19 Composition of Trigonometric Functions

The formula of the linear combination of the cosine function and the sine function is called the composition formula of trigonometric functions. In Section 1, various formulas are derived using these special values. In Section 2, sums of polynomials of cosine and sine functions are derived. And in Section 3, sums of series of cosine and sine functions are derived.

19.1 Basic Formula and its Application

19.1.1 Basic Formula

Formula 19.1.1

(1) Cosine representation

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta-\Phi) \tag{1.1c}$$

(2) Sine representation

$$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\Phi) \tag{1.1s}$$

Where, $\cos\Phi = \dfrac{a}{\sqrt{a^2+b^2}} \quad , \quad \sin\Phi = \dfrac{b}{\sqrt{a^2+b^2}}$ (1.ab)

Proof

From trigonometric addition formulas,

$$\cos(\theta-\Phi) = \cos\Phi\cos\theta + \sin\Phi\sin\theta$$

Here, let

$$\cos\Phi = \frac{a}{\sqrt{a^2+b^2}} \quad , \quad \sin\Phi = \frac{b}{\sqrt{a^2+b^2}} \tag{1.ab}$$

Substituting these for the above,

$$\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta = \cos(\theta-\Phi)$$

From this, (1.1c) is obtained.

Next,

$$\cos\Phi\sin\theta + \sin\Phi\cos\theta = \sin(\theta+\Phi)$$

Substituting (1.ab) for this,

$$\frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \sin(\theta+\Phi)$$

From this, (1.1s) is obtained. Q.E.D.

Inverse tangent function with two variables

In Formula 19.1.1 , Φ is obtained from the proviso two equations. Recently, there is a function that can obtain this with one expression. It is sometimes described as $arctan2(a,b)$, but in ***Mathematica*** it is described as $ArcTan[a,b]$.

This function gives the inverse tangent of b/a in consideration of which quadrant the point (a,b) is in. Specifically, it is as follows.

$$ArcTan[a,b] = \begin{cases} \tan^{-1}\dfrac{b}{a} & if\ a>0 \\ \tan^{-1}\dfrac{b}{a}+\pi & if\ a<0\ and\ b\ge 0 \\ \tan^{-1}\dfrac{b}{a}-\pi & if\ a<0\ and\ b<0 \\ +\dfrac{\pi}{2} & if\ a=0\ and\ b>0 \\ -\dfrac{\pi}{2} & if\ a=0\ and\ b<0 \\ undefined & if\ a=0\ and\ b=0 \end{cases}$$

If Formula 19.1.1 is rewritten using this function, it becomes as follows.

Formula 19.1.1'

(1) Cosine representation

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta-\Phi) \qquad (1.1'c)$$

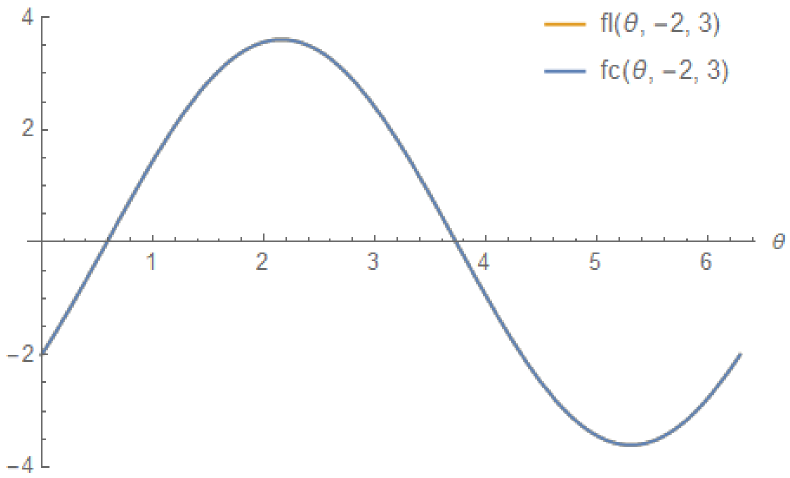
(2) Sine representation

$$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\Phi) \qquad (1.1's)$$

Where, $\Phi = ArcTan[a,b]$

Example : Cosine representation

If the left side of (1.1'c) is fl and the right side is fr and these are drawn in a 2D figure, it is as follows. Orange is the left side and blue is the right side, but both sides are exactly overlapped and orange (left side) cannot be seen.



In Formula 19.1.1 , if $a=1$, $b=\pm 1$ are given, the followings are obtained.

Special Values

$$\cos\theta \pm \sin\theta = \sqrt{2}\cos\left(\theta \mp \frac{\pi}{4}\right) \qquad (1.0c)$$

$$\sin\theta \pm \cos\theta = \sqrt{2}\sin\left(\theta \pm \frac{\pi}{4}\right) \qquad (1.0s)$$

19.1.2 Formulas that makes the sum of cosine and sine a product

Using these special values, we obtain formulas that makes the sum of cosine and sine a product.

Formula 19.1.2

(1) Cosine representation

$$\cos(A+B) + \sin(A-B) = 2\cos\left(A-\frac{\pi}{4}\right)\cos\left(B+\frac{\pi}{4}\right) \qquad (c+s.c)$$

$$\cos(A+B) - \sin(A-B) = 2\cos\left(A+\frac{\pi}{4}\right)\cos\left(B-\frac{\pi}{4}\right) \qquad (c-s.c)$$

$$\sin(A+B) + \cos(A-B) = 2\cos\left(A-\frac{\pi}{4}\right)\cos\left(B-\frac{\pi}{4}\right) \qquad (s+c.c)$$

$$\sin(A+B) - \cos(A-B) = -2\cos\left(A+\frac{\pi}{4}\right)\cos\left(B+\frac{\pi}{4}\right) \qquad (s-c.c)$$

