7 Completed Riemann Zeta

7.1 Even Function & Odd Function

Definition 7.1.0

Let $f(z)$ be a function in the domain D.

(1) When $f(z) = f(-z)$, $f(z)$ is an **even function** .

(2) When $f(z) = -f(-z)$, $f(z)$ is an **odd function** .

Theorem 7.1.1

Let $f_1(z), f_2(z)$ are functions in the domain D.

(1) If $f_1(z), f_2(z)$ are even functions , $f_1(z) \pm f_2(z)$ is also an even function .

(2) If $f_1(z), f_2(z)$ are odd functions , $f_1(z) \pm f_2(z)$ is also an odd function .

Proof

Whent $f_1(z), f_2(z)$ are even functions , since $f_k(z) = f_k(-z)$ $k=1,2$,

$$f_1(z) \pm f_2(z) = f_1(-z) \pm f_2(-z)$$

Whent $f_1(z), f_2(z)$ are odd functions , since $f_k(z) = -f_k(-z)$ $k=1,2$,

$$f_1(z) \pm f_2(z) = -f_1(-z) \pm \{-f_2(-z)\} = -\{f_1(-z) \pm f_2(-z)\}$$

Theorem 7.1.2

Let $f_1(z), f_2(z)$ are functions in the domain D.

(1) If $f_1(z), f_2(z)$ are even functions , $f_1(z)\, f_2(z)$ is an even function .

(2) If $f_1(z), f_2(z)$ are odd functions , $f_1(z)\, f_2(z)$ is an even function .

(3) If $f_1(z)$ is an odd function and $f_2(z)$ is an even function , $f_1(z)\, f_2(z)$ is an odd function .

Proof

Whent $f_1(z), f_2(z)$ are odd functions , since $f_k(z) = -f_k(-z)$ $k=1,2$,

$$f_1(z)\, f_2(z) = -f_1(-z)\{-f_2(-z)\} = f_1(-z) f_2(-z)$$

Thus, **(2)** is proved.

(1) ,**(3)** are also proved in a similar way.

When $f(z)$ is a function in the domain, if the complex variable z is regarded as one variable , the following theorems hold..

Theorem 7.1.3

Let $f(z)$ be a complex function in the domain D.

(1) If $f(z)$ is an even function , both the real part and the imaginary part are even functions.

(2) If $f(z)$ is an odd function , both the real part and the imaginary part are odd functions.

Proof

Let R be a real number space in a domain D and let

$$f(z)=u(z)+iv(z)\ , \qquad u,v \in R$$

Replacing z with $-z$,

$$f(-z)=u(-z)+iv(-z)$$

From these,

if $f(z)=f(-z)$ then $u(z)=u(-z)$, $v(z)=v(-z)$.

if $f(z)=-f(-z)$ then $u(z)=-u(-z)$, $v(z)=-v(-z)$.

Theorem 7.1.4

Let $f(z)$ be a complex function in the domain D. Then,

if $f(z)$ is an even function or an odd function , $|f(z)|^2$ is an even function.

Proof

Let R be a real number space in a domain D and let

$$f(z)=u(z)+iv(z)\ , \qquad u,v \in R$$

Then,

$$|f(z)|^2 = u^2(z)+v^2(z)$$

If $f(z)$ is an even function, both $u(z)$ and $v(z)$ are even functions from Theorem 7.1.3 (1) and if $f(z)$ is an odd function, both $u(z)$ and $v(z)$ are odd functions from Theorem 7.1.3 (2) .

Then, in the case of which, both $u^2(z)$ and $v^2(z)$ become even functions from Theorem 7.1.2 .

Furthermore, $u^2(z)+v^2(z)$ becomes an even function from Theorem 7.1.1 (1) .

Theorem 7.1.5

Let a complex function $f(z)$ be holomorphic in the whole domain D.

(1) If $f(z)$ is an even function , the 1st order derivative $f^{(1)}(z)$ is an odd function.

(2) If $f(z)$ is an odd function , the 1st order derivative $f^{(1)}(z)$ is an even function.

Proof

If a function $f(z)$ is an even function, $f(z) = f(-z)$.

Differentiating both sides with respect to z ,

$$f^{(1)}(z) = f^{(1)}(-z)\cdot(-z)^{(1)} = -f^{(1)}(-z)$$

Therefore, $f^{(1)}(z)$ is an odd function.

If a function $f(z)$ is an odd function, $f(z) = -f(-z)$.

Differentiating both sides with respect to z ,

$$f^{(1)}(z) = -f^{(1)}(-z)\cdot(-z)^{(1)} = f^{(1)}(-z)$$

Therefore, $f^{(1)}(z)$ is an even function.

Using this theorem, we obtain the very important following theorem.

Theorem 7.1.6

Let a complex function $f(z)$ be holomorphic in the whole domain D.

(1) $f(z)$ is an even function if and only if the Maclaurin series includes only even powers.

(2) $f(z)$ is an odd function if and only if the Maclaurin series includes only odd powers.

Proof

By assumption, $f(z)$ is expanded to Maclaurin series as follows.

$$f(z) = \frac{f^{(0)}(0)}{0!}z^0 + \frac{f^{(1)}(0)}{1!}z^1 + \frac{f^{(2)}(0)}{2!}z^2 + \frac{f^{(3)}(0)}{3!}z^3 + \cdots$$

(1) According to Theorem 7.5.1 , if $f(z)$ is an even function, $f^{(2n-1)}(z)$ $n=1,2,3,\cdots$ are odd functions.

