1 Dirichlet Beta Generating Functions

Both of hyperbolic functions and trigonometric functions can be expanded to Fourier series and Taylor series. And if the termwise higher order integration of these is carried out, Dirichlet Beta at natural numbers can be obtained. Where, these are automorphisms which are expressed by lower betas. However, in this chapter, we stop those so far.

The work that obtain the non-automorphism formulas by removing lower betas from these is done in the next chapter " **2 Formulas for Dirichlet Beta** ".

On the other hand, if the termwise higher order differentiation of these is carried out, we obtain the negative integer beta. these are the non-automorphism formulas.

1.1 Generating function of sech x family

The idea of obtaining Beta by integrating the following expression is the most natural.

$$\frac{e^x}{1+e^{2x}} = e^{-1x} - e^{-3x} + e^{-5x} - e^{-7x} +- \cdots \qquad \left(= \frac{\operatorname{sech} x}{2}\right) \tag{1.0}$$

1.1.1 Termwise Higher Integral of Fourier Series of $e^x/\left(e^{2x}+1\right)$

Formula 1.1.1

When $\beta(n) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^n}$, the following expressions hold for $x>0$.

$$\int_0^x \frac{e^x}{e^{2x}+1} dx = -\sum_{r=0}^{\infty} \frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^1} + \frac{x^0}{0!}\beta(1) \tag{1.1}$$

$$\int_0^x\int_0^x \frac{e^x}{e^{2x}+1} dx^2 = \sum_{r=0}^{\infty} \frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^2} - \frac{x^0}{0!}\beta(2) + \frac{x^1}{1!}\beta(1) \tag{1.2}$$

$$\int_0^x\int_0^x\int_0^x \frac{e^x}{e^{2x}+1} dx^3 = -\sum_{r=0}^{\infty} \frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^3} + \frac{x^0}{0!}\beta(3) - \frac{x^1}{1!}\beta(2) + \frac{x^2}{2!}\beta(1)$$

$$\vdots$$

$$\int_0^x \cdots \int_0^x \frac{e^x}{e^{2x}+1} dx^n = (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^n} + \sum_{s=0}^{n-1} (-1)^{n-s} \frac{x^s}{s!}\beta(n-s) \tag{1.n}$$

Proof

$e^x/\left(e^{2x}+1\right)$ can be expanded to a Fourier series as follows.

$$\begin{aligned}\frac{e^x}{e^{2x}+1} &= e^{-1x} - e^{-3x} + e^{-5x} - e^{-7x} +- \cdots \\ &= \cos 1ix - \cos 3ix + \cos 5ix - \cos 7ix +- \cdots \\ &\quad + i(\sin 1ix - \sin 3ix + \sin 5ix - \sin 7ix +- \cdots)\end{aligned}$$

Integrating both sides of this with respect to x from 0 to x ,

$$\int_0^x \frac{e^x}{e^{2x}+1} dx = -\left[\frac{e^{-x}}{1} - \frac{e^{-3x}}{3} + \frac{e^{-5x}}{5} - \frac{e^{-7x}}{7} +- \cdots\right]_0^x$$

$$= -\left(\frac{e^{-x}}{1} - \frac{e^{-3x}}{3} + \frac{e^{-5x}}{5} - \frac{e^{-7x}}{7} +- \cdots\right) + \frac{e^0}{1^1} - \frac{e^0}{3^1} + \frac{e^0}{5^1} - \frac{e^0}{7^1} +- \cdots$$

$$= -\sum_{r=0}^{\infty}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^1} + \frac{x^0}{0!}\beta(1)$$

Next, integrating both sides of this with respect to x from 0 to x ,

$$\int_0^x\int_0^x \frac{e^x}{e^{2x}+1}dx^2 = \left[\sum_{r=0}^{\infty}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^2}\right]_0^x + \frac{x^1}{1!}\beta(1)$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^2} - \frac{x^0}{0!}\beta(2) + \frac{x^1}{1!}\beta(1)$$

Hereafter, in a similar way we obtein the desired expressions. In addition, these are the collateral integrals.

1.1.2 Termwise Higher Integral of Taylor Series of $e^x/(e^{2x}+1)$

Formula 1.1.2

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, E_{10}=50521, \cdots$ are Euler Numbers , the following expressions hold for $|x|< \pi/2$.

$$\int_0^x \frac{e^x}{e^{2x}+1}dx = \frac{1}{2}\sum_{r=0}^{\infty}\frac{E_{2r}}{(2r+1)!}x^{2r+1} \tag{2.1}$$

$$\int_0^x\int_0^x \frac{e^x}{e^{2x}+1}dx^2 = \frac{1}{2}\sum_{r=0}^{\infty}\frac{E_{2r}}{(2r+2)!}x^{2r+2} \tag{2.2}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x \frac{e^x}{e^{2x}+1}dx^n = \frac{1}{2}\sum_{r=0}^{\infty}\frac{E_{2r}}{(2r+n)!}x^{2r+n} \tag{2.n}$$

Proof

$$\frac{e^x}{1+e^{2x}} = \frac{1}{2}\frac{2}{e^x+e^{-x}} = \frac{\operatorname{sech} x}{2} \quad , \quad \operatorname{sech} x = \sum_{r=0}^{\infty}\frac{E_{2r}}{(2r)!}x^{2r}$$

From these,

$$\frac{e^x}{e^{2x}+1} = \frac{1}{2}\sum_{r=0}^{\infty}\frac{E_{2r}}{(2r)!}x^{2r} \qquad |x|<\frac{\pi}{2}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expressions.

