7 Expression of Polynomial with Real Coefficients by Real & Imaginary parts.

Each of polynomial with real coefficients can be easily expressed by real and imaginary parts. However, we have never seen the formula that expresses any polynomial with real coefficients by real and imaginary parts. This chapter presents such a formula.

17.1 Lemma and Formulas

First, we prepare an important lemma. This is a reprint from " **14 Taylor Expansion by Real Part & Imaginary Part** "

Lemma 14.1.0 (Reprint)

When x,y are real numbers and r is a non-negative integer, the following expressions hold.

$$(x+iy)^r = \sum_{s=0}^{\lceil \frac{r-1}{2} \rceil} (-1)^s \binom{r}{2s} x^{r-2s} y^{2s} + i \sum_{s=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^s \binom{r}{2s+1} x^{r-2s-1} y^{2s+1} \tag{1.0}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

Using this Lemma, w can separate any polynomial with real coefficients by real and imaginary parts.

Formula 17.1.1

Let a be a real number and $f_n(z)$ $(z = x+iy)$ be a polynomial with real coefficients as follows.

$$f_n(z) = \sum_{r=0}^{n} \frac{f_n^{(r)}(a)}{r!} (z-a)^r \tag{1.1}$$

Then, the following expressions hold for the real and imaginary parts $u_n(x,y)$, $v_n(x,y)$

$$u_n(x,y) = \sum_{r=0}^{n} \frac{f_n^{(r)}(a)}{r!} \sum_{s=0}^{\lceil \frac{r-1}{2} \rceil} (-1)^s \binom{r}{2s} (x-a)^{r-2s} y^{2s} \tag{1.1u}$$

$$v_n(x,y) = \sum_{r=0}^{n} \frac{f_n^{(r)}(a)}{r!} \sum_{s=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^s \binom{r}{2s+1} (x-a)^{r-2s-1} y^{2s+1} \tag{1.1v}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

Proof

Replacing x with $x-a$ in Lemma 14.1.0 ,

$$\{(x-a)+iy\}^r = \sum_{s=0}^{\lceil \frac{r-1}{2} \rceil} (-1)^s \binom{r}{2s} (x-a)^{r-2s} y^{2s}$$

$$+ i \sum_{s=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^s \binom{r}{2s+1} (x-a)^{r-2s-1} y^{2s+1}$$

Substituting this for (1.1) ,

$$f_n(x,y) = \sum_{r=0}^{n} \frac{f_n^{(r)}(a)}{r!} \sum_{s=0}^{\lceil \frac{r-1}{2} \rceil} (-1)^s \binom{r}{2s} (x-a)^{r-2s} y^{2s}$$

$$+ i \sum_{r=0}^{n} \frac{f_n^{(r)}(a)}{r!} \sum_{s=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^s \binom{r}{2s+1} (x-a)^{r-2s-1} y^{2s+1}$$

Describing the real and imaginary parts as $u_n(x,y), v_n(x,y)$ respectively, we obtain the desired expressions.

Example of Expansion

$u_5(x,y)$ is expanded as follows.

$$u_5(x,y) = \frac{f_5^{(0)}(a)}{0!} \binom{0}{0} (x-a)^{0-0} y^0$$

$$+ \frac{f_5^{(1)}(a)}{1!} \binom{1}{0} (x-a)^{1-0} y^0$$

$$+ \frac{f_5^{(2)}(a)}{2!} \left\{ \binom{2}{0} (x-a)^{2-0} y^0 - \binom{2}{2} (x-a)^{2-2} y^2 \right\}$$

$$+ \frac{f_5^{(3)}(a)}{3!} \left\{ \binom{3}{0} (x-a)^{3-0} y^0 - \binom{3}{2} (x-a)^{3-2} y^2 \right\}$$

$$+ \frac{f_5^{(4)}(a)}{4!} \left\{ \binom{4}{0} (x-a)^{4-0} y^0 - \binom{4}{2} (x-a)^{4-2} y^2 + \binom{4}{4} (x-a)^{4-4} y^4 \right\}$$

$$+ \frac{f_5^{(5)}(a)}{5!} \left\{ \binom{5}{0} (x-a)^{5-0} y^0 - \binom{5}{2} (x-a)^{5-2} y^2 + \binom{5}{4} (x-a)^{5-4} y^4 \right\}$$

As seen from this example, This polynomial is somewhat halfway with respect to both x and y .
So, we rearrange this into a more beautiful polynomial.