So, $f^{(2n+1)}(0) = -f^{(2n+1)}(-0)$ $n=1,2,3,\cdots$

From this, we obtain $f^{(2n+1)}(0) = 0$ $n=1,2,3,\cdots$.

Conversely, if the Maclaurin series includes only even powers, each terms are all even functions.

Then, according to Theorem 7.1.1 , the Maclaurin series becomes an even function.

(2) According to Theorem 7.5.1 , if $f(z)$ is an odd function, $f^{(2n)}(z)$ $n=1,2,3,\cdots$ are odd functions.

So, $f^{(2n)}(0) = -f^{(2n)}(-0)$ $n=1,2,3,\cdots$

From this, we obtain $f^{(2n)}(0) = 0$ $n=1,2,3,\cdots$.

Conversely, if the Maclaurin series includes only odd powers, each terms are all odd functions.

Then, according to Theorem 7.1.1 , the Maclaurin series becomes an odd function.

7.2 Complex Conjugate Property

Definition 7.2.0

When the function $f(z)$ defined in the domain D satisfies

$$f\left(\bar{z}\right) = \overline{f(z)} \qquad z \in D$$

We say that the $f(z)$ has **complex conjugate property**.

Example $\boldsymbol{f(z) = e^z}$

$$\because \quad f(z) = f(x,y) = e^{x+iy} = e^x(\cos y + i \sin y)$$

$$f\left(\bar{z}\right) = f(x,-y) = e^{x-iy} = e^x(\cos y - i \sin y) = \overline{f(z)}$$

Note1

Let

$$f(z) = f(x,y) = u(x,y) + iv(x,y)$$

If this has the complex conjugate property ,

$$u(x,-y) + iv(x,-y) = u(x,y) - iv(x,y)$$

That is

$$u(x,y) = u(x,-y) \ , \ \ v(x,y) = -v(x,-y)$$

These mean as follows.

(1) Real part $u(x,y)$ is an even function with respect to y for any $x \in D$.

(2) Imaginary part $v(x,y)$ is an odd function with respect to y for any $x \in D$.

Note2

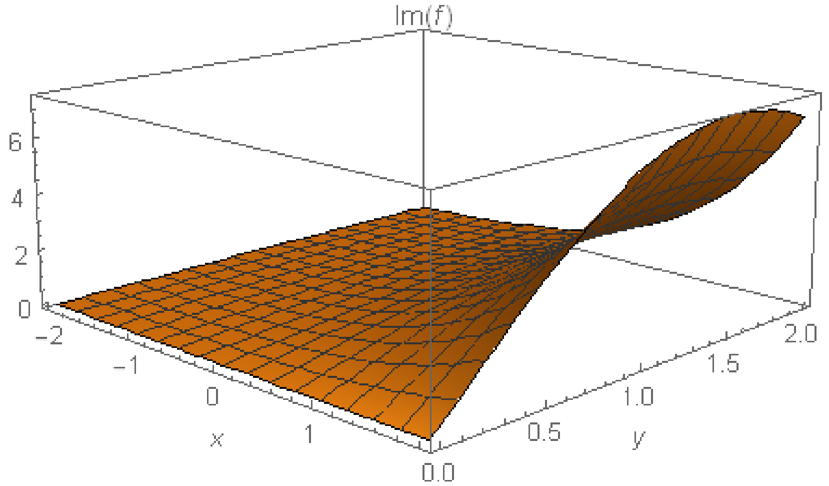
If $f(z)$ has the complex conjugate property, $v(x,0)=0$ for any $x \in D$.

Because, since $v(x,y) = -v(x,-y)$, substituting $y=0$ for this,

$$v(x,0) = -v(x,0)$$

In order to hold this for arbitrary x , $v(x,0) = -v(x,0) = 0$ are required.

For example, when $f(x,y) = e^{x+iy}$, the sectional view taken along $y=0$ of the imaginary part is as follows.



Note3

The complex conjugate property $f(\overline{z}) = \overline{f(z)}$ does not generally hold. For example, $f(z)=(-e)^z$ does not satisfy this. For your reference, this is a holomorphic function. However, if $f(z)$ is a rational function with real coefficients (Coefficients of the Taylor series are real numbers), it is known that $f(z)$ satisfies this Therefore, most of the complex function has the complex conjugate property.

Examples

(1) $log(\overline{z}) = \overline{log(z)}$

(2) $a^{\overline{z}} = \overline{a^z}$ $\quad a$ is a positive number

(3) $\zeta(\overline{s}) = \overline{\zeta(s)}$, $\quad \Gamma(\overline{s}) = \overline{\Gamma(s)}$

The following formula is proved easily.