1.1.3 Dirichlet Beta

Comparing Formula1.1.1 and Formula1.1.2 , we obtain the following Dirichlet Beta.

Formula 1.1.3

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, E_{10}=50521, \cdots$ are Euler Numbers , the following expressions hold for $|x|< \pi/2$.

$$\beta(1) = \sum_{r=0}^{\infty}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^1} + \frac{1}{2}\sum_{r=0}^{\infty}\frac{E_{2r}x^{2r+1}}{(2r+1)!}$$

$$\beta(2) = \sum_{r=0}^{\infty} \frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^2} - \frac{1}{2} \sum_{r=0}^{\infty} \frac{E_{2r} x^{2r+2}}{(2r+2)!} + \frac{x^1}{1!} \beta(1)$$

$$\beta(3) = \sum_{r=0}^{\infty} \frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^3} + \frac{1}{2} \sum_{r=0}^{\infty} \frac{E_{2r} x^{2r+3}}{(2r+3)!} + \frac{x^1}{1!} \beta(2) - \frac{x^2}{2!} \beta(1)$$

$$\beta(4) = \sum_{r=0}^{\infty} \frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^4} - \frac{1}{2} \sum_{r=0}^{\infty} \frac{E_{2r} x^{2r+4}}{(2r+4)!} + \frac{x^1}{1!} \beta(3) - \frac{x^2}{2!} \beta(2) + \frac{x^3}{3!} \beta(1)$$

$$\vdots$$

$$\beta(n) = \sum_{r=0}^{\infty} \frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^n} - \frac{(-1)^n}{2} \sum_{r=0}^{\infty} \frac{E_{2r} x^{2r+n}}{(2r+n)!} - \sum_{s=1}^{n-1} \frac{(-1)^s x^s}{s!} \beta(n-s)$$

Proof

Comparing (1.1) and (2.1) , we obtain

$$\beta(1) = \sum_{r=0}^{\infty} \frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^1} + \frac{1}{2} \sum_{r=0}^{\infty} \frac{E_{2r} x^{2r+1}}{(2r+1)!}$$

Comparing (1.2) and (2.2) , we obtain

$$\beta(2) = \sum_{r=0}^{\infty} \frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^1} - \frac{1}{2} \sum_{r=0}^{\infty} \frac{E_{2r} x^{2r+2}}{(2r+2)!} + \frac{x^1}{1!} \beta(1)$$

Hereafter, in a similar way we obtein the desired expressions.

1.1.4 Lineal Higher Interal of $e^x/\left(e^{2x}+1\right)$

Mentioned above was a result of the collateral higher integral of $e^x/\left(e^{2x}+1\right)$. If this is integrated in the lineal line, it becomes the following.

First of all,

$$\int_{\infty}^{x} \cdots \int_{\infty}^{x} \frac{e^x}{e^{2x}+1} dx^n = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k e^{-(2k+1)x}}{(2k+1)^n}$$

According to Cauchy formula for repeated integration ,

$$\int_{\infty}^{x} \cdots \int_{\infty}^{x} \frac{e^x}{e^{2x}+1} dx^n = \frac{1}{\Gamma(n)} \int_{\infty}^{x} \frac{e^t (x-t)^{n-1}}{e^{2t}+1} dt$$

From these ,

$$(-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k e^{-(2k+1)x}}{(2k+1)^n} = \frac{1}{\Gamma(n)} \int_{\infty}^{x} \frac{e^t (x-t)^{n-1}}{e^{2t}+1} dt$$

Giving this $x=0$,

$$(-1)^n \beta(n) = \frac{1}{\Gamma(n)} \int_{\infty}^{0} \frac{e^t (-t)^{n-1}}{e^{2t}+1} dt = \frac{(-1)^n}{\Gamma(n)} \int_{0}^{\infty} \frac{e^t t^{n-1}}{e^{2t}+1} dt$$

Thus, we obtain

$$\beta(n) = \frac{1}{\Gamma(n)} \int_{0}^{\infty} \frac{e^t t^{n-1}}{e^{2t}+1} dt = \frac{1}{2\Gamma(n)} \int_{0}^{\infty} t^{n-1} \operatorname{sech} t \, dt$$

n does not need to be a natural number any longer.