Formula 17.1.2

Let a be a real number and $f_n(z)$ $(z=x+iy)$ be a polynomial with real coefficients as follows.

$$f_n(z) = \sum_{s=0}^{n} f_n^{(s)}(a) \frac{(z-a)^s}{s!} \tag{1.2}$$

Then, the following expressions hold for the real and imaginary parts $u_n(x,y), v_n(x,y)$

$$u_n(x,y) = \sum_{r=0}^{\lceil \frac{n-1}{2} \rceil} \sum_{s=0}^{n-2r} f_n^{(2r+s)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!} \tag{1.2u}$$

$$v_n(x,y) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{s=0}^{n-2r-1} f_n^{(2r+s+1)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.2v}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

Proof

From Formula 17.1.1 ,

$$u_n(x,y) = \sum_{r=0}^{n} \frac{f_n^{(r)}(a)}{r!} \sum_{s=0}^{\left\lceil \frac{r-1}{2} \right\rceil} (-1)^s \binom{r}{2s} (x-a)^{r-2s} y^{2s} \tag{1.1u}$$

$$v_n(x,y) = \sum_{r=0}^{n} \frac{f_n^{(r)}(a)}{r!} \sum_{s=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^s \binom{r}{2s+1} (x-a)^{r-2s-1} y^{2s+1} \tag{1.1v}$$

Here, let

$$A_r := \frac{f_n^{(r)}(a)}{r!} \quad , \quad X := (x-a)$$

$$a_{rs} := (-1)^s \binom{r}{2s} \quad , \quad b_{rs} := (-1)^s \binom{r}{2s+1}$$

Then,

$$u_n(x,y) = \sum_{r=0}^{n} A_r \sum_{s=0}^{\left\lceil \frac{r-1}{2} \right\rceil} a_{rs} X^{r-2s} y^{2s} \tag{1.1u'}$$

$$v_n(x,y) = \sum_{r=0}^{n} A_r \sum_{s=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} b_{rs} X^{r-2s-1} y^{2s+1} \tag{1.1v'}$$

When $n=5$, (1.1u') is rearranged as follows.

$$\begin{aligned}
u_5(x,y) &= \sum_{r=0}^{5} A_r \sum_{s=0}^{\left\lceil \frac{r-1}{2} \right\rceil} a_{rs} X^{r-2s} y^{2s} \\
&= A_0\left\{a_{00} X^{0-0} y^0\right\} \\
&\quad + A_1\left\{a_{10} X^{1-0} y^0\right\} \\
&\quad + A_2\left\{a_{20} X^{2-0} y^0 + a_{21} X^{2-2} y^2\right\} \\
&\quad + A_3\left\{a_{30} X^{3-0} y^0 + a_{31} X^{3-2} y^2\right\} \\
&\quad + A_4\left\{a_{40} X^{4-0} y^0 + a_{41} X^{4-2} y^2 + a_{42} X^{4-4} y^4\right\} \\
&\quad + A_5\left\{a_{50} X^{5-0} y^0 + a_{51} X^{5-2} y^2 + a_{52} X^{5-4} y^4\right\} \\
&= \left\{A_0 a_{00} X^0 + A_1 a_{10} X^1 + A_2 a_{20} X^2 + A_3 a_{30} X^3 + A_4 a_{40} X^4 + A_5 a_{50} X^5\right\} y^0 \\
&\quad + \left\{A_2 a_{21} X^0 + A_3 a_{31} X^1 + A_4 a_{41} X^2 + A_5 a_{51} X^3\right\} y^2 \\
&\quad + \left\{A_4 a_{42} X^0 + A_5 a_{52} X^1\right\} y^4 \\
&= \left\{\sum_{s=0}^{5} A_{0+s} a_{0+s,0} X^s\right\} y^0 + \left\{\sum_{s=0}^{3} A_{2+s} a_{2+s,1} X^s\right\} y^2 + \left\{\sum_{s=0}^{1} A_{4+s} a_{4+s,2} X^s\right\} y^4
\end{aligned}$$

i.e.

$$u_5(x,y) = \sum_{r=0}^{2} \left\{\sum_{s=0}^{5-2r} A_{2r+s}\, a_{2r+s,r} X^s\right\} y^{2r}$$

When $n=4$, (1.1u') is rearranged as follows.

$$u_4(x,y) = \sum_{r=0}^{n} A_r \sum_{s=0}^{\left\lceil\frac{r-1}{2}\right\rceil} a_{rs} X^{r-2s} y^{2s}$$

