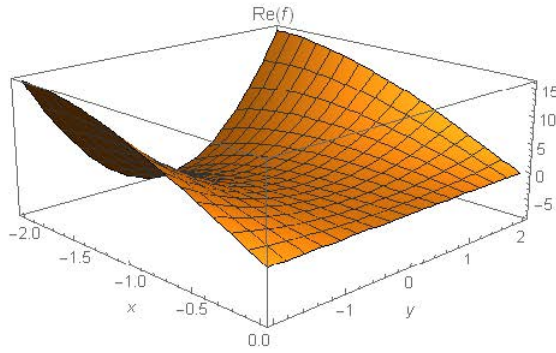
We can see that both cross sections are straight lines.



Example 2 Real part & Imaginary part of the 3rd-degree function

When $f(z) = z^3$, the sectional view taken along $x = 0$ of the real part is drawn on the left, and the sectional view taken along $y = 0$ of the imaginary part is drawn on the right.

In the real part, the cross section is a straight line for arbitrary y . In the imaginary part, the cross section is a straight line for arbitrary x .



Finally, we present and prove the following very important theorem.

Theorem 7.2.4

When $f(z)$ ($z = x + iy$) is a function with the complex conjugate property in the domain D and has a zero $x_1 + iy_1$,

- (1) if $f(z)$ is an even function, $-x_1 - iy_1$, $x_1 - iy_1$, $-x_1 + iy_1$ are also zeros of $f(z)$.
- (2) if $f(z)$ is an odd function, $-x_1 - iy_1$, $x_1 - iy_1$, $-x_1 + iy_1$ are also zeros of $f(z)$.

Proof

- (1) if $f(z)$ is an even function and $z_1 = x_1 + iy_1$ is its zero, from Theorem 7.2.3 (1),

$$u(x_1, y_1) = u(x_1, -y_1) = u(-x_1, y_1) = u(-x_1, -y_1) = 0$$

$$v(x_1, y_1) = -v(x_1, -y_1) = -v(-x_1, y_1) = v(-x_1, -y_1) = 0$$

From these, we can see that $x_1 \pm iy_1$, $-x_1 \pm iy_1$ are zeros of $f(z)$.

- (2) if $f(z)$ is an odd function and $z_1 = x_1 + iy_1$ is its zero, from Theorem 7.2.3 (2),

$$u(x_1, y_1) = u(x_1, -y_1) = -u(-x_1, y_1) = -u(-x_1, -y_1) = 0$$

$$v(x_1, y_1) = -v(x_1, -y_1) = v(-x_1, y_1) = -v(-x_1, -y_1) = 0$$

From these, we can see that $x_1 \pm iy_1$, $-x_1 \pm iy_1$ are zeros of $f(z)$.

Alternative Proof

- (1) if $f(z)$ is an even function and has a zero $x_1 + iy_1$, since $f(z) = f(-z)$, $-x_1 - iy_1$ is also zero of $f(z)$. According to **Note 2** above, these conjugate complex number $x_1 - iy_1$, $-x_1 + iy_1$ are also zeros of $f(z)$.
- (2) if $f(z)$ is an odd function and has a zero $x_1 + iy_1$, since $f(z) = -f(-z)$, $-x_1 - iy_1$ is also zero of $f(z)$. According to **Note 2** above, these conjugate complex number $x_1 - iy_1$, $-x_1 + iy_1$ are also zeros of $f(z)$.

7.3 Symmetric Functional Equation

Riemann functional equation can be transformed to a symmetric form.

Formula 7.3.1 (Riemann)

$$\pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) = \pi^{-\frac{1-z}{2}} \Gamma\left(\frac{1-z}{2}\right) \zeta(1-z) \quad z \neq 0, 1 \quad (3.1)$$

$$\pi^{-\frac{z}{2}} \Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\} \zeta\left(\frac{1}{2}+z\right) = \pi^{\frac{z}{2}} \Gamma\left\{\frac{1}{2}\left(\frac{1}{2}-z\right)\right\} \zeta\left(\frac{1}{2}-z\right) \quad (3.1')$$

Where, $z \neq \pm 1/2$

Proof

From Riemann functional equation,

$$\zeta(z) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \sin \frac{z\pi}{2} \cdot \zeta(1-z) \quad z \neq 0, 1$$

Multiplying both sides by $\pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right)$,

$$\begin{aligned} \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) &= \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \sin \frac{z\pi}{2} \cdot \zeta(1-z) \\ &= \pi^{-\frac{1-z}{2}} \Gamma\left(\frac{z}{2}\right) \sin \frac{z\pi}{2} \cdot 2^z \Gamma(1-z) \pi^{-\frac{1}{2}} \zeta(1-z) \end{aligned}$$