(2) Sine representation

$$\cos(A+B) + \sin(A-B) = -2\sin\left(A+\frac{\pi}{4}\right)\sin\left(B-\frac{\pi}{4}\right) \qquad (c+s.s)$$

$$\cos(A+B) - \sin(A-B) = -2\sin\left(A-\frac{\pi}{4}\right)\sin\left(B+\frac{\pi}{4}\right) \qquad (c-s.s)$$

$$\sin(A+B) + \cos(A-B) = 2\sin\left(A+\frac{\pi}{4}\right)\sin\left(B+\frac{\pi}{4}\right) \qquad s+c.s)$$

$$\sin(A+B) - \cos(A-B) = -2\sin\left(A-\frac{\pi}{4}\right)\sin\left(B-\frac{\pi}{4}\right) \qquad (s-c.s)$$

Proof

$$\begin{aligned}\cos(A+B) + \sin(A-B) &= \cos A\cos B - \sin A\sin B + \sin A\cos B - \cos A\sin B \\ &= \cos A(\cos B - \sin B) + \sin A(\cos B - \sin B)\end{aligned}$$

i.e.

$$\cos(A+B) + \sin(A-B) = (\cos A + \sin A)(\cos B - \sin B) \qquad (1.2c)$$

Applying (1.0c) to each term on the right side ,

$$\cos(A+B) + \sin(A-B) = 2\cos\left(A-\frac{\pi}{4}\right)\cos\left(B+\frac{\pi}{4}\right) \qquad (c+s.c)$$

Repracing B with $-B$,

$$\cos(A-B) + \sin(A+B) = 2\cos\left(A-\frac{\pi}{4}\right)\cos\left(-B+\frac{\pi}{4}\right)$$

i.e.

$$\sin(A+B) + \cos(A-B) = 2\cos\left(A-\frac{\pi}{4}\right)\cos\left(B-\frac{\pi}{4}\right) \qquad (s+c.c.)$$

Applying (1.0s) to (1.2c) ,

$$\cos(A+B) + \sin(A-B) = -2\sin\left(A+\frac{\pi}{4}\right)\sin\left(B-\frac{\pi}{4}\right) \qquad c+s.s)$$

Repracing B with $-B$,

$$\cos(A-B)+\sin(A+B) = -2\sin\left(A+\frac{\pi}{4}\right)\sin\left(-B-\frac{\pi}{4}\right)$$

i.e.

$$\sin(A+B)+\cos(A-B) = 2\sin\left(A+\frac{\pi}{4}\right)\sin\left(B+\frac{\pi}{4}\right) \quad \text{(s+c.s)}$$

Next,

$$\begin{aligned}\cos(A+B)-\sin(A-B) &= \cos A\cos B - \sin A\sin B - \sin A\cos B + \cos A\sin B \\ &= \cos A(\cos B+\sin B) - \sin A(\cos B+\sin B)\end{aligned}$$

i.e.

$$\cos(A+B)-\sin(A-B) = (\cos A-\sin A)(\cos B+\sin B) \quad \text{(1.2s)}$$

Applying (1.0c) to each term on the right side ,

$$\cos(A+B)-\sin(A-B) = 2\cos\left(A+\frac{\pi}{4}\right)\cos\left(B-\frac{\pi}{4}\right) \quad \text{(c-s.c)}$$

Repracing B with $-B$,

$$\sin(A+B)-\cos(A-B) = -2\cos\left(A+\frac{\pi}{4}\right)\cos\left(B+\frac{\pi}{4}\right) \quad \text{(s-c.c)}$$

Applying (1.0s) to (1.2s) ,

$$\cos(A+B)-\sin(A-B) = -2\sin\left(A-\frac{\pi}{4}\right)\sin\left(B+\frac{\pi}{4}\right) \quad \text{(c-s.s)}$$

Repracing B with $-B$,

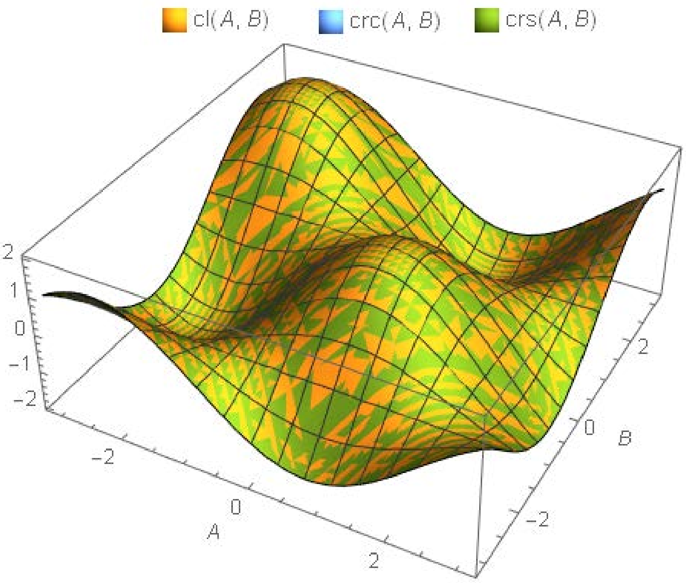
$$\sin(A+B)-\cos(A-B) = -2\sin\left(A-\frac{\pi}{4}\right)\sin\left(B-\frac{\pi}{4}\right) \quad \text{(s-c.s)}$$

Q.E.D.

Example

$$\cos(A+B)+\sin(A-B) = 2\cos\left(A-\frac{\pi}{4}\right)\cos\left(B+\frac{\pi}{4}\right) = -2\sin\left(A+\frac{\pi}{4}\right)\sin\left(B-\frac{\pi}{4}\right)$$

If these 3D figure are drawn together, it is as follows. The left side is cl, the middle is crc, and the right side is crs. The three are exactly overlapped and look like spots.



Note

It is easy once we know it. For example,

$$2\cos A\cos B=\cos(A+B)+\cos(A-B)$$

Replacing $A\,,B$ with $A-\pi/4\,,B-\pi/4$ respectively,

$$2\cos\left(A-\frac{\pi}{4}\right)\cos\left(B-\frac{\pi}{4}\right)=\cos\left(A+B-\frac{\pi}{2}\right)+\cos(A-B)=\sin(A+B)+\cos(A-B)$$

(s+c.c)

Why this is not seen in any formula collection, I wonder.

19.2 Recurrence Formula

In this section, by sequentially calculating the sum of the two terms of trigonometric functions, we calculate the sum of the n terms.

