

4 Euler-Maclaurin Summation Formula

4.1 Bernoulli Number & Bernoulli Polynomial

4.1.1 Definition of Bernoulli Number

Bernoulli numbers B_k ($k=1, 2, 3, \dots$) are defined as coefficients of the following equation.

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$

4.1.2 Expression of Bernoulli Numbers

$$\frac{x}{e^x - 1} = \frac{B_0}{0!} + \frac{B_1}{1!}x + \frac{B_2}{2!}x^2 + \frac{B_3}{3!}x^3 + \frac{B_4}{4!}x^4 + \dots \quad (1.1)$$

$$\frac{e^x - 1}{x} = \frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \dots \quad (1.2)$$

Making the Cauchy product of (1.1) and (1.2),

$$1 = B_0 + \left(\frac{B_1}{1!1!} + \frac{B_0}{0!2!} \right)x + \left(\frac{B_2}{2!1!} + \frac{B_1}{1!2!} + \frac{B_0}{0!3!} \right)x^2 + \dots$$

In order to hold this for arbitrary x ,

$$B_0 = 1, \quad \frac{B_1}{1!1!} + \frac{B_0}{0!2!} = 0, \quad \frac{B_2}{2!1!} + \frac{B_1}{1!2!} + \frac{B_0}{0!3!} = 0, \dots$$

The coefficient of x^n is as follows.

$$\frac{B_n}{n!1!} + \frac{B_{n-1}}{(n-1)!2!} + \frac{B_{n-2}}{(n-2)!3!} + \dots + \frac{B_1}{1!n!} + \frac{B_0}{0!(n+1)!} = 0$$

Multiplying both sides by $(n+1)!$,

$$\frac{B_n(n+1)!}{n!1!} + \frac{B_{n-1}(n+1)!}{(n-1)!2!} + \frac{B_{n-2}(n+1)!}{(n-2)!3!} + \dots + \frac{B_1(n+1)!}{1!n!} + \frac{B_0}{0!} = 0$$

Using binomial coefficients,

$${}_{n+1}C_n B_n + {}_{n+1}C_{n-1} B_{n-1} + {}_{n+1}C_{n-2} B_{n-2} + \dots + {}_{n+1}C_1 B_1 + {}_{n+1}C_0 B_0 = 0$$

Replacing $n+1$ with n ,

$${}_n C_{n-1} B_{n-1} + {}_n C_{n-2} B_{n-2} + {}_n C_{n-3} B_{n-3} + \dots + {}_n C_1 B_1 + {}_n C_0 B_0 = 0 \quad (1.3)$$

Substituting $n=2, 3, 4, \dots$ for this one by one,

$${}_2 C_1 B_1 + {}_2 C_0 B_0 = 0 \quad \longrightarrow \quad B_2 = -\frac{1}{2}$$

$${}_3 C_2 B_2 + {}_3 C_1 B_1 + {}_3 C_0 B_0 = 0 \quad \longrightarrow \quad B_2 = \frac{1}{6}$$

\vdots

Thus, we obtain Bernoulli numbers.

The first few Bernoulli numbers are as follows.

$$B_0 = 1, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \dots$$

$$B_1 = -\frac{1}{2}, \quad B_3 = B_5 = B_7 = \dots = 0$$

4.1.3 Calculation of Bernoulli number

(1) Method of substitution one by one

Generally, it is performed by the method of substituting $n = 2, 3, 4, \dots$ for (1.3) one by one. Since it is necessary to calculate from the small number one by one, it is difficult to obtain a large Bernoulli number directly. This method is suitable for the small number.

(2) Method by double sums.

It is calculated by the following double sums of binomial coefficients

$$B_n = \sum_{k=0}^n \frac{1}{k+1} \sum_{r=0}^k (-1)^r {}_k C_r r^n$$

If the first few are written down, it is as follows.

$$B_0 = \frac{1}{1} {}_0 C_0 0^0$$

$$B_1 = \frac{1}{1} {}_0 C_0 0^1 + \frac{1}{2} ({}_1 C_0 0^1 - {}_1 C_1 1^1)$$

$$B_2 = \frac{1}{1} {}_0 C_0 0^2 + \frac{1}{2} ({}_1 C_0 0^2 - {}_1 C_1 1^2) + \frac{1}{3} ({}_2 C_0 0^2 - {}_2 C_1 1^2 + {}_2 C_2 2^2)$$

⋮

Using this, a large Bernoulli number can be obtained directly. However, since the number of terms increases with accelerating speed, this is not suitable for manual calculation.

4.1.4 Bernoulli Polynomial

(1) Definition

When B_n are Bernoulli numbers, polynomial $B_n(x)$ that satisfies the following three expressions is called **Bernoulli Polynomial**.

$$B_0(x) = 1$$

$$\frac{d}{dx} B_n(x) = n B_{n-1}(x) \quad (n \geq 1)$$

$$\int_0^1 B_n(x) dx = 0 \quad (n \geq 1)$$

(2) Properties

The following expressions follow directly from the definition.

$$B_n(x) = n \int_0^x B_{n-1}(t) dt + B_n \quad (n \geq 1)$$

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}$$

$$B_n(0) = B_n$$

$$B_n(1) = B_n(0) \quad (n \geq 2)$$

$$B_n(x+1) - B_n(x) = n x^{n-1} \quad (n \geq 1)$$

$$B_n(1-x) = (-1)^n B_n(x) \quad (n \geq 1)$$

Furthermore, the followings are known although not directly followed from the definition.

For arbitrary natural number m and interval $[0, 1]$,

$$\begin{aligned} |B_{2m}(x)| &\leq |B_{2m}| \\ |B_{2m+1}(x)| &\leq (2m+1) |B_{2m}| \end{aligned}$$

Examples

$$\begin{aligned} B_1(x) &= x - \frac{1}{2}, & B_2(x) &= x^2 - x + \frac{1}{6}, & B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\ B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}, & B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \\ B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}, & B_7(x) &= x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{2}x^3 + \frac{1}{6}x, \\ B_8(x) &= x^8 - 4x^7 + \frac{14}{3}x^6 - \frac{7}{3}x^4 + \frac{2}{3}x^2 - \frac{1}{30}, \\ B_9(x) &= x^9 - \frac{9}{2}x^8 + 6x^7 - \frac{21}{5}x^5 + 2x^3 - \frac{3}{10}x, \\ B_{10}(x) &= x^{10} - 5x^9 + \frac{15}{2}x^8 - 7x^6 + 5x^4 - \frac{3}{2}x^2 + \frac{5}{66}, \dots \end{aligned}$$

4.1.5 Fourier Expansion of Bernoulli Polynomial

Bernoulli Polynomial $B_m(x)$ can be expanded to Fourier series on $0 \leq x \leq 1$. This means that Bernoulli Polynomial $B_m(x - \lfloor x \rfloor)$ can be expanded to Fourier series on $x \geq 0$.

Formula 4.1.5

When m is a natural number, $\lfloor x \rfloor$ is a floor function and B_m are Bernoulli numbers,

$$B_m(x - \lfloor x \rfloor) = -2m! \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^m} \cos\left(2\pi s x - \frac{m\pi}{2}\right) \quad x \geq 0$$

Proof

According to Formula 5.1.2 ("05 Generalized Bernoulli Polynomials"), the following expression holds.

$$B_m(x) = -2m! \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^m} \cos\left(2\pi s x - \frac{m\pi}{2}\right) \quad 0 \leq x \leq 1$$

Since $B_m(x - \lfloor x \rfloor) = B_m(x)$ on $0 \leq x < 1$,

$$B_m(x - \lfloor x \rfloor) = -2m! \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^m} \cos\left(2\pi s x - \frac{m\pi}{2}\right) \quad 0 \leq x < 1$$

On $1 \leq x < 2$,

$$\text{Left: } B_m(x+1 - \lfloor x+1 \rfloor) = B_m(x+1 - \lfloor x \rfloor - \lfloor 1 \rfloor) = B_m(x - \lfloor x \rfloor)$$

$$\text{Right: } -2m! \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^m} \cos\left\{2\pi s(x+1) - \frac{m\pi}{2}\right\} = -2m! \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^m} \cos\left(2\pi s x - \frac{m\pi}{2}\right)$$

$$\therefore B_m(x - \lfloor x \rfloor) = -2m! \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^m} \cos\left(2\pi s x - \frac{m\pi}{2}\right) \quad 1 \leq x < 2$$