$$= A_0\left\{a_{00} X^{0-0} y^0\right\}$$
$$+A_1\left\{a_{10} X^{1-0} y^0\right\}$$
$$+A_2\left\{a_{20} X^{2-0} y^0 + a_{21} X^{2-2} y^2\right\}$$
$$+A_3\left\{a_{30} X^{3-0} y^0 + a_{31} X^{3-2} y^2\right\}$$
$$+A_4\left\{a_{40} X^{4-0} y^0 + a_{41} X^{4-2} y^2 + a_{42} X^{4-4} y^4\right\}$$
$$= \left\{A_0 a_{00} X^0 + A_1 a_{10} X^1 + A_2 a_{20} X^2 + A_3 a_{30} X^3 + A_4 a_{40} X^4\right\} y^0$$
$$+ \left\{A_2 a_{21} X^0 + A_3 a_{31} X^1 + A_4 a_{41} X^2\right\} y^2$$
$$+ \left\{A_4 a_{42} X^0\right\} y^4$$
$$= \left\{\sum_{s=0}^{4} A_{0+s}\, a_{0+s,0} X^s\right\} y^0 + \left\{\sum_{s=0}^{2} A_{2+s}\, a_{2+s,1} X^s\right\} y^2 + \left\{\sum_{s=0}^{0} A_{4+s}\, a_{4+s,2} X^s\right\} y^4$$

i.e.

$$u_4(x,y) = \sum_{r=0}^{2} \left\{\sum_{s=0}^{4-2r} A_{2r+s}\, a_{2r+s,r} X^s\right\} y^{2r}$$

These can be unified notation as follows.

$$u_n(x,y) = \sum_{r=0}^{\left\lceil\frac{n-1}{2}\right\rceil} \left\{\sum_{s=0}^{n-2r} A_{2r+s}\, a_{2r+s,r} X^s\right\} y^{2r}$$

Since

$$A_r = \frac{f^{(r)}(a)}{r\,!} \qquad \Longrightarrow \qquad A_{2r+s} = \frac{f^{(2r+s)}(a)}{(2r+s)\,!}$$

$$a_{rs} = (-1)^s \binom{r}{2s} \qquad \Longrightarrow \qquad a_{2r+s,r} = (-1)^r \binom{2r+s}{2r}$$

returning to the original symbol ,

$$u_n(x,y) = \sum_{r=0}^{\left\lceil\frac{n-1}{2}\right\rceil} \left\{\sum_{s=0}^{n-2r} \frac{f_n^{(2r+s)}(a)}{(2r+s)\,!} \binom{2r+s}{2r} (x-a)^s\right\} (-1)^r y^{2r}$$

Further,

$$\binom{2r+s}{2r} = \frac{(2r+s)\,!}{(2r)\,!\; s\,!}$$

Substituting this for the above,

$$u_n(x,y) = \sum_{r=0}^{\left\lceil\frac{n-1}{2}\right\rceil} \left\{\sum_{s=0}^{n-2r} \frac{f_n^{(2r+s)}(a)}{(2r+s)\,!} \frac{(2r+s)\,!}{(2r)\,!\; s\,!} (x-a)^s\right\} (-1)^r y^{2r}$$

i.e.

$$u_n(x,y) = \sum_{r=0}^{\lceil \frac{n-1}{2} \rceil} \sum_{s=0}^{n-2r} f_n^{(2r+s)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!} \tag{1.2u}$$

In the same way as above for (1.1v') , we obtain

$$v_n(x,y) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \left\{ \sum_{s=0}^{n-2r} \frac{f_n^{(2r+s)}(a)}{(2r+s)!} \binom{2r+s}{2r+1} (x-a)^{s-1} \right\} (-1)^r y^{2r+1}$$

Here, the first term $(s=0)$ in { } is

$$\frac{f_n^{(2r)}(a)}{(2r)!} \binom{2r}{2r+1} (x-a)^{-1} = 0 \qquad \mathit{for}\ r = 0, 1, \cdots, \left\lfloor \frac{n-1}{2} \right\rfloor$$

So,

$$\sum_{s=0}^{n-2r} \frac{f_n^{(2r+s)}(a)}{(2r+s)!} \binom{2r+s}{2r+1} (x-a)^{s-1} = \sum_{s=1}^{n-2r} \frac{f_n^{(2r+s)}(a)}{(2r+s)!} \binom{2r+s}{2r+1} (x-a)^{s-1}$$

$$= \sum_{s=0}^{n-2r-1} \frac{f_n^{(2r+s+1)}(a)}{(2r+s+1)!} \binom{2r+s+1}{2r+1} (x-a)^{s}$$

Therefore,

$$v_n(x,y) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \left\{ \sum_{s=0}^{n-2r-1} \frac{f_n^{(2r+s+1)}(a)}{(2r+s+1)!} \binom{2r+s+1}{2r+1} (x-a)^{s} \right\} (-1)^r y^{2r+1}$$

Further,

$$\binom{2r+s+1}{2r+1} = \frac{(2r+s+1)!}{(2r+1)!\, s!}$$

Substituting this for the above,

$$v_n(x,y) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \left\{ \sum_{s=0}^{n-2r-1} \frac{f_n^{(2r+s+1)}(a)}{(2r+s+1)!} \frac{(2r+s+1)!}{(2r+1)!\, s!} (x-a)^{s} \right\} (-1)^r y^{2r+1}$$

i.e.