Formula 19.2.1

$$a_1 \cos\theta + a_2 \cos(\theta \pm \phi) = A_2 \cos(\theta \pm \Phi) \tag{2.1c}$$

$$a_1 \sin\theta + a_2 \sin(\theta \pm \phi) = A_2 \sin(\theta \pm \Phi) \tag{2.1s}$$

Where,

$$A_2 = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos\phi}$$

$$\Phi = ArcTan[a_1 + a_2 \cos\phi\ , a_2 \sin\phi]$$

Proof

From trigonometric addition formulas,

$$\cos(\theta - \phi) = \cos\phi \cos\theta + \sin\phi \sin\theta$$

Substituting this for the left side of (2.1c),

$$\begin{aligned} a_1 \cos\theta + a_2 \cos(\theta - \phi) &= a_1 \cos\theta + a_2 (\cos\phi \cos\theta + \sin\phi \sin\theta) \\ &= a_1 \cos\theta + a_2 \cos\phi \cos\theta + a_2 \sin\phi \sin\theta \\ &= (a_1 + a_2 \cos\phi)\cos\theta + (a_2 \sin\phi)\sin\theta \end{aligned}$$

Here, putting $a_1' = a_1 + a_2 \cos\phi$, $a_2' = a_2 \sin\phi$,

$$a_1 \cos\theta + a_2 \cos(\theta - \phi) = a_1' \cos\theta + a_2' \sin\theta$$

Applying Formula 19.1.1' (1.1'c) to this ,

$$a_1 \cos\theta + a_2 \cos(\theta - \phi) = a_1' \cos\theta + a_2' \sin\theta = A \cos(\theta - \Phi) \tag{2.1c-}$$

$$A = \sqrt{(a_1 + a_2 \cos\phi)^2 + (a_2 \sin\phi)^2} = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos\phi}$$

$$\Phi = ArcTan[a_1 + a_2 \cos\phi\ , a_2 \sin\phi]$$

Next, repracing ϕ with $-\phi$,

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos\phi}$$

$$-\Phi = ArcTan[a_1 + a_2 \cos\phi\ , a_2 \sin\phi]$$

Repracing Φ with $-\Phi$,

$$a_1 \cos\theta + a_2 \cos(\theta + \phi) = A \cos(\theta + \Phi) \tag{2.1c+}$$

By combining (2.1c-) and (2.1c+). (2.1c) is obtained.

From trigonometric addition formulas,

$$\sin(\theta + \phi) = \cos\phi \sin\theta + \sin\phi \cos\theta$$

Substituting this for the left side of (2.1s),

$$\begin{aligned} a_1 \sin\theta + a_2 \sin(\theta + \phi) &= a_1 \sin\theta + a_2 (\cos\phi \sin\theta + \sin\phi \cos\theta) \\ &= a_1 \sin\theta + a_2 \cos\phi \sin\theta + a_2 \sin\phi \cos\theta \\ &= (a_1 + a_2 \cos\phi)\sin\theta + (a_2 \sin\phi)\cos\theta \end{aligned}$$

Here, putting $a_1' = a_1 + a_2 \cos\phi$, $a_2' = a_2 \sin\phi$,

$$a_1 \sin\theta + a_2 \sin(\theta + \phi) = a_1' \sin\theta + a_2' \cos\theta$$

Applying Formula 19.1.1' (1.1's) to this ,

$$a_1 \sin\theta + a_2 \sin(\theta+\phi) = a_1' \sin\theta + a_2' \cos\theta = A \sin(\theta+\Phi) \tag{2.1s+}$$

$$A = \sqrt{(a_1 + a_2 \cos\phi)^2 + (a_2 \sin\phi)^2} = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos\phi}$$

$$\Phi = ArcTan[a_1 + a_2 \cos\phi \,,\, a_2 \sin\phi]$$

Repracing ϕ with $-\phi$ in (2.1s+) ,

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(-\phi)} = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos\phi}$$

$$-\Phi = ArcTan[a_1 + a_2 \cos\phi \,,\, a_2 \sin\phi]$$

Repracing Φ with $-\Phi$,

$$a_1 \sin\theta + a_2 \sin(\theta-\phi) = A \sin(\theta-\Phi) \tag{2.1s-}$$

By combining (2.1s+) and (2.1s-). (2.1s) is obtained. Q.E.D.

Formula 19.2.2

$$a_1 \cos(\theta+\phi_1) + a_2 \cos(\theta+\phi_2) = A_2 \cos\{(\theta+\phi_2)+\Phi_2\}$$

$$a_1 \sin(\theta+\phi_1) + a_2 \sin(\theta+\phi_2) = A_2 \sin\{(\theta+\phi_2)+\Phi_2\}$$

Where,

$$A_2 = \sqrt{a_2^2 + a_1^2 + 2a_2 a_1 \cos(\phi_1-\phi_2)}$$

$$\Phi_2 = ArcTan[a_2 + a_1 \cos(\phi_1-\phi_2) \,,\, a_1 \sin(\phi_1-\phi_2)]$$

Proof

$$a_1 \cos(\theta+\phi_1) + a_2 \cos(\theta+\phi_2) = a_2 \cos(\theta+\phi_2) + a_1 \cos\{(\theta+\phi_2)+(\phi_1-\phi_2)\}$$

$$a_1 \sin(\theta+\phi_1) + a_2 \sin(\theta+\phi_2) = a_2 \sin(\theta+\phi_2) + a_1 \sin\{(\theta+\phi_2)+(\phi_1-\phi_2)\}$$