Formula 7.2.1

When $f_1(z), f_2(z)$ are functions with the complex conjugate property in the domain D ,

(1) $f_1(\overline{z}) \pm f_2(\overline{z}) = \overline{f_1(z) \pm f_2(z)}$

(2) $f_1(\overline{z})\ f_2(\overline{z}) = \overline{f_1(z)\ f_2(z)}$

Proof

$$f_1(z) = u_1 + i\, v_1 \quad , \quad f_1(\overline{z}) = u_1 - i\, v_1$$

$$f_2(z) = u_2 + i\, v_2 \quad , \quad f_2(\overline{z}) = u_2 - i\, v_2$$

From these,

$$f_1(\overline{z}) \pm f_2(\overline{z}) = u_1 - i\, v_1 \pm (u_2 - i\, v_2) = u_1 \pm u_2 - i\,(v_1 \pm v_2)$$

$$= \overline{f_1(z) \pm f_2(z)}$$

Next,

$$f_1(z) f_2(z) = u_1 u_2 - v_1 v_2 + i(u_1 v_2 + u_2 v_1)$$

$$\overline{f_1(z) f_2(z)} = u_1 u_2 - v_1 v_2 - i(u_1 v_2 + u_2 v_1)$$

$$f_1(\overline{z}) f_2(\overline{z}) = u_1 u_2 - v_1 v_2 - i(u_1 v_2 + u_2 v_1)$$

$$\therefore\ f_1(\overline{z}) f_2(\overline{z}) = \overline{f(z) g(z)}$$

Even Function & Odd Function with the complex conjugate property

When $f(z)$ is an even function or an odd function and has the complex conjugate property, the following vary usefull theorems hold.

Theorem 7.2.2

When $f(x,y)=u(x,y)+iv(x,y)$ is a function with the complex conjugate property in the domain D ,

(1) if $f(x,y)$ is an even function,

$$u(x,y) = u(-x,y) = u(x,-y)$$

$$v(x,y) = -v(-x,y) = -v(x,-y)$$

(2) if $f(x,y)$ is an odd function,

$$u(x,y) = -u(-x,y) = u(x,-y)$$
$$v(x,y) = v(-x,y) = -v(x,-y)$$

Proof

Since $f(x,y)$ has the complex conjugate property, the following equation holds as mentioned above.

$$u(x,-y) + iv(x,-y) = u(x,y) - iv(x,y) \tag{2.1}$$

i.e.

$$u(x,y) = u(x,-y) \,,\; v(x,y) = -v(x,-y)$$

Next, reversing the sign of x in the (2.1) ,

$$u(-x,-y) + iv(-x,-y) = u(-x,y) - iv(-x,y) \tag{2.2}$$

(1) If $f(x,y)$ is an even function, from Theorem 7.1.3 (1) ,

$$u(-x,-y) = u(x,y) \quad ,\; v(-x,-y) = v(x,y)$$

Substituting these for the left hand side of (2.2) ,

$$u(x,y) + iv(x,y) = u(-x,y) - iv(-x,y)$$

That is,

$$u(x,y) = u(-x,y) \,,\; v(x,y) = -v(-x,y)$$

(2) If $f(x,y)$ is an odd function, from Theorem 7.1.3 (2) ,

$$u(-x,-y) = -u(x,y) \,,\; v(-x,-y) = -v(x,y)$$

Substituting these for the left hand side of (2.2) ,

$$-u(x,y) - iv(x,y) = u(-x,y) - iv(-x,y)$$

That is,

$$u(x,y) = -u(-x,y) \,,\; v(x,y) = v(-x,y)$$

Note

This theorem means the followings.

(1) If $f(x,y)$ is an even function,

Real part $u(x,y)$ is an even function with respect to both x and y .

Imaginary part $v(x,y)$ is an odd function with respect to both x and y .

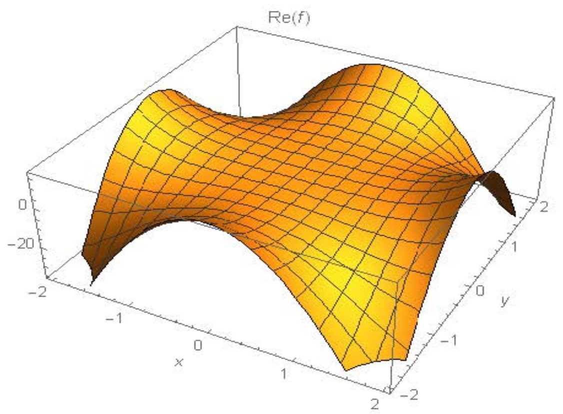
(2) If $f(x,y)$ is an odd function,

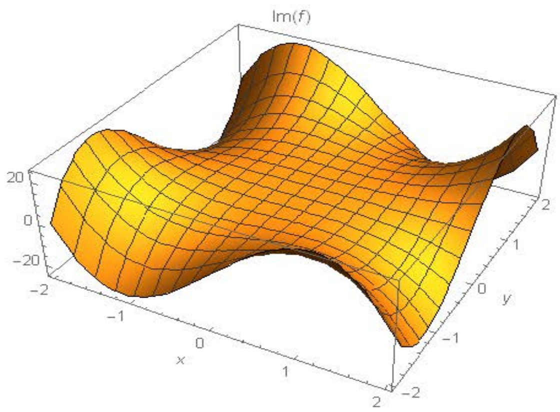
Real part $u(x,y)$ is an odd function with respect to x and is an even function with respect to y

Imaginary part $v(x,y)$ is an even function with respect to x and is an odd function with respect to y

