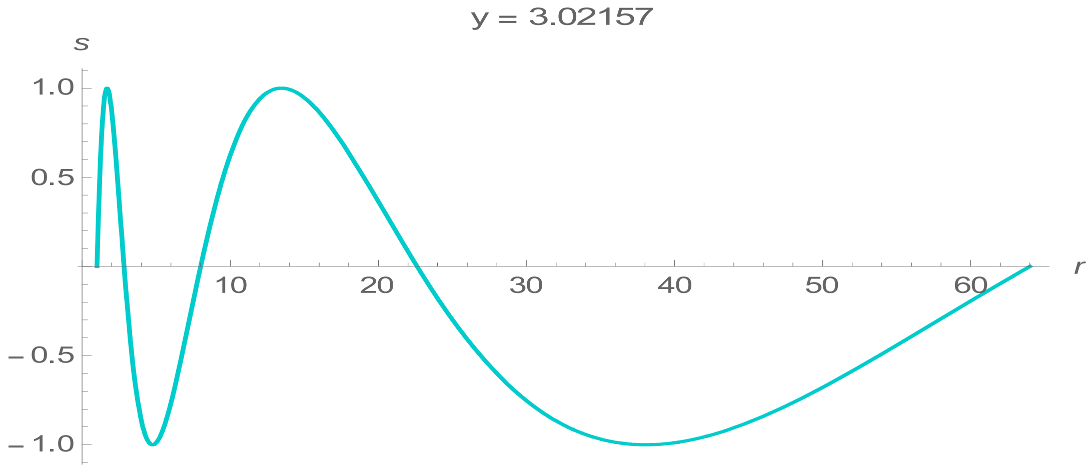
13 Dirichlet Eta type Sine Series

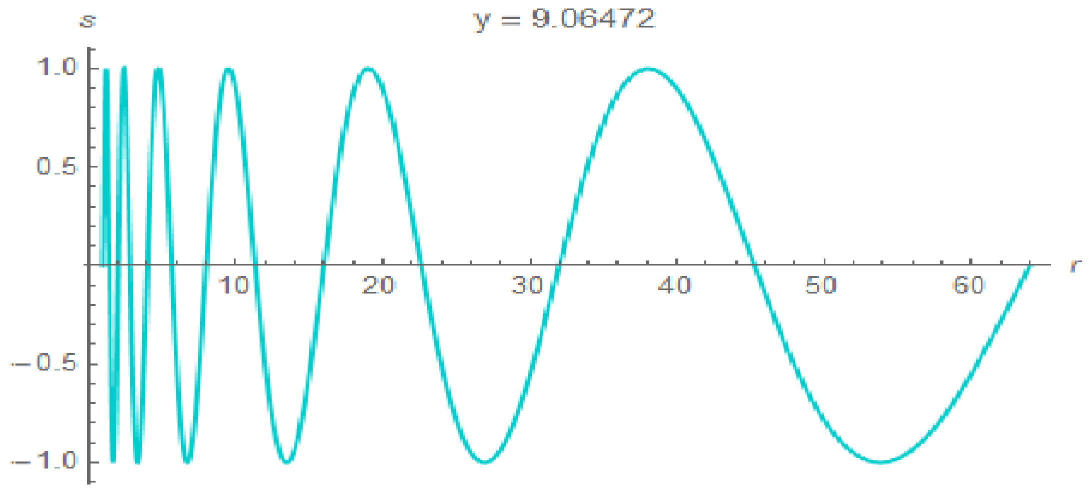
13.1 sin(y log r)

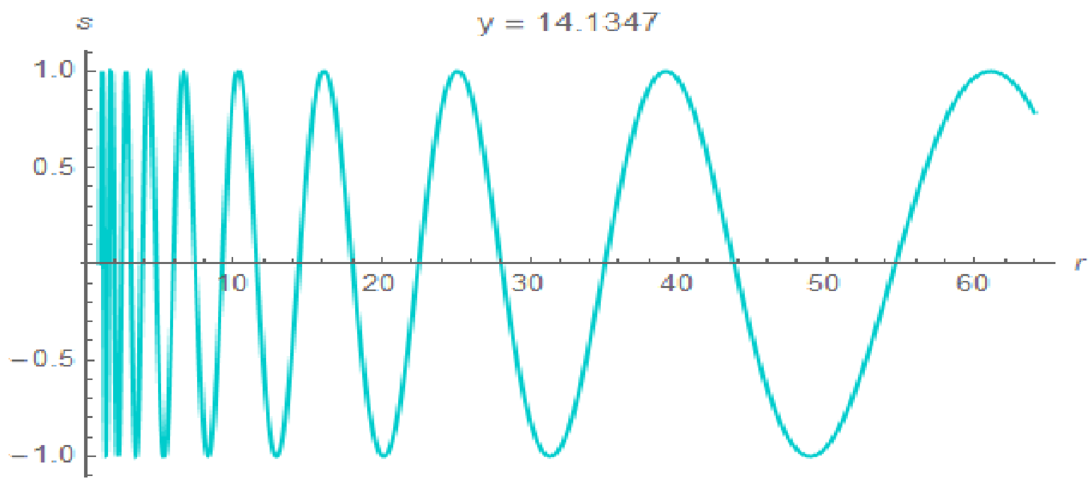
Let r, y are positive numbers respectively, and consider the following function $s(r,y)$.

$$s(r,y) = \sin(y \log r) \tag{1.1}$$

When $y = 3.02157$, 9.06472 , 14.1347 , these 2D figures for $r = 1 \sim 64$ are drawn as follows.







Observation of these reveals the following.

(1) $s(r,y)$ is a variable periodic function with respect to r .

(2) The number of periods within the same interval is approximately proportional to y .

Below, we consider this function in more detail.

Amplitude (A)

The amplitude of this function is $A = 1$.

Period (P)

This function is a periodic function. The first period starts at 0π and ends at 2π , the second period starts at 2π and ends at 4π , so

$$y \log r_0 = 0\pi \ , \ y \log r_1 = 2\pi \ , \ y \log r_2 = 4\pi \ , \ \cdots \ , y \log r_n = 2n\pi \ , \ \cdots$$

From these,

$$r_0 = e^{0\pi/y} \quad , \quad r_1 = e^{2\pi/y} \quad , \quad r_2 = e^{4\pi/y} \quad , \quad \cdots \quad , \ r_n = e^{2n\pi/y} \quad , \quad \cdots$$

Therefore, the function c is separated into the following unit intervals.

$$\left[e^{0\pi/y} \ , \ e^{2\pi/y}\right) \quad , \quad \left[e^{2\pi/y} \ , \ e^{4\pi/y}\right) \quad , \quad \cdots \quad , \quad \left[e^{(2n-2)\pi/y} \ , \ e^{2n\pi/y}\right) \quad , \quad \cdots$$

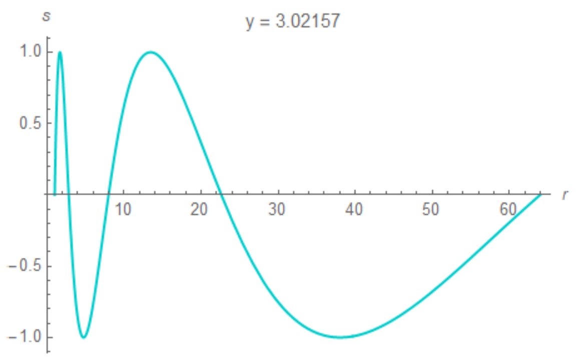
Each of these has one mountain and one valley. We will call these the 1 st period, the 2 nd period, $\cdots$. i.e.

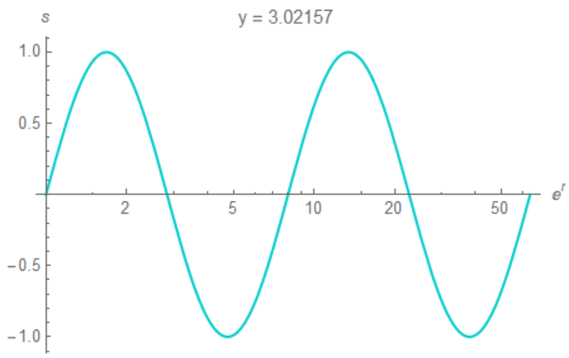
$$P(n,y) = \left[e^{(2n-2)\pi/y} \ , \ e^{2n\pi/y}\right)$$

For example, if $y = 3.02157$, the 1 st period $\left[e^{0\pi/y}, e^{2\pi/y}\right)$ and the 2 nd period $\left[e^{2\pi/y}, e^{4\pi/y}\right)$ are

$$P(1,3.02157) = [1 \ , \ 8) \quad , \quad P(2,3.02157) = [8 \ , \ 64)$$

If these two periods are drawn in succession, it is as follows. The left is normal scale and the right is semi-logarithmic scale. Since these are sinine functions, one cycle is from peak to peak, and there are two peaks in these figures except for the end point.





Wavelength (λ)

The wavelength are the length of these periods. That is,

$$e^{0\pi/y}\left(e^{2\pi/y} - 1\right) \ , \ e^{2\pi/y}\left(e^{2\pi/y} - 1\right) \quad , \quad \cdots \quad , \ e^{(2n-2)\pi/y}\left(e^{2\pi/y} - 1\right) \ , \ \cdots$$

The wavelength is $e^{2\pi/y}$ times longer than the previous period in each period. So, **this function is a variable periodic function**. That is,

$$\lambda(n,y) = e^{(2n-2)\pi/y}\left(e^{2\pi/y} - 1\right)$$

In the figure above, the wavelengths of the 1 st and the 2 nd periods of $c(r,y)$ are

$$\lambda(1,3.02157) = 7 \quad , \quad \lambda(2,3.02157) = 56$$

However, $c(r,y)$ drawn on a semi-logarithmic scale looks like a fixed period at first glance.

When $n=1$, y can be back calculated from λ.

$$y = \frac{2\pi}{\log(\lambda+1)}$$

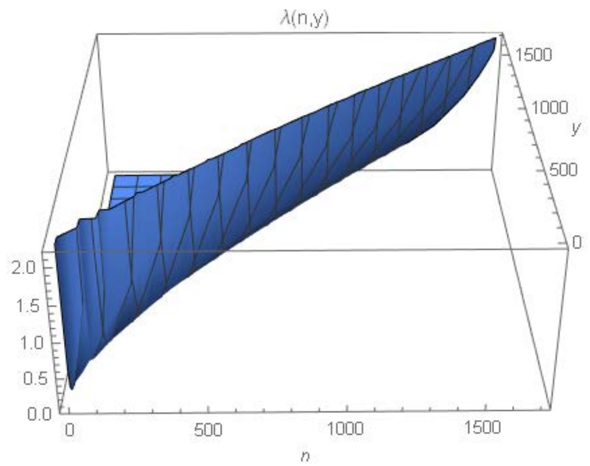
From this,

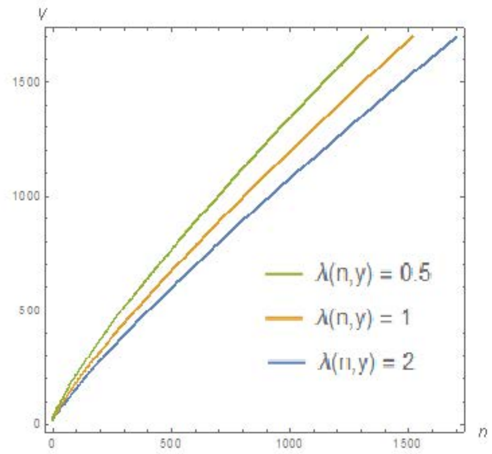
When $\lambda(1,y)=7$, $y = \frac{2\pi}{\log 8} = 3.02157$

When $\lambda(1,y)=1$, $y = \frac{2\pi}{\log 2} = 9.06472$

When $\lambda(1,y)=0.559743$, $y = \frac{2\pi}{\log 1.559743} = 14.1347$

A 3D view of $\lambda(n,y)$ at $n \neq 1$ is shown on the left. And the contour plots at $\lambda=0.5$, $\lambda=1.0$, $\lambda=2.0$ are shown on the right.