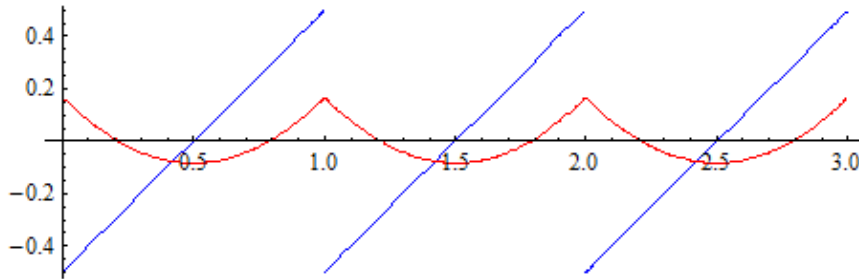
Hereafter by induction, the following expression holds for arbitrary natural number n .

$$B_m(x - \lfloor x \rfloor) = -2m! \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^m} \cos\left(2\pi s x - \frac{m\pi}{2}\right) \quad n \leq x < n+1$$

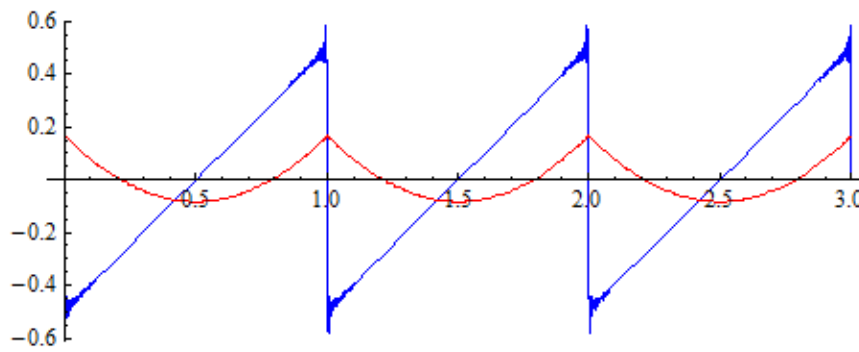
Q.E.D.

Examples

If the left side and the right side are illustrated for $m=1, 2$, it is as follows. Blue is $m=1$ and Red is $m=2$.
Left side



Right side



4.2 Euler-Maclaurin Summation Formula

Formula 4.2.1

When $f(x)$ is a function of class C^m on a closed interval $[a, b]$, $\lfloor x \rfloor$ is the floor function, B_r are Bernoulli numbers and $B_n(x)$ are Bernoulli polynomials, the following expression holds.

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x) dx + \sum_{r=1}^m \frac{B_r}{r!} \{f^{(r-1)}(b) - f^{(r-1)}(a)\} + R_m \quad (1.1)$$

$$R_m = \frac{(-1)^{m+1}}{m!} \int_a^b B_m(x - \lfloor x \rfloor) f^{(m)}(x) dx \quad (1.1r)$$

$$= (-1)^m 2 \int_a^b \left\{ \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^m} \cos\left(2\pi s x - \frac{m\pi}{2}\right) \right\} f^{(m)}(x) dx \quad (1.1r')$$

When $\lim_{m \rightarrow \infty} |R_m| = \infty$ m is a even number s. t. $\frac{|f^{(m)}(x)|}{(2\pi)^m} = \text{minimum for } x \in [a, b]$

Proof

When B_r are Bernoulli numbers and $B_n(x)$ are Bernoulli polynomials,

$$B_0(x) = 1, \quad \int_0^x B_n(x) dx = \frac{1}{n+1} \{B_{n+1}(x) - B_{n+1}\}$$

$$B_{n+1}(1) = B_{n+1}(0) = B_{n+1}$$

Using these,

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 B_0(x) f(x) dx \\ &= \frac{1}{1!} [B_1(x) f(x)]_0^1 - \int_0^1 \frac{B_1(x)}{1!} f'(x) dx \\ &= \frac{1}{1!} [B_1(x) f(x)]_0^1 - \frac{1}{2!} [B_2(x) f'(x)]_0^1 + \int_0^1 \frac{B_2(x)}{2!} f''(x) dx \\ &= \frac{1}{1!} [B_1(x) f(x)]_0^1 - \frac{1}{2!} [B_2(x) f'(x)]_0^1 \\ &\quad + \frac{1}{3!} [B_3(x) f''(x)]_0^1 - \int_0^1 \frac{B_3(x)}{3!} f'''(x) dx \\ &\vdots \\ &= \sum_{r=1}^m \frac{(-1)^{r-1}}{r!} [B_r(x) f^{(r-1)}(x)]_0^1 + (-1)^m \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x) dx \end{aligned}$$

Here,

$$[B_1(x) f(x)]_0^1 = \left(1 - \frac{1}{2}\right) f(1) - \left(0 - \frac{1}{2}\right) f(0) = \frac{1}{2} \{f(1) + f(0)\}$$

And

$$(-1)^{r-1} B_r(1) = (-1)^{r-1} B_r(0) = -B_r \quad \text{for } r \geq 2$$

Therefore,

$$(-1)^{r-1} [B_r(x) f^{(r-1)}(x)]_0^1 = -B_r \{f^{(r-1)}(1) - f^{(r-1)}(0)\}$$

Substituting these for the above,

$$\int_0^1 f(x) dx = \frac{1}{2} \{f(1) + f(0)\} - \sum_{r=2}^m \frac{B_r}{r!} \{f^{(r-1)}(1) - f^{(r-1)}(0)\} \\ + \frac{(-1)^m}{m!} \int_0^1 B_m(x) f^{(m)}(x) dx$$

Replacing $f(x)$ with $f(x+k)$,

$$\int_0^1 f(x+k) dx = \frac{1}{2} \{f(k+1) + f(k)\} - \sum_{r=2}^m \frac{B_r}{r!} \{f^{(r-1)}(k+1) - f^{(r-1)}(k)\} \\ + \frac{(-1)^m}{m!} \int_0^1 B_m(x) f^{(m)}(x+k) dx$$

That is,

$$\int_k^{k+1} f(x) dx = \frac{1}{2} \{f(k+1) + f(k)\} - \sum_{r=2}^m \frac{B_r}{r!} \{f^{(r-1)}(k+1) - f^{(r-1)}(k)\} \\ + \frac{(-1)^m}{m!} \int_k^{k+1} B_m(x-k) f^{(m)}(x) dx$$

Accumulating this from $k=a$ to $k=b-1$,

$$\int_a^b f(x) dx = \frac{1}{2} \sum_{k=a}^{b-1} \{f(k+1) + f(k)\} - \sum_{r=2}^m \frac{B_r}{r!} \sum_{k=a}^{b-1} \{f^{(r-1)}(k+1) - f^{(r-1)}(k)\} \\ + \frac{(-1)^m}{m!} \sum_{k=a}^{b-1} \int_k^{k+1} B_m(x-k) f^{(m)}(x) dx$$

Here,

$$\sum_{k=a}^{b-1} \{f^{(r-1)}(k+1) - f^{(r-1)}(k)\} = f^{(r-1)}(b) - f^{(r-1)}(a)$$

$$\sum_{k=a}^{b-1} \{f(k+1) + f(k)\} + f(a) + f(b) = 2 \sum_{k=a}^b f(k)$$

$$B_m(x-k) = B_m(x - \lfloor x \rfloor) \quad (k \leq x \leq k+1)$$

Therefore,

$$\int_a^b f(x) dx = \sum_{k=a}^b f(k) - \frac{1}{2} \{f(a) + f(b)\} - \sum_{r=2}^m \frac{B_r}{r!} \{f^{(r-1)}(b) - f^{(r-1)}(a)\} \\ + \frac{(-1)^m}{m!} \int_a^b B_m(x - \lfloor x \rfloor) f^{(m)}(x) dx$$

Transposing the terms,

$$\sum_{k=a}^b f(k) = \int_a^b f(x) dx + \frac{1}{2} \{f(a) + f(b)\} + \sum_{r=2}^m \frac{B_r}{r!} \{f^{(r-1)}(b) - f^{(r-1)}(a)\} \\ - \frac{(-1)^m}{m!} \int_a^b B_m(x - \lfloor x \rfloor) f^{(m)}(x) dx$$

Subtracting $f(b)$ from both sides,

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x) dx - \frac{1}{2} \{f(b) - f(a)\} + \sum_{r=2}^m \frac{B_r}{r!} \{f^{(r-1)}(b) - f^{(r-1)}(a)\} - \frac{(-1)^m}{m!} \int_a^b B_m(x - \lfloor x \rfloor) f^{(m)}(x) dx$$

Since $B_1 = -1/2$, including the 2nd term of the right side into Σ , we obtain

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x) dx + \sum_{r=1}^m \frac{B_r}{r!} \{f^{(r-1)}(b) - f^{(r-1)}(a)\} + R_m \quad (1.1)$$

$$R_m = \frac{(-1)^{m+1}}{m!} \int_a^b B_m(x - \lfloor x \rfloor) f^{(m)}(x) dx \quad (1.1r)$$

Last, substituting Formula 4.1.5

$$B_m(x - \lfloor x \rfloor) = -2m! \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^m} \cos\left(2\pi s x - \frac{m\pi}{2}\right) \quad x \geq 0$$

for (1.1r), we obtain

$$R_m = (-1)^m 2 \int_a^b \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^m} \cos\left(2\pi s x - \frac{m\pi}{2}\right) f^{(m)}(x) dx \quad (1.1r')$$

When R_m is divergent, the amplitude is almost determined by $\frac{f^{(m)}(x)}{(2\pi)^m}$. Therefore, at this time,

even number m should be chosen such that $\frac{|f^{(m)}(x)|}{(2\pi)^m}$ becomes minimum for $x \in [a, b]$.