$$v_n(x,y) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{s=0}^{n-2r-1} f_n^{(2r+s+1)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.2v}$$

Maclaurin Expansion by Real and Imaginary Parts

$u_5(x,y), v_5(x,y)$ are expanded to the Maclaurin series. as follows.

$$u_5(x,y) = \left\{ f_5^{(0)}(0)\frac{x^0}{0!} + f_5^{(1)}(0)\frac{x^1}{1!} + f_5^{(2)}(0)\frac{x^2}{2!} + \cdots + f_5^{(5)}(0)\frac{x^5}{5!} \right\} \frac{(-1)^0 y^0}{0!}$$

$$+ \left\{ f_5^{(2)}(0)\frac{x^0}{0!} + f_5^{(3)}(0)\frac{x^1}{1!} + f_5^{(4)}(0)\frac{x^2}{2!} + f_5^{(5)}(0)\frac{x^3}{3!} \right\} \frac{(-1)^1 y^2}{2!}$$

$$+ \left\{ f_5^{(4)}(0)\frac{x^0}{0!} + f_5^{(5)}(0)\frac{x^1}{1!} \right\} \frac{(-1)^2 y^4}{4!}$$

$$v_5(x,y) = \left\{ f_5^{(1)}(0)\frac{x^0}{0!} + f_5^{(2)}(0)\frac{x^1}{1!} + f_5^{(3)}(0)\frac{x^2}{2!} + \cdots + f_5^{(5)}(0)\frac{x^4}{4!} \right\} \frac{(-1)^0 y^1}{1!}$$
$$+ \left\{ f_5^{(3)}(0)\frac{x^0}{0!} + f_5^{(4)}(0)\frac{x^1}{1!} + f_5^{(5)}(0)\frac{x^2}{2!} \right\} \frac{(-1)^1 y^3}{3!}$$
$$+ \left\{ f_5^{(5)}(0)\frac{x^0}{0!} \right\} \frac{(-1)^2 y^5}{5!}$$

A polynomial whose Maclaurin series does not include even-order derivatives is called an odd polynomial. The following holds for the odd polynomial.

Formula 17.1.2 ' (Odd Polynomial)

Let $f_{2n+1}(z)$ $(z = x+iy)$ be a polynomial with real coefficients as follows.

$$f_{2n+1}(z) = \sum_{s=0}^{n} f_{2n+1}^{(2s+1)}(0)\frac{z^{2s+1}}{(2s+1)!} \tag{1.2'}$$

Then, the following expressions hold for the real and imaginary parts $u_{2n+1}(x,y), v_{2n+1}(x,y)$

$$u_{2n+1}(x,y) = \sum_{r=0}^{n}\sum_{s=0}^{n-r} f^{(2r+2s+1)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{1.2u'}$$

$$v_{2n+1}(x,y) = \sum_{r=0}^{n}\sum_{s=0}^{n-r} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.2v'}$$

Where, $0^0 = 1$

A polynomial whose Maclaurin series does not include odd-order derivatives is called an even polynomial. The following holds for the even polynomial.

Formula 17.1.2 " (Even Polynomial)

Let $f_{2n}(z)$ $(z = x+iy)$ be a polynomial with real coefficients as follows.

$$f_{2n}(z) = \sum_{s=0}^{n} f_{2n}^{(2s)}(0)\frac{z^{2s}}{(2s)!} \tag{1.2"}$$

Then, the following expressions hold for the real and imaginary parts $u_{2n}(x,y), v_{2n}(x,y)$.

$$u_{2n}(x,y) = \sum_{r=0}^{n}\sum_{s=0}^{n-r} f^{(2r+2s)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{1.2u"}$$

$$v_{2n}(x,y) = \sum_{r=0}^{n-1}\sum_{s=0}^{n-1-r} f^{(2r+2s+2)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.2v"}$$

Where, $0^0 = 1$

Cauchy–Riemann Equations

When (1.2u) and (1.2v) are partially differentiated with respect to x , y respectively, it is as follows. These are the Cauchy–Riemann equations

$$\frac{\partial u_n(x,y)}{\partial x} = \frac{\partial v_n(x,y)}{\partial y} = \sum_{r=0}^{\left\lceil \frac{n-1}{2} \right\rceil} \sum_{s=0}^{n-2r-1} f_n^{(2r+s+1)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$\frac{\partial u_n(x,y)}{\partial y} = -\frac{\partial v_n(x,y)}{\partial x} = -\sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \sum_{s=0}^{n-2r-2} f_n^{(2r+s+2)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Treatment of 0^0 in *Mathematica*

Formula manipulation soft ***Mathematica*** does not calculate 0^0 as *Indeterminate* . Since it is inconvenient, the following options are specified prior to calculation in this paper .