Applying Formula 19.2.1 ,

$$a_2 \cos(\theta+\phi_2) + a_1 \cos\{(\theta+\phi_2)+(\phi_1-\phi_2)\} = A_2 \cos\{(\theta+\phi_2)+\Phi_2\}$$

$$a_2 \sin(\theta+\phi_2) + a_1 \sin\{(\theta+\phi_2)+(\phi_1-\phi_2)\} = A_2 \sin\{(\theta+\phi_2)+\Phi_2\}$$

$$A_2 = \sqrt{a_2^2 + a_1^2 + 2a_2 a_1 \cos(\phi_1-\phi_2)}$$

$$\Phi_2 = ArcTan[a_2 + a_1 \cos(\phi_1-\phi_2) \,,\, a_1 \sin(\phi_1-\phi_2)]$$

Example1: $\theta = 0$, $a_r(x) = (-1)^{r-1} r^{-x}$, $\phi_r(y) = y \log r$ $(r = 1, 2)$

$$1^{-x} \cos(y \log 1) - 2^{-x} \cos(y \log 2) = A_2(x,y) \cos(y \log 2 + \Phi_2(x,y)) \tag{2c}$$

$$1^{-x} \sin(y \log 1) - 2^{-x} \sin(y \log 2) = A_2(x,y) \sin(y \log 2 + \Phi_2(x,y)) \tag{2s}$$

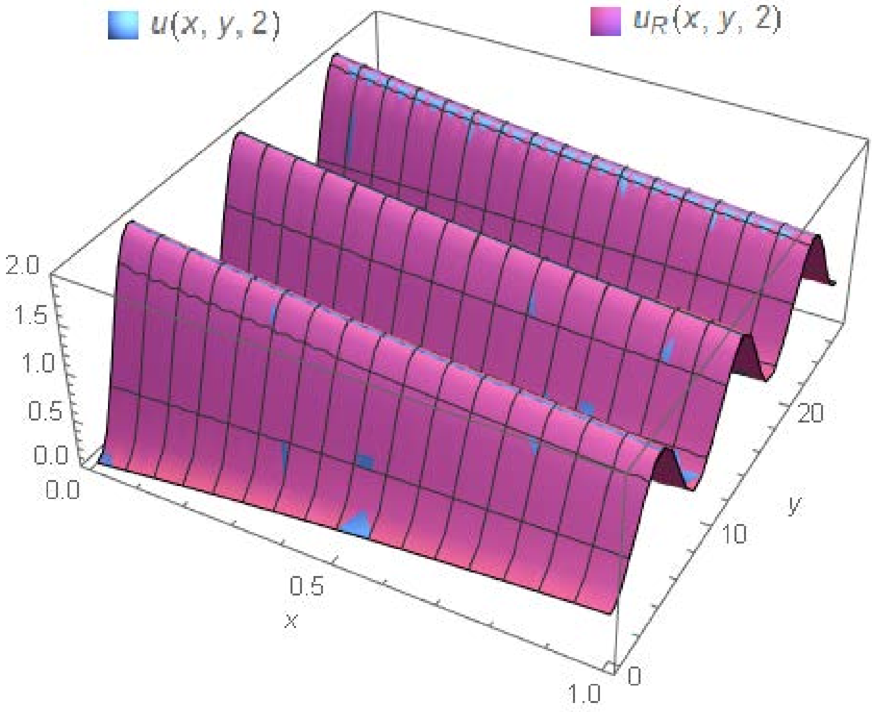
Where,

$$A_2(x,y) = \sqrt{2^{-2x} + 1^{-2x} - 2 \cdot 2^{-x} 1^{-x} \cos(y \log 1 - y \log 2)}$$

$$\Phi_2(x,y) = ArcTan[-2^{-x} + 1^{-x} \cos(y \log 1 - y \log 2) \,,\, 1^{-x} \sin(y \log 1 - y \log 2)]$$

3D view of both sides of (2c) is as follows. Blue is the left side $u(x,y)$ and magenta is the right side $u_R(x,y)$.

The two overlap exactly. So the former is almost invisible.



Formula 19.2.3

$$a_1 \cos(\theta+\phi_1)+a_2 \cos(\theta+\phi_2)+a_3 \cos(\theta+\phi_3) = A_3 \cos\{(\theta+\phi_3)+\Phi_3\}$$

$$a_1 \sin(\theta+\phi_1)+a_2 \sin(\theta+\phi_2)+a_3 \sin(\theta+\phi_3) = A_3 \sin\{(\theta+\phi_3)+\Phi_3\}$$

Where,

$$A_1 = a_1$$

$$\Phi_1 = 0$$

$$A_3 = \sqrt{a_3^2+A_2^2+2a_3A_2 \cos(\phi_2-\phi_3+\Phi_2)}$$

$$\Phi_3 = ArcTan[a_3+A_2 \cos(\phi_2-\phi_3+\Phi_2) , A_2 \sin(\phi_2-\phi_3+\Phi_2)]$$

Proof

$$a_1 \cos(\theta+\phi_1) + a_2 \cos(\theta+\phi_2) + a_3 \cos(\theta+\phi_3)$$

$$= A_2 \cos\{(\theta+\phi_2)+\Phi_2\} + a_3 \cos(\theta+\phi_3)$$

$$= a_3 \cos(\theta+\phi_3) + A_2 \cos\{(\theta+\phi_3)+(\phi_2-\phi_3+\Phi_2)\}$$

$$a_1 \sin(\theta+\phi_1) + a_2 \sin(\theta+\phi_2) + a_3 \sin(\theta+\phi_3)$$

$$= a_3 \sin(\theta+\phi_3) + A_2 \sin\{(\theta+\phi_3)+(\phi_2-\phi_3+\Phi_2)\}$$

Applying Formula 19.2.1 ,

$$a_3 \cos(\theta+\phi_3) + A_2 \cos\{(\theta+\phi_3)+(\phi_2-\phi_3+\Phi_2)\} = A_3 \cos\{(\theta+\phi_3)+\Phi_3\}$$

$$a_3 \sin(\theta+\phi_3) + A_2 \sin\{(\theta+\phi_3)+(\phi_2-\phi_3+\Phi_2)\} = A_3 \sin\{(\theta+\phi_3)+\Phi_3\}$$

$$A_3 = \sqrt{a_3^2+A_2^2+2a_3A_2 \cos(\phi_2-\phi_3+\Phi_2)}$$

$$\Phi_3 = ArcTan[a_3+A_2 \cos(\phi_2-\phi_3+\Phi_2) , A_2 \sin(\phi_2-\phi_3+\Phi_2)]$$