Q.E.D.

Removing the terms of larger odd number than 1 in the Formula 4.2.1, we obtain the following formula.

Formula 4.2.2

When $f(x)$ is a function of class C^{2m} on a closed interval $[a, b]$, $\lfloor x \rfloor$ is the floor function, B_r are Bernoulli numbers and $B_n(x)$ are Bernoulli polynomials, the following expression holds.

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x) dx - \frac{1}{2} \{f(b) - f(a)\} + \sum_{r=1}^m \frac{B_{2r}}{(2r)!} \{f^{(2r-1)}(b) - f^{(2r-1)}(a)\} + R_{2m} \quad (2.1)$$

$$R_{2m} = -\frac{1}{(2m)!} \int_a^b B_{2m}(x - \lfloor x \rfloor) f^{(2m)}(x) dx \quad (2.1r)$$

$$= (-1)^m 2 \int_a^b \left\{ \sum_{s=1}^{\infty} \frac{\cos(2\pi s x)}{(2\pi s)^{2m}} \right\} f^{(2m)}(x) dx \quad (2.1r')$$

When $\lim_{m \rightarrow \infty} |R_{2m}| = \infty$, m is natural number s.t. $\frac{|f^{(2m)}(x)|}{(2\pi)^{2m}} = \text{minimum for } x \in [a, b]$

Proof

Removing B_3, B_5, B_7, \dots from Formula 4.2.1, we obtain (2.1), (2.1r). And, from Formula 4.1.5,

$$\begin{aligned}
B_{2m}(x - \lfloor x \rfloor) &= -2(2m)! \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^{2m}} \cos(2\pi s x - m\pi) \\
&= -(-1)^m 2(2m)! \sum_{s=1}^{\infty} \frac{\cos(2\pi s x)}{(2\pi s)^{2m}}
\end{aligned}$$

Substituting this for (2.1r), we obtain (2.1r').

When R_{2m} is divergent, the amplitude is almost determined by $\frac{f^{(2m)}(x)}{(2\pi)^{2m}}$. Therefore, at this time, natural number m should be chosen such that $\frac{|f^{(2m)}(x)|}{(2\pi)^{2m}}$ becomes minimum for $x \in [a, b]$.

Q.E.D.

Formula 4.2.2'

When $f(x)$ is a function of class C^{2m} on a closed interval $[a, b]$, $\lfloor x \rfloor$ is the floor function, B_r are Bernoulli numbers and $B_n(x)$ are Bernoulli polynomials, the following expression holds.

$$\begin{aligned}
\sum_{k=a}^b f(k) &= \int_a^b f(x) dx + \frac{1}{2} \{f(b) + f(a)\} \\
&\quad + \sum_{r=1}^m \frac{B_{2r}}{(2r)!} \{f^{(2r-1)}(b) - f^{(2r-1)}(a)\} + R_{2m} \tag{2.1}
\end{aligned}$$

$$R_{2m} = -\frac{1}{(2m)!} \int_a^b B_{2m}(x - \lfloor x \rfloor) f^{(2m)}(x) dx \tag{2.1r}$$

$$= (-1)^m 2 \int_a^b \left\{ \sum_{s=1}^{\infty} \frac{\cos(2\pi s x)}{(2\pi s)^{2m}} \right\} f^{(2m)}(x) dx \tag{2.1r'}$$

When $\lim_{m \rightarrow \infty} |R_{2m}| = \infty$, m is natural number s.t. $\frac{|f^{(2m)}(x)|}{(2\pi)^{2m}} = \text{minimum for } x \in [a, b]$

Proof

Adding $f(b)$ to both sides of Formula 4.2.2 (2.1), we obtain the desired expression immediately.

Q.E.D.

4.3 Sum of Elementary Sequence

4.3.1 Sum of Arithmetic Sequence

Formula 4.3.1

$$\sum_{k=0}^{n-1} (a+kd) = \frac{n}{2} \{2a + (n-1)d\} \quad (1.1)$$

Proof

$$\{S_{n-1}\} = a, a+d, a+2d, a+3d, \dots, a+(n-1)d$$

From this,

$$f(x) = a+xd$$

$$f^{(1)}(x) = d, f^{(2)}(x) = f^{(3)}(x) = \dots = 0$$

Substituting these for Formula 4.2.2,

$$\begin{aligned} \sum_{k=0}^{n-1} (a+kd) &= \int_0^n (a+xd) dx - \frac{1}{2} \{ (a+nd) - (a+0d) \} \\ &\quad + \frac{B_2}{2!} \{d-d\} + \sum_{r=2}^m \frac{B_{2r}}{(2r)!} \{0-0\} + R_{2m} \end{aligned}$$

$$= \left[\frac{xa}{1!} + \frac{x^2 d}{2!} \right]_0^n - \frac{nd}{2} + R_{2m}$$

$$= \frac{na}{1!} + \frac{n^2 d}{2!} - \frac{nd}{2} + R_{2m}$$

$$R_{2m} = -\frac{1}{(2m)!} \int_0^n B_{2m}(x-\lfloor x \rfloor) \cdot 0 dx = 0$$

Thus, we obtain the desired expression.

4.3.2 Sum of Geometric Sequence

Formula 4.3.2

$$\sum_{k=0}^{n-1} r^k = (r^n - 1) \sum_{s=0}^m \frac{B_s}{s!} (\log r)^{s-1} + R_m \quad (2.1)$$

$$R_m = (-1)^{m+1} \frac{(\log r)^m}{m!} \int_0^n B_m(x-\lfloor x \rfloor) r^x dx \quad (2.1r)$$

$$\sum_{k=0}^{n-1} r^k = (r^n - 1) \sum_{s=0}^{\infty} \frac{B_s}{s!} (\log r)^{s-1} = \frac{r^n - 1}{r - 1} \quad (2.1)$$

Proof

$$\{S_{n-1}\} = r^0, r^1, r^2, r^3, \dots, r^{n-1}$$

From this,

$$f(x) = r^x$$

$$\int_0^n f(x) = \left[\frac{r^x}{\log r} \right]_0^n = \frac{r^n - r^0}{\log r}$$

$$f^{(s-1)}(x) = r^x (\log r)^{s-1} \quad (s=1, \dots, m+1)$$

Substituting these for Formula 4.2.1 ,

$$\sum_{k=0}^{n-1} r^k = \frac{r^n - r^0}{\log r} + \sum_{s=1}^m \frac{B_s}{s!} \{r^n (\log r)^{s-1} - r^0 (\log r)^{s-1}\} + R_m$$

$$R_m = \frac{(-1)^{m+1}}{m!} \int_0^n B_m(x - [x]) r^x (\log r)^m dx$$

Including the 1st term of the right side into Σ , we obtain

$$\sum_{k=0}^{n-1} r^k = (r^n - 1) \sum_{s=0}^m \frac{B_s}{s!} (\log r)^{s-1} + R_m \quad (2.1)$$

$$R_m = (-1)^{m+1} \frac{(\log r)^m}{m!} \int_0^n B_m(x - [x]) r^x dx \quad (2.1r)$$

This is the sum of geometric sequence by Euler-Maclaurin Summation Formula.