From these figures, we can see that the slope of the $\lambda(n,y)$ contour decreases as n increases. The reason why is,

$$\frac{\partial}{\partial n}\lambda(n,y) = \frac{2\pi}{y} e^{(2n-2)\pi/y}\left(e^{2\pi/y}-1\right) > 0 \qquad for\ n,y > 0$$

Using this contour plot, we can find the n,y pair that gives the desired λ.

Mountain (Ms)

Since $s(r,y)$ is a sine function, the mountain is at $1/4$ of the period plotted on a semilogarithmic scale.

$$Ms(n,y) = e^{\frac{(4n-3)\pi}{2y}}$$

In the figure above, the mountains of the 1 st and the 2 nd periods of $s(r,y)$ are

$$Ms(1,3.02157) = 1.68179 \quad , \quad Ms(2,3.02157) = 13.4544$$

Valley (Vs)

Since $s(r,y)$ is a sine function, the vallay is at $3/4$ of the period plotted on a semilogarithmic scale.

$$Vs(n,y) = e^{\frac{(4n-1)\pi}{2y}}$$

In the figure above, the valleys of the 1 st and the 2 nd periods of $s(r,y)$ are

$$Vs(1,3.02157)=4.756843 \quad , \quad Vs(2,3.02157)=38.0548$$

Zeros (Zs)

Since $s(r,y)$ is the sine function, the zeros are at the left edge and middle of the period plotted on a semi-logarithmic scale.

$$Zs(n,y) = \left\{ e^{\frac{(2n-2)\pi}{y}}, e^{\frac{(2n-1)\pi}{y}} \right\}$$

In the figure above, the zeros of the 1 st and the 2 nd periods of $c(r,y)$ are

$$Zs(1,3.02157) = \{1, 2.82843\} \quad , \quad Zs(2,3.02157) = \{8, 22.6275\}$$

Near zeros (Xs)

When the variable r of $s(r,y)$ is a discrete variable, we will call the integer r within ± 0.5 from the zero point **the neighborhood of the zero point**. That is,

$$Xs(n,y) = \left\{ Round\left(e^{\frac{(2n-2)\pi}{y}}\right), Round\left(e^{\frac{(2n-1)\pi}{y}}\right) \right\}$$

In the figure above,

$$Xs(1,3.02157)=\{1,3\} \quad , \quad Xs(2,3.02157)=\{8, 23\}$$

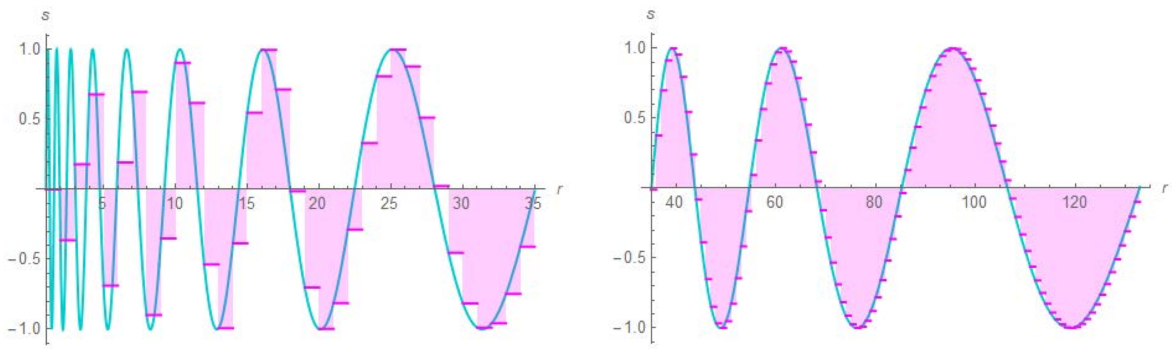
Riemann Zeta type sine Series (when $y = 14.1347\cdots$)

Until now, we have treated r as a continuous variable in the function $s(r,y)$, but in this section, we consider the following Riemann Zeta type sine function with respect to y.

$$v(y) = \sum_{r=1}^{\infty} \sin(y \log r) \tag{1.2}$$

Then, r has to be a discrete variable.

For example, when $y_1=14.1347\cdots$, $r=1,2,\cdots,34$ and $r=35,36,\cdots,132$ are drawn in succession, it is as follows.



The function value of (1.2) is the sum of the areas of magenta. In the left figure, this sum differs greatly from the integral value of (1.1). On the other hand, in the right figure this sum is close to the integral value of (1.1).

Divergence

In the right figure, the area seems to be zero due to cancellation of plus and minus, but it is not. Because, the interval between waves expands and eventually becomes infinite. So, the series in (1.2) diverges. In fact,

$$v(y) \backsim \int_1^\infty sin(y\,log\,r)\,dr$$

But, the right hand side becoms

$$\int_1^\infty sin(y\,log\,r)\,dr = \left[\frac{r\{-y\,cos(y\,log\,r) + sin(y\,log\,r)\}}{1+y^2} \right]_1^\infty$$

$$= \pm\infty$$

Note

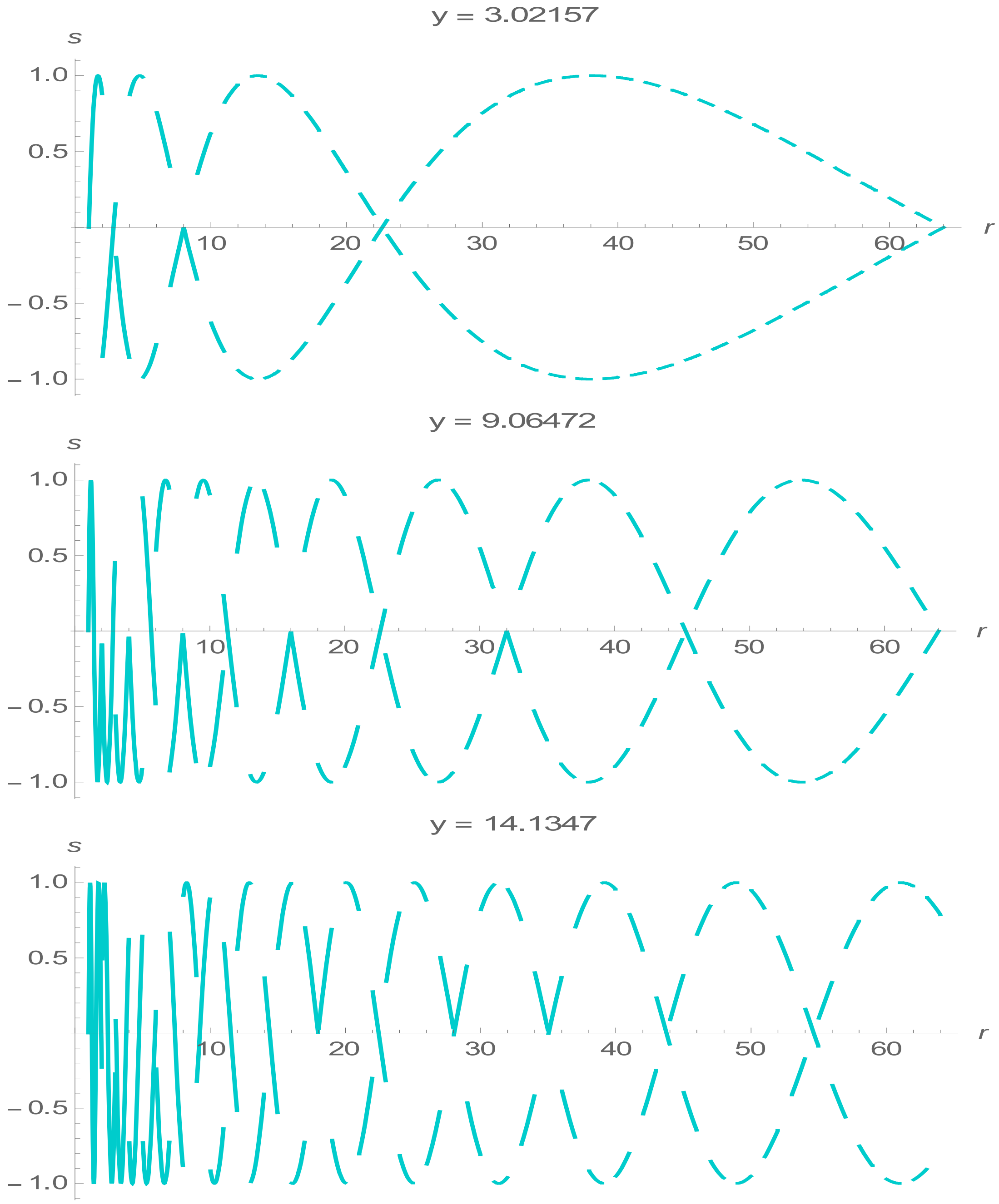
(1.2) becoms only an asymptotic expansion even if it is accelerated.

13.2 ± sin(y log r)

Let r, y are positive numbers respectively, and consider the following function $s(r, y)$.

$$s(r,y) = (-1)^{\lfloor r-1 \rfloor} \sin(y \log r) \qquad (\lfloor\ \rfloor \text{ is floor function }) \tag{2.1}$$

When $y = 3.02157$, 9.06472 , 14.1347 , these 2D figures for $r = 1 \sim 64$ are drawn as follows.



Unlike the previous section, $s(r, y)$ is a discontinuous function with respect to r . However, it is the same as the previous section that $s(r, y)$ is a variable periodic function and that the number of cycles in the same interval is approximately proportional to y .

