05 Factorization of Completed Dirichlet Beta

5.1 Hadamard product of $\omega(z)$

Formula 5.1.1

Let completed beta function be as follows.

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z) \tag{1.d}$$

When non-trivial zeros of $\beta(z)$ are $z_k = x_k \pm i y_k \quad k=1,2,3,\cdots$ and γ is Euler-Mascheroni constant, $\omega(z)$ is expressed by the Hadamard product as follows.

$$\omega(z) = e^{\left(\frac{3\log\pi}{2} - \frac{\gamma}{2} - \log 2 - 4\log\Gamma\left(\frac{3}{4}\right)\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} \tag{1.0}$$

$$\omega(z) = e^{\left(\frac{3\log\pi}{2} - \frac{\gamma}{2} - \log 2 - 4\log\Gamma\left(\frac{3}{4}\right)\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}} \tag{1.1}$$

Proof

As well as in the zeta function, it is known that the Hadamard product of (1.d) is also expressed as follows.

$$\omega(z) = A\,e^{Bz}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} \tag{1.p}$$

Substituting $z=0$ for (1.d),

$$\omega(0) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+0}\Gamma\left(\frac{1+0}{2}\right)\beta(0) = \frac{2}{\sqrt{\pi}}\cdot\sqrt{\pi}\cdot\frac{1}{2} = 1$$

If (1.p) is given $z=0$, $\omega(0)=A$. We obtain $A=1$ from both. Thus,

$$\omega(z) = e^{Bz}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} \tag{1.p'}$$

Next, from (1.d) and (1.p'),

$$\left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z) = e^{Bz}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

From this,

$$\beta(z) = \frac{e^{Bz}}{\left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

Taking the logarithm of both sides,

$$\log\beta(z) = Bz - \log\frac{2}{\sqrt{\pi}} - z\log\frac{2}{\sqrt{\pi}} - \log\Gamma\left(\frac{1+z}{2}\right) + \log\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

Here,

$$log\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} = \sum_{k=1}^{\infty}log\left(1-\frac{z}{z_k}\right) + \sum_{k=1}^{\infty}\frac{z}{z_k}$$

Using this,

$$log\beta(z) = Bz - log\frac{2}{\sqrt{\pi}} - z\,log\frac{2}{\sqrt{\pi}} - log\Gamma\left(\frac{1+z}{2}\right) + \sum_{k=1}^{\infty}log\left(1-\frac{z}{z_k}\right) + \sum_{k=1}^{\infty}\frac{z}{z_k}$$

Differentiating both sides with respect to z ,

$$\frac{\beta'(z)}{\beta(z)} = B - log\frac{2}{\sqrt{\pi}} - \frac{1}{2}\frac{\Gamma'\{(z+1)/2\}}{\Gamma\{(z+1)/2\}} + \sum_{k=1}^{\infty}\frac{-1/z_k}{1-z/z_k} + \sum_{k=1}^{\infty}\frac{1}{z_k}$$

i.e.

$$\frac{\beta'(z)}{\beta(z)} = B - log\,2 + \frac{1}{2}log\,\pi - \frac{1}{2}\frac{\Gamma'\{(z+1)/2\}}{\Gamma\{(z+1)/2\}} + \sum_{k=1}^{\infty}\left(\frac{1}{z-z_k} + \frac{1}{z_k}\right)$$

Here, putting $z=0$,

$$\frac{\beta'(0)}{\beta(0)} = B - log\,2 + \frac{1}{2}log\,\pi - \frac{1}{2}\frac{\Gamma'(1/2)}{\Gamma(1/2)} \tag{1.w}$$

On the other hand, according to " ***Wolfram MathWorld*** Dirichlet Beta Function " ,

$$\beta'(0) = log\left\{\Gamma^2\left(\frac{1}{4}\right)/\left(2\pi\sqrt{2}\right)\right\}$$

From this and $\beta(0) = 1/2$,

$$\frac{\beta'(0)}{\beta(0)} = 4\,log\Gamma\left(\frac{1}{4}\right) - 3log\,2 - 2log\,\pi$$