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Unprotect [Power] ;  Power [0,0] = 1 ;
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17.2 Example1: Cyclotomic Equation

Examples are given in this section and the next section. As the first , we takes up Cyclotomic Equation in this section .

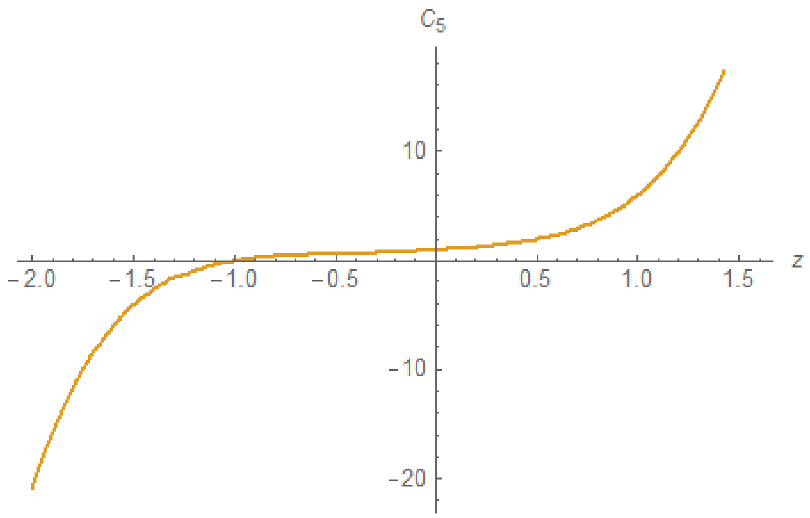
Cyclotomic Equation is as follows.

$$C_n(z) = 1 + z + z^2 + \cdots + z^n = 0 \qquad \left(= \frac{1 - z^{n+1}}{1 - z} \right) \tag{2.0}$$

So, the function $C_n(z)$ is expanded to the Maclaurin series. as

$$C_n(z) = \sum_{s=0}^{n} s! \frac{z^s}{s!} \tag{2.1}$$

When $n=5$, this is drawn as follows.



Expression by real and imaginary parts by Formula 17.1.2

From (2.1) ,

$$C_n^{(s)}(0) = s! \qquad s = 0, 1, \cdots, n$$

$$C_n^{(2r+s)}(0) = (2r+s)! \qquad \begin{array}{l} r = 0, 1, \cdots, \lceil (n-1)/2 \rceil \\ s = 0, 1, \cdots, n-2r \end{array}$$

$$C_n^{(2r+s+1)}(0) = (2r+s+1)! \qquad \begin{array}{l} r = 0, 1, \cdots, \lfloor (n-1)/2 \rfloor \\ s = 0, 1, \cdots, n-2r-1 \end{array}$$

Substituting these for Formula 17.1.2 ,

$$u_n(x,y) = \sum_{r=0}^{\lceil \frac{n-1}{2} \rceil} \sum_{s=0}^{n-2r} (2r+s)! \frac{x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!} \tag{2.2u}$$

$$v_n(x,y) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{s=0}^{n-2r-1} (2r+s+1)! \frac{x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{2.2v}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

When $n=5$, these are expanded as follows.

$$u_5(x,y) = \sum_{r=0}^{\left\lceil\frac{5-1}{2}\right\rceil}\sum_{s=0}^{5-2r}(2r+s)!\frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$= \left(0!\frac{x^0}{0!} + 1!\frac{x^1}{1!} + 2!\frac{x^2}{2!} + 3!\frac{x^3}{3!} + 4!\frac{x^4}{4!} + 5!\frac{x^5}{5!}\right)\frac{y^0}{0!}$$

$$- \left(2!\frac{x^0}{0!} + 3!\frac{x^1}{1!} + 4!\frac{x^2}{2!} + 5!\frac{x^3}{3!}\right)\frac{y^2}{2!}$$

$$+ \left(4!\frac{x^0}{0!} + 5!\frac{x^1}{1!}\right)\frac{y^4}{4!}$$

$$= 1 + x + x^2 + x^3 + x^4 + x^5 - y^2 - 3xy^2 - 6x^2y^2 - 10x^3y^2 + y^4 + 5xy^4$$

$$v_5(x,y) = \sum_{r=0}^{\left\lfloor\frac{5-1}{2}\right\rfloor}\sum_{s=0}^{5-2r-1}(2r+s+1)!\frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