Example2: $\theta = 0$, $a_r(x) = (-1)^{r-1} r^{-x}$, $\phi_r(y) = y \log r$ ($r = 1,2,3$)

$$1^{-x}\cos(y\log 1) - 2^{-x}\cos(y\log 2) + 3^{-x}\cos(y\log 3) = A_3(x,y)\cos(y\log 3 + \Phi_3(x,y)) \quad \text{(3c)}$$

$$1^{-x}\sin(y\log 1) - 2^{-x}\sin(y\log 2) + 3^{-x}\sin(y\log 3) = A_3(x,y)\sin(y\log 3 + \Phi_3(x,y)) \quad \text{(3s)}$$

Where,

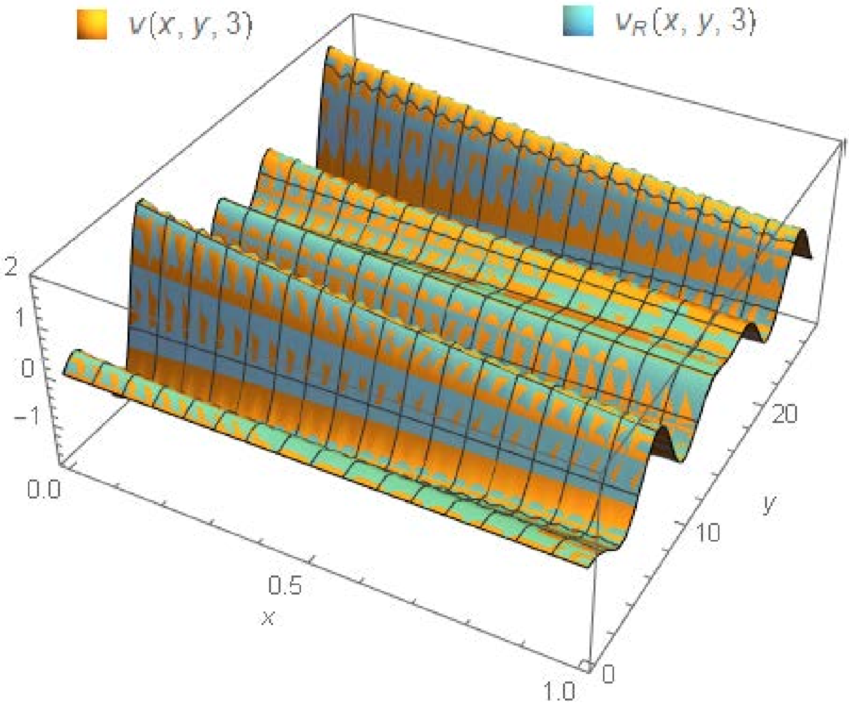
$$A_1(x,y) = a_1(x) \quad , \quad \Phi_1(x,y) = 0$$

$$A_3(x,y) = \sqrt{3^{-2x} + A_2(x,y)^2 + 2\cdot 3^{-x} A_2(x,y)\cos\{y\log 2 - y\log 3 + \Phi_2(x,y)\}}$$

$$\Phi_3(x,y) = ArcTan[3^{-x} + A_2(x,y)\cos\{y\log 2 - y\log 3 + \Phi_2(x,y)\} , A_2(x,y)\sin\{y\log 2 - y\log 3 + \Phi_2(x,y)\}]$$

3D view of both sides of (3s) is as follows. Orange is the left side $v(x,y)$ and cyan is the right side $v_R(x,y)$.

The two overlap exactly and look like spots.



Thus, induction gives the following general formula.

Formula 19.2.n

$$\sum_{r=1}^{n} a_r \cos(\theta + \phi_r) = A_n \cos\{(\theta + \phi_n) + \Phi_n\}$$

$$\sum_{r=1}^{n} a_r \sin(\theta + \phi_r) = A_n \sin\{(\theta + \phi_n) + \Phi_n\}$$

Where,

$$A_1 = a_1$$

$$\Phi_1 = 0$$

$$A_n = \sqrt{a_n^2 + A_{n-1}^2 + 2a_n A_{n-1}\cos(\phi_{n-1} - \phi_n + \Phi_{n-1})}$$

$$\Phi_n = ArcTan[a_n + A_{n-1}\cos(\phi_{n-1} - \phi_n + \Phi_{n-1}) , A_{n-1}\sin(\phi_{n-1} - \phi_n + \Phi_{n-1})]$$

Example3: $\theta=0$, $a_r(x)=(-1)^{r-1}r^{-x}$, $\phi_r(y)=y\log r$ $(r=1,2,\cdots,6)$

$$\sum_{r=1}^{6}(-1)^{r-1}r^{-x}\cos(y\log r)=A_6(x,y)\cos(y\log 6+\Phi_6(x,y)) \tag{6c}$$

$$\sum_{r=1}^{6}(-1)^{r-1}r^{-x}\sin(y\log r)=A_6(x,y)\sin(y\log 6+\Phi_6(x,y)) \tag{6s}$$

I show the source code and the result of formula manipulation software ***Mathematica*** for drawing 3D figures of both sides of (6c). The left and right overlap exactly and look like spots.

a_r & ϕ_r

```
a_r_[x_] := (-1)^(r-1) r^-x        φ_r_[y_] := y Log[r]
```

***u & v* (Left hand side)**

```
u[x_, y_, n_] := Σ_{r=1}^{n} a_r[x] Cos[φ_r[y]]        v[x_, y_, n_] := Σ_{r=1}^{n} a_r[x] Sin[φ_r[y]]
```