Again, according to " ***Wolfram MathWorld*** Gamma Function " ,

$$\frac{\Gamma'(1/2)}{\Gamma(1/2)} = \frac{\Gamma'(1)}{\Gamma(1)} - 2log\,2 = -\gamma - 2log\,2$$

Substituting these for the both sides of (1.w) ,

$$4\,log\Gamma\left(\frac{1}{4}\right) - 3log\,2 - 2log\,\pi = B - log\,2 + \frac{1}{2}log\,\pi + \frac{\gamma}{2} + log\,2$$

From this,

$$B = 4\,log\Gamma\left(\frac{1}{4}\right) - 3log\,2 - \frac{5}{2}log\,\pi - \frac{\gamma}{2}$$

Further,

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$$

Using this,

$$B = \frac{3\,log\,\pi}{2} - \frac{\gamma}{2} - log\,2 - 4\,log\Gamma\left(\frac{3}{4}\right)$$

Substituting this for (1.p') , we obtain (1.0) .

And, if non-trivial zeros are $z_k = x_k \pm iy_k \quad k=1,2,3,\cdots$,

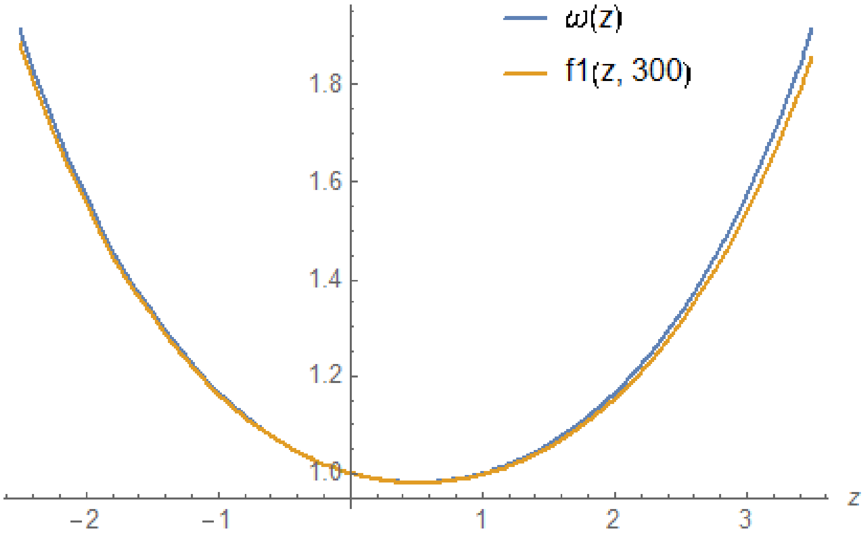
$$\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} = \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2} + \frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}}$$

Using this, we obtain (1.1) .

If $x_n = 1/2 \quad n=1,2,3,\cdots$, (1.1) becomes

$$\omega(z) = e^{\left(\frac{3\log\pi}{2} - \frac{\gamma}{2} - \log 2 - 4\log\Gamma\left(\frac{3}{4}\right)\right)z} \prod_{n=1}^{\infty}\left(1 - \frac{z}{1/4+y_n^2} + \frac{z^2}{1/4+y_n^2}\right) e^{\frac{z}{1/4+y_n^2}} \qquad (1.1')$$

When 300 zeros on the critical line are read into the formula manipulation software ***Mathematica*** and both sides of (1.1') are drawn respectively, it is as follow. Both sides almost overlap at the center.



As a special value for (1.1) , an important formula used in the next section is obtained.

