13 Convergence Acceleration of Multiple Series

Nowadays, with the advancement of functions to be used, opportunities to calculate double series and triple series have increased. So, in this section, we discuss these convergence acceleration methods.

13.1 Series Acceleration Method

If multiple series are converted to a single series in some way, an acceleration method can be applied. Since this is similar to a series circuit in an electric circuit, we call this ***series acceleration method***.

13.1.1 Half Multiple Series

According to Formula 2.1.0 in " **02 Multiple Series & Exponetial Function** " , The multiple series which is absolutely convergent can be converted to half multiple series. The half multiple series is a single series.

Formula 2.1.0

(0) When a multiple series $\displaystyle\sum_{r_1,r_2,\cdots,r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n}$ is absolutely convergent, the following expression holds.

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{r_1-r_2,r_2-r_3,\cdots,r_{n-1}-r_n,r_n} \tag{0.0}$$

(1) When a multiple series $\displaystyle\sum_{r_1,r_2,\cdots,r_n=1}^{\infty} a_{r_1,r_2,\cdots,r_n}$ is absolutely convergent, the following expression holds.

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\cdots\sum_{r_n=1}^{\infty} a_{r_1,r_2,\cdots,r_n} = \sum_{r_1=1}^{\infty}\sum_{r_2=1}^{r_1}\cdots\sum_{r_n=1}^{r_{n-1}} a_{1+r_1-r_2,1+r_2-r_3,\cdots,1+r_{n-1}-r_n,r_n} \tag{0.1}$$

Note

In short, we should just perform the following operations to $\displaystyle\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_{n-1},r_n}$.

Replace r_{n-1} with $r_{n-1}-r_n$, and replace the 1st ∞ with r_{n-1} from the right.

Replace r_{n-2} with $r_{n-2}-r_{n-1}$, and replace the 2nd ∞ with r_{n-2} from the right.

⋮

Replace r_1 with r_1-r_2 , and replace the $(n-1)$th ∞ with r_1 from the right.

When the subscripts starts from 1 (i.e. (1)) , $+1$ is only added.

Example 1.0

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-1)^{r+s+t}x^{2r+2s+2t+3}}{(2r+1)(2s+1)(2t+1)} = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^{r}x^{2r+3}}{(2r-2s+1)(2s-2t+1)(2t+1)}$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{r_3=1}^{\infty}\sum_{r_4=1}^{\infty}\frac{(-1)^{r_1+r_2+r_3+r_4}}{r_1 r_2 r_3 r_4} = \sum_{r_1=1}^{\infty}\sum_{r_2=1}^{r_1}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3}\frac{(-1)^{1+r_1}}{(1+r_1-r_2)(1+r_2-r_3)(1+r_3-r_4)r_4}$$

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{(-1)^{1+r}}{\{(1+r-s)s\}^x}\cos\left(y\log\frac{s}{1+r-s}\right)$$

13.1.2 Acceleration of Half Multiple Series

Thus, since the multiple series is converted to half multiple series (single series), acceleration method can be applied. Although there are various acceleration methods, in this chapter, we use **Knopp Transformation** described in " **10 Convergence Acceleration & Summation Method** ".

Theorem 13.1.2

(0) When a multiple series of functions $\sum_{r_1,r_2,\cdots,r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n}(z)$ is absolutely convergent in the domain D ,

the following expression holds for arbitrary positive number q .

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n}(z) = \sum_{k=0}^{\infty}\sum_{r_1=0}^{k}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} \frac{q^{k-r_1}}{(q+1)^{k+1}}\binom{k}{r_1} a_{r_1-r_2,r_2-r_3,\cdots,r_{n-1}-r_n,r_n}(z) \tag{1.0}$$

(1) When a multiple series of functions $\sum_{r_1,r_2,\cdots,r_n=1}^{\infty} a_{r_1,r_2,\cdots,r_n}(z)$ is absolutely convergent in the domain D ,

the following expression holds for arbitrary positive number q .