Example1 $f(z) = z^4 + 1$

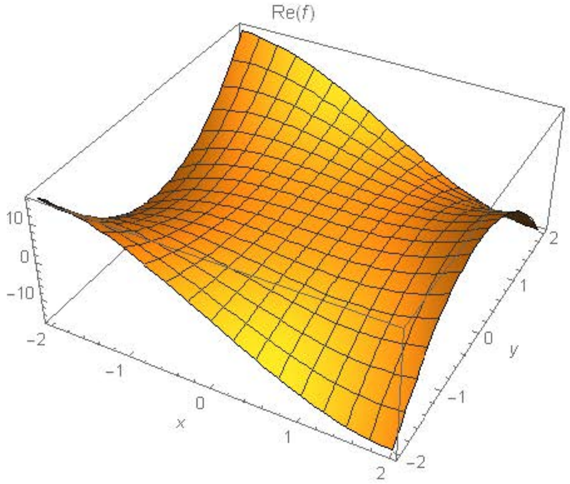
The 3D figures of the real part and the imaginary part are as follows. We can see that these are as Note (1) .

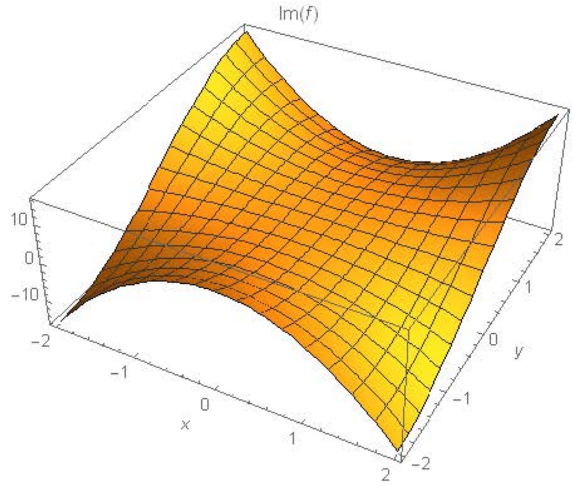




Example2 $f(z)=z^3$

The 3D figures of the real part and the imaginary part are as follows. We can see that these are as Note (2) .





Corollary 7.2.2

Let $f(x,y)=u(x,y)+iv(x,y)$ be a function with the complex conjugate property in the domain D .
Then, the followings hold for any real number $x,y \in D$.

(1) When $f(x,y)$ is an even function , $v(0,y) = 0$, $v(x,0) = 0$.

(2) When $f(x,y)$ is an odd function , $u(0,y) = 0$, $v(x,0) = 0$.

Proof

Since $f(x,y)$ has the complex conjugate property, $v(x,0)=0$ and Theorem 7.2.2 holds. Then

(1) When $f(x,y)$ is an even function , $v(x,y) = -v(-x,y)$

Substituting $x=0$ for this, $v(0,y) = -v(0,y)$

In order to hold this for arbitrary y , $v(0,y) = -v(0,y) = 0$ are required.

(2) When $f(x,y)$ is an odd function , $u(x,y) = -u(-x,y)$

Substituting $x=0$ for this, $u(0,y) = -u(0,y)$

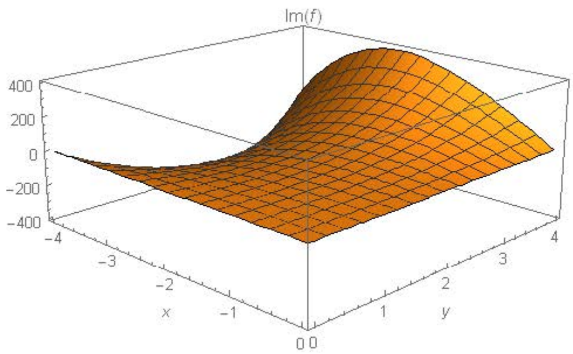
In order to hold this for arbitrary y , $u(0,y) = -u(0,y) = 0$ are required.

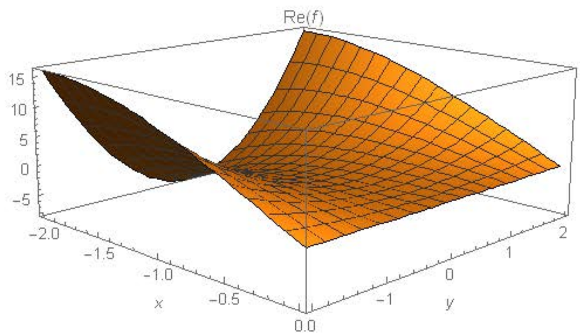
Examples Imaginary part of the 4th-order functions & real part of the 3rd-order function

When $f(z)=z^4+1$, the sectional view taken along $x=0, y=0$ of the imaginary part is drawn on the left.

When $f(z)=z^3$, the sectional view taken along $x=0$ of the real part is drawn on the right.

In both figures, those cut ends exist on the straight line of the height 0.





7.3 Symmetric Functional Equation

Riemann functional equation can be transformed to a symmetric form.