***A_n & Φ_n* (Recurrence formula)**

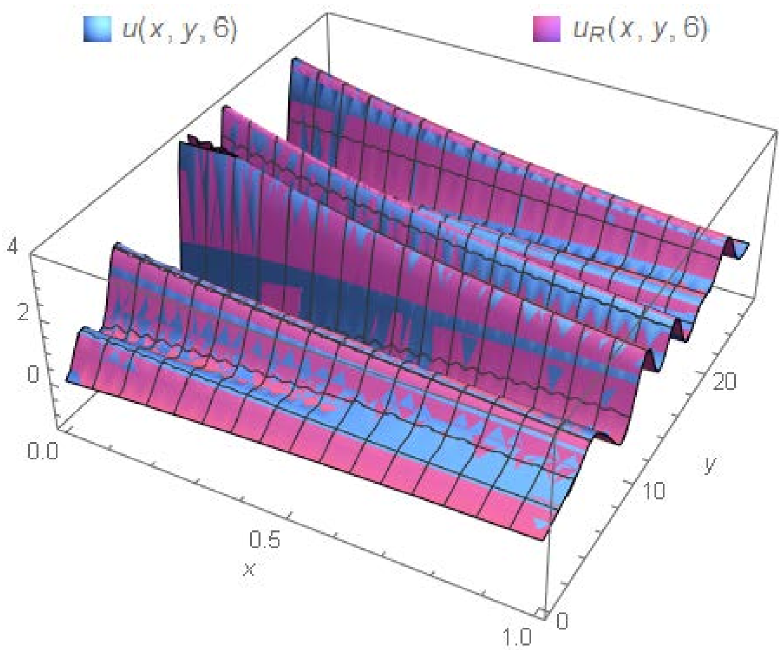
```
A_n_[x_, y_] :=
 If[n == 1, a_1[x], √(a_n[x]^2 + A_(n-1)[x, y]^2 + 2 a_n[x] A_(n-1)[x, y] Cos[φ_(n-1)[y] - φ_n[y] + Φ_(n-1)[x, y]])]

Φ_n_[x_, y_] := If[n == 1, 0, ArcTan[a_n[x] + A_(n-1)[x, y] Cos[φ_(n-1)[y] - φ_n[y] + Φ_(n-1)[x, y]],
                                     A_(n-1)[x, y] Sin[φ_(n-1)[y] - φ_n[y] + Φ_(n-1)[x, y]]]]
```

***u_R & v_R* (Right hand side)**

```
u_R[x_, y_, n_] := A_n[x, y] Cos[φ_n[y] + Φ_n[x, y]]
v_R[x_, y_, n_] := A_n[x, y] Sin[φ_n[y] + Φ_n[x, y]]
```

n = 6 (drawing)

```
dummy[x_, y_] := -10
Plot3D[{dummy[x, y], u[x, y, 6], u_R[x, y, 6]}, {x, 0, 1}, {y, 0, 27.3},
 AxesLabel → Automatic, PlotLegends → "Expressions", ClippingStyle → None,
 PlotStyle → {, , ColorData[96, 4]}, PlotRange → {-1.5, 4}]
```



Advantages and disadvantages of Formula 19.2.n

Formula 19.2.n is expressed as a recurrence formula and is suitable for computers. In fact, the figures of Examples 1 and 2 were drawn by putting $n=2$ and $n=3$ in the source code of Example 3.

However, for 3 or more terms, the explicit expression (in terms of $a_r\ \phi_r$ only) of A_n , Φ_n becomes very complicated. For example,

```
Φ3[x, y]
ArcTan[3^-x + Sqrt[1 + 2^(-2 x) - 2^(1-x) Cos[y Log[2]]] ×
    Cos[ArcTan[-2^-x + Cos[y Log[2]], -Sin[y Log[2]]] + y Log[2] - y Log[3]],
        Sqrt[1 + 2^(-2 x) - 2^(1-x) Cos[y Log[2]]] ×
    Sin[ArcTan[-2^-x + Cos[y Log[2]], -Sin[y Log[2]]] + y Log[2] - y Log[3]]]
```

For 4 or more terms, explicit representations of A_n , Φ_n are hopeless.

19.3 Explicit Formula

In this section, we seek a simpler representation of trigonometric polynomials, and extend it to seek the formula of the trigonometric series.

19.3.1 Explicit sum of two terms

Formula 19.3.1

$$a_1 \cos(\theta+\phi_1) + a_2 \cos(\theta+\phi_2) = A \cos(\theta+\Phi) \tag{3.1c}$$

$$a_1 \sin(\theta+\phi_1) + a_2 \sin(\theta+\phi_2) = A \sin(\theta+\Phi) \tag{3.1s}$$

Where,

$$A = \sqrt{(a_1 \cos\phi_1 + a_2 \cos\phi_2)^2 + (a_1 \sin\phi_1 + a_2 \sin\phi_2)^2}$$

$$\Phi = ArcTan[a_1 \cos\phi_1 + a_2 \cos\phi_2 \ , a_1 \sin\phi_1 + a_2 \sin\phi_2]$$

Proof

Expanding the left side of (3.1s) with trigonometric addition formulas,

$$\begin{aligned} a_1 \sin(\theta+\phi_1) + a_2 \sin(\theta+\phi_2) \\ &= a_1(\sin\theta \cos\phi_1 + \cos\theta \sin\phi_1) + a_2(\sin\theta \cos\phi_2 + \cos\theta \sin\phi_2) \\ &= (a_1 \cos\phi_1 + a_2 \cos\phi_2)\sin\theta + (a_1 \sin\phi_1 + a_2 \cos\phi_2)\cos\theta \end{aligned}$$

Here, let

$$a_1 \cos\phi_1 + a_2 \cos\phi_2 = \alpha \quad , \quad a_1 \sin\phi_1 + a_2 \cos\phi_2 = \beta$$

Then, (3.1s) becomes

$$\alpha \sin\theta + \beta \cos\theta = A \sin(\theta+\Phi)$$

Applying Formula 19.1.1' (1.1's) to this ,

$$A = \sqrt{\alpha^2 + \beta^2} \quad , \quad \Phi = ArcTan[\alpha, \beta]$$

Returning α, β to the original symbols,

$$A = \sqrt{(a_1 \cos\phi_1 + a_2 \cos\phi_2)^2 + (a_1 \sin\phi_1 + a_2 \sin\phi_2)^2}$$

$$\Phi = ArcTan[a_1 \cos\phi_1 + a_2 \cos\phi_2 \ , a_1 \sin\phi_1 + a_2 \sin\phi_2]$$

Next, repracing θ with $\theta+\pi/2$ in (3.1s) ,

$$a_1 \sin\left(\theta+\frac{\pi}{2}+\phi_1\right) + a_2 \sin\left(\theta+\frac{\pi}{2}+\phi_2\right) = A \sin\left(\theta+\frac{\pi}{2}+\Phi\right)$$

This is equal to (3.1c) . Q.E.D.