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\cdots\sum_{r_n=1}^{\infty} a_{r_1,r_2,\cdots,r_n}(z) = \sum_{k=1}^{\infty}\sum_{r_1=1}^{k}\sum_{r_2=1}^{r_1}\cdots\sum_{r_n=1}^{r_{n-1}} \frac{q^{k-r_1}}{(q+1)^{k+1}}\binom{k}{r_1} \times a_{1+r_1-r_2,1+r_2-r_3,\cdots,1+r_{n-1}-r_n,r_n}(z) \tag{1.1}$$

Proof

By assumption, the multiple series of functions can be rearranged as follows.

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n}(z) = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{r_1-r_2,r_2-r_3,\cdots,r_{n-1}-r_n,r_n}(z)$$

Since the right side is a single series with respect to r_1 and is convergent, according to Formula 10.3.2 (1) in " **10 Convergence Acceleration & Summation Method** ", (1.0) holds. And (1.1) holds similarly.

Total number of calculated terms

In the theorem, when the upper limit of Σ is terminated with m , the total number of calculated terms $T_1(n,m)$ is given by the following equation.

(0) $$T_1(n,m) = \sum_{k=0}^{m}\sum_{r_1=0}^{k}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} 1 = \frac{1}{(n+1)!}\prod_{j=1}^{n+1}(j+m)$$

(1) $$T_1(n,m) = \sum_{k=1}^{m}\sum_{r_1=1}^{k}\sum_{r_2=1}^{r_1}\cdots\sum_{r_n=1}^{r_{n-1}} 1 = \frac{1}{(n+1)!}\prod_{j=0}^{n}(j+m)$$

Example 1.1

$$f(x) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{(-1)^{r+s+t} x^{2r+2s+2t+3}}{(2r+1)(2s+1)(2t+1)} = \left(\tan^{-1}x\right)^3 =: g(x)$$

We want to obtain the value at $x=1$ of this series with 6 significant digits. For the time being, when the upper limit of each Σ is calculated as 300 and shown together with the value of $g(1)$, it is as follows.

```
f[x_, m_] := Sum[Sum[Sum[((-1)^(r+s+t) x^(2 r+2 s+2 t+3))/((2 r + 1) (2 s + 1) (2 t + 1)), {t,0,m}], {s,0,m}], {r,0,m}]
N[f[1, 300], 8]
0.48601170
```

```
g[x_] := ArcTan[x]^3
N[g[1]]
0.48447307
```

Althogh the total number of calculated terms is $301^3 = 27,270,901$, only 2 significant digits on the left side are obtained. It is difficult to obtain the target accuracy with a personal computer.

So, we try acceleration of this series. First, we convert this to half triple series. Then, as seen in Example 1.0 it becomes as follows.

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-1)^{r+s+t}x^{2r+2s+2t+3}}{(2r+1)(2s+1)(2t+1)} = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^{r}x^{2r+3}}{(2r-2s+1)(2s-2t+1)(2t+1)}$$

Second, we apply Knopp Transformation mentioned in " **10 Convergence Acceleration & Summation Method** " to this right side. Then,

$$f(x,q) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{q^{k-r}}{(q+1)^{k+1}}\binom{k}{r}\frac{(-1)^{r}x^{2r+3}}{(2r-2s+1)(2s-2t+1)(2t+1)}$$

When this is calculated at $q=1/2$ and $m=12$, it is as follows. (m is the upper limit of Σ. Same as below.)

```
f[x_, q_, m_] := Sum[Sum[Sum[Sum[(q^(k-r))/((q + 1)^(k+1)) (Binomial[k, r] (-1)^r x^(2 r+3))/((2 r - 2 s + 1) (2 s - 2 t + 1) (2 t + 1)), {t,0,s}], {s,0,r}], {r,0,k}], {k,0,m}]
SetPrecision[f[1, 1/2, 12], 8]
0.48447319
```

```
N[g[1]]
0.48447307
```

The result has reached 6 significant figures of target precision. In addition, the total number of calculated terms T_1 is as follows.

$$\sum_{k=0}^{12}\sum_{r=0}^{k}\sum_{s=0}^{r}\sum_{t=0}^{s}1 \qquad 1820$$

Example 1.2

$$f = \sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{r_3=1}^{\infty}\sum_{r_4=1}^{\infty}\frac{(-1)^{r_1+r_2+r_3+r_4}}{r_1 r_2 r_3 r_4} = (\log 2)^4 =: g$$