1.1.5 Higher order differentiation of $e^x/(e^{2x}+1)$

(1) Higher order derivatives of $e^x/(e^{2x}+1)$

Formula 1.1.5

Let us define the 2nd kind coefficients that make powers constant as follows.

$$1^n+3^n x^2+5^n x^4+7^n x^6+\cdots=\frac{\sum_{r=0}^{n}{}_nK_r x^{2r}}{(1-x^2)^{n+1}} \qquad |x^2|\leqq 1 \tag{5.0}$$

Then, the following expressions hold.

$$\frac{e^x(e^{2x}-1)}{(e^{2x}+1)^2}=1^1e^{-x}-3^1e^{-3x}+5^1e^{-5x}-7^1e^{-7x}+-\cdots$$

$$\frac{e^x(e^{4x}-6e^{2x}+1)}{(e^{2x}+1)^3}=1^2e^{-x}-3^2e^{-3x}+5^2e^{-5x}-7^2e^{-7x}+-\cdots$$

$$\frac{e^x(e^{6x}-23e^{4x}+23e^{2x}-1)}{(e^{2x}+1)^4}=1^3e^{-x}-3^3e^{-3x}+5^3e^{-5x}-7^3e^{-7x}+-\cdots$$

$$\frac{e^x(e^{8x}-76e^{6x}+230e^{4x}-76e^{2x}+1)}{(e^{2x}+1)^5}=1^4e^{-x}-3^3e^{-3x}+5^3e^{-5x}-+\cdots$$

$$\vdots$$

$$\frac{e^x\sum_{r=0}^{n}(-1)^r{}_nK_r e^{(2n-2r)x}}{(e^{2x}+1)^{n+1}}=1^ne^{-x}-3^ne^{-3x}+5^ne^{-5x}-7^ne^{-7x}+-\cdots \tag{5.n}$$

Proof

Replacing x^2 with $-e^{-2x}$ in the definition (5.0) ,

$$1^n-3^ne^{-2x}+5^ne^{-4x}-7^ne^{-6x}+-\cdots=\frac{\sum_{r=0}^{n}(-1)^r{}_nK_r e^{-2rx}}{(1+e^{-2x})^{n+1}}$$

Multiplying by e^{-x} the both sides,

$$1^n-3^ne^{-2x}+5^ne^{-4x}-7^ne^{-6x}+-\cdots=\frac{e^{-x}\sum_{r=0}^{n}(-1)^r{}_nK_r e^{-2rx}}{(1+e^{-2x})^{n+1}}$$

Multiplying by $e^{2x(n+1)}$ the numerator and the denominator in the right side,

$$\frac{e^{-x}\sum_{r=0}^{n}(-1)^r{}_nK_r e^{-2rx}e^{2x(n+1)}}{(1+e^{-2x})^{n+1}e^{2x(n+1)}}=\frac{e^x\sum_{r=0}^{n}(-1)^r{}_nK_r e^{(2n-2r)x}}{(e^{2x}+1)^{n+1}}$$

Then we obtain

$$1^n e^{-x} - 3^n e^{-3x} + 5^n e^{-5x} - 7^n e^{-7x} + - \cdots = \frac{e^x \sum_{r=0}^{n} (-1)^r \, {}_nK_r \, e^{(2n-2r)x}}{\left(e^{2x}+1\right)^{n+1}} \tag{5.n}$$

This is corresponding to the n th order derivative of (1.0) .

Note

Although these coefficients are mentioned later , it is given by the following formula.

$${}_nK_r = \sum_{k=0}^{r} (-1)^k \binom{n+1}{k} (2r+1-2k)^n \qquad n=1,2,3,\cdots$$

(2) Dirichlet Beta at a negative integer

Substituting $x=0$ for the above, we obtain Dirichlet Beta at a negative integer as higher order difference quationts.

Formula 1.1.5'

$$\frac{1-1}{2^2} = 1^1-3^1+5^1-7^1+- \cdots = \beta(-1)$$

$$\frac{1-6+1}{2^3} = 1^2-3^2+5^2-7^2+- \cdots = \beta(-2)$$

$$\frac{1-23+23-1}{2^4} = 1^3-3^3+5^3-7^3+- \cdots = \beta(-3)$$

$$\frac{1-76+230-76+1}{2^5} = 1^4-3^4+5^4-7^4+- \cdots = \beta(-4)$$

$$\vdots$$

$$\frac{\sum_{r=0}^{n} (-1)^r \, {}_nK_r}{2^{n+1}} = 1^n-3^n+5^n-7^n+- \cdots = \beta(-n) \tag{5.n'}$$

(3) The 2nd kind coefficients that make powers constant and Euler Numbers

Formula 1.1.5"

There is next relation between the 2nd kind coefficient that make powers constant ${}_nK_r$ and Euler Number E_{2n} .

$$\sum_{r=0}^{2n} (-1)^r \, {}_{2n}K_r = 2^{2n} E_{2n} \tag{5.n''}$$

Proof

Replacing n with $2n$ in (5.n') ,

$$\frac{\sum_{r=0}^{2n} (-1)^r \, {}_{2n}K_r}{2^{2n+1}} = \beta(-2n) = \frac{E_{2n}}{2}$$

From this, we obtain the desired expression.