Here, substituting $x = \log r$ for $\sum_{s=0}^{\infty} \frac{B_s}{s!} x^s = \frac{x}{e^{x-1}}$ (definitional identity of Bernoulli numbers)

$$\sum_{s=0}^{\infty} \frac{B_s}{s!} (\log r)^s = \frac{\log r}{e^{\log r} - 1} = \frac{\log r}{r - 1}$$

From this,

$$\sum_{s=0}^{\infty} \frac{B_s}{s!} (\log r)^{s-1} = \frac{1}{r - 1}$$

Furthermore,

$$\lim_{m \rightarrow \infty} \frac{(\log r)^m}{m!} = 0$$

Then $R_{\infty} = 0$. Thus,

$$\sum_{k=0}^{n-1} r^k = (r^n - 1) \sum_{s=0}^{\infty} \frac{B_s}{s!} (\log r)^{s-1} = \frac{r^n - 1}{r - 1}$$

4.3.3 Sum of integer powers of natural numbers

Formula 4.3.3 (Jacob Bernoulli)

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1} \sum_{r=0}^m \binom{m+1}{r} B_r n^{m+1-r} \quad (3.1)$$

$$= \frac{1}{m+1} \{B_{m+1}(n) - B_{m+1}(0)\} \quad (3.1')$$

Proof

$$\{S_{n-1}\} = 0^m, 1^m, 2^m, 3^m, \dots, (n-1)^m$$

From this,

$$f(x) = x^m$$

$$\int_0^n f(x) dx = \int_0^n x^m dx = \frac{1}{m+1} (n^{m+1} - 0^{m+1})$$

$$f^{(r-1)}(x) = \frac{m!}{(m-r+1)!} x^{m-r+1} \quad (r=1, \dots, m+1)$$

Substituting these for Formula 4.2.1 ,

$$\begin{aligned} \sum_{k=0}^{n-1} k^m &= \frac{1}{m+1} (n^{m+1} - 0^{m+1}) + \sum_{r=1}^m \frac{B_r}{r!} \frac{m!}{(m-r+1)!} (n^{m-r+1} - 0^{m-r+1}) + R_m \\ &= \sum_{r=0}^m \frac{B_r}{r!} \frac{m!}{(m-r+1)!} (n^{m-r+1} - 0^{m-r+1}) + R_m \\ &= \frac{1}{m+1} \sum_{r=0}^m \frac{(m+1)!}{r!(m+1-r)!} B_r (n^{m+1-r} - 0^{m+1-r}) + R_m \\ R_m &= \frac{(-1)^{m+1}}{m!} \int_0^n B_m(x-[x]) \frac{m!}{0!} x^0 dx \end{aligned}$$

i.e.

$$\begin{aligned} \sum_{k=0}^{n-1} k^m &= \frac{1}{m+1} \sum_{r=0}^m \binom{m+1}{r} B_r (n^{m+1-r} - 0^{m+1-r}) + R_m \\ R_m &= \frac{(-1)^{m+1}}{m!} \int_0^n B_m(x-[x]) \end{aligned}$$

Here,

$$\int_0^n B_m(x-[x]) dx = \sum_{r=0}^{n-1} \int_0^1 B_m(x) dx = 0$$

Then $R_m = 0$. Therefore

$$\begin{aligned} \sum_{k=0}^{n-1} k^m &= \frac{1}{m+1} \sum_{r=0}^m \binom{m+1}{r} B_r (n^{m+1-r} - 0^{m+1-r}) \quad (3.w) \\ &= \frac{1}{m+1} \sum_{r=0}^m \binom{m+1}{r} B_r n^{m+1-r} \quad (3.1) \end{aligned}$$

Furthermore

$$\binom{m+1}{m+1} B_{m+1} (n^0 - 0^0) = 0$$

Then, (3.w) is rewritten as follows.

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1} \sum_{r=0}^{m+1} \binom{m+1}{r} B_r (n^{m+1-r} - 0^{m+1-r})$$

Expressing this with the Bernoulli polynomial,

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1} \{B_{m+1}(n) - B_{m+1}(0)\} \quad (3.1')$$

Example: $m=3, n=101$

$$\sum_{k=0}^{101-1} k^3 = 0^3 + 1^3 + 2^3 + \dots + 100^3 = 25502500$$

$$\frac{1}{3+1} \{B_{3+1}(101) - B_{3+1}(0)\} = \frac{1}{4} \left(\frac{3060299999}{30} + \frac{1}{30} \right) = 25502500$$

4.3.4 Sum of alternative integer powers of natural numbers

Formula 4.3.4

$$\begin{aligned} \sum_{k=0}^{n-1} (-1)^{k-1} k^m &= \frac{1}{m+1} \sum_{r=0}^m \binom{m+1}{r} B_r \cdot \left(n^{m+1-r} - 2^{m+1} \left\lfloor \frac{n}{2} \right\rfloor^{m+1-r} \right) \\ &= \frac{1}{m+1} \left\{ B_{m+1}(n) - B_{m+1} - 2^{m+1} \left\{ B_{m+1} \left(\left\lfloor \frac{n}{2} \right\rfloor \right) - B_{m+1} \right\} \right\} \end{aligned}$$

Proof

When n is an even number,

$$\begin{aligned} 1^m - 2^m + 3^m - 4^m + \dots + (n-1)^m - n^m \\ &= 1^m + 3^m + 5^m + \dots + (n-1)^m - (2^m + 4^m + 6^m + \dots + n^m) \\ &= 1^m + 2^m + 3^m + \dots + n^m - 2(2^m + 4^m + 6^m + \dots + n^m) \\ &= 1^m + 2^m + 3^m + \dots + n^m - 2^{m+1} \left\{ 1^m + 2^m + 3^m + \dots + \left(\frac{n}{2} \right)^m \right\} \end{aligned}$$

i.e.

$$\sum_{k=1}^n (-1)^{k-1} k^m = \sum_{r=1}^n r^m - 2^{m+1} \sum_{r=1}^{\frac{n}{2}} r^m$$

Adding $(n+1)^m$ to both sides,

$$\sum_{k=1}^{n+1} (-1)^{k-1} k^m = \sum_{r=1}^{n+1} r^m - 2^{m+1} \sum_{r=1}^{\frac{n}{2}} r^m$$

Let $n+1 \rightarrow n$, $\frac{n+1}{2} \rightarrow \left\lfloor \frac{n}{2} \right\rfloor$. Then

$$\sum_{k=1}^n (-1)^{k-1} k^m = \sum_{r=1}^n r^m - 2^{m+1} \sum_{r=1}^{\left\lfloor \frac{n}{2} \right\rfloor} r^m$$

When $m \neq 0$,

$$\sum_{k=0}^n (-1)^{k-1} k^m = \sum_{r=0}^n r^m - 2^{m+1} \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} r^m$$

Replacing n with $n+1$,

$$\sum_{k=0}^{n-1} (-1)^{k-1} k^m = \sum_{r=0}^{n-1} r^m - 2^{m+1} \sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} r^m = \sum_{r=0}^{n-1} r^m - 2^{m+1} \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} r^m$$

Applying Formula 4.3.3 to this,

$$\begin{aligned} \sum_{r=0}^{n-1} r^m &= \frac{1}{m+1} \sum_{r=0}^m \binom{m+1}{r} B_r n^{m+1-r} = \frac{1}{m+1} \{ B_{m+1}(n) - B_{m+1}(0) \} \\ \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} r^m &= \frac{1}{m+1} \sum_{r=0}^m \binom{m+1}{r} B_r \left\lfloor \frac{n}{2} \right\rfloor^{m+1-r} = \frac{1}{m+1} \left\{ B_{m+1} \left(\left\lfloor \frac{n}{2} \right\rfloor \right) - B_{m+1}(0) \right\} \end{aligned}$$

Using these,

$$\begin{aligned}\sum_{k=0}^{n-1} (-1)^{k-1} k^m &= \frac{1}{m+1} \sum_{r=0}^m \binom{m+1}{r} B_r \cdot \left(n^{m+1-r} - 2^{m+1} \left\lceil \frac{n}{2} \right\rceil^{m+1-r} \right) \\ &= \frac{1}{m+1} \left\{ B_{m+1}(n) - B_{m+1} - 2^{m+1} \left\{ B_{m+1} \left(\left\lceil \frac{n}{2} \right\rceil \right) - B_{m+1} \right\} \right\}\end{aligned}$$

Example: $m=3, n=101$

$$\begin{aligned}\sum_{k=0}^{101-1} (-1)^{k-1} k^3 &= 1^3 - 2^3 + 3^3 - 4^3 + \dots + 99^3 - 100^3 &&= -507500 \\ \frac{1}{3+1} \left\{ B_{3+1}(101) - B_{3+1} - 2^{3+1} \left\{ B_{3+1} \left(\left\lceil \frac{101}{2} \right\rceil \right) - B_{3+1} \right\} \right\} &&= -507500\end{aligned}$$

Especially in the case of $m=2$, the following interesting formula holds.