19.3.2 Trigonometric Polynomials

What should be noted in Formula 19.3.1 is the explicit form of the absolute value A and the argument Φ. By observing these, we can easily guess what the sum of the n terms should be. In fact, the following holds.

Formula 19.3.2

$$c(\theta) = \sum_{r=1}^{n} a_r \cos(\theta+\phi_r) = A \cos(\theta+\Phi) \tag{3.2c}$$

$$s(\theta) = \sum_{r=1}^{n} a_r \sin(\theta+\phi_r) = A \sin(\theta+\Phi) \tag{3.2s}$$

Where,

$$A = \sqrt{\left(\sum_{r=1}^{n} a_r \cos\phi_r\right)^2 + \left(\sum_{r=1}^{n} a_r \sin\phi_r\right)^2} \qquad \left(= \sqrt{c^2(0) + s^2(0)}\right)$$

$$\Phi = ArcTan\left[\sum_{r=1}^{n} a_r \cos\phi_r \,,\ \sum_{r=1}^{n} a_r \sin\phi_r\right] \qquad \left(= ArcTan[c(0), s(0)]\right)$$

Proof

Though proof of Formula 19.3.1 is also possible, here, we adopt a simpler way using the identity.
Let

$$c(\theta) = \sum_{r=1}^{n} a_r \cos(\theta + \phi_r) \quad , \quad s(\theta) = \sum_{r=1}^{n} a_r \sin(\theta + \phi_r)$$

$$f(\theta) = c(\theta) + i\, s(\theta)$$

From these,

$$f(\theta) = \sqrt{c^2(\theta) + s^2(\theta)} \left\{ \frac{c(\theta)}{\sqrt{c^2(\theta) + s^2(\theta)}} + i \frac{s(\theta)}{\sqrt{c^2(\theta) + s^2(\theta)}} \right\}$$

Suppose this is represented in polar form as follows.

$$f(\theta) = A\{\cos(\theta + \Phi) + i \sin(\theta + \Phi)\}$$

$$A = \sqrt{c^2(\theta) + s^2(\theta)}$$

$$\theta + \Phi = ArcTan[c(\theta) \,, s(\theta)]$$

Returning c, s to the original symbols,

$$A = \sqrt{\left\{\sum_{r=1}^{n} a_r \cos(\theta + \phi_r)\right\}^2 + \left\{\sum_{r=1}^{n} a_r \sin(\theta + \phi_r)\right\}^2} \tag{a}$$

$$\theta + \Phi = ArcTan\left[\sum_{r=1}^{n} a_r \cos(\theta + \phi_r) \,,\ \sum_{r=1}^{n} a_r \sin(\theta + \phi_r)\right] \tag{t}$$

Here,

$$\left\{\sum_{r=1}^{n} a_r \cos(\theta + \phi_r)\right\}^2 + \left\{\sum_{r=1}^{n} a_r \sin(\theta + \phi_r)\right\}^2$$

$$= \sum_{r=1}^{n} a_r^2 + 2\sum_{r \neq s} a_r a_s \{\cos(\theta + \phi_r)\cos(\theta + \phi_s) + \sin(\theta + \phi_r)\sin(\theta + \phi_s)\}$$

$$= \sum_{r=1}^{n} a_r^2 + 2\sum_{r \neq s} a_r a_s \cos\{(\theta + \phi_r) - (\theta + \phi_s)\}$$

$$= \sum_{r=1}^{n} a_r^2 + 2\sum_{r \neq s} a_r a_s \cos(\phi_r - \phi_s)$$

$$= \sum_{r=1}^{n} a_r^2 + 2\sum_{r \neq s} a_r a_s (\cos\phi_r \cos\phi_s + \sin\phi_r \sin\phi_s)$$

$$= \left(\sum_{r=1}^{n} a_r \cos\phi_r\right)^2 + \left(\sum_{r=1}^{n} a_r \sin\phi_r\right)^2$$

Substituting this for (a) ,

$$A = \sqrt{\left(\sum_{r=1}^{n} a_r \cos\phi_r\right)^2 + \left(\sum_{r=1}^{n} a_r \sin\phi_r\right)^2} \qquad \left(= \sqrt{c^2(0) + s^2(0)}\right)$$

Further, substituting $\theta = 0$ for (t), we obtain

$$\Phi = ArcTan\left[\sum_{r=1}^{n} a_r \cos\phi_r \ , \ \sum_{r=1}^{n} a_r \sin\phi_r \right] \qquad (= ArcTan[c(0), s(0)])$$

Example: $a_r(x) = (-1)^{r-1} r^{-x}$, $\phi_r(y) = y \log r$, $r = 1 \sim 100$

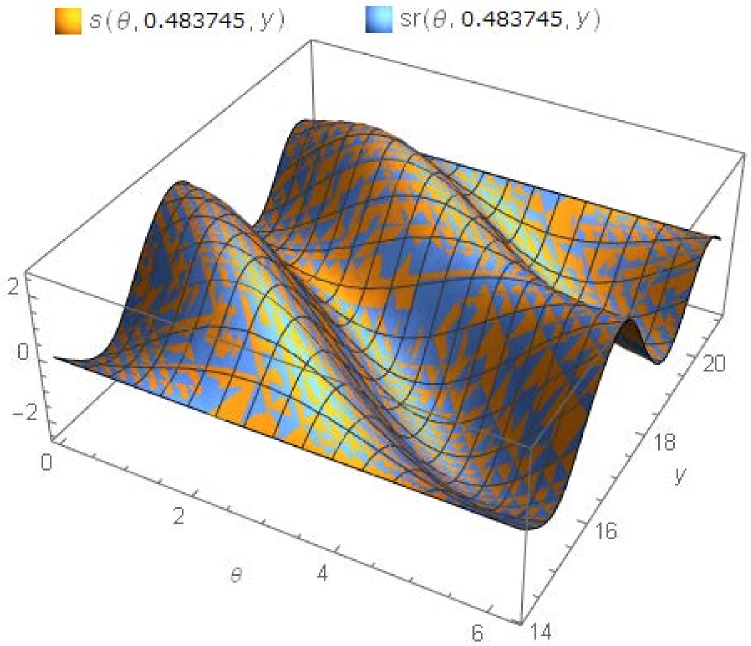
$$s(\theta, x, y) = \sum_{r=1}^{100} \frac{(-1)^{r-1}}{r^x} \sin(\theta + y \log r) = A(x,y) \sin\{\theta + \Phi(x,y)\} \tag{3.2sx}$$