1.2 Generating function of sec x family

The idea of obtaining Beta by integrating the following expression is also natural.

$$\frac{e^{ix}}{1+e^{2ix}} = e^{ix} - e^{3ix} + e^{5ix} - e^{7ix} + - \cdots \qquad \left(= \frac{1}{2} sec x\right) \tag{1.0}$$

1.2.1 Termwise Higher Integral of Fourier Series of $e^{ix}/\left(1+e^{2ix}\right)$

Formula 1.2.1

When $\beta(n) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^n}$, the following expressions hold for $|x| < \pi/2$.

$$\int_0^x \frac{e^{ix}}{1+e^{2ix}} dx = i^{-1} \sum_{r=0}^{\infty} \frac{(-1)^r e^{(2r+1)ix}}{(2r+1)^1} - i^{-1} \frac{x^0}{0!} \beta(1) \tag{1.1}$$

$$\int_0^x \int_0^x \frac{e^{ix}}{1+e^{2ix}} dx^2 = i^{-2} \sum_{r=0}^{\infty} \frac{(-1)^r e^{(2r+1)ix}}{(2r+1)^2} - i^{-2} \frac{x^0}{0!} \beta(2) - i^{-1} \frac{x^1}{1!} \beta(1) \tag{1.2}$$

$$\int_0^x \int_0^x \int_0^x \frac{e^{ix}}{1+e^{2ix}} dx^3 = i^{-3} \sum_{r=0}^{\infty} \frac{(-1)^r e^{(2r+1)ix}}{(2r+1)^3} - i^{-3} \frac{x^0}{0!} \beta(3) - i^{-2} \frac{x^1}{1!} \beta(2) - i^{-1} \frac{x^2}{2!} \beta(1) \tag{1.3}$$

$$\vdots$$

$$\int_0^x \cdots \int_0^x \frac{e^{ix}}{1+e^{2ix}} dx^n = i^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r e^{(2r+1)ix}}{(2r+1)^n} - \sum_{s=0}^{n-1} \frac{i^{-(n-s)} x^s}{s!} \beta(n-s) \tag{1.n}$$

Proof

$e^{ix}/\left(1+e^{2ix}\right)$ can be expanded to a Fourier series in a broad sense as follows.

$$\begin{aligned} \frac{e^{ix}}{1+e^{2ix}} &= e^{ix} - e^{3ix} + e^{5ix} - e^{7ix} + - \cdots \\ &= cos x - cos 3x + cos 5x - cos 7x + - \cdots \\ &\quad + i(sin x - sin 3x + sin 5x - sin 7x + - \cdots) \end{aligned} \tag{1.0}$$

(1.0) does not hold as an equation. However, the higher order integral of both sides of (1.0) holds as an equation. Integrating both sides of (1.0) with respect to x from 0 to x ,

$$\int_0^x \frac{e^{ix}}{1+e^{2ix}} dx = i^{-1} \left[\sum_{r=0}^{\infty} \frac{(-1)^r e^{(2r+1)ix}}{(2r+1)^1} \right]_0^x - i^{-1} \left(1 - \frac{e^0}{3^1} + \frac{e^0}{5^1} - \frac{e^0}{7^1} + - \cdots \right)$$

$$= i^{-1} \sum_{r=0}^{\infty} \frac{(-1)^r e^{(2r+1)ix}}{(2r+1)^1} - i^{-1} \frac{x^0}{0!} \beta(1) \tag{1.1}$$

Next, integrating both sides of (1.1) with respect to x from 0 to x ,

$$\int_0^x \int_0^x \frac{e^{ix}}{1+e^{2ix}} dx^2 = i^{-2} \left[\sum_{r=0}^{\infty} \frac{(-1)^r e^{(2r+1)ix}}{(2r+1)^2} \right]_0^x - i^{-1} \frac{x^1}{1!} \beta(1)$$

$$= i^{-2} \sum_{r=0}^{\infty} \frac{(-1)^r e^{(2r+1)ix}}{(2r+1)^2} - i^{-2} \frac{x^0}{0!} \beta(2) - i^{-1} \frac{x^1}{1!} \beta(1) \tag{1.2}$$

Hereafter, in a similar way we obtein the desired expressions. Of course, these are the collateral integrals.

