08 Factorization of Completed Riemann Zeta

8.1 Hadamard product of $\xi(z)$

Formula 8.1.1 (Hadamard product of $\xi(z)$)

Let completed zeta function be as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) \tag{1.d}$$

When non-trivial zeros of $\zeta(z)$ are $z_k = x_k \pm i y_k \quad k=1,2,3,\cdots$ and γ is Euler-Mascheroni constant, $\xi(z)$ is expressed by the Hadamard product as follows.

$$\xi(z) = e^{\left(\log 2 + \frac{\log\pi}{2} - 1 - \frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} \tag{1.0}$$

$$\xi(z) = e^{\left(\log 2 + \frac{\log\pi}{2} - 1 - \frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}} \tag{1.1}$$

Proof

Suppose that the Hadamard product is expressed as follows.

$$\xi(z) = A e^{Bz}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} \tag{1.p}$$

And let us find A and B of this. As first,

$$\Gamma\left(\frac{z}{2}\right) = \frac{2}{z}\Gamma\left(\frac{z}{2}+1\right)$$

Substituting this for (1.d) ,

$$\xi(z) = -2(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}+1\right)\zeta(z) \tag{1.d'}$$

Substituting $z=0$ for (1.d') ,

$$\xi(0) = -2(1-0)\pi^{-\frac{0}{2}}\Gamma\left(\frac{0}{2}+1\right)\zeta(0) = -2\cdot 1\cdot\left(-\frac{1}{2}\right) = 1$$

Substituting $z=0$ for (1.p) , $\xi(0)=A$. We obtain $A=1$ from both. Thus,

$$\xi(z) = e^{Bz}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} \tag{1.p'}$$

Next, from (1.d') and (1.p') ,

$$-2(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}+1\right)\zeta(z) = e^{Bz}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

From this,

$$\zeta(z) = -\frac{\pi^{\frac{z}{2}} e^{Bz}}{2(1-z)\Gamma\left(\frac{z}{2}+1\right)} \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

Taking the logarithm of both sides,

$$\log\zeta(z) = \frac{z}{2}\log\pi + Bz - \log 2 - \log(z-1) - \log\Gamma\left(\frac{z}{2}+1\right) + \log\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

Here,

$$\log\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} = \sum_{k=1}^{\infty}\log\left(1-\frac{z}{z_k}\right) + \sum_{k=1}^{\infty}\frac{z}{z_k}$$

Using this,

$$\log\zeta(z) = \frac{z}{2}\log\pi + Bz - \log 2 - \log(z-1) - \log\Gamma\left(\frac{z}{2}+1\right) + \sum_{k=1}^{\infty}\log\left(1-\frac{z}{z_k}\right) + \sum_{k=1}^{\infty}\frac{z}{z_k}$$

Differentiating both sides with respect to z ,

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{\log\pi}{2} + B - \frac{1}{z-1} - \frac{1}{2}\frac{\Gamma'(z/2+1)}{\Gamma(z/2+1)} + \sum_{k=1}^{\infty}\frac{-\frac{1}{z_k}}{1-\frac{z}{z_k}} + \sum_{k=1}^{\infty}\frac{1}{z_k}$$

i.e.

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{\log\pi}{2} + B - \frac{1}{z-1} - \frac{1}{2}\frac{\Gamma'(z/2+1)}{\Gamma(z/2+1)} + \sum_{k=1}^{\infty}\left(\frac{1}{z-z_k}+\frac{1}{z_k}\right)$$

Putting $z=0$,

$$\frac{\zeta'(0)}{\zeta(0)} = \frac{\log\pi}{2} + B + 1 - \frac{1}{2}\frac{\Gamma'(1)}{\Gamma(1)}$$

Here, the following special values are known for $\zeta(z)$ and $\Gamma(z)$.

$$\zeta(0) = -\frac{1}{2} \quad , \quad \zeta'(0) = -\frac{\log 2\pi}{2} \quad , \quad \Gamma(1) = 1 \quad , \quad \Gamma'(1) = -\gamma$$