Formula 5.1.2 (Special value)

When non-trivial zeros of Dirichlet beta function are $x_n \pm i y_n \quad n=1,2,3,\cdots$, the following expression holds.

$$\prod_{n=1}^{\infty}\left(1 - \frac{2x_n - 1}{x_n^2+y_n^2}\right) e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{4\log\Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3\log\pi}{2}} = 1.08088915\cdots \qquad (1.2)$$

Proof

Giving $z = 1$ to (1.1) ,

$$\omega(1) = e^{\frac{3\log\pi}{2} - \frac{\gamma}{2} - \log 2 - 4\log\Gamma\left(\frac{3}{4}\right)} \prod_{n=1}^{\infty}\left(1 - \frac{2x_n - 1}{x_n^2+y_n^2}\right) e^{\frac{2x_n}{x_n^2+y_n^2}} = 1$$

From this,

$$\prod_{n=1}^{\infty}\left(1 - \frac{2x_n - 1}{x_n^2+y_n^2}\right) e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{4\log\Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3\log\pi}{2}} = 1.08088915\cdots \qquad (1.2)$$

If $x_n = 1/2 \quad n=1,2,3,\cdots$, (1.2) becomes

$$\prod_{r=1}^{\infty}\left(1-\frac{2\cdot 1/2-1}{(1/2)^2+y_r^2}\right)e^{\frac{2\cdot 1/2}{(1/2)^2+y_r^2}} = e^{\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}} = e^{4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2}}$$

From this,

$$\sum_{n=1}^{\infty}\frac{1}{1/4+y_n^2} = 4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2} = 0.07778398\cdots \qquad \text{(1.2')}$$

As a special value for (1.0) , the following formulas is obtained.

Formula 5.1.3 (Special value)

When non-trivial zeros of Dirichlet beta function are $x_k \pm i y_k \quad k=1,2,3,\cdots$, the following expression holds.

$$\prod_{n=1}^{\infty}\left\{1-\frac{1}{(x_n+iy_n)^2}\right\}\left\{1-\frac{1}{(x_n-iy_n)^2}\right\} = \omega(-1) = 1.16624361\cdots \qquad (1.3)$$

Proof

Same as the proof of Formula 8.1.3 in " **08 Factorization of Completed Riemann Zeta** " (Riemann Zeta Func.)

Left side of (1.3) was computed using 10000 zeros on the critical line in an attempt. Both sides were equal up to 2 decimal places.

```
y := ReadList["BetaZeros.prn", Number]
```

$$g[m\_] := \prod_{n=1}^{m}\left(1-\frac{1}{(1/2+i\,y[[n]])^2}\right)\left(1-\frac{1}{(1/2-i\,y[[n]])^2}\right)$$

```
N[g[10000]]
1.16582 + 0. i
```

5.2 Non-trivial zeros whose real part is not 1/2

According to Theorem 4.2.1 in " **04 Completed Dirichlet beta** ", if Dirichlet beta function $\beta(z)$ has non-trivial zero whose real part is not $1/2$, the one set have to consist of the following four.

$$1/2+\alpha_s \pm i\delta_s \ , \ 1/2-\alpha_s \pm i\delta_s \qquad (0<\alpha_s<1/2)$$

In this section, we will consider how the formulas in the previous section are expressed when non-trivial zeros whose real part is 1/2 and non-trivial zeros whose real part is not 1/2 are mixed.

Lemma 5.2.1

Let γ be Euler-Mascheroni constant, non-trivial zeros of Dirichlet beta function are $x_n+iy_n \quad n=1,2,3,\cdots$. Among them, zeros whose real part is 1/2 are $1/2 \pm iy_r \quad r=1,2,3,\cdots$ and zeros whose real parts is not 1/2 are $1/2 \pm \alpha_s \pm i\delta_s \ (0<\alpha_s<1/2) \quad s=1,2,3,\cdots$. Then Formual 5.1.1 (1.1) is expressed as follows.

$$\omega(z) = e^{\left(4\log\Gamma\left(\frac{1}{4}\right)-3\log 2-\frac{5}{2}\log\pi-\frac{\gamma}{2}\right)z} \prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)e^{\frac{z}{1/4+y_r^2}}$$

$$\times \prod_{s=1}\left\{1-\frac{(1+2\alpha_s)z}{(1/2+\alpha_s)^2+\delta_s^2}+\frac{z^2}{(1/2+\alpha_s)^2+\delta_s^2}\right\}e^{\frac{(1+2\alpha_s)z}{(1/2+\alpha_s)^2+\delta_s^2}}$$

$$\times \prod_{s=1}\left\{1-\frac{(1-2\alpha_s)z}{(1/2-\alpha_s)^2+\delta_s^2}+\frac{z^2}{(1/2-\alpha_s)^2+\delta_s^2}\right\}e^{\frac{(1-2\alpha_s)z}{(1/2-\alpha_s)^2+\delta_s^2}} \qquad (2.1)$$

Proof

Same as the proof of Lemma 8.2.1 in " **08 Factorization of Completed Riemann Zeta** " (Riemann Zeta Func.)