Amplitude (A)

The amplitude of this function is

$$A(r) = \left| (-1)^{\lfloor r-1 \rfloor} \right| = 1$$

Period (P)

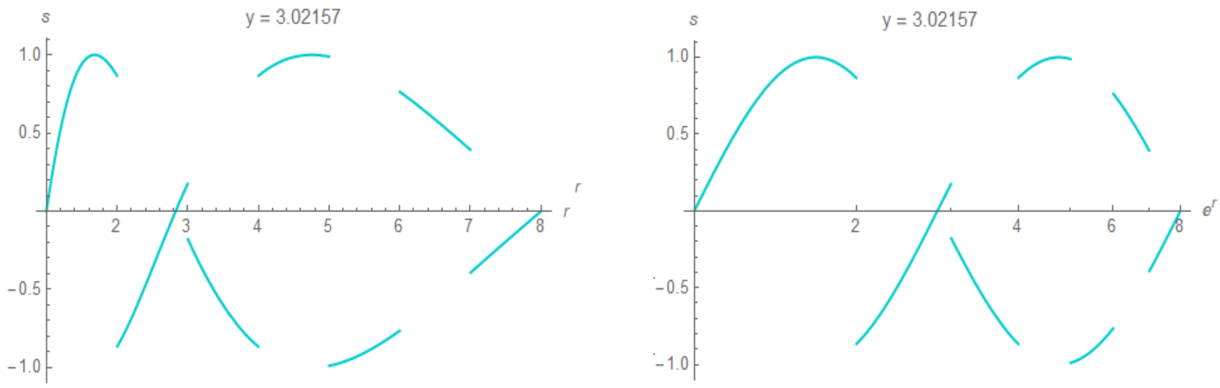
The period of this function is the same as in the previous section, That is,

$$P(n,y) = \left[e^{(2n-2)\pi/y}, e^{2n\pi/y}\right)$$

For example, if $y=3.02157$, the 1 st period $\left[e^{0\pi/y}, e^{2\pi/y}\right)$ is

$$P(1,3.02157) = [1, 8)$$

This is drawn as follows. The left is normal scale and the right is semi-logarithmic scale.



Wavelength (λ)

The wavelength of this function is the same as in the previous section, That is,

$$\lambda(n,y) = e^{(2n-2)\pi/y}\left(e^{2\pi/y}-1\right)$$

In the figure above,

$$\lambda(1,3.02157) = 7$$

Mountain or Valley (MVs)

Unlike the previous section, this function $s(r,y)$ changes sign. For this reason, mountains and valleys exist at most twice as many as in the previous section.

$$MVs(n,y) = \left\{e^{\frac{(4n-3)\pi}{2y}}, e^{\frac{(4n-1)\pi}{2y}}\right\}$$

The mountain or valley is determined by the sign of $s(r,y)$ at $r = MVs(n,y)$.

In the figure above,

$$MVs(1,3.02157) = \{1.68179, 4.75684\}$$

$$\{s(1.68179, 3.02157), s(4.75684, 3.02157)\} = (1,1)$$

So, both the former and the latter are mountains.

Zeros (Zs)

The zeros of this function are the same as in the previous section, That is,

$$Zs(n,y) = \left\{e^{\frac{(2n-2)\pi}{y}}, e^{\frac{(2n-1)\pi}{y}}\right\}$$

In the figure above,

$$Zs(1,3.02157) = \{1, 2.82843\}$$

Constriction (Xs)

Since this function $s(r,y)$ changes sign, the zero point looks like a constriction. So, we will call the integer r within ± 0.5 from the zero point **constriction**. That is,

$$Xs(n,y) = \left\{ Round\left(e^{\frac{(2n-2)\pi}{2y}} \right), Round\left(e^{\frac{(2n-1)\pi}{2y}} \right) \right\}$$

In the figure above,

$$Xs(1,3.02157) = \{1,3\}$$

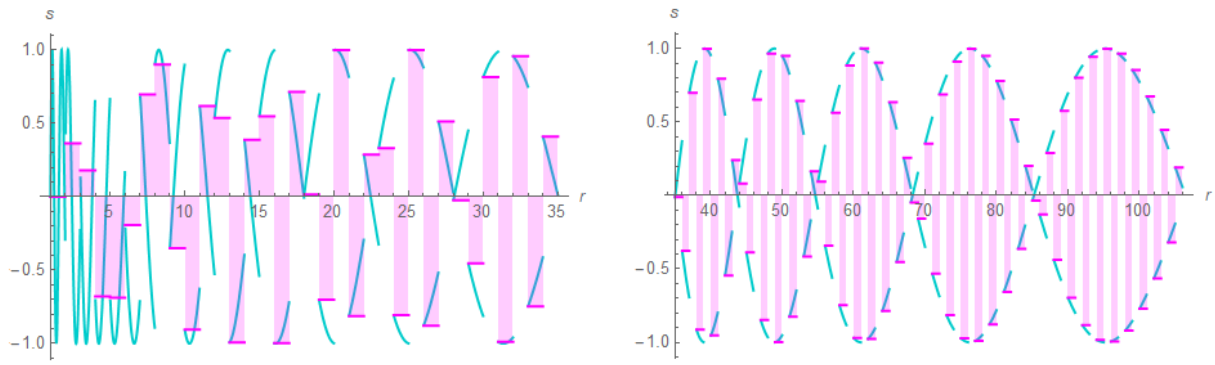
Dirichlet Eta type sine Series (when $y = 14.1347\cdots$)

We consider the following Dirichlet Eta type sine function with respect to y .

$$v(y) = \sum_{r=1}^{\infty} (-1)^{r-1} \sin(y \log r) \tag{2.2}$$

Then, r has to be a discrete variable.

For example, when $y_1 = 14.1347\cdots$, $r = 1, 2, \cdots, 34$ and $r = 35, 36, \cdots, 105$ are drawn in succession, it is as follows.



The function value of (2.2) is the sum of the areas of magenta. In the left figure, this sum differs greatly from the integral value of (2.1). On the other hand, in the right figure this sum is close to the integral value of (2.1) .

Convergence ?

The last two constrictions in the right figure belong to the 11 th period. The area between two constrictions seems to cancel out to zero. As a trial, when $y_1 = 14.1347\cdots$, the area between each constriction in the 11 th and the 25 th periods are calculated as follows.

The 11 th period $\quad Xs(11, y_1) = (85, 106)$, $\lambda(11, y_1) = 47.7$

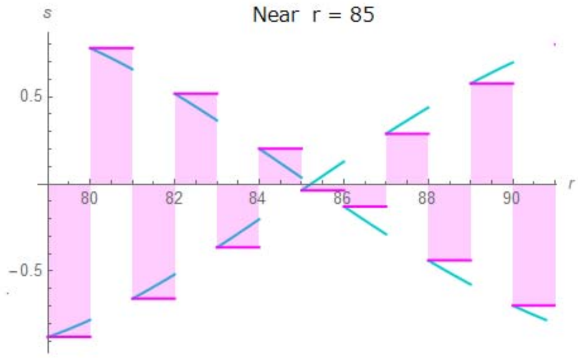
$$v_{11}(y_1) = \sum_{r=85}^{105} (-1)^{r-1} \sin(y_1 \log r) = 0.00208785$$

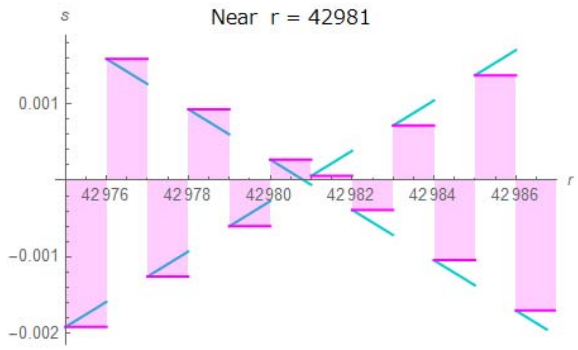
The 25 th period $\quad Xs(25, y_1) = (42981, 53679)$, $\lambda(25, y_1) = 24058.2$

$$v_{25}(y_1) = \sum_{r=42981}^{53678} (-1)^{r-1} \sin(y_1 \log r) = -0.0000684506$$

Certainly, the area between two constrictions approaches 0 as r increases.

To find out the cause of this, let us compare the enlarged images near $r = 85$ and $r = 42981$.





Then, at a glance, it can be seen that the scale of the vertical axis is an order of magnitude. Why? The reason is simple. Because, the wavelength becomes longer as r moves away from the origin. Since the amplitude is 1, the longer the wavelength, the slower the slope of the variable-length sine curve. In fact, if we differentiate the unsigned part of (2.1) with respect to r,

$$\frac{\partial}{\partial r}s(r,y) = \frac{y}{r}cos(y\log r)$$

Let $r \to \infty$ then $\partial s(r,y)/\partial r \to 0$.

Thus, the area between two adjacent constrictions approaches 0 as r increases.

Maximum Error

However, even so, this series $v(y)$ is a divergent series. i.e. it just oscillates and never converges. Therefore, if this series is truncated at mountain or valley, there will be a maximum error of ± 0.5. In fact, in the example above, truncating r at the 30 th period valley of 553771 yielded an error of -0.5 as follows. In addition the correct value is the Dirichlet Eta function value $-Im(\eta(0+i\,14.1347\cdots)) = -0.0657473$.

```
v[y_, m_] := Σ_{r=1}^{m} (-1)^(r-1) Sin[y Log[r]]

N[MVs[30, y1]]        {443408., 553771.}

N[v[y1, 553771]]      N[-Im[DirichletEta[i y1]]]
-0.565747             -0.0657473

-0.565747 - (-0.0657473) = -0.5
```

Minimum Error

On the other hand, a better approximation is obtained if the series is truncated at an appropriate constriction. In fact, in the example above, when r was truncated at the 30 th period constriction 495526, it was in consistent with the Dirichlet Eta function value $-Im(\eta(0+i\,14.1347\cdots)) = -0.0657473$.

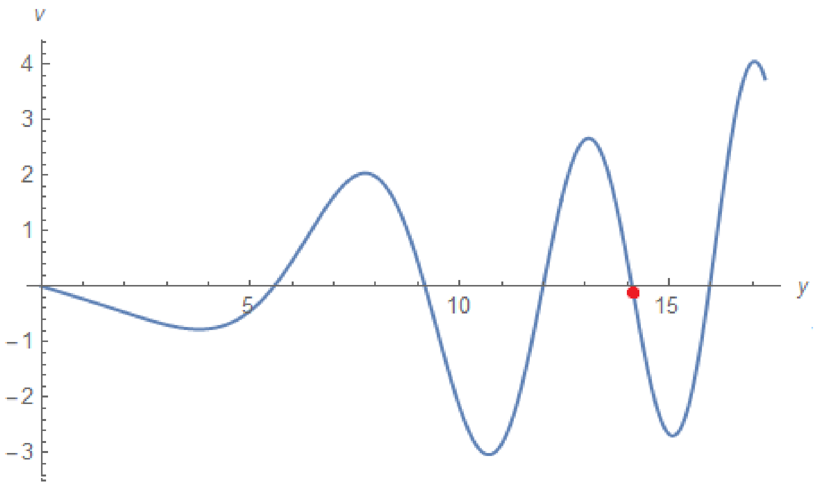
```
Xs[30, y1]          {396771, 495526}

N[v[y1, 495526]]    -0.0657471
```

Summation Method and Convergence Acceleration

After all, this series (2.2) oscillates within an error of ± 0.5. A summation method insists that an error of ± 0.5 should be regarded as 0 on average. One of the simplest summation methods is Euler Transformation. The Euler transformation accelerates the convergence of the series and also applies the summation method. Applying the Euler transformation to (2.2) ,

$$v(y,m) = \sum_{k=1}^{m} \sum_{r=1}^{k} \frac{1}{2^{k+1}} \binom{k}{r} (-1)^{r-1} \sin(y \log r) \qquad (2.2')$$

And the 2D figure is



The value of (2.2') at y_1=14.1347⋯ in this figure is as follows. It is in consistent with the Dirichlet Eta function value $-Im(\eta(0+i\,14.1347\cdots))$.