Here,

$$\Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

From this

$$\Gamma\left(\frac{z}{2}\right) \sin \frac{z\pi}{2} = \frac{\pi}{\Gamma\left(1-\frac{z}{2}\right)}$$

Substituting this for the above,

$$\begin{aligned} \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) &= \pi^{-\frac{1-z}{2}} \frac{\pi}{\Gamma\left(1-\frac{z}{2}\right)} \cdot 2^z \Gamma(1-z) \pi^{-\frac{1}{2}} \zeta(1-z) \\ &= \pi^{-\frac{1-z}{2}} \frac{\sqrt{\pi}}{\Gamma\left(1-\frac{z}{2}\right)} \cdot 2^z \Gamma(1-z) \zeta(1-z) \end{aligned}$$

Furthermore,

$$\Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right) = 2^{1-z} \sqrt{\pi} \Gamma(z)$$

Replacing z with $1-z$,

$$\Gamma\left(\frac{1-z}{2}\right)\Gamma\left(\frac{2-z}{2}\right) = 2^z\sqrt{\pi}\Gamma(1-z)$$

From this,

$$\frac{\sqrt{\pi}}{\Gamma\left(1-\frac{z}{2}\right)} \cdot 2^z\Gamma(1-z) = \Gamma\left(\frac{1-z}{2}\right)$$

Substituting this for the above ,

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) \quad z \neq 0, 1 \quad (3.1)$$

Next, replacing z with $z+1/2$,

$$\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) = \pi^{-\frac{1}{2}\left(\frac{1}{2}-z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}-z\right)\right\}\zeta\left(\frac{1}{2}-z\right)$$

Multiplying both sides by $\pi^{1/4}$,

$$\pi^{-\frac{z}{2}}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) = \pi^{\frac{z}{2}}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}-z\right)\right\}\zeta\left(\frac{1}{2}-z\right) \quad (3.1')$$

Symmetric functional equation is obtained also about Dirichlet Eta Function.

Formula 7.3.2

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}(1-2^z)\eta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}(1-2^{1-z})\eta(1-z)$$

Where, $z \neq 0, 1$ (3.3)

$$\pi^{-\frac{z}{2}}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\left(1-2^{\frac{1}{2}+z}\right)\eta\left(\frac{1}{2}+z\right) = \pi^{\frac{z}{2}}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}-z\right)\right\}\left(1-2^{\frac{1}{2}-z}\right)\eta\left(\frac{1}{2}-z\right)$$

Where, $z \neq \pm 1/2$ (3.3')

Proof

Substituting $\zeta(z) = \eta(z)/(1-2^{1-z})$ for (3.1) ,

$$\Gamma\left(\frac{z}{2}\right)\frac{\pi^{-\frac{z}{2}}}{1-2^{1-z}}\eta(z) = \Gamma\left(\frac{1-z}{2}\right)\frac{\pi^{-\frac{1-z}{2}}}{1-2^{1-(1-z)}}\eta(1-z)$$

i.e.

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}(1-2^z)\eta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}(1-2^{1-z})\eta(1-z) \quad (3.3)$$

Next, replacing z with $z+1/2$ and multiplying both sides by $\pi^{1/4}$, we obtain (3.3').

7.4 Completed Riemann Zeta

Let us define the functions $\xi(z)$, $\Xi(z)$ on the complex plane as follows.

$$\xi(z) = z(1-z) \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) \quad (4.1)$$

$$\Xi(z) = \left(\frac{1}{2}+z\right) \left(\frac{1}{2}-z\right) \pi^{-\frac{z}{2}} \Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\} \zeta\left(\frac{1}{2}+z\right) \quad (4.1')$$

Then, from Formula 7.3.1, the following expressions hold on the whole complex plane.

$$\xi(z) = \xi(1-z) \quad (4.2)$$

$$\Xi(z) = \Xi(-z) \quad (4.2')$$

These are called **Completed Riemann Zeta**. When $z = x + iy$, the real part of $\xi(z)$ is line symmetry with respect to $x=1/2$ and the real part of $\Xi(z)$ is line symmetry with respect to $x=0$. In this chapter, we investigate the properties of $\Xi(z)$.

cf.