1.2.2 Termwise Higher Integral of Taylor Series of $e^{ix}/\left(1+e^{2ix}\right)$

Formula 1.2.2

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, E_{10}=50521, \cdots$ are Euler Numbers, the following expressions hold for $|x|<\pi/2$.

$$\int_0^x \frac{e^{ix}}{1+e^{2ix}}dx = \frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+1)!}x^{2r+1} \tag{2.1}$$

$$\int_0^x\int_0^x \frac{e^{ix}}{1+e^{2ix}}dx^2 = \frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+2)!}x^{2r+2} \tag{2.2}$$

$$\int_0^x\int_0^x\int_0^x \frac{e^{ix}}{1+e^{2ix}}dx^3 = \frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+3)!}x^{2r+3} \tag{2.3}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x \frac{e^{ix}}{1+e^{2ix}}dx^n = \frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+n)!}x^{2r+n} \tag{2.n}$$

Proof

$$\frac{e^{ix}}{1+e^{2ix}} = \frac{1}{2}\frac{2}{e^{ix}+e^{-ix}} = \frac{1}{2}\sec x \quad , \quad \sec x = \sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r)!}x^{2r}$$

From these,

$$\frac{e^{ix}}{1+e^{2ix}} = \frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r)!}x^{2r} \qquad 0<x<\frac{\pi}{2}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expressions.

1.2.3 Dirichlet Beta Polynomials

Comparing Formula1.2.1 and Formula1.2.2 , we obtain the following Dirichlet Beta Polynomials

Formula 1.2.3

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, E_{10}=50521, \cdots$ are Euler Numbers, the following expressions hold for $|x|<\pi/2$.

Even Beta

$$\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^1} - \frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+1)!}x^{2r+1} = 0 \tag{3.0es}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^3} + \frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|x^{2r+3}}{(2r+3)!} = \frac{x^1}{1!}\beta(2) \tag{3.1es}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^5} - \frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|x^{2r+5}}{(2r+5)!} = \frac{x^1}{1!}\beta(4) - \frac{x^3}{3!}\beta(2) \tag{3.2es}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^7} + \frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|x^{2r+7}}{(2r+7)!} = \frac{x^1}{1!}\beta(6) - \frac{x^3}{3!}\beta(4) + \frac{x^5}{5!}\beta(2) \tag{3.3es}$$

$$\vdots$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^{2n+1}}-\frac{(-1)^n}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|\,x^{2n+1+2r}}{(2n+1+2r)\,!}=\sum_{s=0}^{n-1}\frac{(-1)^s x^{2s+1}}{(2s+1)\,!}\beta(2n-2s) \tag{3.nes}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^2}+\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|\,x^{2r+2}}{(2r+2)\,!}=\frac{x^0}{0!}\beta(2) \tag{3.1ec}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^4}-\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|\,x^{2r+4}}{(2r+4)\,!}=\frac{x^0}{0!}\beta(4)-\frac{x^2}{2!}\beta(2) \tag{3.2ec}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^6}+\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|\,x^{2r+6}}{(2r+6)\,!}=\frac{x^0}{0!}\beta(6)-\frac{x^2}{2!}\beta(4)+\frac{x^4}{4!}\beta(2) \tag{3.3ec}$$

$$\vdots$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^{2n}}-\frac{(-1)^n}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|\,x^{2n+2r}}{(2n+2r)\,!}=\sum_{s=0}^{n-1}\frac{(-1)^s x^{2s}}{(2s)\,!}\beta(2n-2s) \tag{3.nec}$$

Odd Beta

$$\sum_{r=0}^{\infty}\frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^1}=\frac{x^0}{0!}\beta(1) \tag{3.0oc}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^3}=\frac{x^0}{0!}\beta(3)-\frac{x^2}{2!}\beta(1) \tag{3.1oc}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^5}=\frac{x^0}{0!}\beta(5)-\frac{x^2}{2!}\beta(3)+\frac{x^4}{4!}\beta(1) \tag{3.2oc}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^7}=\frac{x^0}{0!}\beta(7)-\frac{x^2}{2!}\beta(5)+\frac{x^4}{4!}\beta(3)-\frac{x^6}{6!}\beta(1) \tag{3.3oc}$$

$$\vdots$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^{2n+1}}=\sum_{s=0}^{n}\frac{(-1)^s x^{2s}}{(2s)\,!}\beta(2n+1-2s) \tag{3.noc}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^2}=\frac{x^1}{1!}\beta(1) \tag{3.1os}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^4}=\frac{x^1}{1!}\beta(3)-\frac{x^3}{3!}\beta(1) \tag{3.2os}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^6}=\frac{x^1}{1!}\beta(5)-\frac{x^3}{3!}\beta(3)+\frac{x^5}{5!}\beta(1) \tag{3.3os}$$

$$\vdots$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^{2n}}=\sum_{s=0}^{n-1}\frac{(-1)^s x^{2s+1}}{(2s+1)\,!}\beta(2n+1-2s) \tag{3.nos}$$

Proof

Comparing (1.1) and (2.1), we obtain

$$\sum_{r=0}^{\infty}\frac{(-1)^r e^{(2r+1)ix}}{(2r+1)^1}-\frac{i}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+1)\,!}x^{2r+1}=\frac{x^0}{0!}\beta(1)$$

Substituting $e^{irx} = \cos rx + i\sin rx$ for this ,

$$\sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^1}+i\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^1}-\frac{i}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+1)!}x^{2r+1}$$
$$=\frac{x^0}{0!}\beta(1)$$