We want to obtain the value of this series with 6 significant digits. For the time being, when the upper limit of each Σ is calculated as 100 and shown together with the value of g, it is as follows.

```
f[m_] := Sum[Sum[Sum[Sum[(-1)^(r1+r2+r3+r4)/(r1 r2 r3 r4), {r4,1,m}], {r3,1,m}], {r2,1,m}], {r1,1,m}]
N[f[100], 8]
0.22427891
```

```
g := Log[2]^4
N[g, 8]
0.23083510
```

Althogh the total number of calculated terms is $100^4 = 100{,}000{,}000$, only 1 significant digits on the left side is obtained. It is almost impossible to obtain the target accuracy with a personal computer.

So, we try acceleration of this series. First, we convert this to half quadruple series. Then, as seen in Example 1.0 , it becomes as follows.

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{r_3=1}^{\infty}\sum_{r_4=1}^{\infty}\frac{(-1)^{r_1+r_2+r_3+r_4}}{r_1 r_2 r_3 r_4} = \sum_{r_1=1}^{\infty}\sum_{r_2=1}^{r_1}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3}\frac{(-1)^{1+r_1}}{(1+r_1-r_2)(1+r_2-r_3)(1+r_3-r_4)r_4}$$

Second, applying Knopp Transformation to this right side,

$$f(q) = \sum_{k=1}^{\infty}\sum_{r_1=1}^{k}\sum_{r_2=1}^{r_1}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3}\frac{q^{k-r_1}}{(q+1)^{k+1}}\binom{k}{r_1}\frac{(-1)^{1+r_1}}{(1+r_1-r_2)(1+r_2-r_3)(1+r_3-r_4)r_4}$$

When this is calculated at $q=1/2$ and $m=16$, it is as follows.

$$\texttt{f[}q\_\texttt{, }m\_\texttt{] :=}\sum_{k=1}^{m}\sum_{r_1=1}^{k}\sum_{r_2=1}^{r_1}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3}\frac{q^{k-r_1}}{(q+1)^{k+1}}\frac{(-1)^{1+r_1}\,\texttt{Binomial}[k, r_1]}{(1+r_1-r_2)(1+r_2-r_3)(1+r_3-r_4)\,r_4}$$

$\texttt{N}\left[\texttt{f}\left[\frac{1}{2}, 16\right], 8\right]$

0.23083508

N[g, 8]

0.23083510

The result has reached 6 significant figures of target precision. The total number of calculated terms T_1 is

$$\sum_{k=1}^{16}\sum_{r_1=1}^{k}\sum_{r_2=1}^{r_1}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3} 1 \qquad 15\,504$$

Example 1.3

$$f(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) = |\eta(x,y)|^2 =: g(x,y)$$

Where, $$\eta(x,y) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{(2r-1)^{x+iy}}$$

That is, the right side is the square of the absolute value of the Dirichlet eta function. This function has common zeros with the Riemann zeta function. Now, we want to obtain the value at $x_1=1/2$, $y_1=14.13472514\cdots$ of this series with 6 significant digits. For the time being, when the upper limit of each Σ is calculated as 1000 and shown together with the value of $g(x_1, y_1)$, it is as follows.

$$\texttt{f[}x\_\texttt{, }y\_\texttt{, }m\_\texttt{] :=}\sum_{r=1}^{m}\sum_{s=1}^{m}\frac{(-1)^{r+s}}{(r\,s)^x}\,\texttt{Cos}\left[y\,\texttt{Log}\left[\frac{s}{r}\right]\right]$$

$\texttt{g[}x\_\texttt{, }y\_\texttt{] := Abs[DirichletEta[}x + i\,y\texttt{]]}^2$ $\quad y_{n\_}$:= Im[ZetaZero[n]]

$\texttt{SetPrecision}\left[\texttt{f}\left[\frac{1}{2}, y_1, 1000\right], 8\right]$

$0.\times 10^{-4}$

$\texttt{SetPrecision}\left[\texttt{g}\left[\frac{1}{2}, y_1\right], 8\right]$

$0.\times 10^{-10}$

The total number of calculated terms is $1000^2 = 1{,}000{,}000$. And 4 significant digits on the left side are obtained. Although it is possible to obtain the target accuracy with a personal computer, it takes a lot of times.