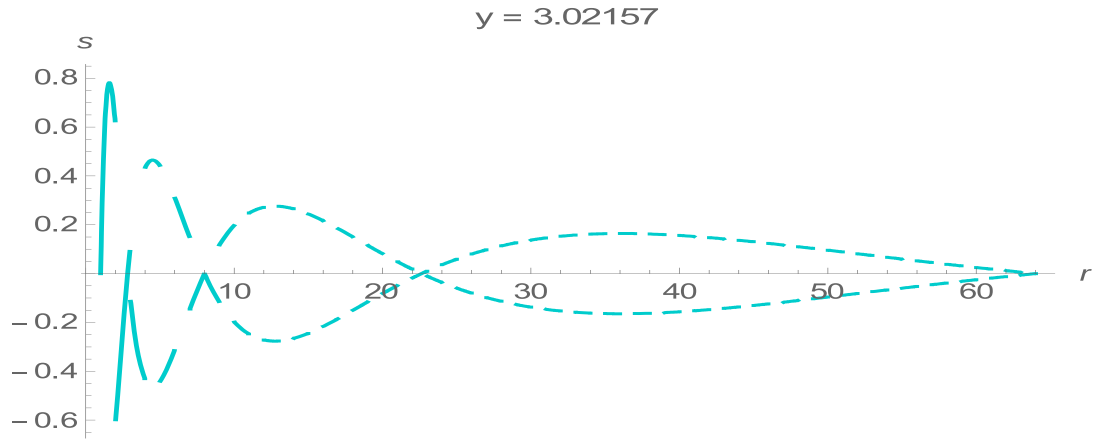
```
N[v[y1, 42]]          -0.0657473
```

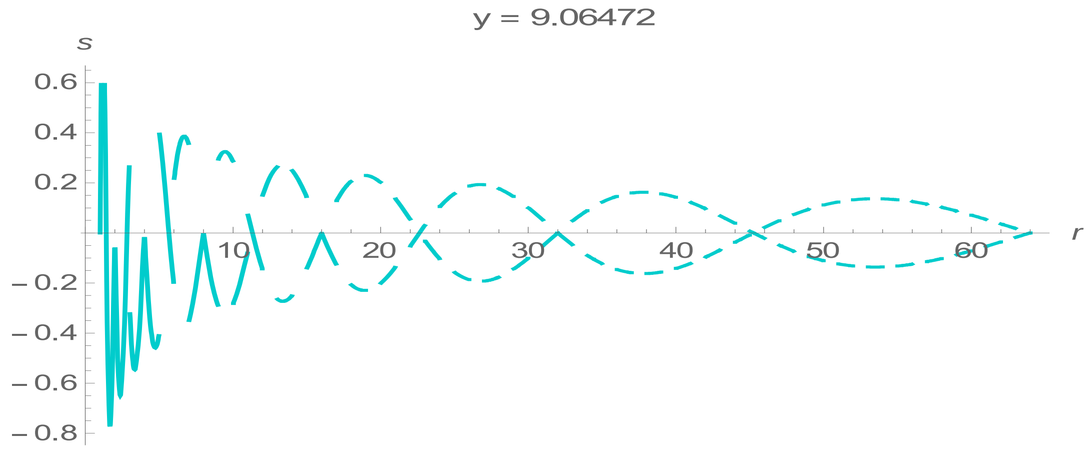
13.3 ± sin(y log r) / r^x

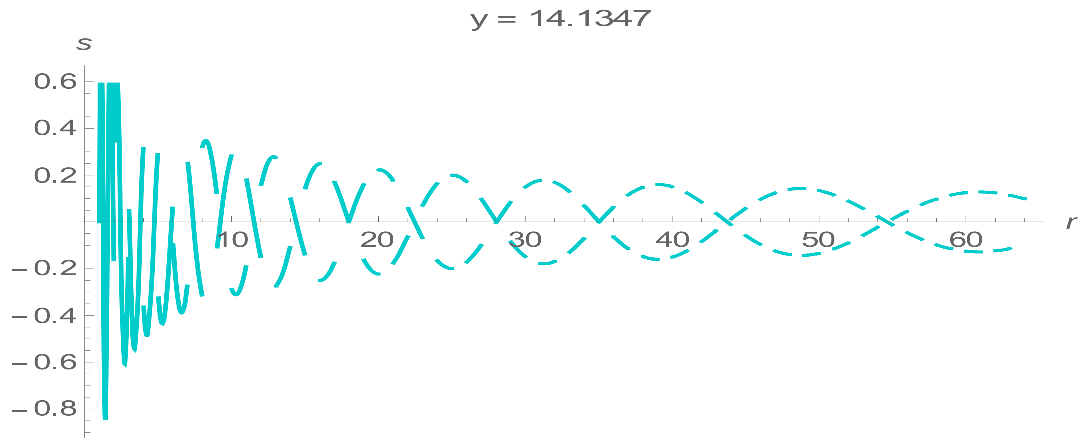
Let r, x, y are positive numbers respectively, and consider the following function $s(r,x,y)$.

$$s(r,x,y) = \frac{(-1)^{\lfloor r-1 \rfloor}}{r^x} \sin(y \log r) \qquad (\lfloor\ \rfloor \text{ is floor function }) \tag{3.1}$$

When $x = 1/2$, $y = 3.02157$, 9.06472, 14.1347, these 2D figures for $r = 1 \sim 64$ are drawn as follows.







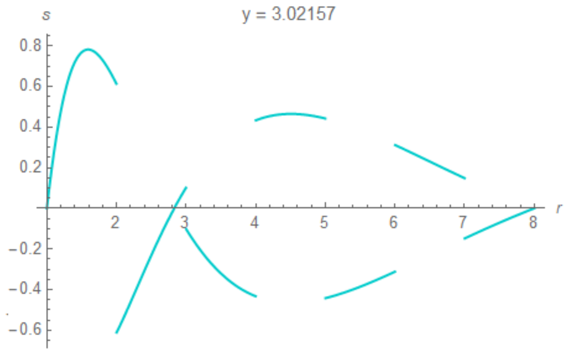
Amplitude (A)

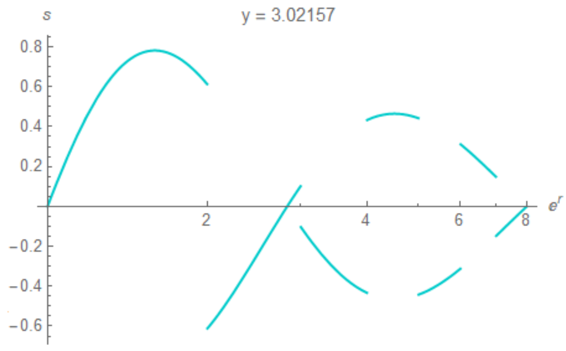
The amplitude of this function is

$$A(r,x) = \left| \frac{(-1)^{\lfloor r-1 \rfloor}}{r^x} \right| = \frac{1}{r^x}$$

When $x > 0$ $\lim_{r\to\infty} 1/r^x = 0$. Therefore, this function represents a damped oscillation.

For example, when $x=1/2, y=3.02157$, the 1st period $\left[e^{0\pi/y}, e^{2\pi/y}\right)$ of $s(r,x,y)$ is drawn as follows. The left is normal scale and the right is semi-logarithmic scale.





Period (P)

The period of this function is the same as in the previous section, That is,

$$P(n,y) = \left[e^{(2n-2)\pi/y}, e^{2n\pi/y}\right)$$

In the figure above,

$$P(1,3.02157) = [1, 8)$$

Wavelength (λ)

The wavelength of this function is the same as in the previous section, That is,

$$\lambda(n,y) = e^{(2n-2)\pi/y}\left(e^{2\pi/y} - 1\right)$$

In the figure above,

$$\lambda(1,3.02157) = 7$$

Mountain or Valley (MVs)

The mountains or valleys of this function are the same as in the previous section, That is,

$$MVs(n,y) = \left\{e^{\frac{(4n-3)\pi}{2y}}, e^{\frac{(4n-1)\pi}{2y}}\right\}$$

In the figure above,

$$MVs(1,3.02157) = \{1.68179, 4.75684\}$$

$$\{s(1.68179, 1/2, 3.02157), s(4.75684, 1/2, 3.02157)\}$$
$$= \{0.771106, 0.458501\}$$

So, both the former and the latter are mountains.

Zeros (Zs)

The zeros of this function are the same as in the previous section, That is,

$$Zs(n,y) = \left\{e^{\frac{(2n-2)\pi}{y}}, e^{\frac{(2n-1)\pi}{y}}\right\}$$

In the figure above,

$$Zs(1,3.02157) = \{1, 2.82843\}$$

Constriction (Xs)

The constrictions of this function are the same as in the previous section, That is,

$$Xs(n,y) = \left\{ Round\left(e^{\frac{(2n-2)\pi}{2y}} \right), Round\left(e^{\frac{(2n-1)\pi}{2y}} \right) \right\}$$

In the figure above,

$$Xs(1,3.02157) = \{1,3\}$$

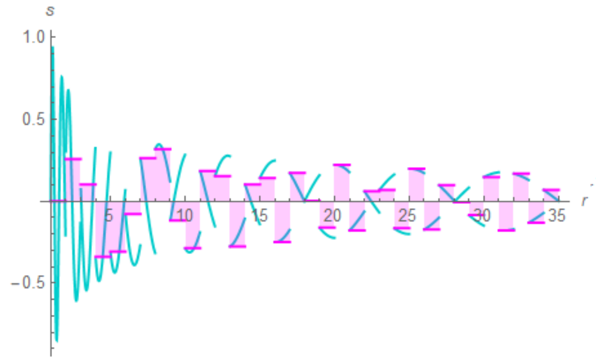
Dirichlet Eta type sine Series (when $x = 1/2$, $y = 14.1347\cdots$)

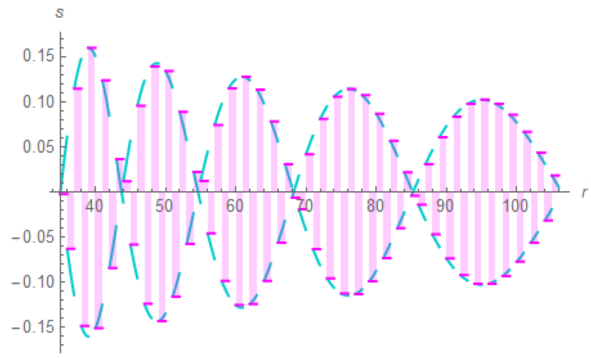
We consider the following Dirichlet Eta type sinine function with respect to x,y .

$$v(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^x} \sin(y \log r) \tag{3.2}$$

Then, r has to be a discrete variable.

For example, when $x=1/2$, $y=14.1347\cdots$, $r=1,2,\cdots,34$ and $r=35,\cdots,105$ are drawn in succession, it is as follows.





The function value of (3.2) is the sum of the areas of magenta. In the left figure, this sum differs greatly from the integral value of (3.1). On the other hand, in the right figure this sum is close to the integral value of (3.1) .

Convergence

The last two constrictions in the right figure belong to the 11 th period. The area between two constrictions seems to cancel out to zero. As a trial, when $x=1/2, y=14.1347\cdots$, the area between each constriction in the 11 th and the 25 th periods are calculated as follows.