Formula 4.3.4'

$$\sum_{k=1}^n (-1)^{k-1} k^2 = (-1)^{n-1} \sum_{k=1}^n k = (-1)^{n-1} \frac{n(n+1)}{2}$$

Proof

When n is an even number,

$$\begin{aligned}1^2 - 2^2 + 3^2 - 4^2 + \dots - (n-1)^2 - n^2 \\ &= - \{ (1+1)^2 - 1^2 \} - \{ (3+1)^2 - 3^2 \} - \dots - [\{ (n-1)+1 \}^2 - (n-1)^2] \\ &= -(2 \cdot 1 + 1) - (2 \cdot 3 + 1) - (2 \cdot 5 + 1) - \dots - \{ 2(n-1) + 1 \} \\ &= -2 \{ 1 + 3 + 5 + \dots + (n-1) \} - (1 + 1 + 1 \dots + 1) \\ &= - \{ 1 + 3 + 5 + \dots + (n-1) \} - (2 + 4 + 6 + \dots + n) \\ &= -(1 + 2 + 3 + \dots + n)\end{aligned}$$

i.e.

$$\sum_{k=1}^n (-1)^{k-1} k^2 = - \sum_{k=1}^n k = - \frac{n(n+1)}{2}$$

When n is an odd number,

$$\begin{aligned}-0^2 + 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \dots + n^2 \\ &= - \{ (1-1)^2 - 1^2 \} - \{ (3-1)^2 - 3^2 \} - \dots - \{ (n-1)^2 - n^2 \} \\ &= -(-2 \cdot 1 + 1) - (-2 \cdot 3 + 1) - (-2 \cdot 5 + 1) - \dots - (-2n + 1) \\ &= 2 \{ 1 + 3 + 5 + \dots + n \} - (1 + 1 + 1 \dots + 1) \\ &= 1 + 3 + 5 + \dots + n + 0 + 2 + 4 + \dots + (n-1) \\ &= 1 + 2 + 3 + \dots + n\end{aligned}$$

i.e.

$$\sum_{k=1}^n (-1)^{k-1} k^2 = \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

Combining both, we obtain the desired expression.

Example: $n = 999$

$$\sum_{k=1}^{999} (-1)^{k-1} k^2 = 1^2 - 2^2 + 3^2 - 4^2 + \dots + 999^2 = 499500$$

$$(-1)^{999-1} (1+2+3+\dots+999) = \frac{999 \cdot 1000}{2} = 499500$$

4.3.5 Sum of Trigonometric Sequence

Formula 4.3.5s

$$\sum_{k=0}^{n-1} \sin k = -\frac{\sin n}{2} - (\cos n - 1) \left\{ 1 + \sum_{r=1}^m (-1)^r \frac{B_{2r}}{(2r)!} \right\} + R_{2m} \quad (5.s)$$

$$R_{2m} = \frac{(-1)^{m+1}}{(2m)!} \int_0^n B_{2m}(x - \lfloor x \rfloor) \sin x \, dx \quad (5.sr)$$

$$\sum_{k=0}^{n-1} \sin k = -\frac{\sin n}{2} - (\cos n - 1) \left(\frac{1}{2} \cot \frac{1}{2} \right) \quad (5.s')$$

$$= \sin \frac{n-1}{2} \sin \frac{n}{2} / \sin \frac{1}{2} \quad (5.s'')$$

Proof

$$\{S_{n-1}\} = \sin 1, \sin 2, \sin 3, \dots, \sin(n-1)$$

From this,

$$f(x) = \sin x$$

$$\int_0^n f(x) \, dx = [-\cos x]_0^n = -\cos n + \cos 0 = -\cos n + 1$$

$$f^{(2r-1)}(x) = (-1)^{r-1} \cos x \quad (r=1, \dots, m)$$

$$f^{(2m)}(x) = (-1)^m \sin x$$

Substituting these for Formula 4.2.2 ,

$$\sum_{k=0}^{n-1} \sin k = -\cos n + \cos 0 - \frac{1}{2} (\sin n - \sin 0) + \sum_{r=1}^m \frac{B_{2r}}{(2r)!} (-1)^{r-1} \{\cos n - \cos 0\} + R_{2m}$$

$$R_{2m} = -\frac{1}{(2m)!} \int_0^n B_{2m}(x - \lfloor x \rfloor) (-1)^m \sin x \, dx$$

i.e.

$$\sum_{k=0}^{n-1} \sin k = -\frac{\sin n}{2} - (\cos n - 1) \left\{ 1 + \sum_{r=1}^m (-1)^r \frac{B_{2r}}{(2r)!} \right\} + R_{2m} \quad (5.s)$$

$$R_{2m} = \frac{(-1)^{m+1}}{(2m)!} \int_0^n B_{2m}(x - \lfloor x \rfloor) \sin x \, dx \quad (5.sr)$$

This is the sum of sine sequence by Euler-Maclaurin Summation Formula.

Here, let $m \rightarrow \infty$. Then,

$$\lim_{m \rightarrow \infty} \left\{ 1 + \sum_{r=1}^m (-1)^r \frac{B_{2r}}{(2r)!} \right\} = \frac{1}{2} \cot \frac{1}{2}, \quad \lim_{m \rightarrow \infty} \frac{B_{2m}(x - \lfloor x \rfloor) \sin x}{(2m)!} = 0$$

Therefore, (5.s) becomes as follows.

$$\sum_{k=0}^{n-1} \sin k = -\frac{\sin n}{2} - (\cos n - 1) \left(\frac{1}{2} \cot \frac{1}{2} \right) \quad (5.s')$$

Moreover, using formulus of trigonometric functions,

$$\begin{aligned} \sum_{k=0}^{n-1} \sin k &= -\frac{1}{2} \cot \frac{1}{2} \cdot (\cos n - 1) - \frac{1}{2} \sin n \\ &= -\frac{1}{2} \left\{ \cos \frac{1}{2} \cdot (\cos n - 1) - \sin \frac{1}{2} \cdot \sin n \right\} / \sin \frac{1}{2} \\ &= -\frac{1}{2} \left\{ -2 \sin \frac{n-1}{2} \sin \frac{n}{2} \right\} / \sin \frac{1}{2} \end{aligned}$$

i.e.

$$\sum_{k=0}^{n-1} \sin k = \sin \frac{n-1}{2} \sin \frac{n}{2} / \sin \frac{1}{2} \quad (5.s'')$$

This is consistent with the result of having applied Trigonometric Addition Formulas to $\sum_{k=0}^{n-1} \sin k$ directly.

Formula 4.3.5c

$$\sum_{k=0}^{n-1} \cos k = \sin n \cdot \left\{ 1 + \sum_{r=1}^m (-1)^r \frac{B_{2r}}{(2r)!} \right\} - \frac{1}{2} (\cos n - 1) \quad (5.c)$$

$$R_{2m} = \frac{(-1)^{m+1}}{(2m)!} \int_0^n B_{2m}(x - \lfloor x \rfloor) \cos x \, dx \quad (5.cr)$$

$$\sum_{k=0}^{n-1} \cos k = \sin n \cdot \frac{1}{2} \cot \frac{1}{2} - \frac{1}{2} (\cos n - 1) \quad (5.c')$$

$$= \cos \frac{n-1}{2} \sin \frac{n}{2} / \sin \frac{1}{2} \quad (5.c'')$$

Proof

$$\{S_{n-1}\} = \cos 1, \cos 2, \cos 3, \dots, \cos(n-1)$$

From this,

$$f(x) = \cos x$$

$$\int_0^n f(x) \, dx = [\sin x]_0^n = \sin n - \sin 0 = \sin n$$

$$f^{(2r-1)}(x) = (-1)^r \sin x \quad (r=1, \dots, m)$$

$$f^{(2m)}(x) = (-1)^m \cos x$$

Substituting these for Formula 4.2.2 ,

$$\begin{aligned} \sum_{k=0}^{n-1} \cos k &= \sin n - \frac{1}{2} (\cos n - \cos 0) \\ &\quad + \sum_{r=1}^m \frac{B_{2r}}{(2r)!} \{ (-1)^r \sin n - (-1)^r \sin 0 \} + R_{2m} \end{aligned}$$

$$= \sin n - \frac{1}{2} (\cos n - \cos 0) + \sin n \sum_{r=1}^m (-1)^r \frac{B_{2r}}{(2r)!} + R_{2m}$$

$$R_{2m} = \frac{(-1)^{m+1}}{(2m)!} \int_0^n B_{2m}(x - \lfloor x \rfloor) \cos x \, dx$$

i.e.

$$\sum_{k=0}^{n-1} \cos k = \sin n \left\{ 1 + \sum_{r=1}^m (-1)^r \frac{B_{2r}}{(2r)!} \right\} - \frac{1}{2} (\cos n - 1) + R_{2m} \quad (5.c)$$

$$R_{2m} = \frac{(-1)^{m+1}}{(2m)!} \int_0^n B_{2m}(x - \lfloor x \rfloor) \cos x \, dx \quad (5.cr)$$

This is the sum of cosine sequence by Euler-Maclaurin Summation Formula.