So, we try acceleration of this series. First, we convert this to half double series. Then, as seen in Example 1.0 , it becomes as follows.

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{(-1)^{1+r}}{\{(1+r-s)s\}^x}\cos\left(y\log\frac{s}{1+r-s}\right)$$

Second, applying Knopp Transformation to this right side,

$$f(x,y,q) = \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{s=1}^{r}\frac{q^{k-r}}{(q+1)^{k+1}}\binom{k}{r}\frac{(-1)^{1+r}}{\{(1+r-s)s\}^x}\cos\left(y\log\frac{s}{1+r-s}\right)$$

When this is calculated at $q=1/2$ and $m=24$, it is as follows.

$$\texttt{f[x\_, y\_, q\_, m\_] :=} \sum_{k=1}^{m}\sum_{r=1}^{k}\sum_{s=1}^{r}\frac{q^{k-r}}{(q+1)^{k+1}}\frac{(-1)^{1+r}\,\texttt{Binomial}[k,r]}{((1+r-s)\,s)^x}\,\texttt{Cos}\left[y\,\texttt{Log}\left[\frac{s}{1+r-s}\right]\right]$$

$$\texttt{SetPrecision}\left[\texttt{f}\left[\frac{1}{2}, y_1, \frac{1}{3}, 24\right], 8\right] \qquad \texttt{SetPrecision}\left[\texttt{g}\left[\frac{1}{2}, y_1\right], 8\right]$$

$$0.\times 10^{-6} \qquad 0.\times 10^{-10}$$

The result has reached 6 significant figures of target precision. The total number of calculated terms T_1 is

$$\sum_{k=1}^{24}\sum_{r=1}^{k}\sum_{s=1}^{r}1 \qquad 2600$$

13.2 Parallel Acceleration Method

The series acceleration method in the previous section was orthodox and the acceleration effect was high. However, it was annoying to have to rearrange multiple series to half multiple series. In this chapter, I present the method of accelerating as it is, without rearranging a multiple series. Since this is similar to a parallel circuit in an electric circuit, we call this ***parallel acceleration method***.

13.2.1 Parallel Accelerator

For this purpose, I developed (invented ?) a new accelerator. It is as follows.

Formula 13.2.1 (Parallel Accelerator)

$$b(r_1,r_2,\cdots,r_n) = \sum_{k=r_1+r_2+\cdots+r_n}^{\infty} \frac{q^{k-r_1-r_2-\cdots-r_n}}{(q+1)^{k+1}} \binom{k}{r_1+r_2+\cdots+r_n} = 1 \quad \text{for} \quad \begin{array}{l} r_s = 0,1,2,\cdots \\ s = 1,2,\cdots,n \\ q > 0 \end{array}$$

Proof

Formula 10.2.1 in " **10 Convergence Acceleration & Summation Method** " was as follows.

$$b(r) = \sum_{k=r}^{\infty} \frac{q^{k-r}}{(q+1)^{k+1}} \binom{k}{r} = 1 \quad \text{for} \quad \begin{array}{l} r = 0,1,2,\cdots \\ q > 0 \end{array}$$

Replacing r with $r_1+r_2+\cdots+r_n$ in this formula, we obtain the desired expression.

13.2.2 Acceleration by Parallel Accelerator

Though it is an accelerator like a joke, if this is used as follows, we can accelerate as it is, without rearranging a multiple series.

Proposition 13.2.2

(0) When a multiple series of functions $\displaystyle\sum_{r_1,r_2,\cdots,r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n}(z)$ is absolutely convergent in the domain D, the following expression holds for arbitrary positive number q.

