10 Dirichlet Series & Taylor Series

Abstract

(1) Dirichlet series can be converted to Taylor series within its convergence area.

(2) If the coefficients of the Dirichlet series and the center of the Taylor expansion are limited to real numbers, the Taylor series for each real part and imaginary part can be obtained.

(3) Hyperbolic functions are represented by finite or infinite General Dirichlet Series.

10.1 General Dirichlet Series & Taylor Series

Definition 10.1.0 (General Dirichlet Series)

When R is a real number set, $\lambda_t \in R$ $s.t.$ $\lambda_t < \lambda_{t+1}$ $t=1,2,3,\cdots$ and a_t are complex numbers, we call the following series **General Dirichlet Series**.

$$f(z) = \sum_{t=1}^{\infty} a_t e^{-\lambda_t z}$$

General Dirichlet series can be converted to Taylor series within its convergence area. The following shows the conversion formula.

Formula 10.1.1

When a function $f(z)$ that is holomorphic on domain D is expanded into a general Dirichlet series, the following expression holds for arbitrary complex numbers z, c belonging to the convergence area.

$$f(z) = \sum_{t=1}^{\infty} a_t e^{-\lambda_t z} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t} (-\lambda_t)^s \frac{(z-c)^s}{s!} \tag{1.1}$$

Proof

$$f(z) = \sum_{t=1}^{\infty} a_t e^{-\lambda_t z}$$

Expanding $e^{-\lambda_t z}$ into Taylor series with respect to z around c,

$$e^{-\lambda_t z} = \sum_{s=0}^{\infty} e^{-c\lambda_t} (-\lambda_t)^s \frac{(z-c)^s}{s!}$$

Substituting this for the right-hand side and swapping Σ, we obtain the desired expression.

Note

When the s-th order differential coefficient at c of the function $f(z)$ is expressed as $f^{(s)}(c)$, the following expression clearly holds.

$$f^{(s)}(c) = \sum_{t=1}^{\infty} a_t e^{-c\lambda_t} (-\lambda_t)^s \qquad s=0,1,2,\cdots \tag{1.1c}$$

Example 1 $coth z - 1$

$coth\ z$ is expanded into Fourier series for $Re(z) > 0$ as follows.

$$coth\, z = \frac{e^z + e^{-z}}{e^z - e^{-z}} = \frac{1+e^{-2z}}{1-e^{-2z}} = \left(1+e^{-2z}\right)\left(1 + e^{-2z} + e^{-4z} + e^{-6z} + \cdots\right)$$

$$= 1 + \left(2e^{-2z} + 2e^{-4z} + 2e^{-6z} + 2e^{-8z} + - \cdots\right)$$

From this,

$$coth\, z - 1 = \sum_{t=1}^{\infty} 2\, e^{-2tz} \qquad Re(z) > 0$$

Put $a_t = 2$, $\lambda_t = 2t$ in the right hand. Then $\lambda_t < \lambda_{t+1}$ This is a general Dirichlet series. So, from the formula,

$$coth\, z - 1 = \sum_{t=1}^{\infty} 2\, e^{-2tz} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} 2e^{-2tc}(-2t)^s \frac{(z-c)^s}{s\,!}$$

The function values of the three parties at $z = 2+1.5i$ are calculated as follows. The center is the general Dirichlet series and the right end is the Taylor series at $c = 1+i$. All three are exactly the same.

N[Coth[2 + 1.5 ⅈ] - 1]	N[f[2 + 1.5 ⅈ, 20]]	N[f[2 + 1.5 ⅈ, 1 + ⅈ, 77]]
-0.0356315 - 0.00498689 ⅈ	-0.0356315 - 0.00498689 ⅈ	-0.0356315 - 0.00498689 ⅈ

In addition, the following has to be hold from (1.1c) .

$$(coth\, c)^{(s)} = \sum_{t=1}^{\infty} 2e^{-2tc}(-2t)^s \quad \left(=: Ch_s(c)\right) \qquad s = 1, 2, 3, \cdots$$

When $c = 1+i$, both sides are calculated for $s = 1, 2$ as follows. Both sides are the same for the 1st and 2nd orders.