Theorem 5.2.2

Let γ be Euler-Mascheroni constant, non-trivial zeros of Dirichlet beta function are $x_n+iy_n \quad n=1,2,3,\cdots$. Among them, zeros whose real part is 1/2 are $1/2 \pm iy_r \quad r=1,2,3,\cdots$ and zeros whose real parts is not 1/2 are $1/2 \pm \alpha_s \pm i\delta_s \ (0<\alpha_s<1/2) \quad s=1,2,3,\cdots$. Then the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)=1 \qquad (2.2)$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2}=\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\delta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\delta_s^2}\right\} \qquad (2.3)$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2}=4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2}=0.07778398\cdots \qquad (2.4)$$

Proof

Same as the proof of Formula 8.2.2 in " **08 Factorization of Completed Riemann Zeta** " (Riemann Zeta Func.)

Although the story goes a little aside, using (2.2) , we obtain the following special values.

Formula 5.2.3 (Special values)

When non-trivial zeros of Dirichlet beta function are $x_k \pm iy_k \quad k=1,2,3,\cdots$, the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right) = 1 \tag{2.5$_+$}$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right) = \omega(-1) = 1.1662436\cdots \tag{2.5$_-$}$$

Proof

Same as the proof of Formula 8.2.3 in " **08 Factorization of Completed Riemann Zeta** " (Riemann Zeta Func.)

Left side of (2.5$_-$) was computed using 10000 zeros on the critical line in an attempt. Both sides were equal up to 2 decimal places.

```
y := ReadList["BetaZeros.prn", Number]
```

$$g_+[m\_] := \prod_{n=1}^{m}\left(1+\frac{1}{1/2+i\,y[[n]]}\right)\left(1+\frac{1}{1/2-i\,y[[n]]}\right)$$

```
N[g+[10000]]
1.16582 + 0. i
```

Well, let us return to the subject. By using Theorem 5.2.2 , the very important following theorem is obtained.

Theorem 5.2.4

Let non-trivial zeros of Dirichlet beta function are $x_n+iy_n \quad n=1,2,3,\cdots$ and γ be Euler-Mascheroni constant. If the following expression holds, non-trivial zeros whose real parts is not $1/2$ do not exist.

$$\sum_{n=1}^{\infty}\frac{1}{1/4+y_n^2} = 4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2} = 0.07778398\cdots \tag{1.2'}$$

Proof

Although non-trivial zeros $1/2 \pm iy_r \quad r=1,2,3,\cdots$ exist in fact, assume non-trivial zeros $1/2+\alpha_s \pm i\delta_s$, $1/2-\alpha_s \pm i\delta_s \quad (0<\alpha_s<1/2)$ exist in addition. Then, the following expression holds from Theorem 5.2.2 (2.3) , (2.4) .

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\delta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\delta_s^2}\right\}$$
$$= 4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2}$$

Here, the following inequality holds for $0<\alpha_s<1/2$ and arbitrary real number δ_s ,

$$\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\delta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\delta_s^2} = \frac{1/2-2\alpha_s^2+2\delta_s^2}{\left\{(1/2+\alpha_s)^2+\delta_s^2\right\}\left\{(1/2-\alpha_s)^2+\delta_s^2\right\}} > 0$$

So,

$$\sum_{s=1} \left\{ \frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\delta_s^2} + \frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\delta_s^2} \right\} > 0 \qquad for\ \ 0 < \alpha_s < 1/2$$

Thus,

$$\sum_{r=1}^{\infty} \frac{1}{1/4+y_r^2} < 4\log\Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3\log\pi}{2}$$

i.e.

$$\sum_{r=1}^{\infty} \frac{1}{1/4+y_r^2} \neq 4\log\Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3\log\pi}{2}$$

As the contrapositive to the above, this theorem holds.