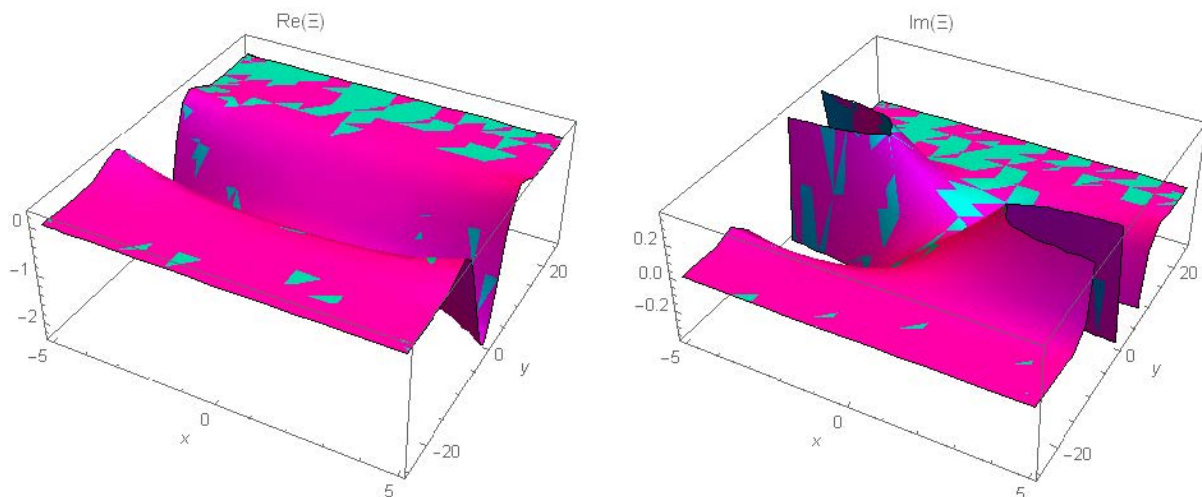
Definitions of $\xi(z)$ and $\Xi(z)$ in this chapter are different from Landau's following definitions.

$$\xi(z) = \frac{1}{2} z(z-1) \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) \quad : \text{xi function}$$

$$\Xi(z) = \xi\left(\frac{1}{2} + iz\right) \quad : \text{Xi function}$$

7.4.1 Properties of Completed Riemann Zeta

If the real part and the imaginary part of $\Xi(z)$, $\Xi(-z)$ are drawn on 3D figure, it is as follows. $\Xi(z)$ is cyan and $\Xi(-z)$ is magenta.



From this figure, we can see the following.

- (1) The function $\Xi(z)$ is holomorphic on the whole complex plane. A singular point is not seen anywhere.
- (2) The function $\Xi(z)$ is an even function with respect to the complex number z . ($\Xi(z) = \Xi(-z)$)

In the above figure, $\Xi(z)$ (cyan) and $\Xi(-z)$ (magenta) overlap and are visible in spots.

(3) The real part $u(x, y)$ is an even function for both x and y . (See the above left figure.)

i.e. $u(x, y) = u(-x, y) = u(x, -y)$

It is because $\Xi(z)$ is an even function and has the complex conjugate property. (Theorem 7.2.3 (1))

(4) The imaginary part $v(x, y)$ is an odd function for both x and y . (See the above right figure.)

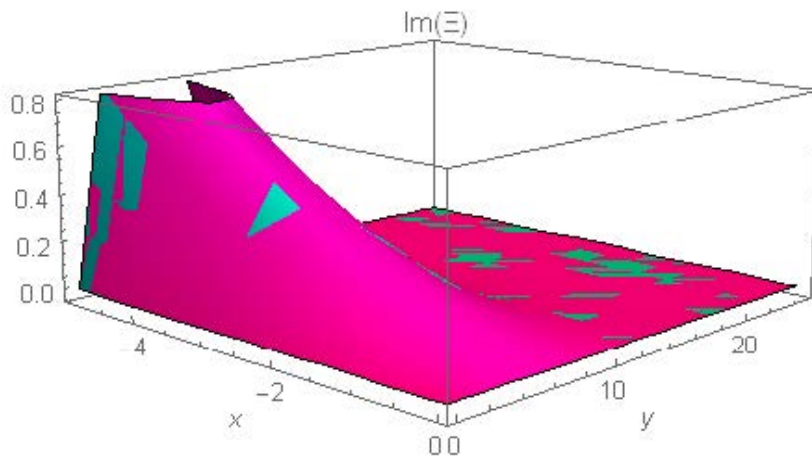
i.e. $v(x, y) = -v(-x, y) = -v(x, -y)$

It is also because $\Xi(z)$ is an even function and has the complex conjugate property. (Theorem 7.2.3 (1))

(5) $v(0, y) = 0$, $v(x, 0) = 0$ regardless of the values of x, y .