The 1st order	N[{Coth'[1 + ⅈ]	, Ch₁[1 + ⅈ, 12]}]
	{0.293911 + 0.377797 ⅈ	, 0.293911 + 0.377797 ⅈ}
The 2nd order	N[{Coth''[1 + ⅈ]	, Ch₂[1 + ⅈ, 12]}]
	{-0.674671 - 0.527944 ⅈ	, -0.674671 - 0.527944 ⅈ}

In a similar way, the following examples are obtained.

Example 2 $tanh z - 1$

$$tanh\, z - 1 = \sum_{t=1}^{\infty} 2(-1)^t e^{-2tz} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} 2(-1)^t e^{-2tc}(-2t)^s \frac{(z-c)^s}{s\,!}$$

Example 3 $csch z$

$$csch\, z = \sum_{t=1}^{\infty} 2e^{-(2t-1)z} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} 2e^{-c(2t-1)}\{-(2t-1)\}^s \frac{(z-c)^s}{s\,!}$$

Example 4 $sech z$

$$sech\, z = \sum_{t=1}^{\infty} 2(-1)^{t-1} e^{-(2t-1)z} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} 2(-1)^{t-1} e^{-c(2t-1)}\{-(2t-1)\}^s \frac{(z-c)^s}{s\,!}$$

If coefficients of the Dirichlet series a_t and the center c of the Taylor expansion are limited to real numbers, the Taylor series for each real part and imaginary part can be obtained.

Formula 10.1.2

When $f(z)=\sum_{t=1}^{\infty} a_t e^{-\lambda_t z}$ $(z=x+iy)$ is general Dirichlet serie, u, v are real and imaginary parts of $f(z)$ and c , a_t $t=1,2,3,\cdots$ are arbitrary real numbers, the following expressions hold in the convergence area.

$$f(z) = \sum_{s=0}^{\infty}\sum_{t=1}^{\infty} a_t e^{-c\lambda_t}(-\lambda_t)^s \frac{(z-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{\infty} a_t e^{-c\lambda_t}(-\lambda_t)^{2r+s} \frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{\infty} a_t e^{-c\lambda_t}(-\lambda_t)^{2r+s+1} \frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0=1$

Proof

$$f(z) = \sum_{t=1}^{\infty} a_t e^{-\lambda_t z}$$

$$f(x,y) = \sum_{t=1}^{\infty} a_t e^{-\lambda_t (x+iy)} = \sum_{t=1}^{\infty} a_t e^{-\lambda_t x}\{\cos(\lambda_t y) - i\sin(\lambda_t y)\}$$

i.e.

$$f(x,y) = \sum_{t=1}^{\infty} a_t e^{-\lambda_t x}\cos(\lambda_t y) - i\sum_{t=1}^{\infty} a_t e^{-\lambda_t x}\sin(\lambda_t y)$$

From this,

$$u(x,y) = \sum_{t=1}^{\infty} a_t e^{-\lambda_t x}\cos(\lambda_t y)$$

$$v(x,y) = -\sum_{t=1}^{\infty} a_t e^{-\lambda_t x}\sin(\lambda_t y)$$

Expanding $\cos(\lambda_s y)$, $\sin(\lambda_s y)$ into Maclaurin series with respect to y respectively,

$$\cos(\lambda_t y) = \sum_{r=0}^{\infty} \lambda_t^{2r}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$\sin(\lambda_s y) = \sum_{r=0}^{\infty} \lambda_t^{2r+1}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Next, expanding $e^{-\lambda_t x}$ into Taylor series with respect to x around real number c ,

$$e^{-\lambda_t x} = \sum_{s=0}^{\infty} e^{-c\lambda_t}(-1)^s\lambda_t^s\frac{(x-c)^s}{s!}$$

Substituting these for the above,

$$u(x,y) = \sum_{t=1}^{\infty} a_t \sum_{s=0}^{\infty} e^{-c\lambda_t}(-1)^s\lambda_t^s\frac{(x-c)^s}{s!}\sum_{r=0}^{\infty}\lambda_t^{2r}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\sum_{t=1}^{\infty} a_t \sum_{s=0}^{\infty} e^{-c\lambda_t}(-1)^s \lambda_t^s \frac{(x-c)^s}{s!} \sum_{r=0}^{\infty} \lambda_t^{2r+1} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Rearranging Σ,