Formula 7.3.1 (Riemann)

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) \qquad z\neq 0,1 \tag{1.0}$$

$$\pi^{-\frac{z}{2}}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) = \pi^{\frac{z}{2}}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}-z\right)\right\}\zeta\left(\frac{1}{2}-z\right) \tag{1.1}$$

Where, $z\neq \pm 1/2$

Proof

From Riemann functional equation,

$$\zeta(z) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}}\sin\frac{z\pi}{2}\cdot\zeta(1-z) \qquad z\neq 1$$

Multiplying both sides by $\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)$,

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\frac{2\Gamma(1-z)}{(2\pi)^{1-z}}\sin\frac{z\pi}{2}\cdot\zeta(1-z)$$

$$= \pi^{-\frac{1-z}{2}}\Gamma\left(\frac{z}{2}\right)\sin\frac{z\pi}{2}\cdot 2^{z}\Gamma(1-z)\pi^{-\frac{1}{2}}\zeta(1-z)$$

Here,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$$

From this

$$\Gamma\left(\frac{z}{2}\right)\sin\frac{z\pi}{2} = \frac{\pi}{\Gamma\left(1-\frac{z}{2}\right)}$$

Substituting this for the above,

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\frac{\pi}{\Gamma\left(1-\frac{z}{2}\right)}\cdot 2^{z}\Gamma(1-z)\pi^{-\frac{1}{2}}\zeta(1-z)$$

$$= \pi^{-\frac{1-z}{2}}\frac{\sqrt{\pi}}{\Gamma\left(1-\frac{z}{2}\right)}\cdot 2^{z}\Gamma(1-z)\zeta(1-z)$$

Furthermore,

$$\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) = 2^{1-z}\sqrt{\pi}\,\Gamma(z)$$

Replacing z with $1-z$,

$$\Gamma\left(\frac{1-z}{2}\right)\Gamma\left(\frac{2-z}{2}\right) = 2^z\sqrt{\pi}\,\Gamma(1-z)$$

From this,

$$\frac{\sqrt{\pi}}{\Gamma\left(1-\frac{z}{2}\right)}\cdot 2^z\Gamma(1-z) = \Gamma\left(\frac{1-z}{2}\right)$$

Substituting this for the above ,

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) \qquad (1.0)$$

Next, replacing z with $z+1/2$,

$$\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) = \pi^{-\frac{1}{2}\left(\frac{1}{2}-z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}-z\right)\right\}\zeta\left(\frac{1}{2}-z\right)$$

Multiplying both sides by $\pi^{1/4}$,

$$\pi^{-\frac{z}{2}}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) = \pi^{\frac{z}{2}}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}-z\right)\right\}\zeta\left(\frac{1}{2}-z\right) \qquad (1.1)$$

Symmetric functional equation is obtained also about Dirichlet Eta Function.

Formula 7.3.2

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left(1-2^z\right)\eta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\left(1-2^{1-z}\right)\eta(1-z)$$

Where, $z\neq 0,1$ (2.0)

$$\pi^{-\frac{z}{2}}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\left(1-2^{\frac{1}{2}+z}\right)\eta\left(\frac{1}{2}+z\right) = \pi^{\frac{z}{2}}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}-z\right)\right\}\left(1-2^{\frac{1}{2}-z}\right)\eta\left(\frac{1}{2}-z\right)$$

Where, $z\neq \pm 1/2$ (2.1)

Proof

Substituting $\zeta(z)=\eta(z)/\left(1-2^{1-z}\right)$ for (1.0) ,

$$\Gamma\left(\frac{z}{2}\right)\frac{\pi^{-\frac{z}{2}}}{1-2^{1-z}}\eta(z) = \Gamma\left(\frac{1-z}{2}\right)\frac{\pi^{-\frac{1-z}{2}}}{1-2^{1-(1-z)}}\eta(1-z)$$

i.e.

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left(1-2^z\right)\eta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\left(1-2^{1-z}\right)\eta(1-z) \qquad (2.0)$$

Next, replacing z with $z+1/2$ and multiplying both sides by $\pi^{1/4}$, we obtain (2.1) .

7.4 Completed Riemann Zeta

Let us describe the left sides of Formula 7.3.1 as follows respectively.

$$\xi(z) = \pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) \tag{1.0}$$

$$\Xi(z) = \pi^{-\frac{z}{2}}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) \tag{1.1}$$

Then, Formula 7.3.1 is expressed as follows.

$$\xi(z) = \xi(1-z)$$

$$\Xi(z) = \Xi(-z)$$

These are called **Completed Riemann Zeta**. When $z=x+iy$, the real part of $\xi(z)$ is line symmetry with respect to $x=1/2$ and the real part of $\Xi(z)$ is line symmetry with respect to $x=0$. In this chapter, we investigate the properties of $\Xi(z)$.

cf.

Definitions of $\xi(z)$ and $\Xi(z)$ in this chapter are different from **Landau**'s following definitions.

$$\xi(z) = \frac{1}{2}z(z-1)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) \qquad : \text{ xi function}$$

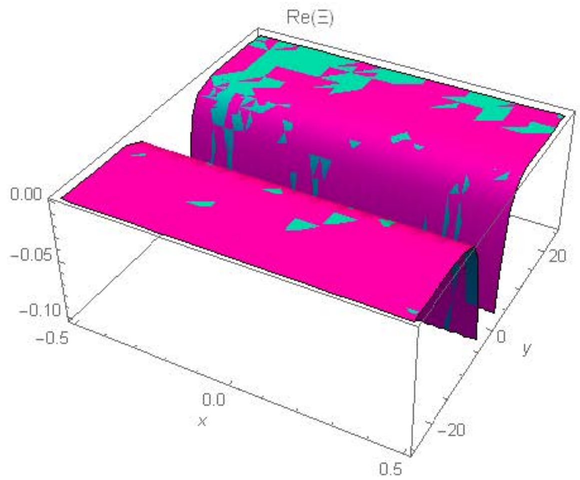
$$\Xi(z) = \xi\left(\frac{1}{2}+iz\right) \qquad : \text{Xi function}$$