Note

This theorem shows that the equation (1.2') is equivalent to the Riemann hypothesis. However, for the proof of (1.2'), the imaginary part y_r of the non-trivial zeros $1/2 \pm i y_r \quad r=1,2,3,\cdots$ have to be obtained as a formula.

Both sides of (1.2') were calculated with the formula manipulation software *Mathematica* using 10000 zeros on the critical line. Both sides coincided with 3 decimal places.

```
y := ReadList["BetaZeros.prn", Number]          γ := EulerGamma

gl[m_] := Sum[1/(1/4 + y[[r]]^2), {r, 1, m}]    gr := 4 Log[Gamma[3/4]] + γ/2 + Log[2] - 3 Log[π]/2

N[gl[10 000]]                                   N[gr]
0.0776004                                       0.077784
```

5.3 Factorization of $\omega(z)$

Formula 5.1.1 (Hadamard product) is what the completed beta function $\omega(z)$ is incompletely factored at the non-trivial zeros. However, using Theorem 5.2.2 , the compensation terms disappear and $\omega(z)$ is completely factorized at the non-trivial zeros.

Theorem 5.3.1 (Factorization of $\omega(z)$)

Let Dirichlet beta function be $\beta(z)$, the non-trivial zeros are $z_n = x_n \pm i y_n \quad n=1,2,3,\cdots$ and completed beta function be as follows.

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z} \Gamma\left(\frac{1+z}{2}\right)\beta(z)$$

Then, $\omega(z)$ is factorized as follows.

$$\omega(z) = \prod_{n=1}^{\infty}\left(1 - \frac{2x_n z}{x_n^2+y_n^2} + \frac{z^2}{x_n^2+y_n^2}\right) \tag{3.1}$$

Proof

From Formula 5.1.1 (1.1) ,

$$\omega(z) = e^{\left(\frac{3\log\pi}{2} - \frac{\gamma}{2} - \log 2 - 4\log\Gamma\left(\frac{3}{4}\right)\right)z} \prod_{n=1}^{\infty}\left(1 - \frac{2x_n z}{x_n^2+y_n^2} + \frac{z^2}{x_n^2+y_n^2}\right) \cdot e^{\sum_{n=1}^{\infty}\frac{2x_n z}{x_n^2+y_n^2}}$$

On the other hand, from Theorem 5.2.2 (2.4) ,

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = 4\log\Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3\log\pi}{2}$$

From this,

$$e^{\sum_{n=1}^{\infty}\frac{2x_n z}{x_n^2+y_n^2}} = e^{\left(4\log\Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3\log\pi}{2}\right)z}$$

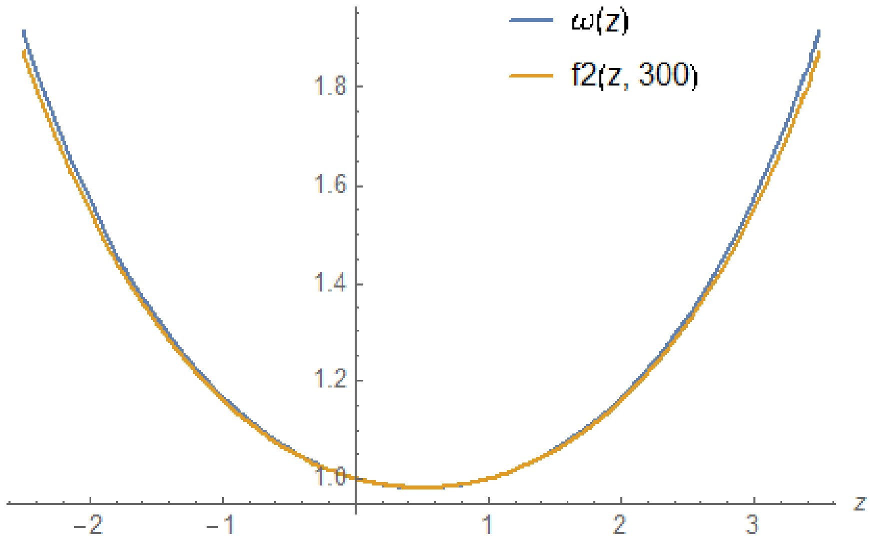
Substituting this for the right side of the above,