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t}(-1)^s \lambda_t^{2r+s} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t} e^{-c\lambda_t}(-1)^{s+1} \lambda_t^{2r+s+1} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Here, let

$$\sum_{t=1}^{\infty} a_t e^{-c\lambda_t}(-1)^s \lambda_t^{2r+s} := f^{(2r+s)}(c)$$

$$\sum_{t=1}^{\infty} a_t e^{-c\lambda_t} e^{-c\lambda_t}(-1)^{s+1} \lambda_t^{2r+s+1} := f^{(2r+s+1)}(c)$$

Then,

$$\sum_{t=1}^{\infty} a_t e^{-c\lambda_t}(-1)^s \lambda_t^s := f^{(s)}(c)$$

Therefore, according to Taylor's theorem

$$f(z) = \sum_{s=0}^{\infty} f^{(s)}(c) \frac{(z-c)^s}{s!} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t}(-1)^s \lambda_t^s \frac{(z-c)^s}{s!}$$

$$= \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t}(-\lambda_t)^s \frac{(z-c)^s}{s!}$$

And

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t}(-1)^s \lambda_t^{2r+s} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t}(-\lambda_t)^{2r+s} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t}(-1)^{s+1} \lambda_t^{2r+s+1} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t}(-\lambda_t)^{2r+s+1} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Example $a_t = (-1)^{t-1}$, $\lambda_t = 1.1^t$

General Dirichlet series is

$$f(z) = \sum_{t=1}^{\infty} (-1)^{t-1} e^{-1.1^t z}$$

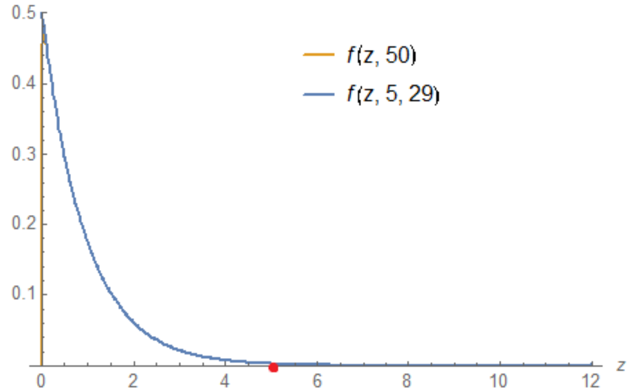
Taylor series are

$$f(z) = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} (-1)^{t-1} e^{-c\,1.1^t} \left(-1.1^t\right)^s \frac{(z-c)^s}{s!}$$

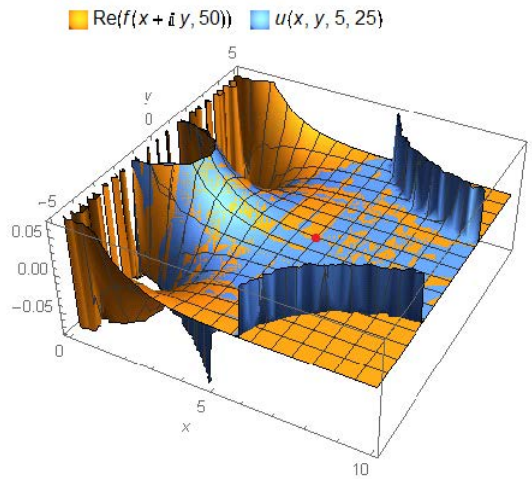
$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} (-1)^{t-1} e^{-c\,1.1^t} \left(-1.1^t\right)^{2r+s} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

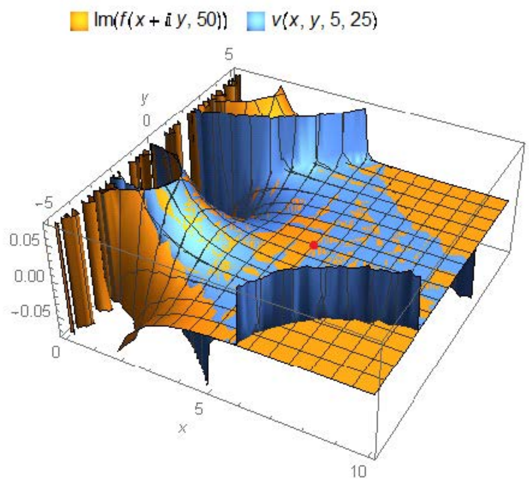
$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{\infty}(-1)^{t-1}e^{-c\,1.1^t}\left(-1.1^t\right)^{2r+s+1}\frac{(x-c)^s}{s\,!}\frac{(-1)^r y^{2r+1}}{(2r+1)\,!}$$