Here, let $m \rightarrow \infty$. Then,

$$\lim_{m \rightarrow \infty} \left\{ 1 + \sum_{k=1}^m (-1)^k \frac{B_{2k}}{(2k)!} \right\} = \frac{1}{2} \cot \frac{1}{2}, \quad \lim_{m \rightarrow \infty} \frac{B_{2m}(x - \lfloor x \rfloor) \cos x}{(2m)!} = 0$$

Therefore, (5.c) becomes as follows.

$$\sum_{k=0}^{n-1} \cos k = \sin n \cdot \frac{1}{2} \cot \frac{1}{2} - \frac{1}{2} (\cos n - 1) \quad (5.c')$$

Moreover, using formuluses of trigonometric functions,

$$\sum_{k=0}^{n-1} \cos k = \cos \frac{n-1}{2} \sin \frac{n}{2} / \sin \frac{1}{2} \quad (5.c'')$$

This is consistent with the result of having applied Trigonometric Addition Formulas to $\sum_{k=0}^{n-1} \cos k$ directly.

4.4 Sum of Harmonic Sequence & Euler-Mascheroni Constant

4.4.1 Sum of Harmonic Sequence

Formula 4.4.1

When γ is Euler-Mascheroni Constant,

$$\sum_{k=1}^{n-1} \frac{1}{k} = \gamma + \log n - \frac{1}{2n} - \sum_{r=1}^m \frac{B_{2r}}{2r \cdot n^{2r}} + R_{2m} \quad (1.1)$$

$$R_{2m} = \int_n^{\infty} \frac{B_{2m}(x - \lfloor x \rfloor)}{x^{2m+1}} dx \quad (1.1r)$$

Where, $2 \leq m < \infty$

Proof

$$\{S_{n-1}\} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-1}$$

From this,

$$f(x) = \frac{1}{x}$$

$$\int_n^h f(x) dx = [\log x]_n^h = \log h - \log n \quad (1 < n < h, \ h, n \text{ are integers})$$

$$f^{(2r-1)}(x) = (-1)^{2r-1} \frac{(2r-1)!}{x^{2r}}, \quad f^{(2m)}(x) = \frac{(2m)!}{x^{2m+1}}$$

Substituting these for Formula 4.2.2 ,

$$\sum_{k=n}^{h-1} \frac{1}{k} = \log h - \log n - \frac{1}{2} \left(\frac{1}{h} - \frac{1}{n} \right) - \sum_{r=1}^m \frac{B_{2r}}{2r} \left(\frac{1}{h^{2r}} - \frac{1}{n^{2r}} \right) + R_{2m}$$

$$R_{2m} = - \int_n^h \frac{B_{2m}(x - \lfloor x \rfloor)}{x^{2m+1}} dx$$

From this,

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{1}{k} &= \sum_{k=1}^{h-1} \frac{1}{k} - \sum_{k=n}^{h-1} \frac{1}{k} \\ &= \sum_{k=1}^{h-1} \frac{1}{k} - \log h + \log n + \frac{1}{2} \left(\frac{1}{h} - \frac{1}{n} \right) + \sum_{r=1}^m \frac{B_{2r}}{2r} \left(\frac{1}{h^{2r}} - \frac{1}{n^{2r}} \right) - R_{2m} \end{aligned}$$

When $h \rightarrow \infty$, since $r \geq 1$,

$$\lim_{h \rightarrow \infty} \left(\sum_{k=1}^{h-1} \frac{1}{k} - \log h \right) = \gamma, \quad \lim_{h \rightarrow \infty} \frac{1}{h} = 0, \quad \lim_{h \rightarrow \infty} \frac{1}{h^{2r}} = 0$$

Then,

$$\sum_{k=1}^{n-1} \frac{1}{k} = \gamma + \log n - \frac{1}{2n} - \sum_{r=1}^m \frac{B_{2r}}{2r \cdot n^{2r}} - R_{2m}$$

$$R_{2m} = - \int_n^{\infty} \frac{B_{2m}(x - \lfloor x \rfloor)}{x^{2m+1}} dx$$

Reversing the sign of the remainder term, we obtain the desired expression.

Example: $\sum_{k=1}^{100} 1/k$

When $m = 2$,

$$\gamma + \log 101 - \frac{1}{2 \cdot 101} - \left(\frac{B_2}{2 \cdot 101^2} + \frac{B_4}{4 \cdot 11^4} \right) = 5.18737751763962\dots$$

This all digits (14 digits below the decimal point) are significant digits. Furthermore, if the Σ is calculated to $m = 8$, the significant digit reaches 34 digits below the decimal point.

c.f.

If Formula 4.2.2 is applied straight to $f(x) = 1/x$, it is as follows.

$$\sum_{k=1}^{n-1} \frac{1}{k} = \log n - \log 1 - \frac{1}{2} \left(\frac{1}{n} - \frac{1}{1} \right) - \sum_{r=1}^m \frac{B_{2r}}{2r} \left(\frac{1}{n^{2r}} - \frac{1}{1^{2r}} \right) + R_{2m}$$

$$R_{2m} = - \int_1^n \frac{B_{2m}(x - \lfloor x \rfloor)}{x^{2m+1}} dx$$

However, the degree of the approximation of this formula is very bad. If Σ is calculated to $m = 3$ in the above example, the significant digit reaches 2 digits below the decimal point. And, it is the best approximation of the above example by this formula.

4.4.2 Calculation of Euler-Mascheroni Constant

Transposing the terms of (1.1), we obtain as follows.

$$\gamma = \sum_{k=1}^{n-1} \frac{1}{k} - \log n + \frac{1}{2n} + \sum_{r=1}^m \frac{B_{2r}}{2r \cdot n^{2r}} + R_{2m} \quad (2.1)$$

$$R_{2m} = - \int_n^\infty \frac{B_{2m}(x - \lfloor x \rfloor)}{x^{2m+1}} dx \quad (2.1r)$$

Where, $m = n$: (number of significant digits + ?) / 2

That is, we can calculate Euler-Mascheroni Constant γ conversely from (1.1). When $m = n$, the number of significant digits is roughly given by $2n - ?$.

Example: $m = n = 10$

$$\sum_{k=1}^{10-1} \frac{1}{k} - \log 10 + \frac{1}{2 \cdot 10} + \sum_{r=1}^{10} \frac{B_{2r}}{2r \cdot 10^{2r}} = 0.577215664901532860\dots$$

This all digits (18 digits below the decimal point) are significant digits. (2.1) is a quite good approximate expression of γ . However, regrettably $\lim_{m \rightarrow \infty} R_{2m} \neq 0$ for definite n . That is, (2.1) is only an asymptotic expansion.

4.5 Sum of Zeta Sequence & Zeta Function

4.5.1 Sum of Zeta Sequence

Formula 4.5.1

When $\zeta(p)$ is the Riemann Zeta Function and $B(p, q)$ is the beta function, the following expression holds for $p \neq 1$.

$$\sum_{k=1}^{n-1} \frac{1}{k^p} = \zeta(p) + \frac{1}{1-p} \sum_{r=0}^m \binom{1-p}{r} B_r n^{1-p-r} + R_m \quad (1.1)$$

$$R_m = \frac{1}{m B(m, p)} \int_n^\infty \frac{B_m(x - \lfloor x \rfloor)}{x^{p+m}} dx \quad (1.1r)$$

Where, m is an even number s.t. $\lceil p \rceil \leq m < \infty$.

Proof

$$\{S_{n-1}\} = 1^{-p}, 2^{-p}, 3^{-p}, \dots, (n-1)^{-p} \quad (p > 1)$$

From this,

$$f(x) = x^{-p}$$

$$\int_n^h f(x) dx = \left[\frac{x^{1-p}}{1-p} \right]_n^h = \frac{h^{1-p} - n^{1-p}}{1-p} \quad (1 < n < h, h, n \text{ are integers})$$

$$f^{(r-1)}(x) = -(-1)^r \frac{\Gamma(p+r-1)}{\Gamma(p)} x^{1-p-r} \quad (r=1, \dots, m+1)$$

Substituting these for Formula 4.2.1 ,

$$\sum_{k=n}^{h-1} \frac{1}{k^p} = \frac{h^{1-p} - n^{1-p}}{1-p} - \sum_{r=1}^m (-1)^r \frac{B_r}{r!} \frac{\Gamma(p+r-1)}{\Gamma(p)} (h^{1-p-r} - n^{1-p-r}) + R_m$$

Here,

$$\begin{aligned} (-1)^r \frac{B_r}{r!} \frac{\Gamma(p+r-1)}{\Gamma(p)} &= (-1)^r \frac{B_r}{\Gamma(1+r)} \frac{(-1)^r \Gamma\{1+r+(p-2)\}}{(p-1)\Gamma\{1+(p-2)\}} \\ &= -\frac{B_r}{1-p} \frac{(-1)^r \Gamma\{1+r+(p-2)\}}{\Gamma(1+r)\Gamma\{1+(p-2)\}} \\ &= -\frac{B_r}{1-p} (-1)^r \binom{p-2+r}{p-2} \\ &= -\frac{1}{1-p} \sum_{r=1}^m \binom{-p+1}{r} B_r \end{aligned}$$