The 11 th period $\quad Xs(11, y_1) = \{85, 106\}$, $\lambda(11, y_1) = 47.7$

$$v_{11}\left(\frac{1}{2}, y_1\right) = \sum_{r=85}^{105} \frac{(-1)^{r-1}}{r^{1/2}} \sin(y_1 \log r) = -0.000473572$$

The 25 th period $\quad Xs(25, y_1) = (42981, 53679)$, $\lambda(25, y_1) = 24058.2$

$$v_{25}\left(\frac{1}{2}, y_1\right) = \sum_{r=42981}^{53678} \frac{(-1)^{r-1}}{r^{1/2}} \sin(y_1 \log r) = -0.00000032251$$

Comparing the two, the area between the two constrictions rapidly converges to 0 as r increases. The reason for this is clear. Even when the amplitude is 1 , the area between constrictions decreases as r increases. Moreover, the amplitude approaches 0 . The synergistic effect forces the area between constrictions to approach

0 even more rapidly. In this case, $\sum 1/r^x$ does not have to converge. It only needs to be $\lim_{r\to\infty} 1/r^x = 0$.
Thus, (3.2) converges if $x > 0$.

Minimum Error

Therefore, in calculating the series $v(x,y)$, a better approximation must be obtained by truncating r at an appropriate constriction.

In fact, when $x=1/2$, $y_1=14.1347\cdots$, if r was truncated at the 25 th period constriction 42981 and compared with the Dirichlet Eta function value $-Im(\eta(1/2+i\,14.1347\cdots) = 0)$, both coincided up to 6 decimal places.

$$v[x\_, y\_, m\_] := \sum_{r=1}^{m} \frac{(-1)^{r-1}}{r^x} \mathrm{Sin}[y\,\mathrm{Log}[r]]$$

$\mathrm{Xs}[25, y_1]$ $\{42981, 53679\}$

$N\left[v\left[\frac{1}{2}, y_1, 42981\right]\right]$ 5.35779×10^{-7}

Maximum Error

This series $v(x,y)$ converges. However, if this series is truncated at mountains or valley, an error of at most $\pm 1/r^x \times 1/2$ will occur. In fact, in the example above, when r was truncated at the valley 59988 of the 25 th period, both coincided up to 2 decimal places. This error is $3{,}800$ times larger than the minimum error above.

$N[\mathrm{MVs}[25, y_1]]$ $\{48032.8, 59988.\}$

$N\left[v\left[\frac{1}{2}, y_1, 59988\right]\right]$ $N\left[\frac{1}{59988^{1/2}} \times \frac{1}{2}\right]$

0.00204144 0.00204145

13.4 Amplitude of v(y)

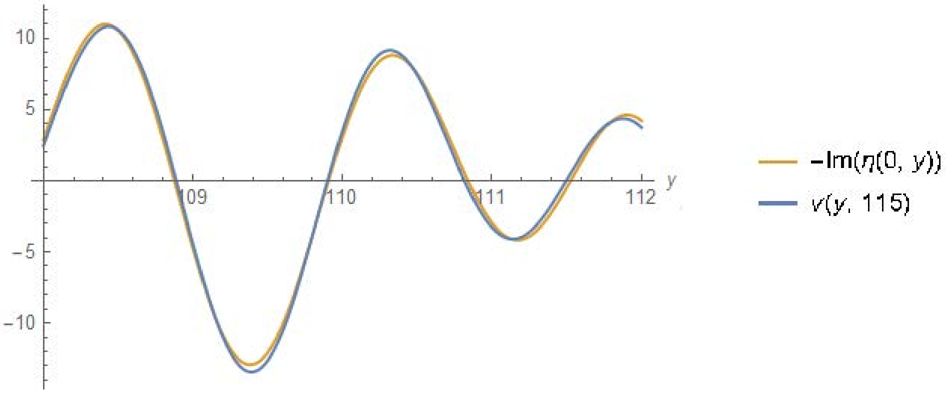
The Dirichlet-Eta type sine function and the series dealt with in Section 2 were as follows.

$$s(r,y) = (-1)^{\lfloor r-1 \rfloor} \sin(y \log r) \qquad (\lfloor\ \rfloor \text{ is floor function }) \tag{2.1}$$

$$v(y) = \sum_{r=1}^{\infty} (-1)^{r-1} \sin(y \log r) \tag{2.2}$$

In this section, we study the amplitudes (mountains, valleys, zeros) of (2.2) using (2.1) .

As a numerical example, drawing the imaginary part of $\eta(0,y)$ near $y=110$ together with $v(y)$ is



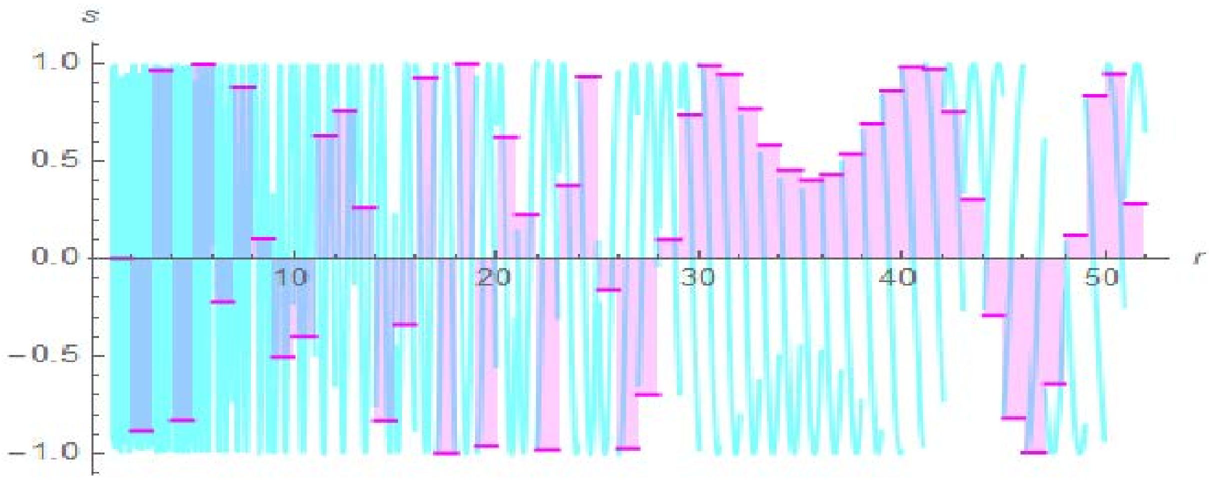
Both are almost the same, but $\eta(0,y)$ with high precision is used in the following calculations.

13.4.1 Mountain of v(y) (near y = 110.5)

Accurate calculation of the mountain near $y=110.5$ using $\eta(0,y)$ is as follows.

```
FindMaximum[-Im[η[0, y]], {y, 110.5}]
{8.81197, {y → 110.337}}      yM := 110.337
```

Drawing $s(r,y_M)$ by (2.1) is as follows. The horizontal axis is r . Cyan is drawn as a continuous variable and magenta as a discrete variable. The sum of the area of magenta becoms mountain 8.812 of (2.2) .



In this figure, three intervals are observed in which magenta is continuously positive. That is,

(1) $r = 28 \sim 43$ are positive for 16 consecutive terms. These are included in the 59 ~ 67 th period.

The wavelengths of the periods are 1.59 ~ 2.51 , and the central value is 2/1 .

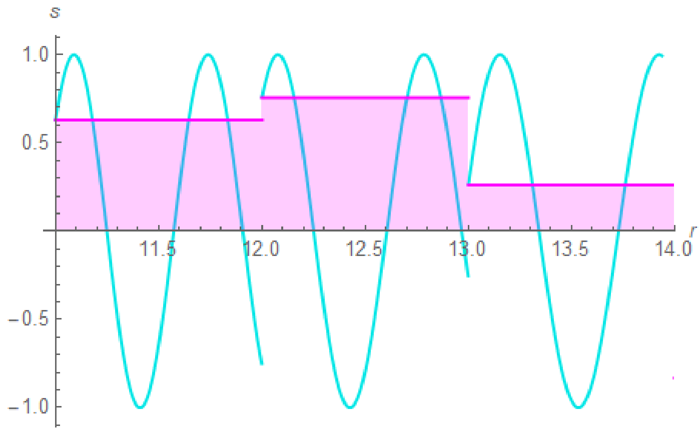
(2) $r = 11 \sim 13$ are positive for 3 consecutive terms. These are included in the 43 ~ 45 th period.

The wavelengths of the periods are 0.64 ~ 0.72 , and the central value is 2/3 (=0.67) .

(3) $r = 7,8$ are positive for 2 consecutive terms. These are included in the $35 \sim 37$ th period.

The wavelengths of the periods are $0.41 \sim 0.46$, and these are close to $2/5\ (=0.4)$.

After $r = 52$, the positive and negative gradually balance out, so these three continuous intervals almost determine the height of the mountain near $y=110.5$. In addition, the same sign continues in the vicinity of the wavelength $\lambda=2/(2k-1)\ k=1,2,3,\cdots$. I will explain this reason by the enlarged view of **(2)** (the central value is $2/3$) which is the easiest to understand.



When $r=11$, $s(11,110.337) = (-1)^{\lfloor 10 \rfloor} sin(110.337\ log\,11) = +0.630$

When $r=11+2/3$, $s(11.67,110.337) = (-1)^{\lfloor 10.67 \rfloor} sin(110.337\ log\,11.67) = +0.778$

When $r=11+3/3$, $s(12,110.337) = (-1)^{\lfloor 11 \rfloor} sin(110.337\ log\,12) = -(-0.756)$

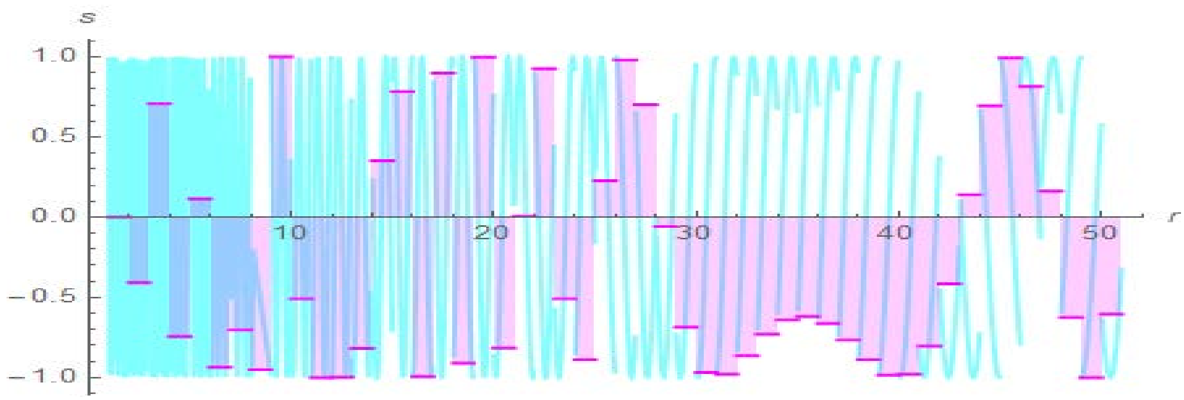
i.e. because the end of the half wavelength is close to the end of the interval of length 1 . This also applies to $\lambda=2/5, 2/7, \cdots$.