When $c=5$, Dirichlet series and Taylor series are drawn in the 2D figure as follows. Orange is Dirichlet series and blue is Taylor series. The red dot is the center of the expansion. Although the abscissa of convergence is observed at $x=0$, both are exactly overlapped and Dirichlet series (orange) is almost invisible.



When $c=5$, the real part and the imaginary part are drawn in the 3D figure as follows. The left is the real part and the right is the imaginary part. In both figures, orange is Dirichlet series and blue is Taylor series.





The convergence area of Taylor series is a square inscribed in a circle with a radius of 5. And the right outside is asymptotic expansion.

10.2 Finite General Dirichlet Series & Taylor Series

The formulas in the previous section also hold for the finite general Dirichlet series.

Formula 10.2.1

When a function $f(z)$ that is holomorphic on domain D is expanded into a finite general Dirichlet series, the following expression holds for arbitrary complex numbers z, c

$$f(z) = \sum_{t=1}^{n} a_t e^{-\lambda_t z} = \sum_{s=0}^{\infty} \sum_{t=1}^{n} a_t e^{-c\lambda_t} (-\lambda_t)^s \frac{(z-c)^s}{s!} \tag{2.1}$$

Example 1 $\cosh z$

$$\cosh z = \frac{e^z + e^{-z}}{2} = \frac{e^z}{2} + \frac{e^{-z}}{2}$$

Let $a_t = 1/2$, $\lambda_t = (-1)^t$ on the right side. Since $\lambda_1 < \lambda_2$, this is a General Dirichlet Series.

So, according to the formula,

$$\cosh z = \sum_{t=1}^{2} \frac{1}{2} e^{-(-1)^t z} = \sum_{s=0}^{\infty} \sum_{t=1}^{2} \frac{1}{2} e^{-c(-1)^t} \{-(-1)^t\}^s \frac{(z-c)^s}{s!}$$

The function values of the three parties at $z = 3+1.2i$ are calculated as follows. The center is the general Dirichlet series and the right end is the Taylor series at $c = 1+i$. All three are exactly the same.

N[Cosh[3 + 1.2 ⅈ]]	N[f[3 + 1.2 ⅈ]]	N[f[3 + 1.2 ⅈ, 1 + ⅈ, 12]]
3.6481 + 9.33705 ⅈ	3.6481 + 9.33705 ⅈ	3.6481 + 9.33705 ⅈ

In addition, $\cosh z$ has the following as zeros. That is, this is a kind of the Riemann Xi function.

$$z = \pm\frac{(2r-1)\pi i}{2} \qquad r=1, 2, 3, \cdots$$

Example 2 $\sinh z$

$$\sinh z = \frac{e^z - e^{-z}}{2} = \frac{e^z}{2} - \frac{e^{-z}}{2}$$

Let $a_t = (-1)^{t-1}/2$, $\lambda_t = (-1)^t$ on the right side. Since $\lambda_1 < \lambda_2$, this is a General Dirichlet Series.

So, according to the formula,

$$\sinh z = \sum_{t=1}^{2} \frac{(-1)^{t-1}}{2} e^{-(-1)^t z} = \sum_{s=0}^{\infty} \sum_{t=1}^{2} \frac{(-1)^{t-1}}{2} e^{-c(-1)^t} \{-(-1)^t\}^s \frac{(z-c)^s}{s!}$$

The function values of the three parties at $z = 3+1.2i$ are calculated as follows. The center is the general Dirichlet series and the right end is the Taylor series at $c = 1+i$. All three are exactly the same.

N[Sinh[3 + 1.2 ⅈ]]	N[f[3 + 1.2 ⅈ]]	N[f[3 + 1.2 ⅈ, 1 + ⅈ, 12]]
3.63005 + 9.38345 ⅈ	3.63005 + 9.38345 ⅈ	3.63005 + 9.38345 ⅈ

Similar to the previous section, the Taylor series for each real part and imaginary part can be obtained.