$$\begin{aligned}\omega(z) &= e^{\left(\frac{3\log\pi}{2} - \frac{\gamma}{2} - \log 2 - 4\log\Gamma\left(\frac{3}{4}\right)\right)z} \prod_{n=1}^{\infty}\left(1 - \frac{2x_n z}{x_n^2+y_n^2} + \frac{z^2}{x_n^2+y_n^2}\right) \\ &\quad \times e^{\left(4\log\Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3\log\pi}{2}\right)z} \\ &= \prod_{n=1}^{\infty}\left(1 - \frac{2x_n z}{x_n^2+y_n^2} + \frac{z^2}{x_n^2+y_n^2}\right) \end{aligned} \tag{3.1}$$

If $x_n = 1/2 \quad n=1,2,3,\cdots$, (3.1) becomes

$$\omega(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{1/4+y_n^2} + \frac{z^2}{1/4+y_n^2}\right) \tag{3.1'}$$

When both sides of (3.1') are drawn respectively using 300 zeros on the critical line, it is as follows. This is exactly the same as the one in Formula 5.1.1 (1.1') .



ω(z)
f2(z, 300)
1.8
1.6
1.4
1.2
1.0
−2
−1
1
2
3
z

5.4 Factorization of $\Omega(z)$

By replacing z with $z+1/2$ in Theorem 5.3.1, completed beta function $\Omega(z)$ that is an even function is obtained.

Theorem 5.4.1 (Factorization of $\Omega(z)$)

Let Dirichlet beta function be $\beta(z)$, the non-trivial zeros are $z_n = x_n \pm iy_n \quad n=1,2,3,\cdots$ and completed beta function be as follows.

$$\Omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}+z} \Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+z\right)\right\}\beta\left(\frac{1}{2}+z\right)$$

Then, $\Omega(z)$ is factorized as follows.

$$\Omega(z) = \Omega(0)\prod_{n=1}^{\infty}\left\{1-\frac{2(x_n-1/2)z}{(x_n-1/2)^2+y_n^2}+\frac{z^2}{(x_n-1/2)^2+y_n^2}\right\} \tag{4.1}$$

Where, $$\Omega(0) = \prod_{n=1}^{\infty}\frac{(x_n-1/2)^2+y_n^2}{x_n^2+y_n^2} = \left(\frac{2}{\sqrt{\pi}}\right)^{3/2}\Gamma\left(\frac{3}{4}\right)\beta\left(\frac{1}{2}\right) = 0.98071361\cdots \tag{4.1$_0$}$$

Proof

From Theorem 5.3.1 ,

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z)$$

$$\omega(z) = \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right) \tag{3.1}$$

Replacing z with $1/2+z$ in the first expression,

$$\omega\left(\frac{1}{2}+z\right) = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}+z}\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+z\right)\right\}\beta\left(\frac{1}{2}+z\right) =: \Omega(z)$$

Substituting $z=0$ for this,

$$\Omega(0) = \left(\frac{2}{\sqrt{\pi}}\right)^{3/2}\Gamma\left(\frac{3}{4}\right)\beta\left(\frac{1}{2}\right) = 0.98071361\cdots$$

Replacing z with $1/2+z$ in (3.1) ,

$$\omega\left(\frac{1}{2}+z\right) = \prod_{n=1}^{\infty}\left\{1-\frac{2x_n}{x_n^2+y_n^2}\left(\frac{1}{2}+z\right)+\frac{1}{x_n^2+y_n^2}\left(\frac{1}{2}+z\right)^2\right\}$$

i.e.

$$\omega\left(\frac{1}{2}+z\right) = \prod_{n=1}^{\infty}\frac{(x_n-1/2)^2+y_n^2}{x_n^2+y_n^2}\left\{1-\frac{2(x_n-1/2)z}{(x_n-1/2)^2+y_n^2}+\frac{z^2}{(x_n-1/2)^2+y_n^2}\right\}$$