From this,

$$\sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^1}=\frac{x^0}{0!}\beta(1) \tag{3.0oc}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^1}-\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+1)!}x^{2r+1}=0 \tag{3.0es}$$

Comparing (1.2) and (2.2) , we obtain

$$\sum_{r=0}^{\infty}\frac{(-1)^r e^{(2r+1)ix}}{(2r+1)^2}+\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+2)!}x^{2r+2}=\frac{x^0}{0!}\beta(2)+i\frac{x^1}{1!}\beta(1)$$

Substituting $e^{irx} = \cos rx + i\sin rx$ for this ,

$$\sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^2}+i\sum_{r=0}^{\infty}\frac{(-1)^r\{(2r+1)x\}}{(2r+1)^2}+\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+2)!}x^{2r+2}$$
$$=\frac{x^0}{0!}\beta(2)+i\frac{x^1}{1!}\beta(1)$$

From this,

$$\sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^2}+\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+2)!}x^{2r+2}=\frac{x^0}{0!}\beta(2) \tag{3.1ec}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^2}=\frac{x^1}{1!}\beta(1) \tag{3.1os}$$

Comparing (1.3) and (2.3) , we obtain

$$\sum_{r=0}^{\infty}\frac{(-1)^r e^{(2r+1)ix}}{(2r+1)^3}+\frac{i}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+3)!}x^{2r+3}=\frac{x^0}{0!}\beta(3)+i\frac{x^1}{1!}\beta(2)-\frac{x^2}{2!}\beta(1)$$

Substituting $e^{irx} = \cos rx + i\sin rx$ for this ,

$$\sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^3}+i\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^3}+\frac{i}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+3)!}x^{2r+3}$$
$$=\frac{x^0}{0!}\beta(3)+i\frac{x^1}{1!}\beta(2)-\frac{x^2}{2!}\beta(1)$$

From this ,

$$\sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^3}=\frac{x^0}{0!}\beta(3)-\frac{x^2}{2!}\beta(1) \tag{3.1oc}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^3}+\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+3)!}x^{2r+3}=\frac{x^1}{1!}\beta(2) \tag{3.1es}$$

Hereafter, in a similar way we obtein the desired expressions.

1.2.5 Higher order differentiation of $e^{ix}/\left(1+e^{2ix}\right)$

Although the higher order differentiation of (1.0) does not hold as an equation, it is useful to generate Dirichlet Beta at a negative integer. And using the 2nd kind of the coefficients that make powers constant, we obtain the same results as **1.1.5** .

1.3 Generating function of csc x family

Dirichlet Even Beta is obtained also from the following expression.

$$\frac{ie^{ix}}{1-e^{2ix}} = i\left(e^{ix}+e^{3ix}+e^{5ix}+e^{-7ix}+\cdots\right) \qquad \left(=-\frac{\csc x}{2}\right) \tag{1.0}$$

Formula 1.3.3

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers, $H_s=\sum_{t=1}^{s}1/t$ is a harmonic number and $\lambda(x)=\sum_{r=1}^{\infty}(2r-1)^{-x}$ is Dirichlet Lambda Function , the following expression holds for $0<x<\pi$.

$$\beta(2n) = \frac{(-1)^n}{2(2n-1)!}\left(\frac{\pi}{2}\right)^{2n-1}\left(\log\frac{\pi}{4}-H_{2n-1}\right) + \frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|}{2r(2r+2n-1)!}\left(\frac{\pi}{2}\right)^{2n-1+2r} - \sum_{s=1}^{n-1}\frac{(-1)^s}{(2s-1)!}\left(\frac{\pi}{2}\right)^{2s-1}\lambda(2n+1-2s)$$

Proof

We obtained the following odd lambda from (1.0) in Formula1.6.3 in " **1 Zeta Generating Functions** "

$$\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n}} - \frac{(-1)^n x^{2n-1}}{2(2n-1)!}\left(\log\frac{x}{2}-H_{2n-1}\right) - \frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|\,x^{2r+2n-1}}{2r(2r+2n-1)!} = -\sum_{s=1}^{n-1}\frac{(-1)^s x^{2s-1}}{(2s-1)!}\lambda(2n+1-2s)$$

Substituting $x=\pi/2$ for this, we obtain

$$\beta(2n) - \frac{(-1)^n}{2(2n-1)!}\left(\frac{\pi}{2}\right)^{2n-1}\left(\log\frac{\pi}{4}-H_{2n-1}\right) - \frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|}{2r(2r+2n-1)!}\left(\frac{\pi}{2}\right)^{2n-1+2r} = -\sum_{s=1}^{n-1}\frac{(-1)^s}{(2s-1)!}\left(\frac{\pi}{2}\right)^{2s-1}\lambda(2n+1-2s)$$

Note

There is a relation $\lambda(n)=\left(1-2^{-n}\right)\zeta(n)$ between Dirichlet Lambda and Riemann Zeta.