Formula 10.2.2

When $f(z)=\sum_{t=1}^{n} a_t e^{-\lambda_t z}$ $(z=x+iy)$ is finite general Dirichlet serie, u, v are real and imaginary parts of $f(z)$ and c , a_t $t=1,2,3,\cdots$ are arbitrary real numbers, the following expressions hold.

$$f(z) = \sum_{s=0}^{\infty}\sum_{t=1}^{n} a_t e^{-c\lambda_t}(-\lambda_t)^s \frac{(z-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{n} a_t e^{-c\lambda_t}(-\lambda_t)^{2r+s} \frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{n} a_t e^{-c\lambda_t}(-\lambda_t)^{2r+s+1} \frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$

Example $a_t = (-1)^{t-1}$, $\lambda_t = 1.1^t$

Finite general Dirichlet series is

$$f(z) = \sum_{t=1}^{n} (-1)^{t-1} e^{-1.1^t z}$$

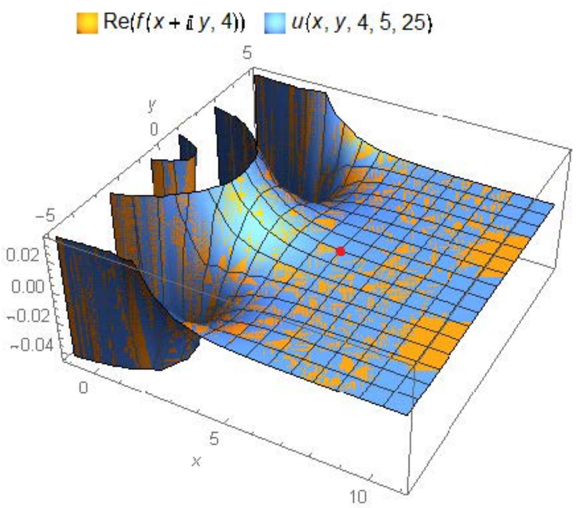
Taylor series are

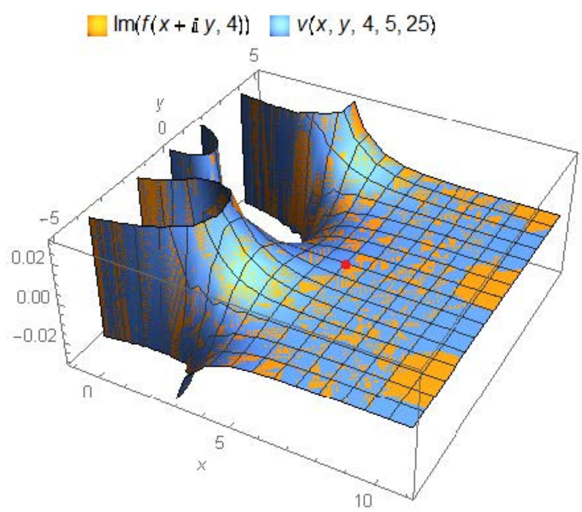
$$f(z) = \sum_{s=0}^{\infty}\sum_{t=1}^{n} (-1)^{t-1} e^{-c\,1.1^t}(-1.1^t)^s \frac{(z-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{n} (-1)^{t-1} e^{-c\,1.1^t}(-1.1^t)^{2r+s} \frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{n} (-1)^{t-1} e^{-c\,1.1^t}(-1.1^t)^{2r+s+1} \frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

When $n=4$, $c=5$, these are drawn in the 3D figure as follows. The left is the real par and the right is the imaginary part. In both figures, orange is Dirichlet series and blue is Taylor series. The red dot is the center of the expansion.





10.3 Ordinary Dirichlet Series & Taylor Series

Ordinary Dirichlet Series is what is put $\lambda_t = \log t$ at General Dirichlet Series, and is defined as follows.

Definition 10.3.0 (Ordinary Dirichlet Series)

When z, a_n $(n=1,2,3,\cdots)$ are complex numbers, we call the following **Ordinary Dirichlet Series**.

$$f(z) = \sum_{t=1}^{\infty} \frac{a_t}{t^z} = \frac{a_1}{1^z} + \frac{a_2}{2^z} + \frac{a_3}{3^z} + \frac{a_4}{4^z} + \cdots$$

Ordinary Dirichlet series can be converted to Taylor series within its convergence area.