Since $\omega(1/2+z) = \Omega(z)$,

$$\Omega(z) = \prod_{n=1}^{\infty} \frac{(x_n - 1/2)^2 + y_n^2}{x_n^2 + y_n^2} \left\{ 1 - \frac{2(x_n - 1/2)z}{(x_n - 1/2)^2 + y_n^2} + \frac{z^2}{(x_n - 1/2)^2 + y_n^2} \right\}$$

Substituting $z=0$ for this,

$$\Omega(0) = \prod_{n=1}^{\infty} \frac{(x_n - 1/2)^2 + y_n^2}{x_n^2 + y_n^2}$$

Substituting this for the right side of $\Omega(z)$, we obtain (4.1) ,

Lemma 5.4.2

Among Theorem 5.4.1 (4.1) , the product $\Omega_h(z)$ of the factor whose real part x_n is 1/2 is expressed as follows.

$$\Omega_h(z) = \Omega_h(0) \prod_{r=1} \left(1 + \frac{z^2}{y_r^2} \right) \tag{4.2}$$

Where, $$\Omega_h(0) = \prod_{r=1}^{\infty} \frac{y_r^2}{1/4 + y_r^2} \tag{4.2$_0$}$$

Proof

Replacing a part of x_n with $1/2$ in Theorem 5.4.1 , we obtain the desired expression immediately.

Note

It is the Riemann hypothesis that $\Omega(z) = \Omega_h(z)$ must be.

Lemma 5.4.3

Assume that the factor whose real part is not 1/2 exists among Theorem 5.4.1 (4.1) . Then, when two real numbers are α_s , δ_s $s.t.$ $0 < \alpha_s < 1/2$ & $|\delta_s| > \sqrt{1/8}$, the product $\Omega_\alpha(z)$ of these factors is expressed as follows.

$$\Omega_\alpha(z) = \Omega_\alpha(0) \prod_{s=1} \left\{ 1 + \frac{2(\delta_s^2 - \alpha_s^2)z^2}{(\alpha_s^2 + \delta_s^2)^2} + \frac{z^4}{(\alpha_s^2 + \delta_s^2)^2} \right\} \tag{4.3}$$

Where, $$\Omega_\alpha(0) = \prod_{s=1} \frac{\alpha_s^2 + \delta_s^2}{(1/2 + \alpha_s)^2 + \delta_s^2} \frac{\alpha_s^2 + \delta_s^2}{(1/2 - \alpha_s)^2 + \delta_s^2} < 1 \tag{4.3$_0$}$$

Proof

Same as the proof of Lemma 8.4.3 in " **08 Factorization of Completed Riemann Zeta** " (Riemann Zeta Func.)

Note

The conditional expression $|\delta_s| > \sqrt{1/8}$ is valid. It is because the zero of $\beta(z)$ does not exist in the domain $0 < \alpha_s < 1/2$ & $|\delta_s| \le \sqrt{1/8}$.

Theorem 5.4.4

When Dirichlet beta function is $\beta(z)$ and the non-trivial zeros are $z_n = x_n \pm iy_n \quad n=1,2,3,\cdots$,
If the following expression holds, non-trivial zeros whose real parts is not $1/2$ do not exist.

$$\prod_{r=1}^{\infty} \frac{y_r^2}{1/4+y_r^2} = \left(\frac{2}{\sqrt{\pi}}\right)^{3/2} \Gamma\left(\frac{3}{4}\right)\beta\left(\frac{1}{2}\right) = 0.98071361\cdots \tag{4.4$_0$}$$

Proof

Although non-trivial zeros $1/2 \pm iy_r \quad r=1,2,3,\cdots$ exist in fact, assume non-trivial zeros $1/2+\alpha_s \pm i\delta_s$, $1/2-\alpha_s \pm i\delta_s \quad (0<\alpha_s<1/2 \ \& \ |\delta_s| > \sqrt{1/8})$ exist in addition. Then, the following expression holds from Theorem 5.4.1 , Lemma 5.4.2 and Lemma 5.4.3 .