So, substituting these for both sides,

$$\log 2\pi = \frac{\log\pi}{2} + B + 1 + \frac{\gamma}{2}$$

From this,

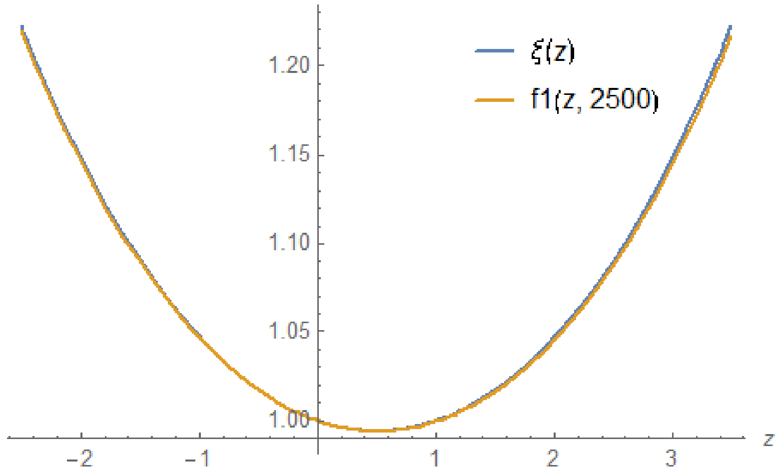
$$B = \log 2 + \frac{\log\pi}{2} - 1 - \frac{\gamma}{2}$$

Substituting this for (1.p') , we obtain (1.0) . Q.E.D.

If $x_n = 1/2 \quad n=1,2,3,\cdots$, (1.1) becomes

$$\xi(z) = e^{\left(\log 2 + \frac{\log\pi}{2} - 1 - \frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{z}{1/4+y_n^2}+\frac{z^2}{1/4+y_n^2}\right)e^{\frac{z}{1/4+y_n^2}} \qquad (1.1')$$

Althogh a general formula for generating zeros whose real part is 1/2 is not known, *Mathematica* has a function $y_n = Im[ZetaZero[n]]$ that generates this numerically. When both sides of (1.1') are drawn overlapping using known non-trivial 2,500 zeros, it is as follows..



As a special value for (1.1) , an important formula used in the next section is obtained.

Formula 8.1.2 (Special value)

When non-trivial zeros of Riemann zeta function are $x_n \pm i y_n \quad n=1,2,3,\cdots$, the following expression holds.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}} = 1.02336448\cdots \tag{1.2}$$

Proof

Giving $z=1$ to (1.1) ,

$$\xi(1) = e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}}\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}} = 1 \tag{1.1$_1$}$$

From this,

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}} \quad (=1.02336448\cdots) \tag{1.2}$$

If $x_n = 1/2 \quad n=1,2,3,\cdots$, (1.2) becomes

$$\prod_{r=1}^{\infty}\left(1-\frac{2\cdot 1/2-1}{(1/2)^2+y_r^2}\right)e^{\frac{2\cdot 1/2}{(1/2)^2+y_r^2}} = e^{\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}} = e^{1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}}$$

From this,

$$\sum_{n=1}^{\infty}\frac{1}{1/4+y_n^2} = 1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2} = 0.0230957\cdots \tag{1.2'}$$

As a special value for (1.0) , the following formulas is obtained.