13.4.2 Valley of v(y) (near y= 109.5)

Accurate calculation of the valley near $y=109.5$ using $\eta(0,y)$ is as follows.

```
FindMinimum[-Im[η[0, y]], {y, 109.5}]
{-12.9166, {y → 109.382}}      yv := 109.382
```

Drawing $s(r,y_V)$ by (2.1) is as follows. The horizontal axis is r . Cyan is drawn as a continuous variable and magenta as a discrete variable. The sum of the area of magenta becoms valley -12.917 in (2.2) .



In this figure, three intervals are observed in which magenta is continuously negative. That is,

(1) $r = 29 \sim 42$ are negative for 14 consecutive terms. These are included in the $59 \sim 66$ th period.

The wavelengths of the periods are $1.65 \sim 2.47$, and the central value is $2/1$.

(2) $r = 10 \sim 13$ are negative for 4 consecutive terms. These are included in the $41 \sim 45$ th period.

The wavelengths of the periods are $0.59 \sim 0.74$, and the central value is $2/3\ (=0.67)$.

(3) $r = 6 \sim 8$ are negative for 3 consecutive terms. These are included in the $33 \sim 36$ th period.

The wavelengths of the periods are $0.37 \sim 0.44$, and the central value is $2/5\ (=0.4)$.

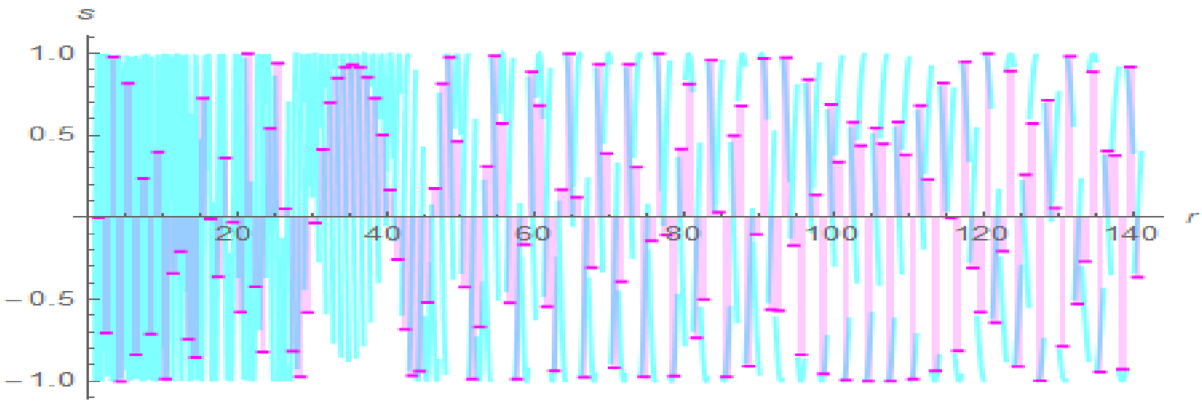
After $r = 51$, the positive and negative gradually balance out, so these three continuous intervals almost determine the depth of the valley near $y = 109.5$.

Re: Zeros of v(y) (near y = 110)

Accurate calculation of the zero near $y = 109.5$ using $\eta(0,y)$ is as follows.

```
FindRoot[-Im[η[0, y]], {y, 110}]
{y → 109.907}                    y_z := 109.907
```

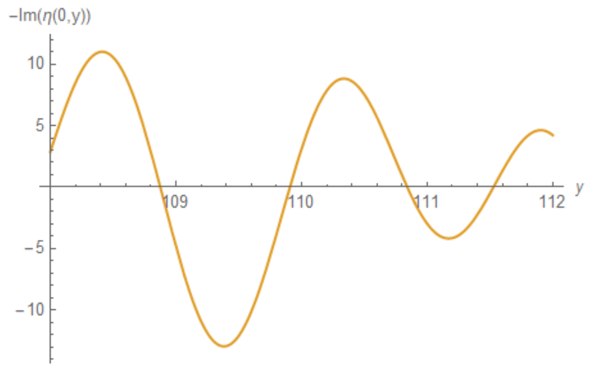
And drawing $s(r, y_Z)$ by (2.1) is as follows.

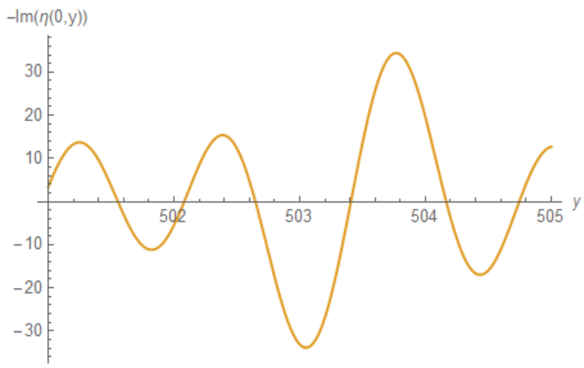


The same sign continues near wavelengths $\lambda = 2/1,\ 2/3$. However, the positive and negative values are canceled as a whole, and the total area of magenta at $r = 1 \sim 140$ is 0.01. So, the sum of the areas after $r = 140$ becomes -0.01.

13.4.3 Amplitude of v(y) and Variable y

If $v(y)\ \ (= -Im\{\eta(0,y)\})$ at $y = 108 \sim 112$ and at $y = 501 \sim 505$ are drawn side by side, it is as follows. The left is the former and the right is the latter.





Comparing the two figures reveals the following.

(1) The amplitudes at $y = 501 \sim 505$ are generally larger than those at $y = 108 \sim 112$.

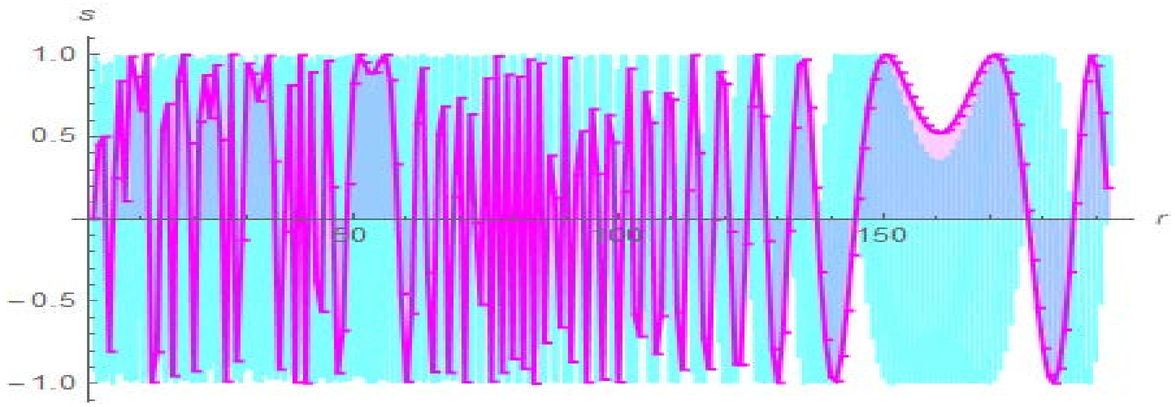
(2) The cycles at $y = 501 \sim 505$ are generally shorter than those at $y = 108 \sim 112$.

Both of these are difficult to prove analytically. However, a graphical proof of **(1)** is possible using $s(r,y)$
It is shown below.

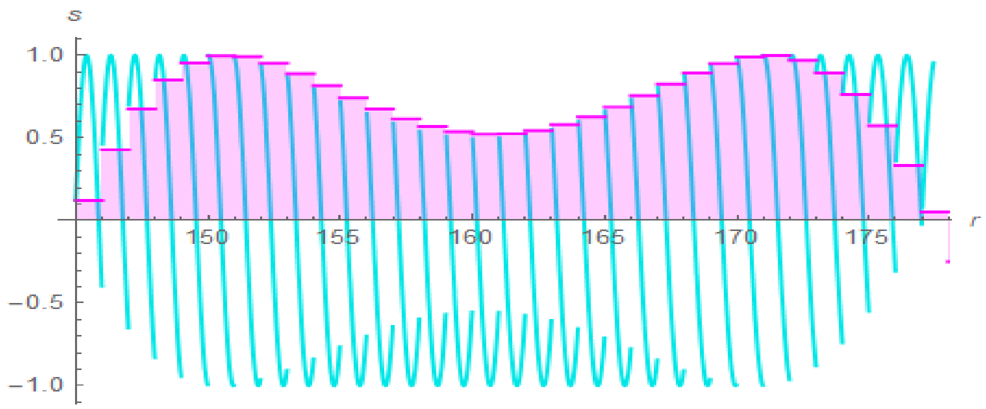
Accurate calculation of the mountain near $y=504$ using $\eta(0,y)$ is as follows.

```
FindMaximum[-Im[η[0, y]], {y, 504}]
{34.4319, {y → 503.767}}        yM := 503.767
```

Drawing $s(r,y_M)$ by (2.1) is as follows. The horizontal axis is r . Cyan is drawn as a continuous variable and magenta as a discrete variable. The sum of the area of magenta becoms mountain 34.432 in (2.2) .

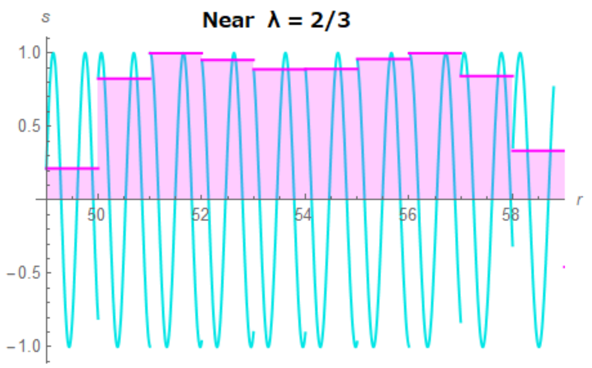


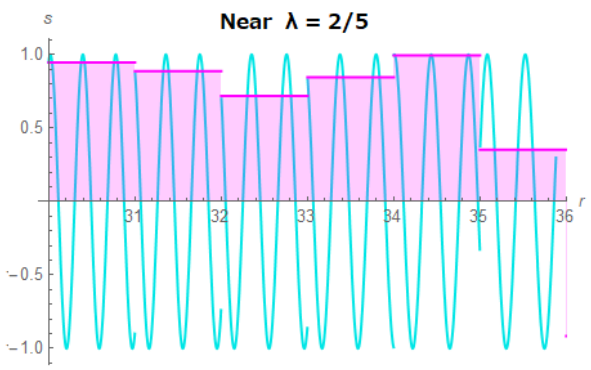
An enlarged view of the wavelength around 2/1 is as follows.



(1) The wavelength in this figure are $1.82 \sim 2.22$. These are included in the $400 \sim 416$ th period.
$r = 145 \sim 177$ are positive for 33 consecutive terms. The interval length is about twice one of 13.4.1 (1) .

An enlarged views of the wavelength around 2/3 and 2/5 are as follows.





(2) The wavelength in the left figure are $0.61 \sim 0.72$. These are included in the $313 \sim 326$ th period. $r = 49 \sim 58$ are positive for 10 consecutive terms. The interval length is about 3 times one of 13.4.1 (2) .