Using this,

$$\sum_{k=n}^{h-1} \frac{1}{k^p} = \frac{h^{1-p} - n^{1-p}}{1-p} + \frac{1}{1-p} \sum_{r=1}^m \binom{1-p}{r} B_r (h^{1-p-r} - n^{1-p-r}) + R_m$$

$$R_m = -\frac{1}{m!} \int_a^b B_m(x - \lfloor x \rfloor) \frac{\Gamma(p+m)}{\Gamma(p)} x^{-p-m} dx$$

i.e.

$$\sum_{k=n}^{h-1} \frac{1}{k^p} = \frac{1}{1-p} \sum_{r=0}^m \binom{1-p}{r} B_r (h^{1-p-r} - n^{1-p-r}) + R_m$$

$$R_m = -\frac{1}{mB(m,p)} \int_n^h \frac{B_m(x-\lfloor x \rfloor)}{x^{p+m}} dx$$

From this,

$$\sum_{k=1}^{n-1} \frac{1}{k^p} = \sum_{k=1}^{h-1} \frac{1}{k^p} - \sum_{k=n}^{h-1} \frac{1}{k^p}$$

$$= \sum_{k=1}^{h-1} \frac{1}{k^p} - \frac{1}{1-p} \sum_{r=0}^m \binom{1-p}{r} B_r (h^{1-p-r} - n^{1-p-r}) - R_m$$

$$R_m = -\frac{1}{mB(m,p)} \int_n^h \frac{B_m(x-\lfloor x \rfloor)}{x^{p+m}} dx$$

Here, let $h \rightarrow \infty$. since $p > 1$, $\lim_{h \rightarrow \infty} h^{p-1-r} = 0$. Therefore,

$$\sum_{k=1}^{n-1} \frac{1}{k^p} = \zeta(p) + \frac{1}{1-p} \sum_{r=0}^m \binom{1-p}{r} B_r n^{1-p-r} - R_m \quad (1.1)$$

$$R_m = -\frac{1}{mB(m,p)} \int_n^\infty \frac{B_m(x-\lfloor x \rfloor)}{x^{p+m}} dx \quad (1.1)r$$

Reversing the sign of the remainder term, we obtain the desired expression.

Example: $\sum_{k=1}^{100} 1/k^{1.1}$

When $m = 1.1 \uparrow = 2$,

$$\sum_{k=1}^{101-1} \frac{1}{k^{1.1}} \doteq \zeta(1.1) + \frac{1}{1-1.1} \sum_{r=0}^2 \binom{1-1.1}{r} B_r n^{1-1.1-r} = 4.278024023 \dots$$

This all digits (9 digits below the decimal point) are significant digits. On practical use, it seems that this is enough. When $m=6$, 14 digits (13 digits below the decimal point) are significant digits.

c.f.

If Formula 4.2.1 is applied straight to $f(x) = x^{-p}$, it is as follows.

$$\sum_{k=1}^{n-1} \frac{1}{k^p} = \frac{1}{1-p} \sum_{r=0}^m \binom{1-p}{r} B_r (n^{1-p-r} - 1) + R_m$$

$$R_m = -\frac{1}{mB(m,p)} \int_1^n \frac{B_m(x-\lfloor x \rfloor)}{x^{p+m}} dx$$

However, the degree of the approximation of this formula is very bad. If ζ is calculated to $m=4$ in the above example, the significant digit reaches 2 digits below the decimal point. And, it is the best approximation of the above example by this formula.

4.5.2 Calculation of Riemann Zeta Function

Transposing the terms of (1.1), we obtain as follows.

$$\zeta(p) = \sum_{k=1}^{n-1} \frac{1}{k^p} - \frac{1}{1-p} \sum_{r=0}^m \binom{1-p}{r} B_r n^{1-p-r} + R_m \quad (2.1)$$

$$R_m = -\frac{1}{m B(m,p)} \int_n^\infty \frac{B_m(x - \lfloor x \rfloor)}{x^{p+m}} dx \quad (2.1r)$$

Where, $m = n$: number of significant digits + ?

That is, we can calculate Riemann Zeta Function $\zeta(s)$ conversely from (1.1) . When $m = n$, the number of significant digits is roughly given by $n - ?$.

Example: $\zeta(1.3)$

When $m = n = 10$,

$$\sum_{k=1}^{10-1} \frac{1}{k^{1.3}} - \frac{1}{1-1.3} \sum_{r=0}^{10} \binom{1-1.3}{r} B_r 10^{1-1.3r} = 3.9319492118095\dots$$

This all digits (13 digits after the decimal point) are significant digits. (2.1) is a quite good approximate expression of $\zeta(p)$. However, regrettably $\lim_{m \rightarrow \infty} R_m \neq 0$ for definite n . That is, (2.1) is only an asymptotic expansion.

4.6 Sum of real number powers of natural numbers

Formula 4.6.1

When $\zeta(p)$ is the Riemann Zeta Function and $B(p, q)$ is the beta function, the following expression holds for $p \neq -1$.

$$\sum_{k=1}^{n-1} k^p = \zeta(-p) + \frac{1}{1+p} \sum_{r=0}^m \binom{1+p}{r} B_r n^{1+p-r} + R_m$$

$$R_m = \frac{1}{m B(m, -p)} \int_n^\infty \frac{B_m(x - \lfloor x \rfloor)}{x^{-p+m}} dx$$

Where, m is an even number s.t. $\lceil p \rceil \leq m < \infty$.

Proof

Reversing the sign of p in Formula 4.5.1, we obtain the desired expression.

Example 1 : $\sum_{k=1}^{100} k^{0.1}$

When $m = 2$,

$$\sum_{k=1}^{101-1} k^{0.1} \doteq \zeta(-0.1) + \frac{1}{1+0.1} \sum_{r=0}^2 \binom{1+0.1}{r} B_r 101^{1+0.1-r} = 144.456549944\dots$$

This all digits (9 digits below the decimal point) are significant digits. On practical use, it seems that this is enough. When $m=6$, 15 digits (12 digits after the decimal point) are significant digits.

Example 2 : $\sum_{k=1}^{100} k^3$

When $m = 2$,

$$\sum_{k=1}^{101-1} k^3 \doteq \zeta(-3) + \frac{1}{1+3} \sum_{r=0}^2 \binom{1+3}{r} B_r n^{1+3-r} = 25502500$$

This result is consistent with the example by Formula 4.3.3.

Example 3 : $\sum_{k=1}^{100} k^{-1.1}$

When $m = 2$,

$$\sum_{k=1}^{101-1} k^{-1.1} \doteq \zeta(1.1) + \frac{1}{1-1.1} \sum_{r=0}^2 \binom{1-1.1}{r} B_r 101^{1-1.1-r} = 4.278024023\dots$$

This reduce to the example of Formula 4.5.1.

c.f.

If Formula 4.2.1 is applied straight to $f(x) = x^p$, it is as follows.

$$\sum_{k=1}^{n-1} k^p = \frac{1}{1+p} \sum_{r=0}^m \binom{1+p}{r} B_r (n^{1+p-r} - 1) + R_m$$

$$R_m = -\frac{1}{m B(m, -p)} \int_1^n B_m(x - \lfloor x \rfloor) x^{p-m} dx$$

However, the degree of the approximation of this formula is very bad. If Σ is calculated to $m=6$ in **Example1** the significant digit reaches 4 digits below the decimal point. And, it is the best approximation of Example1 by this formula.

4.7 Sum of alternative real powers

4.7.1 Sum of alternative positive powers of natural numbers

Formula 4.7.1

When $\zeta(p)$ is the Riemann Zeta Function and $B(p, q)$ is the beta function, the following expression holds for $p \neq -1$.

$$\sum_{k=0}^{n-1} (-1)^{k-1} k^p = (1 - 2^{1+p}) \zeta(-p) + \frac{1}{1+p} \sum_{r=0}^m \binom{1+p}{r} B_r \cdot \left(n^{1+p-r} - 2^{1+p} \left\lceil \frac{n}{2} \right\rceil^{1+p-r} \right) + R_m$$

$$R_m = \frac{1}{m B(m, -p)} \left\{ \int_n^\infty \frac{B_m(x - \lfloor x \rfloor)}{x^{-p+m}} dx - 2^{1+p} \int_{\lceil n/2 \rceil}^\infty \frac{B_m(x - \lfloor x \rfloor)}{x^{-p+m}} dx \right\}$$

Where, m is an even number s.t. $\lceil p \rceil \leq m < \infty$.