Then, this formula means that even beta is expressed by odd zetas. In fact, when $n=2$,

$$\beta(4) - \frac{1}{2\cdot 3!}\left(\frac{\pi}{2}\right)^3\left(\log\frac{\pi}{4}-H_3\right) - \frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|}{2r(2r+3)!}\left(\frac{\pi}{2}\right)^{2r+3} = \frac{\pi}{2}\frac{7}{8}\zeta(3)$$

1.4 The 2nd kind coefficients that make powers constant

1.4.1 Definition

We call "*The 2nd kind polynomial that makes powers of geometric series constant*" the polynomial that satisfies the following equation. And we call "*The 2nd kind coefficients* that make powers constant" the coefficients ${}_nK_r$.

$$1^n + 3^n x^2 + 5^n x^4 + 7^n x^6 + \cdots = \frac{\sum_{r=0}^{n} {}_nK_r x^{2r}}{\left(1-x^2\right)^{n+1}} \qquad \left|x^2\right| \leqq 1 \tag{1.0}$$

These are the following numbers concretely.

$$\begin{array}{c} {}_0K_0 \\ {}_1K_0 \ {}_1K_1 \\ {}_2K_0 \ {}_2K_1 \ {}_2K_2 \\ {}_3K_0 \ {}_3K_1 \ {}_3K_2 \ {}_3K_3 \\ {}_4K_0 \ {}_4K_1 \ {}_4K_2 \ {}_4K_3 \ {}_4K_4 \\ {}_5K_0 \ {}_5K_1 \ {}_5K_2 \ {}_5K_3 \ {}_5K_4 \ {}_5K_5 \\ {}_6K_0 \ {}_6K_1 \ {}_6K_2 \ {}_6K_3 \ {}_6K_4 \ {}_6K_5 \ {}_6K_6 \\ \vdots \end{array} = \begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 6 \quad 1 \\ 1 \quad 23 \quad 23 \quad 1 \\ 1 \quad 76 \quad 230 \quad 76 \quad 1 \\ 1 \quad 237 \quad 1682 \quad 1682 \quad 237 \quad 1 \\ 1 \quad 722 \quad 10543 \quad 23548 \quad 10543 \quad 722 \quad 1 \\ \vdots \end{array}$$

Explanation

Assume a geometric series as follows.

$$1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2} \qquad \left|x^2\right| \leqq 1 \tag{0}$$

Multiplying both sides of (0) by x and differentiating it ,

$$1^1x^0 + 3^1x^2 + 5^1x^4 + 7^1x^6 + \cdots = \frac{1+x^2}{\left(1-x^2\right)^2} \qquad \left|x^2\right| \leqq 1 \tag{1}$$

The coefficients of this series are all the 1st power.

Multiplying both sides of (1) by x and differentiating it ,

$$1^2x^0 + 3^2x^2 + 5^2x^4 + 7^2x^6 + \cdots = \frac{1+6x^2+x^4}{\left(1-x^2\right)^3} \qquad \left|x^2\right| \leqq 1 \tag{2}$$

The coefficients of this series are all the 2nd power.

Multiplying both sides of (2) by x and differentiating it ,

$$1^3x^0 + 3^3x^2 + 5^3x^4 + 7^3x^6 + \cdots = \frac{1+23x^2+23x^4+x^6}{\left(1-x^2\right)^4} \qquad \left|x^2\right| \leqq 1 \tag{3}$$

Then, the coefficients of this series become all the 3rd power.

In this way, raising (0) to the n nd power and multilplying this by $\sum_{r=0}^{n} {}_nK_r x^{2r}$, we can make the powers of coefficients of the geometric series constant such that $1^n, 3^n, 5^n, 7^n, \cdots$.

1.4.2 Properties of the 2nd kind coefficients

Properties of the 2nd kind coefficients ${}_nK_r$ are as follows.

$$\sum_{r=0}^{n} {}_nK_r = (2n)!! \qquad \{= 2^n n!\} \tag{2.1}$$

$$\sum_{r=0}^{2n} (-1)^r {}_{2n}K_r = (-1)^n (\tan x)^{(2n)}\Big|_{x=\frac{\pi}{4}} \tag{2.2}$$

$$= (-1)^n \sum_{k=n+1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)\left|B_{2k}\right|}{2k\,(2k-2n-1)!}\left(\frac{\pi}{4}\right)^{2k-2n-1} \tag{2.2'}$$

$${}_nK_r = (2n-2r+1)\,{}_{n-1}K_{r-1} + (2r+1)\,{}_{n-1}K_r \qquad \begin{pmatrix} n=1,2,3,4,\cdots \\ r=1,2,\cdots,n-1 \end{pmatrix} \tag{2.3}$$

(2.1) is illustrated as follows.

$$\begin{array}{rcl} 1 &=& 0!! \\ 1+1 &=& 2!! \\ 1+6+1 &=& 4!! \\ 1+23+23+1 &=& 6!! \\ 1+76+230+76+1 &=& 8!! \\ 1+237+1682+1682+237+1 &=& 10!! \\ 1+722+10543+23548+10543+722+1 &=& 12!! \\ \vdots & & \vdots \end{array}$$