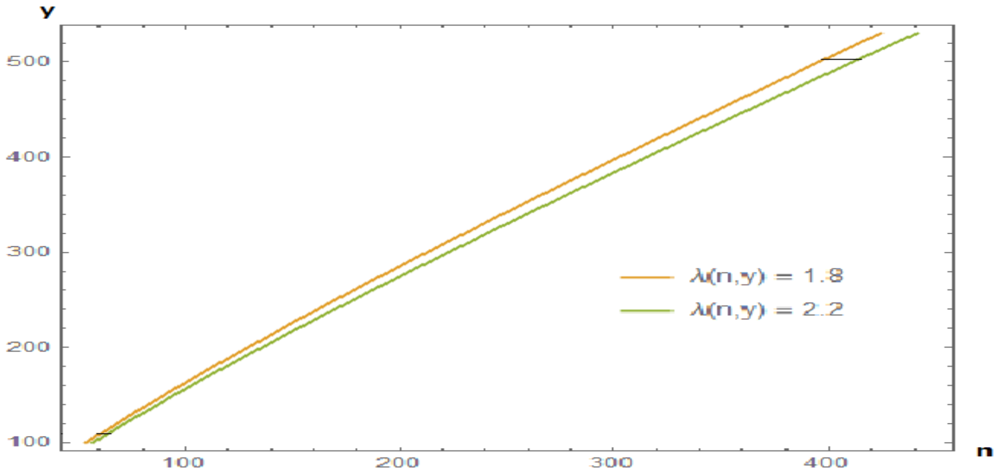
(3) The wavelength in the right figure are $0.37 \sim 0.43$. These are included in the $273 \sim 285$ th period. $r = 30 \sim 35$ are positive for 6 consecutive terms. The interval length is 3 times one of 13.4.1 (3) .

(4) In (1) ~ (3) , it is observed that the number of positive continuous terms increases as y increases.

The reason lies in the definition of wavelength. that is,

$$\lambda(n,y) = e^{(2n-2)\pi/y}\left(e^{2\pi/y} - 1\right)$$

When the near of $\lambda=2$ is ± 2 , the contour plots of $\lambda(n,y)=1.8$ and $\lambda(n,y)=2.2$ are drawn as follows. The vertical axis is y and the horizontal axis is the period number n



The allowable range for the wavelength λ near $y=110$ is the lower left black horizontal line, and the one for the λ near $y=504$ is the upper right black horizontal line. Since the contour at $\lambda=2.2$ has a smaller slope than the contour at $\lambda=1.8$, the upper right horizontal line is considerably longer than the lower left horizontal line. That is, the range of n near $y=504$ is wider than the one near $y=110$. This means that the number of positive terms near $y=504$ is greater than the number of ones near $y=110$. As a result, the mountain near $y=504$ tends to be higher than the one near $y=110$. This also applies to $\lambda=2/3$, $\lambda=2/5$.

(5) As y increases, the period number n that gives the wavelength $\lambda=2$ also increases. Then, the influence of $\lambda = 2/7$, $2/9$, ⋯ also increases. In fact, the near of $\lambda=2/7$ is $r = 21 \sim 25$ and these are 5 consecutive positive terms. Furthermore, $r = 5 \sim 11$ are 7 consecutive positive terms, and these wavelengths are close to $2/15 \sim 2/29$.

(6) As a result of the above, the mountain in $v(y)$ near $y=504$ is higher than the one near $y=110.5$

(7) Although mountains have been used as examples so far, the same applies to valleys. So, the amplitudes of $v(y)$ at $y = 501 \sim 505$ are generally larger than those at $y = 108 \sim 112$.

Note

(7) is similar to ***Bergmann's Law*** (Bears in high latitudes are generally larger than bears in low latitudes.).

13.5 Amplitude of v(x, y)

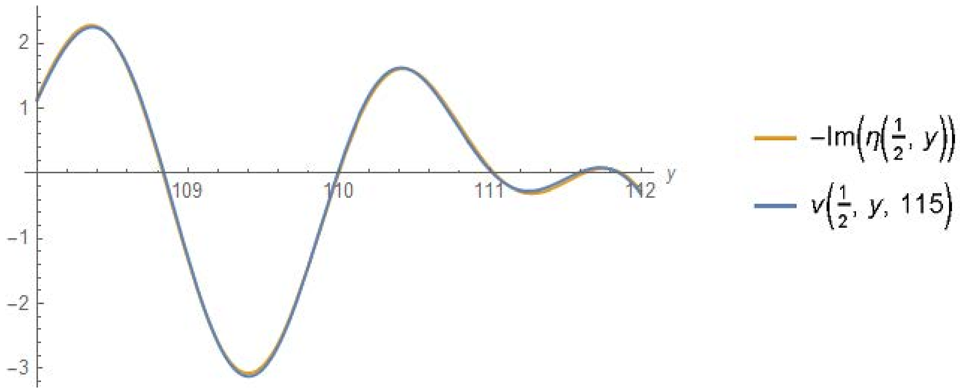
The Dirichlet-Eta type sine function and the series dealt with in Section 3 were as follows.

$$s(r,x,y) = \frac{(-1)^{\lfloor r-1 \rfloor}}{r^x} \sin(y \log r) \qquad (\lfloor\ \rfloor \text{ is floor function }) \tag{3.1}$$

$$v(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^x} \sin(y \log r) \tag{3.2}$$

In this section, we study the amplitudes (mountains, valleys, zeros) on the critical line $(x=1/2)$ of (3.2) using (3.1) .

As a numerical example, drawing the imaginary part of $\eta(1/2, y)$ near $y=110$ together with $v(1/2, y)$ is as follows.



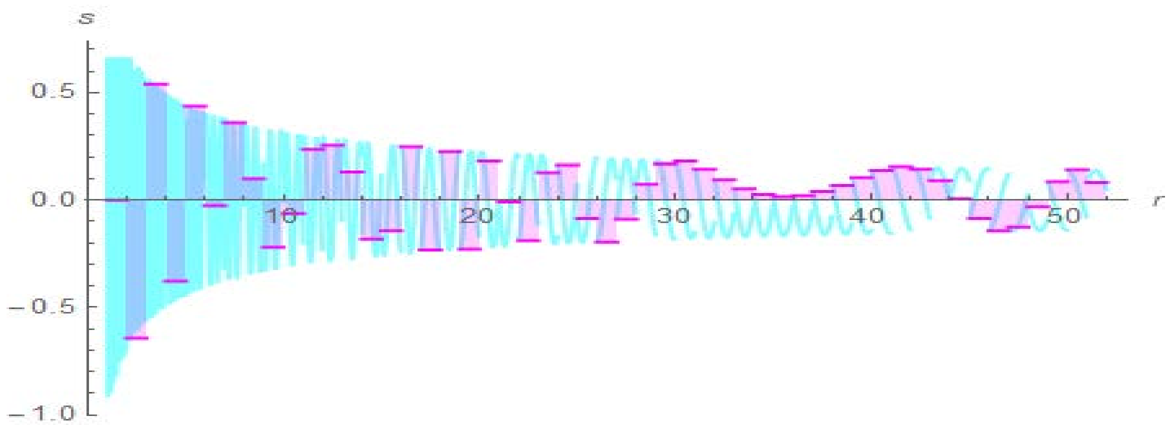
Both are almost the same, but $\eta(1/2, y)$ with high precision is used in the following calculations.

13.5.1 Mountain of v(1/2, y) (near y = 110.5)

Accurate calculation of the mountain near $y=110.5$ using $\eta(1/2, y)$ is as follows.

```
FindMaximum[-Im[η[1 / 2, y]], {y, 110.5}]
{1.61381, {y → 110.426}}          yM := 110.426
```

Drawing $s(r, 1/2, y_M)$ by (3.1) is as follows. The horizontal axis is r . Cyan is drawn as a continuous variable and magenta as a discrete variable. The sum of the area of magenta becoms mountain 1.614 of (3.2)
A big difference from 13.4.1 is that the amplitude decreases as r increases.



The order of positive and negative terms is almost the same as in 13.4.1 . That is,

(1) $r = 28 \sim 43$ are positive for 16 consecutive terms. These are included in the $59 \sim 67$ th period.
The wavelengths of the periods are $1.59 \sim 2.50$, and the central value is $2/1$.

(2) $r = 11 \sim 13$ are positive for 3 consecutive terms. These are included in the $43 \sim 45$ th period.

The wavelengths of the periods are $0.64 \sim 0.72$, and the central value is $2/3\ (=0.67)$.

(3) $r = 7, 8$ are positive for 2 consecutive terms. These are included in the $35 \sim 37$ th period.

The wavelengths of the periods are $0.41 \sim 0.45$, and these are close to $2/5\ (=0.4)$.

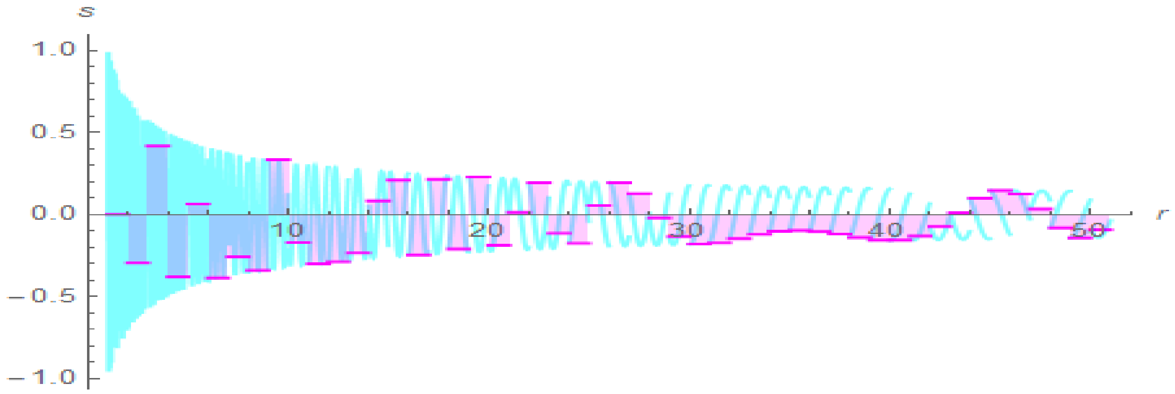
After $r = 51$, the positive and negative gradually balance out, so these three continuous intervals almost determine the height of the mountain near $y=110.5$. The difference from the previous section is that the amplitudes near $\lambda=2/5$, $\lambda=2/3$ are larger than those near $\lambda=2/1$. In addition, the reason why the same signs continue near wavelengths $\lambda=2/(2k-1)$ $k=1,2,3,\cdots$ is the same as in the previous section.

13.5.2 Valley of v(1/2,y) (near y=109.5)

Accurate calculation of the valley near $y=109.5$ using $\eta(1/2,y)$ is as follows.

```
FindMinimum[-Im[η[1 / 2, y]], {y, 109.5}]
{-3.0718, {y → 109.399}}          yv := 109.399
```

Drawing $s(r,1/2,y_V)$ by (3.1) is as follows. The horizontal axis is r . Cyan is drawn as a continuous variable and magenta as a discrete variable. The sum of the area of magenta becoms valley -3.072 of (3.2) .