Tthe conversion formula is obtained immediately by putting $\lambda_t = \log t$ in Formula 10.1.1 ..

Formula 10.3.1

When a function $f(z)$ that is holomorphic on domain D is expanded into a ordinary Dirichlet series, the following expression holds for arbitrary complex numbers z, c belonging to the convergence area.

$$f(z) = \sum_{t=1}^{\infty} \frac{a_t}{t^z} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{a_t}{t^c} (-\log t)^s \frac{(z-c)^s}{s!} \tag{3.1}$$

Note

When the s -th order differential coefficient at c of the function $f(z)$ is expressed as $f^{(s)}(c)$, the following expression clearly holds.

$$f^{(s)}(c) = \sum_{t=1}^{\infty} \frac{a_t}{t^c} (-\log t)^s \qquad s=0,1,2,\cdots \tag{3.1c}$$

Example Dirichlet Eta Series

Dirichlet eta series is

$$\eta(z) = \sum_{t=1}^{\infty} \frac{(-1)^t}{t^z}$$

Since $a_t = (-1)^t$, according to the formula,

$$\eta(z) = \sum_{t=1}^{\infty} \frac{(-1)^t}{t^z} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{(-1)^t}{t^c} (-\log t)^s \frac{(z-c)^s}{s!}$$

The function values of the three parties at $z=3+4i$ are calculated as follows. The center is the general Dirichlet series and the right end is the Taylor series at $c=2+2i$. Three are almost the same.

N[η[3 + ⅈ 4]]	N[η[3 + ⅈ 4, 190]]	N[η[3 + ⅈ 4, 2 + 2 ⅈ, 185]]
1.09894 + 0.0703438 ⅈ	1.09894 + 0.0703438 ⅈ	1.09894 + 0.0703437 ⅈ

In addition, the following has to be hold from (3.1c) .

$$\eta^{(s)}(c) = \sum_{t=1}^{\infty} \frac{(-1)^{t-1}}{t^c} (-\log t)^s \qquad (=: Et_s(c)) \qquad s=1,2,3,\cdots$$

When $c=2+2i$, both sides are calculated for $s=1,2$ as follows. Both sides are the same for the 1st and 2nd orders.

The 1st order	N[{ η'[2 + 2 ⅈ]	, Et₁[2 + 2 ⅈ, 15 000]}]
	{0.0597288 - 0.0993641 ⅈ	, 0.0597288 - 0.0993641 ⅈ}
The 2nd order	N[{ η''[2 + 2 ⅈ]	, Et₂[2 + 2 ⅈ, 220 000]}]
	{-0.0453472 + 0.0468542 ⅈ	, -0.0453472 + 0.0468542 ⅈ}

If coefficients of the Dirichlet series a_t and the center c of the Taylor expansion are limited to real numbers, Taylor series for each real part and imaginary part can be obtained. This is also obtained by putting $\lambda_t = \log t$ in Formula 10.1.2 ..

Formula 10.3.2

When $f(z) = \sum_{t=1}^{\infty} a_t / t^z$ ($z=x+iy$) is ordinary Dirichlet serie, u, v are real and imaginary parts of $f(z)$ and c , a_t $t=1,2,3,\cdots$ are arbitrary real numbers, the following expressions hold in the convergence area.

$$f(z) = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{a_t}{t^c} (-\log t)^s \frac{(z-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{a_t}{t^c} (-\log t)^{2r+s} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{a_t}{t^c} (-\log t)^{2r+s+1} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$

Example Dirichlet Beta Series

When Dirichlet character is $\chi(m,j,t)$, Dirichlet Beta Series is expressed

$$\beta(z) = \sum_{t=1}^{\infty} \frac{\chi(4,2,t)}{t^z} \qquad \left\{ = \sum_{t=1}^{\infty} \frac{(-1)^{t-1}}{(2t-1)^z} \right\}$$

Since $a_t = \chi(4,2,t)$, according to the formula,

$$\beta(z) = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{\chi(4,2,t)}{t^c} (-\log t)^s \frac{(x-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{\chi(4,2,t)}{t^c} (-\log t)^{2r+s} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{\chi(4,2,t)}{t^c} (-\log t)^{2r+s+1} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Though there is no problem as it is, since $\chi(4,2,t)$ $t=1,2,3,\cdots$ is $1, 0, -1, 0, \cdots$, these can be further rewritten as follows.