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n}(z) = \sum_{k=0}^{\infty}\sum_{r_1=0}^{k}\sum_{r_2=0}^{k}\cdots\sum_{r_n=0}^{k} \frac{q^{k-r_1-r_2-\cdots-r_n}}{(q+1)^{k+1}} \binom{k}{r_1+r_2+\cdots+r_n} \times a_{r_1,r_2,\cdots,r_n}(z) \tag{2.0}$$

(1) When a multiple series of functions $\displaystyle\sum_{r_1,r_2,\cdots,r_n=1}^{\infty} a_{r_1,r_2,\cdots,r_n}(z)$ is absolutely convergent in the domain D, the following expression holds for arbitrary positive number q.

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\cdots\sum_{r_n=1}^{\infty} a_{r_1,r_2,\cdots,r_n}(z) = \sum_{k=1}^{\infty}\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\cdots\sum_{r_n=1}^{k} \frac{q^{k-r_1-r_2-\cdots-r_n}}{(q+1)^{k+1}} \binom{k}{r_1+r_2+\cdots+r_n} \times a_{r_1,r_2,\cdots,r_n}(z) \tag{2.1}$$

Proof

Logical proof is difficult. If I had to say, considering multiple series as higher order tensor and applying the corresponding accelerator, we obtain the desired expression.

Total number of calculated terms

In the proposition, when the upper limit of Σ is terminated with m, the total number of calculated terms $T_2(n,m)$ is given by the following equation.

$$\textbf{(0)}\quad T_2(n,m) = \sum_{k=0}^{m}\sum_{r_1=0}^{k}\sum_{r_2=0}^{k}\cdots\sum_{r_n=0}^{k}1 = \sum_{j=1}^{m+1}j^n$$

$$\textbf{(1)}\quad T_2(n,m) = \sum_{k=1}^{m}\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\cdots\sum_{r_n=1}^{k}1 = \sum_{j=1}^{m}j^n$$

Speed comparison with series acceleration method

This is roughly determined by the ratio of the total number of calculated terms as follows.

$$\lim_{m\to\infty}\frac{T_2(n,m)}{T_1(n,m)} = n!$$

That is, when the upper limit of Σ is the same m, the series acceleration method is $n!$ times faster than the parallel acceleration method at most.

Example 2.1

$$f(x) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-1)^{r+s+t}x^{2r+2s+2t+3}}{(2r+1)(2s+1)(2t+1)} = \left(\tan^{-1}x\right)^3 =: g(x)$$

This is the same as Example 1.1 in the previous section. We want to obtain the value at $x=1$ of this series with 6 significant digits.

Applying Proposition 13.2.2 (0) to the left side,

$$f(x,q) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{k}\sum_{t=0}^{k}\frac{q^{k-r-s-t}}{(q+1)^{k+1}}\binom{k}{r+s+t}\frac{(-1)^{r+s+t}x^{2r+2s+2t+3}}{(2r+1)(2s+1)(2t+1)}$$

When this is calculated at $q=1/2$ and $m=12$, it is as follows. (m is the upper limit of Σ. Same as below.)

```
c[k_, r_] := Binomial[k, r]
g[x_] := ArcTan[x]^3
```

$$\texttt{f[x\_, q\_, m\_]} := \sum_{k=0}^{m}\sum_{r=0}^{k}\sum_{s=0}^{k}\sum_{t=0}^{k}\frac{q^{k-r-s-t}}{(q+1)^{k+1}}\texttt{c[k, r+s+t]}\frac{(-1)^{r+s+t}x^{2r+2s+2t+3}}{(2r+1)(2s+1)(2t+1)}$$

```
SetPrecision[f[1, 1/2, 12], 8]
0.48447319

N[g[1]]
0.48447307
```

The result has reached 6 significant figures of target precision. In addition, the total number of calculated terms T_2 is as follows.

$$\sum_{k=0}^{12}\sum_{r=0}^{k}\sum_{s=0}^{k}\sum_{t=0}^{k}1 \qquad 8281$$

Let us compare the acceleration result of Example1.1 with this. Then, though the upper limit of Σ is $12:12$, the total number of calculated terms T is $1820:8281$. So, at calculation speed, Example1.1 is 4.5 times faster than this example. However, in the formula, this example is simpler. It is annoying to choose either.