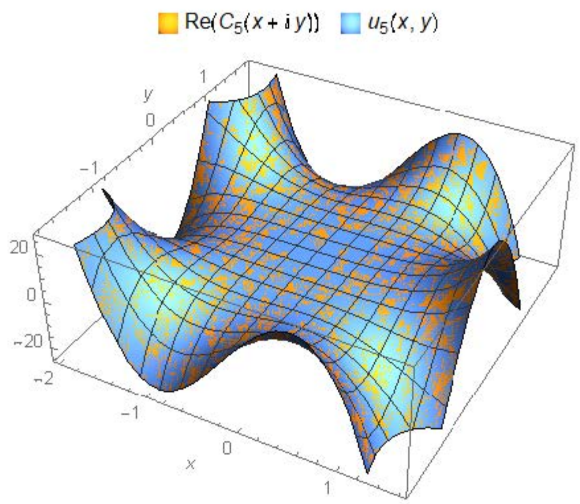
$$= \left(1!\frac{x^0}{0!} + 2!\frac{x^1}{1!} + 3!\frac{x^2}{2!} + 4!\frac{x^3}{3!} + 5!\frac{x^4}{4!}\right)\frac{y^1}{1!}$$

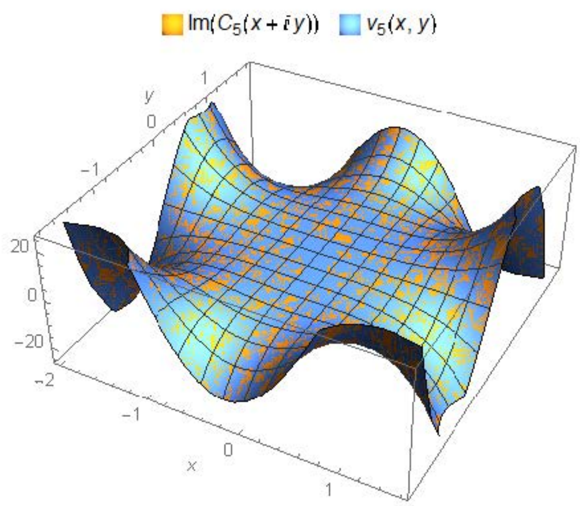
$$- \left(3!\frac{x^0}{0!} + 4!\frac{x^1}{1!} + 5!\frac{x^2}{2!}\right)\frac{y^3}{3!}$$

$$+ \left(5!\frac{x^0}{0!}\right)\frac{y^5}{5!}$$

$$= y + 2xy + 3x^2y + 4x^3y + 5x^4y - y^3 - 4xy^3 - 10x^2y^3 + y^5$$

When $n=5$, both sides are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side.





Zeros

The zeros of the function $C_n(z)$ are given by the real solution of the following simultaneous equations.

$$\begin{cases} u_n(x,y) = \sum_{r=0}^{\lceil \frac{n-1}{2} \rceil} \sum_{s=0}^{n-2r} (2r+s)!\, \frac{x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!} = 0 \\ v_n(x,y) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{s=0}^{n-2r-1} (2r+s+1)!\, \frac{x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \end{cases}$$

When $n=5$, this can be solved by the formula manipulation software *Mathematica* as follows.

```
Solve[u5[x, y] == 0 && v5[x, y] == 0, Element[{x, y}, Reals]]
```

$$\left\{\{x \to -1,\ y \to 0\},\ \left\{x \to -\frac{1}{2},\ y \to -\frac{\sqrt{3}}{2}\right\},\ \left\{x \to -\frac{1}{2},\ y \to \frac{\sqrt{3}}{2}\right\},\ \left\{x \to \frac{1}{2},\ y \to -\frac{\sqrt{3}}{2}\right\},\ \left\{x \to \frac{1}{2},\ y \to \frac{\sqrt{3}}{2}\right\}\right\}$$

These are five of six solutions of the 6th degree equation $1 - z^6 = 0$.

17.3 Example2: Bernoulli Polynomial

In this section, we take the Bernoulli polynomial as a second example.

Bernoulli polynomial is as follows.

$$B_n(z) = \sum_{s=0}^{n} \binom{n}{s} B_{n-s} z^s \tag{3.0}$$

Where,

$$B_0 = 1\,,\ B_2 = \frac{1}{6}\,,\ B_4 = -\frac{1}{30}\,,\ B_6 = \frac{1}{42}\,,\ B_8 = -\frac{1}{30}\,,\ B_{10} = \frac{5}{66}\,,\ \cdots$$