Proof

The following equation was obtained in the proof of Formula 4.3.4.

$$\sum_{k=0}^{n-1} (-1)^{k-1} k^m = \sum_{r=0}^{n-1} r^m - 2^{m+1} \sum_{r=0}^{\lceil \frac{n}{2} \rceil - 1} r^m$$

This equation holds even if the natural number m is extended to the real number p . That is,

$$\sum_{k=0}^{n-1} (-1)^{k-1} k^p = \sum_{r=0}^{n-1} r^p - 2^{p+1} \sum_{r=0}^{\lceil \frac{n}{2} \rceil - 1} r^p$$

Applying Formula 4.6.1 to this,

$$\sum_{r=0}^{n-1} r^p = \zeta(-p) + \frac{1}{1+p} \sum_{r=0}^m \binom{1+p}{r} B_r n^{1+p-r} + R_{m1}$$

$$\sum_{r=0}^{\lceil \frac{n}{2} \rceil - 1} r^p = \zeta(-p) + \frac{1}{1+p} \sum_{r=0}^m \binom{1+p}{r} B_r \left\lceil \frac{n}{2} \right\rceil^{1+p-r} + R_{m2}$$

$$R_{m1} = \frac{1}{m B(m, -p)} \int_n^\infty \frac{B_m(x - \lfloor x \rfloor)}{x^{-p+m}} dx$$

$$R_{m2} = \frac{1}{m B(m, -p)} \int_{\lceil n/2 \rceil}^\infty \frac{B_m(x - \lfloor x \rfloor)}{x^{-p+m}} dx$$

Then

$$\sum_{k=0}^{n-1} (-1)^{k-1} k^p = \zeta(-p) + \frac{1}{1+p} \sum_{r=0}^m \binom{1+p}{r} B_r n^{1+p-r} - 2^{1+p} \left\{ \zeta(-p) + \frac{1}{1+p} \sum_{r=0}^m \binom{1+p}{r} B_r \left\lceil \frac{n}{2} \right\rceil^{1+p-r} + R_{m2} \right\}$$

$$= (1 - 2^{1+p}) \zeta(-p) + \frac{1}{1+p} \sum_{r=0}^m \binom{1+p}{r} B_r \cdot \left(n^{1+p-r} - 2^{1+p} \left\lceil \frac{n}{2} \right\rceil^{1+p-r} \right) + R_{m1} - 2^{1+p} R_{m2}$$

i.e.

$$\sum_{k=0}^{n-1} (-1)^{k-1} k^p = (1 - 2^{1+p}) \zeta(-p) + \frac{1}{1+p} \sum_{r=0}^m \binom{1+p}{r} B_r \cdot \left(n^{1+p-r} - 2^{1+p} \left\lceil \frac{n}{2} \right\rceil^{1+p-r} \right) + R_m$$

$$R_m = \frac{1}{m B(m, -p)} \left\{ \int_n^\infty \frac{B_m(x - \lfloor x \rfloor)}{x^{-p+m}} dx - 2^{p+1} \int_{\lceil n/2 \rceil}^\infty \frac{B_m(x - \lfloor x \rfloor)}{x^{-p+m}} dx \right\}$$

Where, m is an even number s.t. $\lceil p \rceil \leq m < \infty$.

Example : $p=0.6$, $n=1001$

$$\text{fl}[p_, n_] := \sum_{k=0}^{n-1} (-1)^{k-1} k^p$$

$$\text{fr}[p_, n_, m_] := (1 - 2^{1+p}) \text{Zeta}[-p] + \frac{1}{1+p} \sum_{r=0}^m \text{Binomial}[1+p, r] \text{BernoulliB}[r] \left(n^{1+p-r} - 2^{1+p} \text{Ceiling}\left[\frac{n}{2}\right]^{1+p-r} \right)$$

```
SetPrecision[{fl[0.6, 1001], fr[0.6, 1001, 4]}, 15]
{-31.2026520696209, -31.2026520696212}
```

In this case, $m=4$ gives the best approximation (11 digits below the decimal point).

c.f.

If Formula 4.2.1 is applied straight to $\sum_{k=0}^{n-1} (-1)^{k-1} k^p$, it is as follows.

$$\sum_{k=0}^{n-1} (-1)^{k-1} k^p = \frac{1}{1+p} \sum_{r=0}^m \binom{1+p}{r} B_r \cdot \left\{ n^{1+p-r} - 2^{1+p} \left\lceil \frac{n}{2} \right\rceil^{1+p-r} - (1 - 2^{1+p}) \right\} + R_m$$

$$R_m = -\frac{1}{m B(m, -p)} \left\{ \int_1^n \frac{B_m(x - \lfloor x \rfloor)}{x^{-p+m}} dx - 2^{1+p} \int_1^{\lceil n/2 \rceil} \frac{B_m(x - \lfloor x \rfloor)}{x^{-p+m}} dx \right\}$$

However, the degree of the approximation of this formula is very bad. If Σ is calculated to $m=6$ in the above example, the significant digit reaches 3 digits below the decimal point. And, it is the best approximation of the above example by this formula.

4.7.2 Sum of Eta Sequence and Dirichlet Eta Function

Formula 4.7.2

When $\zeta(p)$ is the Riemann Zeta Function, $\eta(p)$ is the Dirichlet Eta Function and $B(p, q)$ is the Beta Function,

the following expression holds for $p \neq -1$.

$$\sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k^p} = (1 - 2^{1-p}) \zeta(p) + \frac{1}{1-p} \sum_{r=0}^m \binom{1-p}{r} B_r \cdot \left(n^{1-p-r} - 2^{1-p} \left\lceil \frac{n}{2} \right\rceil^{1-p-r} \right) + R_m$$

$$R_m = \frac{1}{m B(m, p)} \left\{ \int_n^\infty \frac{B_m(x - \lfloor x \rfloor)}{x^{p+m}} dx - 2^{1-p} \int_{\lceil n/2 \rceil}^\infty \frac{B_m(x - \lfloor x \rfloor)}{x^{p+m}} dx \right\}$$

Where, m is an even number s.t. $\lceil p \rceil \leq m < \infty$.

Especially, when $n \rightarrow \infty$,

$$\eta(p) = (1 - 2^{1-p}) \zeta(p)$$

Proof

Reversing the sign of p in Formula 4.7.1, we obtain the desired expression.

When $n \rightarrow \infty$, since the range of the integral is from ∞ to ∞ , $R_m = 0$ for any m .

Examples : $p=1.7, n=1001, n=\infty$

$$fl[p_, n_] := \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k^p}$$

$$fr[p_, n_, m_] := (1 - 2^{1-p}) \text{Zeta}[p]$$

$$+ \frac{1}{1-p} \sum_{r=0}^m \text{Binomial}[1-p, r] \text{BernoulliB}[r] \left(n^{1-p-r} - 2^{1-p} \text{Ceiling}\left[\frac{n}{2}\right]^{1-p-r} \right)$$

```
SetPrecision[{fl[1.7, 1001]      , fr[1.7, 1001, 4]}, 15]
{0.789721725383435          , 0.789721725383434 }
```

```
SetPrecision[{fl[1.7, \infty]    , fr[1.7, \infty, 4]}, 30]
{0.789725693648715920680558610911 , 0.789725693648715920680558610911 }
```

Reference

Let

$$\sum_{k=1}^{n-1} f(k), f(x) = -x^p \cos \pi x, E_n(x) = \int_1^\infty \frac{e^{-tx}}{t^n} dt, \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

Then the following formula also holds.

$$\begin{aligned} \sum_{k=1}^{n-1} (-1)^{k-1} k^p &= \frac{n^{1+p}}{2} \{E_{-p}(-n\pi i) + E_{-p}(n\pi i)\} - \frac{1}{2} \{E_{-p}(-\pi i) + E_{-p}(\pi i)\} \\ &\quad - \sum_{r=1}^m \frac{B_r}{r!} \sum_{s=0}^{r-1} \binom{r-1}{s} \frac{\Gamma(1+p)}{\Gamma(1+p-s)} \{(-1)^n n^{p-s} + 1\} \pi^{r-1-s} \cos \frac{(r-1-s)\pi}{2} + R_m \\ R_m &= \frac{(-1)^m}{m!} \int_1^n B_m(x - \lfloor x \rfloor) \sum_{s=0}^m \binom{m}{s} \frac{\Gamma(1+p)}{\Gamma(1+p-s)} x^{p-s} \pi^{m-s} \cos \left\{ \pi x + \frac{\pi(m-s)}{2} \right\} dx \end{aligned}$$

However, this is too complicated and does not have a great merit, either..

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Alien's Mathematics