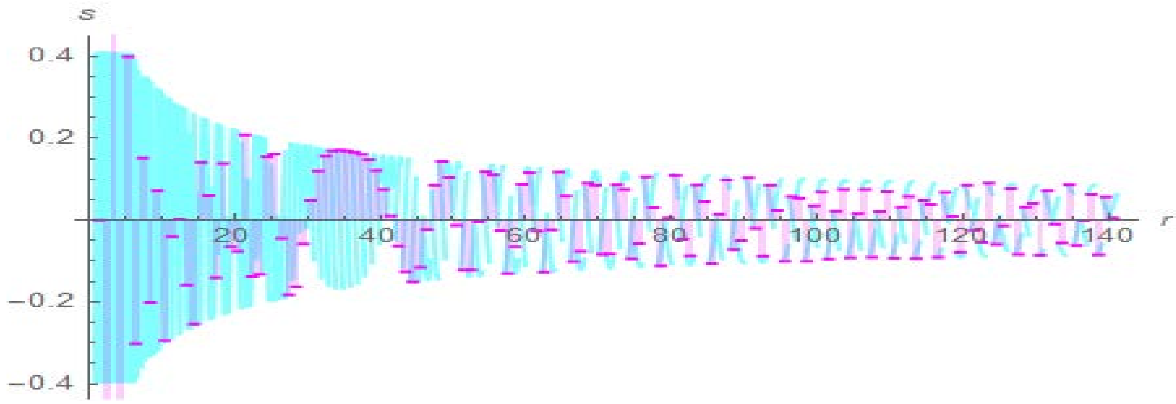
The amplitude decreases with increasing r , the order of the positive and negative terms is almost the same as in 13.4.2 , and the amplitude near $\lambda=2/5$, $\lambda=2/3$ is greater than one near $\lambda=2/1$. These are the same as in mountain.

Re: Zeros of v(1/2,y) (near y=110)

Accurate calculation of the zero near $y=110$ using $\eta(1/2,y)$ is as follows.

```
FindRoot[-Im[η[1 / 2, y]], {y, 110}]
{y → 109.995}                yz := 109.995
```

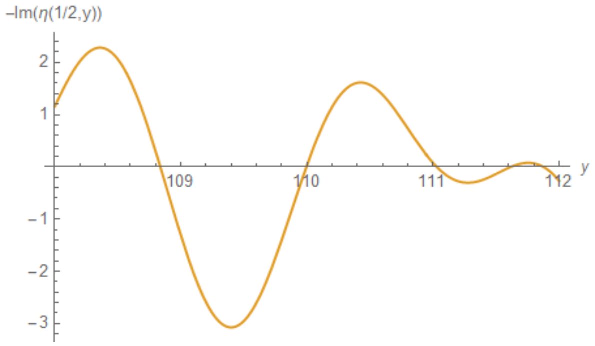
And drawing $s(r,1/2,y_Z)$ by (3.1) is as follows.

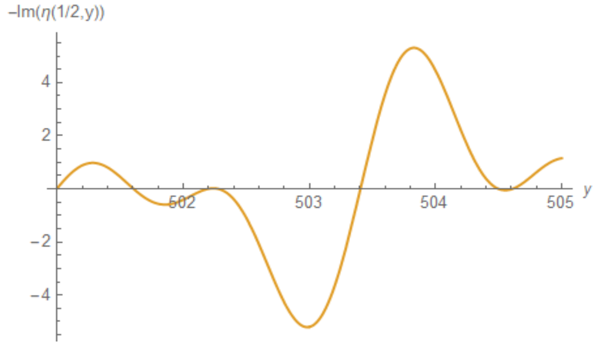


The same sign continues near wavelengths $\lambda = 2/1$, $2/3$. However, the positive and negative values are canceled as a whole, and the total area of magenta at $r = 1 \sim 140$ is 0.02 . So, the sum of the areas after $r = 140$ becomes -0.02 .

13.5.3 Amplitude of v(1/2 , y) and Variable y

If $v(1/2, y)$ $(= -Im\{\eta(1/2, y)\})$ at $y = 108 \sim 112$ and at $y = 501 \sim 505$ are drawn side by side, it is as follows. The left is the former and the right is the latter.





Comparing the two figures reveals the following.

(1) The amplitudes at $y = 501 \sim 505$ are generally larger than those at $y = 108 \sim 112$.

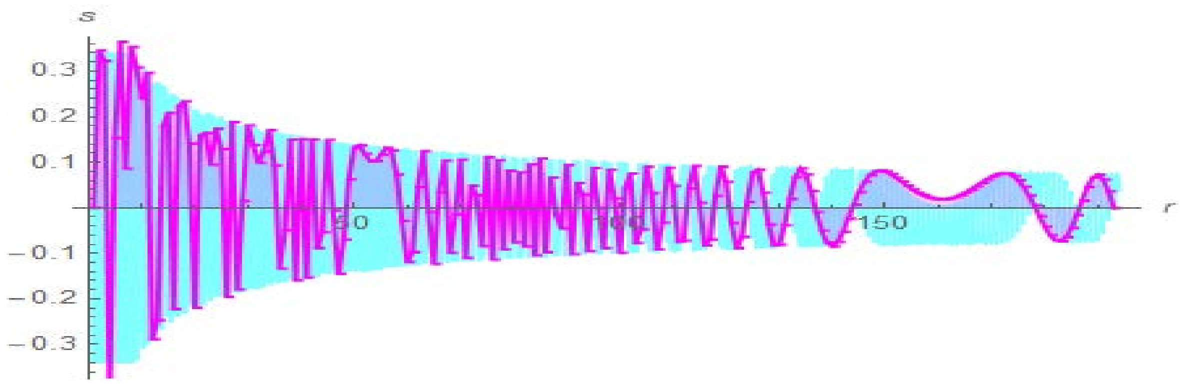
(2) The cycles at $y = 501 \sim 505$ are generally shorter than those at $y = 108 \sim 112$.

For (1) of these, we try to prove it graphically below.

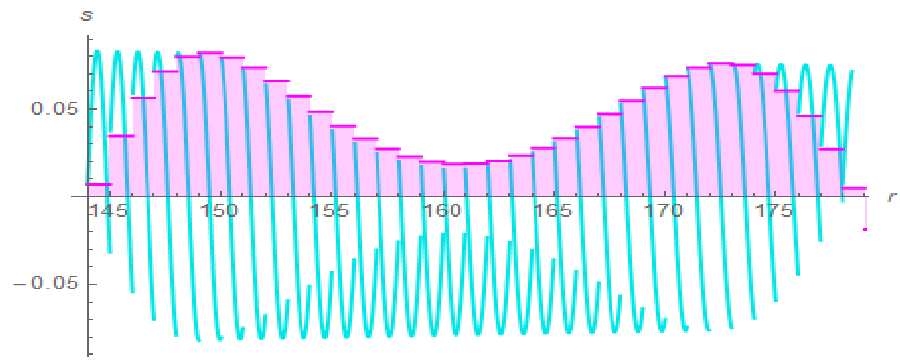
Accurate calculation of the mountain near $y = 504$ using $\eta(1/2, y)$ is as follows.

```
FindMaximum[-Im[η[1 / 2, y]], {y, 504}]
{5.31059, {y → 503.829}}          yM := 503.829
```

Drawing $s(r, 1/2, y_M)$ by (3.1) is as follows. The horizontal axis is r . Cyan is drawn as a continuous variable and magenta as a discrete variable. The sum of the area of magenta becoms mountain 5.31 of (3.2)



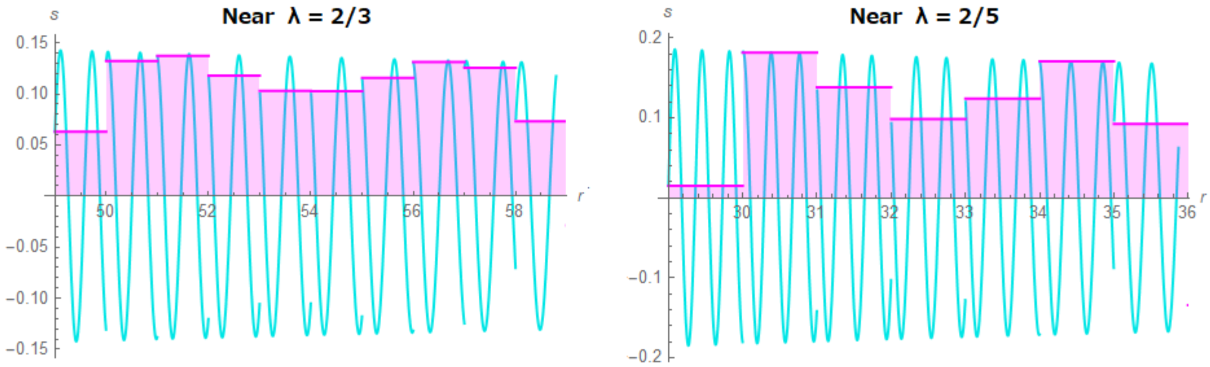
An enlarged view of the wavelength around $2/1$ is as follows.



(1) The wavelength in this figure are $1.80 \sim 2.22$. These are included in the $399 \sim 416$ th period.

$r = 144 \sim 178$ are positive for 35 consecutive terms. The interval length is about twice one of 13.5.1 (1) .

An enlarged views of the wavelength around $2/3$ and $2/5$ are as follows.



(2) The wavelength in the left figure are $0.61 \sim 0.72$. These are included in the $312 \sim 326$ th period.

$r = 49 \sim 58$ are positive for 10 consecutive terms. The interval length is 3.3 times one of 13.5.1 (2)

Moreover, the amplitude is larger than near $\lambda = 2/1$.

(3) The wavelength in the right figure are $0.37 \sim 0.43$. These are included in the $271 \sim 285$ th period.

$r = 29 \sim 35$ are positive for 7 consecutive terms. The interval length is 2.5 times one of 13.5.1 (3) .

Moreover, the amplitude is larger than near $\lambda = 2/3$.

(4) In (1) ~ (3) , it is observed that the number of positive continuous terms increases as y increases.

The reason for this is described in the previous section (4) .

(5) As y increases, the period number n that gives the wavelength $\lambda = 2$ also increases. Then, the influence of $\lambda = 2/7, 2/9, \cdots$ also increases. In fact, the near of $\lambda = 2/7$ is $r = 21 \sim 25$ and these are 5 consecutive positive terms. Furthermore, $r = 5 \sim 11$ are 7 consecutive positive terms, and these wavelengths are close to $2/15 \sim 2/29$. Moreover, A interval with a shorter wavelength has a larger amplitude.

(6) As a result of the above, the mountain in $u(1/2, y)$ near $y = 504$ is higher than the one near $y = 110.5$

(7) Although mountains have been used as examples so far, the same applies to valleys. So, the amplitudes of $v(1/2, y)$ at $y = 501 \sim 505$ are generally larger than those at $y = 108 \sim 112$.

Note1

(7) is similar to ***Bergmann's Law*** (Bears in high latitudes are generally larger than bears in low latitudes.).

Note2

In th case of $x > 1/2$, the influence of $1/r^x$ is too great and (1) ~ (7) do not seem to hold.

2023.04.16

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Alien's Mathematics