$$B_1 = -\frac{1}{2}\ ,\quad B_3 = B_5 = B_7 = \cdots = 0$$

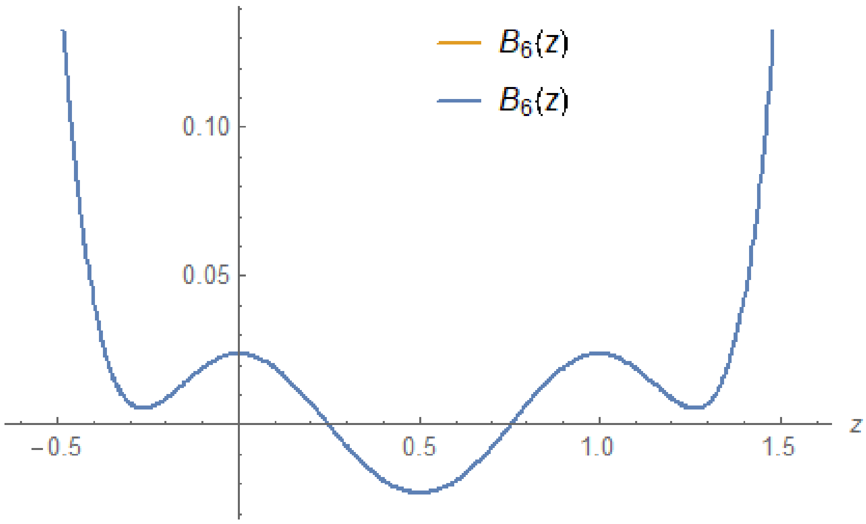
From this,

$$B_n(z) = \sum_{s=0}^{n} \binom{n}{s} B_{n-s}\, s!\, \frac{z^s}{s!} = \sum_{s=0}^{n} \frac{n!}{s!\,(n-s)!} s!\, B_{n-s} \frac{z^s}{s!}$$

i.e.

$$B_n(z) = n! \sum_{s=0}^{n} \frac{B_{n-s}}{(n-s)!} \frac{z^s}{s!} \tag{3.1}$$

When $n=6$, both sides are drawn as follows. Orange is the left side and blue is the right side.



Expression by real and imaginary parts by Formula 17.1.2

From (3.1) ,

$$B_n^{(s)}(0) = \frac{n!\,B_{n-s}}{(n-s)!} \qquad s = 0, 1, \cdots, n$$

$$B_n^{(2r+s)}(0) = \frac{n!\,B_{n-2r-s}}{(n-2r-s)!} \qquad \begin{array}{l} r = 0, 1, \cdots, \lceil (n-1)/2 \rceil \\ s = 0, 1, \cdots, n-2r \end{array}$$

$$B_n^{(2r+s+1)}(0) = \frac{n!\,B_{n-2r-s-1}}{(n-2r-s-1)!} \qquad \begin{array}{l} r = 0, 1, \cdots, \lceil (n-1)/2 \rceil \\ s = 0, 1, \cdots, n-2r-1 \end{array}$$

Substituting these for Formula 17.1.2 ,

$$u_n(x,y) = n!\sum_{r=0}^{\left\lceil\frac{n-1}{2}\right\rceil}\sum_{s=0}^{n-2r}\frac{B_{n-2r-s}}{(n-2r-s)!}\frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{3.2u}$$

$$v_n(x,y) = n!\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\sum_{s=0}^{n-2r-1}\frac{B_{n-2r-s-1}}{(n-2r-s-1)!}\frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{3.2v}$$

Where, $0^0 = 1$, $\lceil x\rceil$ is the ceiling function , $\lfloor x\rfloor$ is the floor function.

When $n=6$, these are expanded as follows.

$$u_6(x,y) = 6!\sum_{r=0}^{\left\lceil\frac{6-1}{2}\right\rceil}\sum_{s=0}^{6-2r}\frac{B_{6-2r-s}}{(6-2r-s)!}\frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$= 6!\left(\frac{B_6}{6!}\frac{x^0}{0!}+\frac{B_5}{5!}\frac{x^1}{1!}+\frac{B_4}{4!}\frac{x^2}{2!}+\frac{B_3}{3!}\frac{x^3}{3!}+\frac{B_2}{2!}\frac{x^4}{4!}+\frac{B_1}{1!}\frac{x^5}{5!}+\frac{B_0}{0!}\frac{x^6}{6!}\right)\frac{y^0}{0!}$$

$$- 6!\left(\frac{B_4}{4!}\frac{x^0}{0!}+\frac{B_3}{3!}\frac{x^1}{1!}+\frac{B_2}{2!}\frac{x^2}{2!}+\frac{B_1}{1!}\frac{x^3}{3!}+\frac{B_0}{0!}\frac{x^4}{4!}\right)\frac{y^2}{2!}$$

$$+ 6!\left(\frac{B_2}{2!}\frac{x^0}{0!}+\frac{B_1}{1!}\frac{x^1}{1!}+\frac{B_0}{0!}\frac{x^2}{2!}\right)\frac{y^4}{4!}$$

$$- 6!\left(\frac{B_0}{0}\frac{x^0}{0}\right)\frac{y^6}{6!}$$

$$= 720\left(\frac{1}{30240}-\frac{x^2}{1440}+\frac{x^4}{288}-\frac{x^5}{240}+\frac{x^6}{720}+\frac{y^2}{1440}-\frac{x^2y^2}{48}+\frac{x^3y^2}{24}-\frac{x^4y^2}{48}+\frac{y^4}{288}-\frac{xy^4}{48}+\frac{x^2y^4}{48}-\frac{y^6}{720}\right)$$

$$v_6(x,y) = 6!\sum_{r=0}^{\left\lfloor\frac{6-1}{2}\right\rfloor}\sum_{s=0}^{6-2r-1}\frac{B_{6-2r-s-1}}{(6-2r-s-1)!}\frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