Example 2.2

$$f=\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{r_3=1}^{\infty}\sum_{r_4=1}^{\infty}\frac{(-1)^{r_1+r_2+r_3+r_4}}{r_1 r_2 r_3 r_4}=(\log 2)^4=:g$$

This is the same as Example 1.2 in the previous section. We want to obtain the value of this series with 6 significant digits.

Applying Proposition 13.2.2 (1) to the left side,

$$f(x,q)=\sum_{k=1}^{\infty}\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\sum_{r_3=1}^{k}\sum_{r_4=1}^{k}\frac{q^{k-r_1-r_2-r_3-r_4}}{(q+1)^{k+1}}\binom{k}{r_1+r_2+r_3+r_4}\frac{(-1)^{r_1+r_2+r_3+r_4}}{r_1 r_2 r_3 r_4}$$

When this is calculated at $q=1/2$ and $m=19$, it is as follows.

```
c[k_, r_] := Binomial[k, r]                    g := Log[2]^4
```

$$f[q\_,m\_]:=\sum_{k=1}^{m}\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\sum_{r_3=1}^{k}\sum_{r_4=1}^{k}\frac{q^{k-r_1-r_2-r_3-r_4}}{(q+1)^{k+1}}c[k,r_1+r_2+r_3+r_4]\frac{(-1)^{r_1+r_2+r_3+r_4}}{r_1 r_2 r_3 r_4}$$

$$N\left[f\left[\frac{1}{2},19\right],8\right] \qquad N[g,8]$$

```
0.23083504                                     0.23083510
```

The result has reached 6 significant figures of target precision. The total number of calculated terms T_2 is

$$\sum_{k=1}^{19}\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\sum_{r_3=1}^{k}\sum_{r_4=1}^{k}1 \qquad 562\,666$$

Let us compare the acceleration result of Example1.2 with this. Then, though the upper limit of Σ is $16:19$, the total number of calculated terms T is $15504:562666$. So, at calculation speed, Example1.2 is 36 times faster than this example. Indeed, when the calculation speed is different like this, the application of this proposition will be hesitated

Example 2.3

$$f(x,y)=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right)=|\eta(x,y)|^2=:g(x,y)$$

Where, $$\eta(x,y)=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{(2r-1)^{x+iy}}$$

This is the same as Example 1.3, We want to obtain the value at $x_1=1/2$, $y_1=14.13472514\cdots$ with 6 significant digits.

Applying Proposition 13.2.2 (1) to the left side,

$$f(x,y,q)=\sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{s=1}^{k}\frac{q^{k-r-s}}{(q+1)^{k+1}}\binom{k}{r+s}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right)$$

When this is calculated at $q=1/2$ and $m=26$, it is as follows.

$c[k\_, r\_] := \text{Binomial}[k, r]$ $y_{n\_} := \text{Im}[\text{ZetaZero}[n]]$

$$f[x\_, y\_, q\_, m\_] := \sum_{k=1}^{m}\sum_{r=1}^{k}\sum_{s=1}^{k} \frac{q^{k-r-s}}{(q+1)^{k+1}} c[k, r+s] \frac{(-1)^{r+s}}{(r\, s)^{x}} \text{Cos}\left[y \text{ Log}\left[\frac{s}{r}\right]\right]$$

$\text{SetPrecision}\left[f\left[\frac{1}{2}, y_1, \frac{1}{3}, 26\right], 8\right]$ $\text{SetPrecision}\left[g\left[\frac{1}{2}, y_1\right], 8\right]$

$0.\times 10^{-6}$ $0.\times 10^{-10}$

The result has reached 6 significant figures of target precision. The total number of calculated terms T_2 is

$\sum_{k=1}^{26}\sum_{r=1}^{k}\sum_{s=1}^{k} 1$ 6201

Let us compare the acceleration result of Example1.3 with this. Then, though the upper limit of Σ is $24:26$, the total number of calculated terms T is $2600:6201$. So, at calculation speed, Example1.3 is 2.4 times faster than this example. However, when the difference in speed is this degree, the application of this proposition seems good.

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