(2.2) and (2.2') are obtained from Super Calculus 10.1.1 (2) . These are illustrated as follows.

$$1 = \left(\tan\frac{\pi}{4}\right)^{(0)} = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)\left|B_{2k}\right|}{2k\,(2k-1)!}\left(\frac{\pi}{4}\right)^{2k-1}$$

$$1-1 = 0$$

$$1-6+1 = -\left(\tan\frac{\pi}{4}\right)^{(2)} = -\sum_{k=2}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)\left|B_{2k}\right|}{2k\,(2k-3)!}\left(\frac{\pi}{4}\right)^{2k-3}$$

$$1-23+23-1 = 0$$

$$1-76+230-76+1 = \left(\tan\frac{\pi}{4}\right)^{(4)} = \sum_{k=3}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)\left|B_{2k}\right|}{2k\,(2k-5)!}\left(\frac{\pi}{4}\right)^{2k-5}$$

$$1-237+1682-1682+237-1 = 0$$

$$1-722+10543-23548+10543-722+1 = -\left(\tan\frac{\pi}{4}\right)^{(6)} = -\sum_{k=4}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)\left|B_{2k}\right|}{2k\,(2k-7)!}\left(\frac{\pi}{4}\right)^{2k-7}$$

(2.3) shows that the coefficients are recursive.

1.4.3 Calculation method one by one for the 2nd kind coefficients .

The calculation method that the author devised is shown as follows.

		Calculating formula
1	1 1	
	3,3	
2	1 6 1	$6 = 1\times 3 + 1\times 3$
	5,3 3,5	
3	1 23 23 1	$23 = 1\times 5 + 6\times 3$
	7,3 5,5 3,7	
4	1 76 230 76 1	$76 = 1\times 7 + 23\times 3$, $230 = 23\times 5 + 23\times 5$
	9,3 7,5 5,7 3,9	
5	1 237 1682 1682 237 1	$237 = 1\times 9 + 76\times 7$, $1682 = 76\times 7 + 230\times 5$
	11,3 9,5 7,7 5,9 3,11	
6	1 722 10543 23548 10543 722 1	$722 = 1\times 11 + 237\times 3$, $10543 = 237\times 9 + 1682\times 5$, $23548 = 1682\times 7 + 1682\times 7$
	⋮	⋮

1.4.4 Direct calculation method for the 2nd kind coefficients

Formula1.4.4

$$ {}_nK_r = \sum_{k=0}^{r} (-1)^k \binom{n+1}{k} (2r+1-2k)^n \qquad n=1,2,3,\cdots \tag{4.1}$$

Proof

From the definition (1.0) ,

$$ \left(1-x^2\right)^{n+1}\left(1^n + 3^n x^2 + 5^n x^4 + 7^n x^6 + \cdots\right) = \sum_{r=0}^{n} {}_nK_r x^{2r} $$

Here,

$$ \left(1-x^2\right)^{n+1} = \binom{n+1}{0}x^0 - \binom{n+1}{1}x^2 + \cdots + (-1)^{n+1}\binom{n+1}{n+1}x^{2n+2} $$

Substitute this for the above. Then, the left side is

$$ \left\{\binom{n+1}{0}x^0 - \binom{n+1}{1}x^2 + \binom{n+1}{2}x^4 - \binom{n+1}{3}x^6 + \cdots + (-1)^{n+1}\binom{n+1}{n+1}x^{2n+2}\right\} $$
$$ \times\left(1^n x^0 + 3^n x^2 + 5^n x^4 + 7^n x^6 + \cdots\right) $$
$$ = 1^n\binom{n+1}{0}x^0 + \left\{\binom{n+1}{0}3^n - \binom{n+1}{1}1^n\right\}x^2 $$
$$ + \left\{\binom{n+1}{0}5^n - \binom{n+1}{1}3^n + \binom{n+1}{2}1^n\right\}x^4 $$
$$ + \left\{\binom{n+1}{0}7^n - \binom{n+1}{1}5^n + \binom{n+1}{2}3^n - \binom{n+1}{3}1^n\right\}x^6 $$
$$ \vdots $$

The right side is ${}_nK_0 x^0 + {}_nK_1 x^2 + {}_nK_2 x^4 + \cdots + {}_nK_n x^{2n}$. In order for the both sides to be equal for arbitrary x , (4.1) is necessary.

Example

$$ {}_3K_2 = \sum_{k=0}^{2} (-1)^k \binom{4}{k} (5-k)^3 = \binom{4}{0} 5^3 - \binom{4}{1} 3^3 + \binom{4}{2} 1^3 = 23 $$

$$ {}_5K_3 = \sum_{k=0}^{3} (-1)^k \binom{6}{k} (7-k)^6 = \binom{6}{0} 7^5 - \binom{6}{1} 5^5 + \binom{6}{2} 3^5 - \binom{6}{3} 1^5 = 1682 $$

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