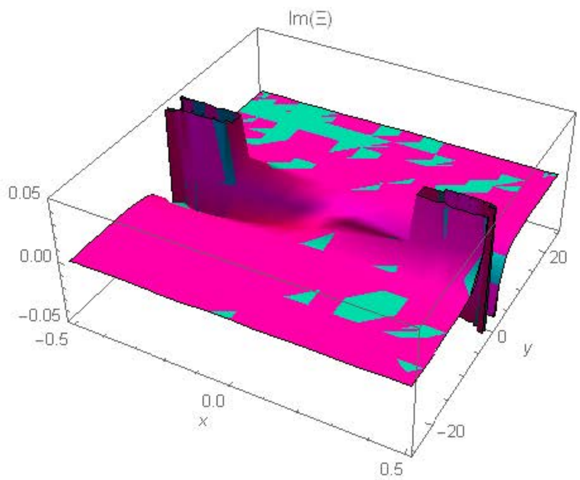
7.4.1 Properties of Completed Riemann Zeta

(1) $\Xi(z)$ is an even function about the complex number z . ($\Xi(z)=\Xi(-z)$)

If the real part and the imaginary part of $\Xi(z),\Xi(-z)$ are drawn on 3D figure, it is as follow.

$\Xi(z)$ (cyan) and $\Xi(-z)$ (magenta) have overlapped and are visible in spots.



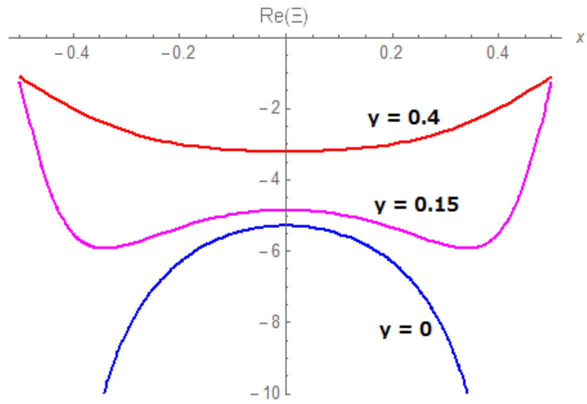


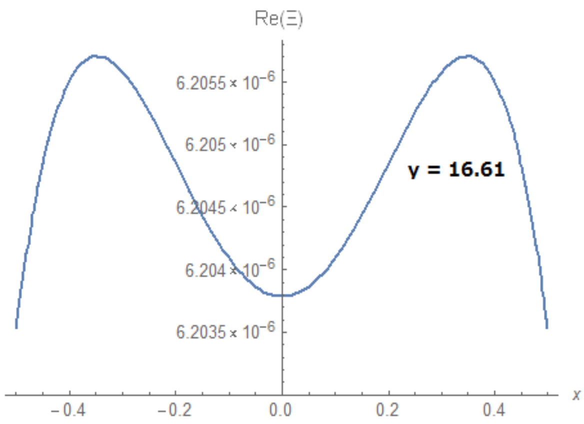
(2) The real part $u(x,y)$ is an even function about both x and y . (See the above left figure.)

i.e. $u(x,y) = u(-x,y) = u(x,-y)$

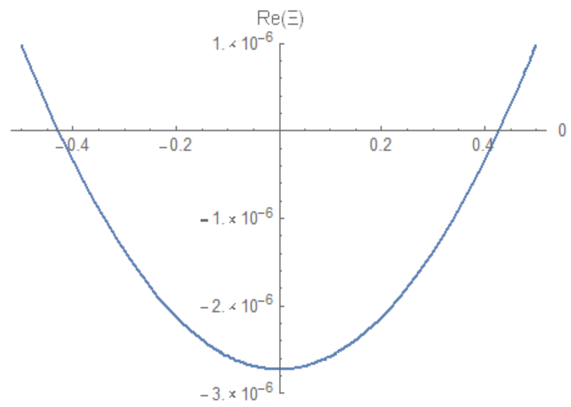
It is because $\Xi(z)$ is an even function and has the complex conjugate property. (Theorem 7.2.2 (1))

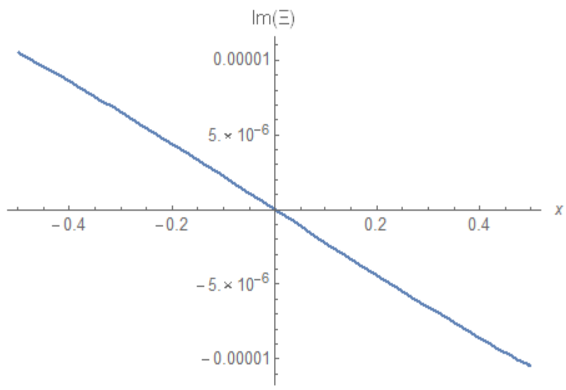
(3) When y is varied parametrically, $u(x,y)$ changes the shape as ∩ , ∪ ., M , W .