Where,

$$A(x,y) = \sqrt{\left(\sum_{r=1}^{100} \frac{(-1)^{r-1}}{r^x} \cos(y \log r) \right)^2 + \left(\sum_{r=1}^{100} \frac{(-1)^{r-1}}{r^x} \sin(y \log r) \right)^2}$$

$$\Phi(x,y) = ArcTan\left[\sum_{r=1}^{100} \frac{(-1)^{r-1}}{r^x} \cos(y \log r) \ , \ \sum_{r=1}^{100} \frac{(-1)^{r-1}}{r^x} \sin(y \log r) \right]$$

When $x = 0.483745\cdots$, both sides of (3.2sx) are drawn in the 3D figure as follows. Orange is the left side and blue is the right side. The mottled pattern indicates that both sides overlap.



The starting line of y is $y = 14.111606\cdots$, but both sides of (3.2sx) are 0 on this line regardless of the value of θ. This is because $x = 0.483745\cdots$, $y = 14.111606\cdots$ is the zero of A . Naturally, Φ is undefined on this line. When $x = 0.48$, $y = 14.11$, $A = 0.00734\cdots$, $\Phi = 2.87029\cdots$.

19.3.3 Trigonometric Series

What should be noted in Formula 19.3.2 is the way of the proof. By observing this, it can be seen that there is no problem even if the upper limit of Σ is ∞ . Thus, the following holds.

Formula 19.3.3

$$c(\theta) = \sum_{r=1}^{\infty} a_r \cos(\theta + \phi_r) = A \cos(\theta + \Phi) \tag{3.3c}$$

$$s(\theta) = \sum_{r=1}^{\infty} a_r \sin(\theta + \phi_r) = A \sin(\theta + \Phi) \tag{3.3s}$$

Where,

$$A = \sqrt{\left(\sum_{r=1}^{\infty} a_r \cos\phi_r\right)^2 + \left(\sum_{r=1}^{\infty} a_r \sin\phi_r\right)^2} \qquad \left(= \sqrt{c^2(0)+s^2(0)} \right)$$

$$\Phi = ArcTan\left[\sum_{r=1}^{\infty} a_r \cos\phi_r \;,\; \sum_{r=1}^{\infty} a_r \sin\phi_r\right] \qquad \left(= ArcTan[c(0), s(0)] \right)$$

Note

The inside of $\sqrt{\;}$ of the absolute value A is generally described as $\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} a_r a_s \cos(\phi_r - \phi_s)$.
However, double series is not adopted in this chapter because it takes time to calculate.

Example: $a_r(x) = (-1)^{r-1} r^{-x}$, $\phi_r(y) = y \log r$, $r = 1 \sim \infty$

$$c(\theta,x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^x} \cos(\theta + y \log r) = A(x,y)\cos\{\theta + \Phi(x,y)\} \quad \{= cr(\theta,x,y)\}$$

$$s(\theta,x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^x} \sin(\theta + y \log r) = A(x,y)\sin\{\theta + \Phi(x,y)\} \quad \{= sr(\theta,x,y)\}$$

Where,

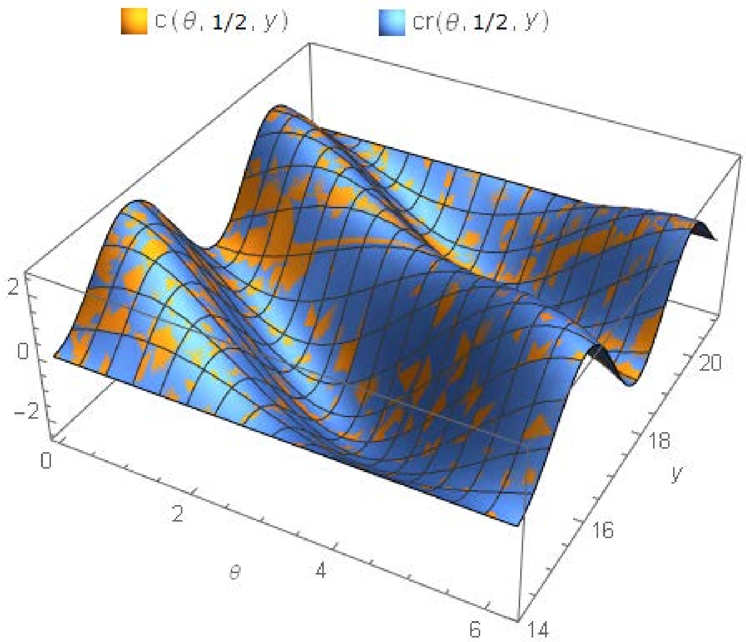
$$A(x,y) = \sqrt{\{c(0,x,y)\}^2 + \{s(0,x,y)\}^2}$$

$$\Phi(x,y) = ArcTan[c(0,x,y), s(0,x,y)]$$

$c(\theta,x,y)$, $cr(\theta,x,y)$ are shown, but it is difficult to draw an accurate figure because these series converges very slowly. So, the following Euler Transformation is applied to $c(\theta,x,y), s(\theta,x,y)$.

$$c(\theta,x,y) = \sum_{k=1}^{m} \sum_{r=1}^{k} \frac{1}{2^{k+1}} \binom{k}{r} \frac{(-1)^r}{r^x} \cos(\theta + y \log r)$$

When $x = 1/2$, $c(\theta,x,y)$, $cr(\theta,x,y)$ are drawn in the 3D figure as follows. Orange is the left side and blue is the right side. The mottled pattern indicates that both sides overlap.



The starting line of y is $y=14.132745\cdots$, but both sides are 0 on this line regardless of the value of θ
This is because $x=1/2$, $y=14.132745\cdots$ is the zero of A . Naturally, Φ is undefined on this line.

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