$$\beta(z) = \sum_{s=0}^{\infty}\sum_{t=1}^{\infty}\frac{(-1)^{t-1}}{(2t-1)^c}\{-log(2t-1)\}^s\frac{(z-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{\infty}\frac{(-1)^{t-1}}{(2t-1)^c}\{-log(2t-1)\}^{2r+s}\frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{\infty}\frac{(-1)^{t-1}}{(2t-1)^c}\{-log(2t-1)\}^{2r+s+1}\frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$

When $2+3i$, $c=0.5$, the real and imaginary parts of $\beta(z)$ and the values of u , v are calculated as follows. The real part are completely the same, and the imaginary part are almost the same.

Real part	N[{Re[β[2 + ⅈ 3]], u[2, 3, 0.5, 55]}] {1.10389, 1.10389}
Imaginary part	N[{Im[β[2 + ⅈ 3]], v[2, 3, 0.5, 78]}] {0.0133583, 0.0133692}

10.4 Finite Ordinary Dirichlet Series & Taylor Series

The formulas in the previous section also hold for the finite ordinary Dirichlet series.

Formula 10.4.1

When a function $f(z)$ that is holomorphic on domain D is expanded into a ordinary Dirichlet series, the following expression holds for arbitrary complex numbers z,c belonging to the convergence area.

$$f(z) = \sum_{t=1}^{n} \frac{a_t}{t^z} = \sum_{s=0}^{\infty} \sum_{t=1}^{n} \frac{a_t}{t^c} (-\log t)^s \frac{(z-c)^s}{s!} \tag{4.1}$$

Example $a_t = (-1)^{t-1}$, $n=6$

$$f(z) = \sum_{t=1}^{6} \frac{(-1)^{t-1}}{t^z} = \sum_{s=0}^{\infty} \sum_{t=1}^{6} \frac{(-1)^{t-1}}{t^c} (-\log t)^s \frac{(z-c)^s}{s!}$$

These are the first 6 terms of the Dirichlet eta series and its Taylor series. When $c=1/2+15i$, the values near these zeros are calculated as follows. The top two lines are the finite Dirichlet eta series and the bottom two lines are the Taylor series. Both are almost 0 .

```
N[f[0.5468208018 + 13.958984203 i, 6]]
-6.26561 × 10^-11 + 3.45629 × 10^-10 i
N[f[0.5468208018 + 13.958984203 i , 6, 1/2 + 15 i, 100]]
-6.26554 × 10^-11 + 3.45629 × 10^-10 i
```

Formula 10.4.2

When $f(z) = \sum_{t=1}^{n} a_t / t^z$ $(z=x+iy)$ is finite ordinary Dirichlet serie, u,v are real and imaginary parts of $f(z)$ and c , a_t $t=1,2,3,\cdots$ are arbitrary real numbers, the following expressions hold.

$$f(z) = \sum_{s=0}^{\infty} \sum_{t=1}^{n} \frac{a_t}{t^c} (-\log t)^s \frac{(z-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{n} \frac{a_t}{t^c} (-\log t)^{2r+s} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{n} \frac{a_t}{t^c} (-\log t)^{2r+s+1} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$

Example $a_t = (-1)^{t-1} t$, $n = 2$

Finite ordinary Dirichlet series is

$$f(z) = \sum_{t=1}^{2} \frac{(-1)^{t-1} t}{t^z} \quad \left(= \frac{1}{1^z} - \frac{2}{2^z} = 1 - 2^{1-z} \right)$$