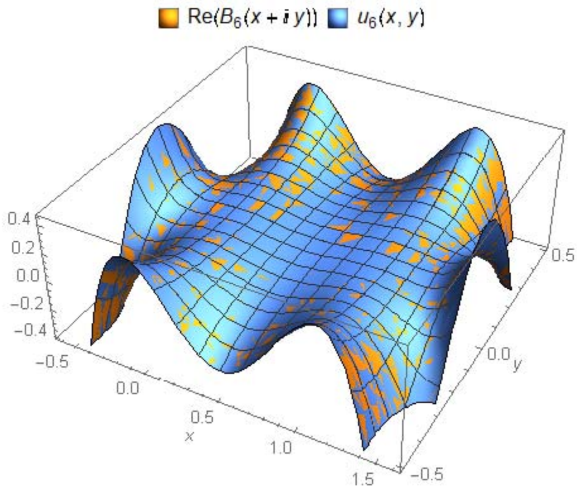
$$= 6!\left(\frac{B_5}{5!}\frac{x^0}{0!}+\frac{B_4}{4!}\frac{x^1}{1!}+\frac{B_3}{3!}\frac{x^2}{2!}+\frac{B_2}{2!}\frac{x^3}{3!}+\frac{B_1}{1!}\frac{x^4}{4!}+\frac{B_0}{0!}\frac{x^5}{5!}\right)\frac{y^1}{1!}$$

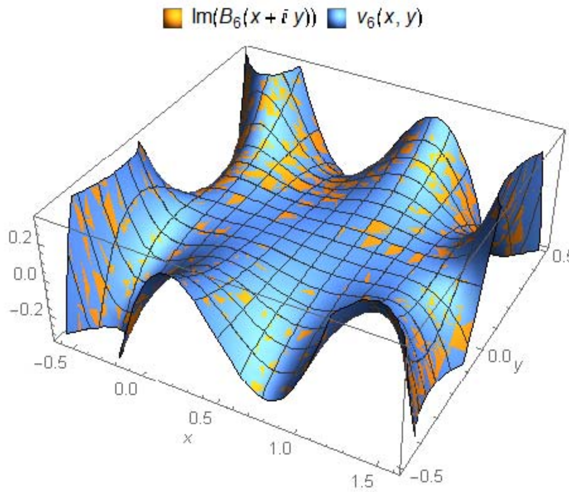
$$- 6!\left(\frac{B_3}{3!}\frac{x^0}{0!}+\frac{B_2}{2!}\frac{x^1}{1!}+\frac{B_1}{1!}\frac{x^2}{2!}+\frac{B_0}{0!}\frac{x^3}{3!}\right)\frac{y^3}{3!}$$

$$+ 6!\left(\frac{B_1}{1!}\frac{x^0}{0!}+\frac{B_0}{0!}\frac{x^1}{1!}\right)\frac{y^5}{5!}$$

$$= 720\left(-\frac{xy}{720}+\frac{x^3y}{72}-\frac{x^4y}{48}+\frac{x^5y}{120}-\frac{xy^3}{72}+\frac{x^2y^3}{24}-\frac{x^3y^3}{36}-\frac{y^5}{240}+\frac{xy^5}{120}\right)$$

When $n=6$, both sides are drawn as follows. The left is the real part and the right is the imaginary part.
In both figures, orange is the left side and blue is the right side.





Zeros

The zeros of the function $B_n(z)$ are given by the real solution of the following simultaneous equations.

$$\begin{cases} u_n(x,y) = n!\displaystyle\sum_{r=0}^{\left\lceil \frac{n-1}{2} \right\rceil}\sum_{s=0}^{n-2r} \frac{B_{n-2r-s}}{(n-2r-s)!}\frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!} = 0 \\ v_n(x,y) = n!\displaystyle\sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor}\sum_{s=0}^{n-2r-1} \frac{B_{n-2r-s-1}}{(n-2r-s-1)!}\frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \end{cases}$$

When $n=6$, this can be solved by the formula manipulation software *Mathematica* as follows.

```
NSolve[u6[x, y] == 0 && v6[x, y] == 0, Element[{x, y}, Reals]]
{{x → 0.752459 , y → 0 } , {x → 0.247541 , y → 0 },
 {x → 1.27289 , y → -0.0649729 }, {x → 1.27289 , y → 0.0649729 },
 {x → -0.272887 , y → 0.0649729 } , {x → -0.272887 , y → -0.0649729 }}
```

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Alien's Mathematics