So, $u(x,y)$ can have two or more zeros within $|x|<1/2$. However, these zeros do not seem to be zeros of Ξ. For example, when $y=14$, the real part and the imaginary part of Ξ are drawn as folloes. Although $x=\pm 0.4278\cdots$ are two zeros of the real part, the imaginary part is not zero for these values.



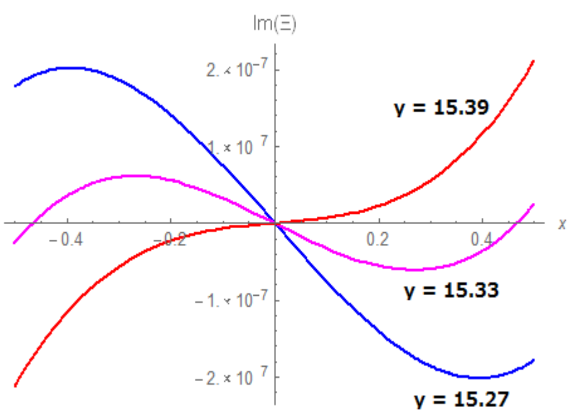


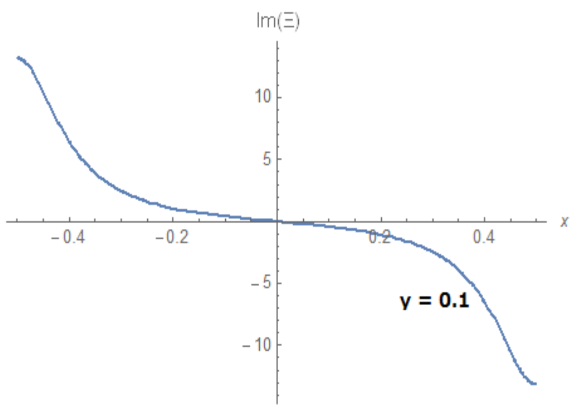
(4) The imaginary part $v(x,y)$ is an odd function about both x and y. (See the above right figure.)

i.e. $v(x,y) = -v(-x,y) = -v(x,-y)$

It is also because $\Xi(z)$ is an even function and has the complex conjugate property. (Theorem 7.2.2 (1))

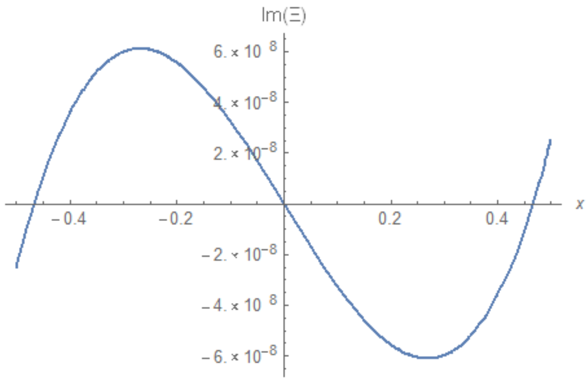
(5) When y is varied parametrically, $v(x,y)$ changes the shape as / , \ , ~ , ∽ .

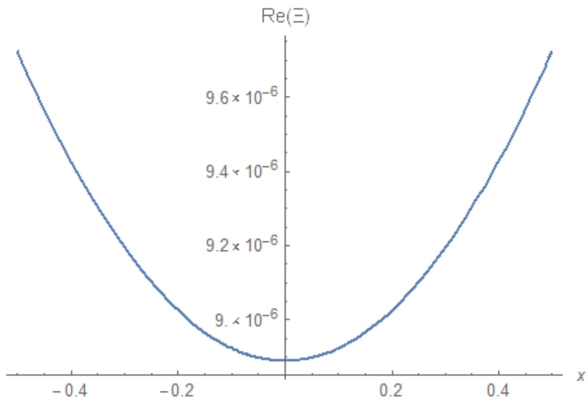




So, $v(x,y)$ can have two or more zeros within $|x|<1/2$. However, these zeros do not seem to be zeros of Ξ. For example, when $y=15.33$, the imaginary part and the real part of Ξ are drawn as folloes.

Although $x=\pm 0.4662\cdots$ are two zeros of the imaginary part, the real part is not zero for these values.

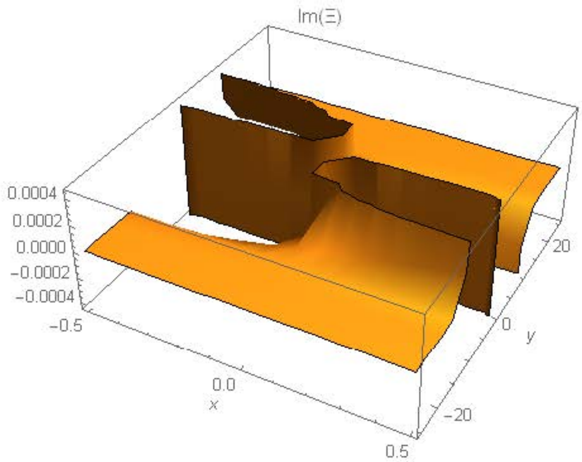


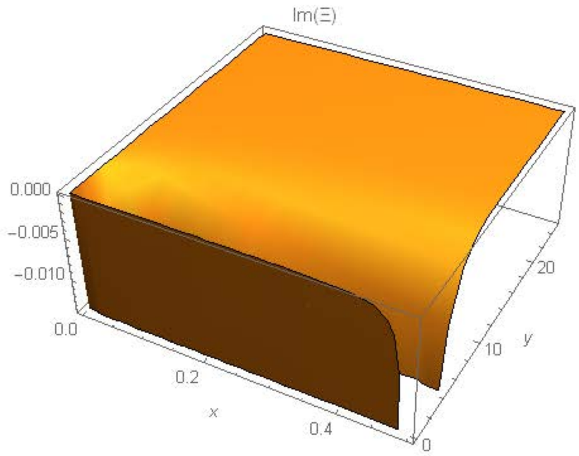


(6) $v(0,y)=0$, $v(x,0)=0$ regardless of the values of x,y .