$$\Omega(z) = \Omega_h(z)\Omega_\alpha(z)$$

$$= \Omega(0) \prod_{r=1} \left(1+\frac{z^2}{y_r^2}\right) \cdot \prod_{s=1} \left\{1+\frac{2\left(\delta_s^2-\alpha_s^2\right)z^2}{\left(\alpha_s^2+\delta_s^2\right)^2}+\frac{z^4}{\left(\alpha_s^2+\delta_s^2\right)^2}\right\} \tag{4.1'}$$

$$\Omega(0) = \Omega_h(0)\Omega_\alpha(0)$$

$$= \prod_{r=1}^{\infty} \frac{y_r^2}{1/4+y_r^2} \cdot \prod_{s=1} \frac{\alpha_s^2+\delta_s^2}{(1/2+\alpha_s)^2+\delta_s^2}\frac{\alpha_s^2+\delta_s^2}{(1/2-\alpha_s)^2+\delta_s^2} = \left(\frac{2}{\sqrt{\pi}}\right)^{3/2}\Gamma\left(\frac{3}{4}\right)\beta\left(\frac{1}{2}\right)$$

$$= 0.98071361\cdots \tag{4.1$_0$'}$$

And, according to Lemma 5.4.3 , when $0<\alpha_s<1/2 \ \& \ |\delta_s| > \sqrt{1/8}$,

$$\prod_{s=1} \frac{\alpha_s^2+\delta_s^2}{(1/2+\alpha_s)^2+\delta_s^2}\frac{\alpha_s^2+\delta_s^2}{(1/2-\alpha_s)^2+\delta_s^2} < 1 \tag{4.3$_0$}$$

Then, from (4.1$_0$') ,

$$\prod_{r=1}^{\infty} \frac{y_r^2}{1/4+y_r^2} > \left(\frac{2}{\sqrt{\pi}}\right)^{3/2}\Gamma\left(\frac{3}{4}\right)\beta\left(\frac{1}{2}\right)$$

i.e.

$$\prod_{r=1}^{\infty} \frac{y_r^2}{1/4+y_r^2} \neq \left(\frac{2}{\sqrt{\pi}}\right)^{3/2}\Gamma\left(\frac{3}{4}\right)\beta\left(\frac{1}{2}\right) = 0.98071361\cdots$$

As the contrapositive to the above, this theorem holds.

Both sides of (4.4$_0$) were calculated with the formula manipulation software *Mathematica* using 10000 zeros on the critical line. Both sides coincided with 4 decimal places.

```
y := ReadList["BetaZeros.prn", Number]

Ω0[m_] := Product[y[[r]]^2 / (1/4 + y[[r]]^2), {r, 1, m}]      Ω[0] := (2/Sqrt[π])^(3/2) Gamma[3/4] DirichletBeta[1/2]

N[Ω0[10000]]                                                   N[Ω[0]]
0.980759                                                       0.980714
```

cf.

If the square root of (4.4_0) is taken, it is as follows. This is also equivalent to the Riemann hypothesis.

$$\prod_{r=1}^{\infty}\frac{y_r}{\sqrt{1/4+y_r^2}} = \left(\frac{2}{\sqrt{\pi}}\right)^{3/4}\sqrt{\Gamma\left(\frac{3}{4}\right)\beta\left(\frac{1}{2}\right)} = 0.99030985\cdots \tag{4.5}$$

Each factor on the left side is an imaginary part when non-trivial zero $z_r = 1/2 + i\,y_r$ is converted to polar coordinates. That is,

$$\prod_{r=1}^{\infty}\sin\theta_r = \left(\frac{2}{\sqrt{\pi}}\right)^{3/4}\sqrt{\Gamma\left(\frac{3}{4}\right)\beta\left(\frac{1}{2}\right)} = 0.99030985\cdots \tag{4.5θ}$$

Data of the zeros of $\beta(z)$

I downloaded the zeros on the critical line of Dirichlet beta function $\beta(z)$ from the following website. I am sincerely thankful to the webmaster.

Tables of approximate values of the first zeros on the critical line of some primitive Dirichlet L-series

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Alien's Mathematics