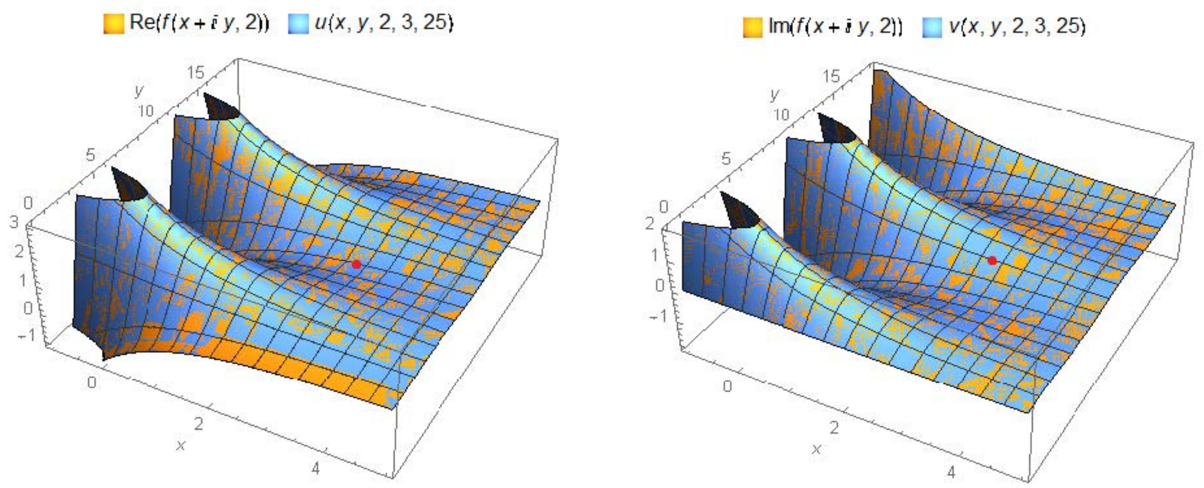
Taylor series are

$$f(z) = \sum_{s=0}^{\infty}\sum_{t=1}^{2}\frac{(-1)^{t-1}t}{t^c}(-\log t)^s\frac{(z-c)^s}{s!}$$

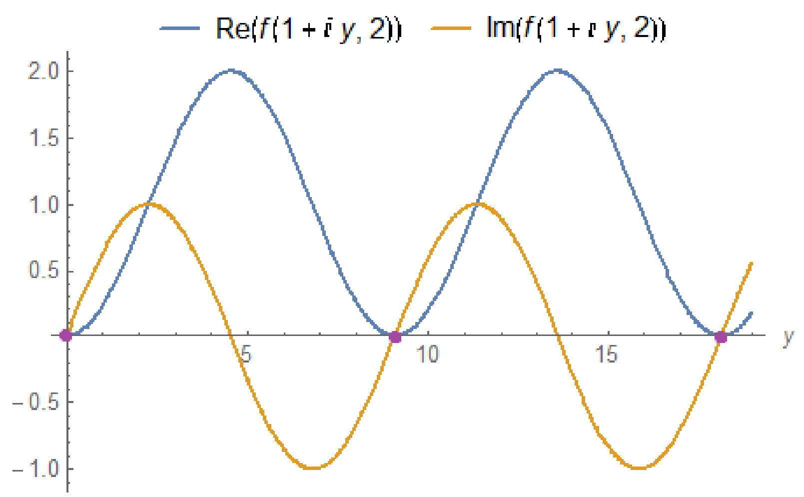
$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{2}\frac{(-1)^{t-1}t}{t^c}(-\log t)^{2r+s}\frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{2}\frac{(-1)^{t-1}t}{t^c}(-\log t)^{2r+s+1}\frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

When $c=3$, these are drawn in the 3D figure as follows. The left is the real part and the right is the imaginary part. In both figures, orange is finite ordinary Dirichlet series and blue is Taylor series. Red dot is the center of the of the expansion.



This finite ordinary Dirichlet series has zeros on $x=1$, In fact, the real and imaginary parts on this line are drawn in the 2D figure as follows. The purple dots are the zeros.



These zeros can be easily obtained from $1-2^{1-z}=0$ to $x=1, y=2n\pi/\log 2 \ \ n=0,1,2,\cdots$.

These are the zeros specific to the Dirichlet eta function $\eta(z)$. Because, there is the following relationship between Riemann zeta function $\zeta(z)$ and Dirichlet eta function $\eta(z)$.

$$\left(\frac{1}{1^z}-\frac{2}{2^z}\right)\left(\frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\cdots\right) = \frac{1}{1^z}-\frac{1}{2^z}+\frac{1}{3^z}-\frac{1}{4^z}+-\cdots$$

2020.10.03
2022.07.21 Updated

Kano Kono
Hiroshima, Japan

Alien's Mathematics