It is because $\Xi(z)$ is an even function and has the complex conjugate property. (Corollary 7.2.2 (1))

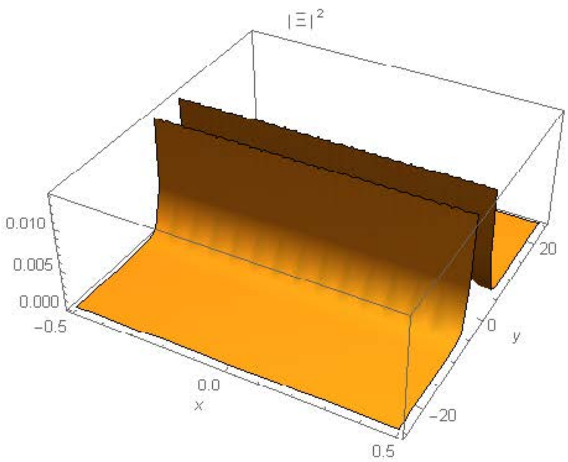
The right is the sectional view along $x=0$, $y=0$ of the left figure.

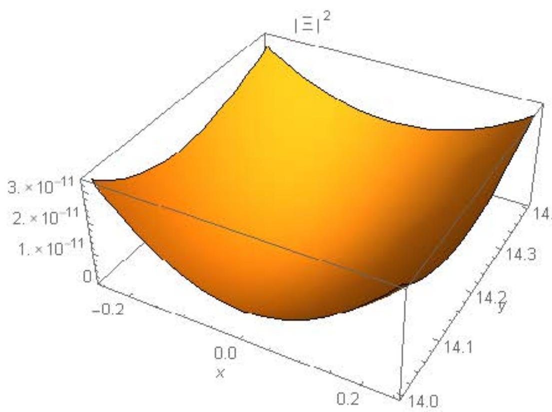




(7) $|\Xi(z)|^2$ is an even function about both x and y .

It is because $\Xi(z)$ is an even function. (Theorem 7.1.4) The complex conjugate property is not required. If this is drawn on 3D figure, it is as the left figure. And the minimum point as shown in the right figure seems to be dotted along $x=0$.





From the above **(2)** and **(4)** . we obtain the important following theorem about the zero of Riemann Zeta.

Theorem 7.4.1

If $1/2+x_0+iy_0$ is a zero of $\zeta(z)$, $1/2-x_0+iy_0$ is also zero of $\zeta(z)$.

Proof

It is synonymous that $1/2+x_0+iy_0$ is a zero of $\zeta(z)$ and that x_0+iy_0 is a zero of $\zeta(1/2+z)$. Then, x_0+iy_0 has to be zero of $\Xi(z)$ from (1.1) .

Here, let $\Xi(z)=u(x,y)+i\,v(x,y)$.

From the above **(2)** , $u(x,y)$ is an even function with respect to x . So, if x_0+iy_0 is a zero of $u(x,y)$, $-x_0+iy_0$ is also zero of $u(x,y)$

From the above **(4)** , $v(x,y)$ is an odd function with respect to x . So, if x_0+iy_0 is a zero of $v(x,y)$, $-x_0+iy_0$ is also zero of $v(x,y)$

Therefore, if x_0+iy_0 is a zero of $\Xi(z)$, $-x_0+iy_0$ is also zero of $\Xi(z)$. And it have to be a zero of $\zeta(1/2+z)$. It is because both the power function of π and the gamma function contained in $\Xi(z)$ do not have zeros. This is synonymous with that $1/2-x_0+iy_0$ is zero of $\zeta(z)$.

Note

This is proved by **Riemann** in 1859. And he expected that it would always be $x_0=0$ regardless of the value of y_0 .

7.4.2 Meanings of Completed Riemann Zeta

The above **(2)** and **(4)** were fruition in Theorem 7.4.1 . The conclusion that the real part of zeros of Riemann zeta function is line symmetry with respect to $x=1/2$ is a big result.

(6) is also the result should be noted. However, the imaginary part of $\Xi(z)$ and the imaginary part of $\zeta(z)$ do not correspond. That is, this can not be directly related to the analysis of $\zeta(z)$.

The diversity of the illustrated curve in **(3)** and **(5)** must let you foresee with the difficulty of obtaining the zeros of the real part and the imaginary part of $\Xi(z)$ by simultaneous equations.

(7) may be promising. $|\Xi(z)|^2$ is represented by real valued function with two variables x,y and is an even function about both x and y . In addition, the shape appears to be much simpler than $\Xi(z)$.

2015.02.26

2017.10.03 Added Theorem 7.1.5, Theorem 7.1.6

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Alien's Mathematics