It is because $\Xi(z)$ is an even function and has the complex conjugate property. (Corollary 7.2.3 (1))

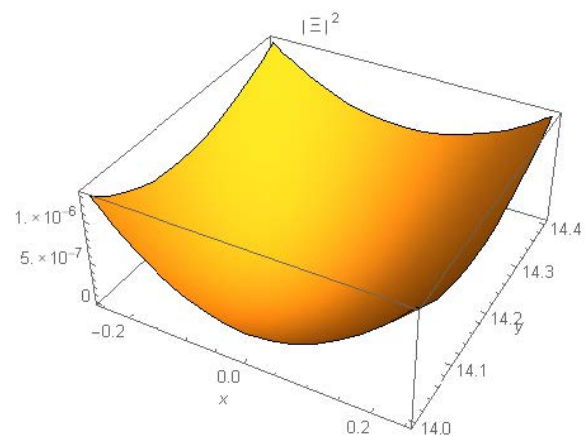
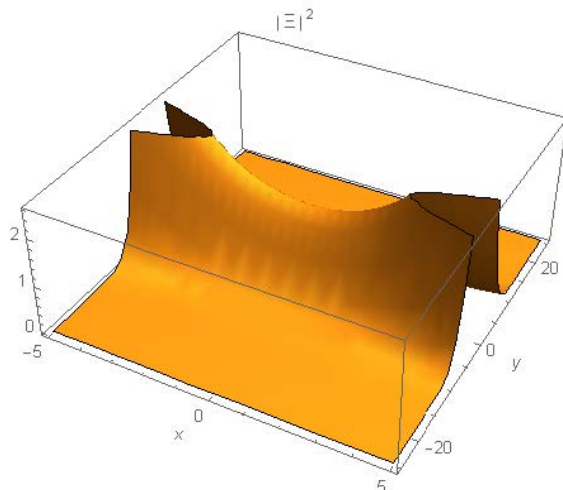
The sectional view along $x = 0$, $y = 0$ of the right figure above is as follows. We can see that both cross sections are straight lines.



(5) $|\Xi(z)|^2$ is an even function for both x and y .

It is because $\Xi(z)$ is an even function. (Theorem 7.1.4) The complex conjugate property is not required.

If this is drawn on 3D figure, it is as the left figure. And the minimum point as shown in the right figure seems to be dotted along $x=0$.



From the above properties, the following important theorem on the zeros of the Riemann zeta function is obtained.

Theorem 7.4.1

If Riemann zeta function $\zeta(z)$ ($z = x + iy$) has zeros, the one set consists of the following four or two.

$$1/2 + \alpha_1 \pm i\beta_1, \quad 1/2 - \alpha_1 \pm i\beta_1$$

Where, i α_1 is a real number s.t. $0 \leq \alpha_1 < 1/2$, and β_1 is an arbitrary real number.

ii When $\alpha_1 = 0$, duplication is excluded.

Proof

It is equivalent that $1/2 + \alpha_1 + i\beta_1$ is a zero of $\zeta(z)$ and that $\alpha_1 + i\beta_1$ is a zero of $\zeta(1/2+z)$.

Here, let us observe (4.1').

$$\Xi(z) = \left(\frac{1}{2} + z\right) \left(\frac{1}{2} - z\right) \pi^{-\frac{z}{2}} \Gamma\left\{\frac{1}{2}\left(\frac{1}{2} + z\right)\right\} \zeta\left(\frac{1}{2} + z\right) \tag{4.1'}$$

Then, we can see the followings.

- (1) Trivial zeros of $\zeta\left(\frac{1}{2} + z\right)$ and zero of $\frac{1}{2} + z$ are offset with the singular points of $\Gamma\left\{\frac{1}{2}\left(\frac{1}{2} + z\right)\right\}$

At those points, $\Xi(z)$ becomes non-zero and holomorphic.

- (2) Zero of $\frac{1}{2} - z$ are offset with the singular point of $\zeta\left(\frac{1}{2} + z\right)$.

At this point, $\Xi(z)$ becomes non-zero and holomorphic.

- (3) $\pi^{-z/2}$ has no zero.

As the result, the zeros of $\zeta(1/2+z)$ coincide with the zeros of $\Xi(z)$.

Since $\Xi(z) = \Xi(-z)$ from (4.2'), $\Xi(z)$ is an even function. And as seen in Theorem 7.2.2 Note1, all the functions that constitute (4.1') have complex conjugate property. So, $\Xi(z)$ also has complex conjugate property from Formula 7.2.1. Then, according to Theorem 7.2.4 (1), $\Xi(z)$ has the following 4 zeros.

$$\alpha_1 \pm i\beta_1, \quad -\alpha_1 \pm i\beta_1$$

This is equivalent to that $\zeta(1/2+z)$ has the following 4 zeros.

$$1/2 + \alpha_1 \pm i\beta_1, \quad 1/2 - \alpha_1 \pm i\beta_1$$

And, if $\alpha_1 = 0$, these result in 2 zeros $1/2 \pm i\beta_1$.

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Kano Kono