Formula 8.1.3 (Special value)

When non-trivial zeros of Riemann zeta function are $x_k \pm iy_k \quad k=1,2,3,\cdots$, the following expression holds.

$$\prod_{n=1}^{\infty}\left\{1-\frac{1}{(x_n+iy_n)^2}\right\}\left\{1-\frac{1}{(x_n-iy_n)^2}\right\} = \frac{\pi}{3} \tag{1.3}$$

Proof

Giving $z = -1,\ 1$ to (1.0) respectively,

$$e^{-\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)}\prod_{k=1}^{\infty}\left(1+\frac{1}{z_k}\right)e^{-\frac{1}{z_k}} = \xi(-1) = \frac{\pi}{3}$$

$$e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)}\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right)e^{\frac{1}{z_k}} = \xi(1) = 1$$

Multiplying both sides respectively,

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right)\left(1+\frac{1}{z_k}\right) = \frac{\pi}{3}$$

Let $z_k = x_k \pm iy_k \quad k=1,2,3,\cdots$. Then,

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right) = \prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right)$$

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{z_k}\right) = \prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right)$$

From these,

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right)\left(1+\frac{1}{z_k}\right) = \prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right)$$

$$= \prod_{n=1}^{\infty}\left\{1-\frac{1}{(x_n+iy_n)^2}\right\}\left\{1-\frac{1}{(x_n-iy_n)^2}\right\}$$

Therefore,

$$\prod_{n=1}^{\infty}\left\{1-\frac{1}{(x_n+iy_n)^2}\right\}\left\{1-\frac{1}{(x_n-iy_n)^2}\right\} = \frac{\pi}{3} \tag{1.3}$$

Both sides of (1.3) were computed using known non-trivial 20000 zeros in an attempt. Both sides was equal up to 3 decimal places.

```
zo_n_ := ZetaZero[n]        zc_n_ := Conjugate[zo_n]
```

$$gl[m\_] := \prod_{n=1}^{m}\left(1-\frac{1}{zo_n^{\,2}}\right)\left(1-\frac{1}{zc_n^{\,2}}\right) \qquad gr := \frac{\pi}{3}$$

```
N[gl[20 000]]               N[gr]
1.04703 + 0. i              1.0472
```

8.2 Non-trivial zeros whose real part is not 1/2

According to Theorem 7.4.1 in " **07 Completed Riemann Zeta** ", if Riemann zeta function $\zeta(z)$ has non-trivial zero whose real part is not $1/2$, the one set have to consist of the following four.

$$1/2+\alpha_s \pm i\beta_s \ , \ 1/2-\alpha_s \pm i\beta_s \qquad (0<\alpha_s<1/2)$$

In this section, we will consider how the formulas in the previous section are expressed when non-trivial zeros whose real part is 1/2 and non-trivial zeros whose real part is not 1/2 are mixed.

Lemma 8.2.1

Let γ be Euler-Mascheroni constant, non-trivial zeros of Riemann zeta function are $x_n+iy_n \quad n=1,2,3,\cdots$. Among them, zeros whose real part is 1/2 are $1/2\pm iy_r \quad r=1,2,3,\cdots$ and zeros whose real parts is not 1/2 are $1/2\pm\alpha_s\pm i\beta_s \ (0<\alpha_s<1/2) \quad s=1,2,3,\cdots$. Then Formual 8.1.1 (1.1) is expressed as follows.

$$\xi(z)=e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)e^{\frac{z}{1/4+y_r^2}}$$

$$\times\prod_{s=1}\left\{1-\frac{(1+2\alpha_s)z}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{z^2}{(1/2+\alpha_s)^2+\beta_s^2}\right\}e^{\frac{(1+2\alpha_s)z}{(1/2+\alpha_s)^2+\beta_s^2}}$$

$$\times\prod_{s=1}\left\{1-\frac{(1-2\alpha_s)z}{(1/2-\alpha_s)^2+\beta_s^2}+\frac{z^2}{(1/2-\alpha_s)^2+\beta_s^2}\right\}e^{\frac{(1-2\alpha_s)z}{(1/2-\alpha_s)^2+\beta_s^2}} \qquad (2.1)$$

Proof

Formula 8.1.1 (1.1) was as follows.

$$\xi(z)=e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}} \qquad (1.1)$$

Regarding non-trivial zeros $1/2\pm iy_r \quad r=1,2,3,\cdots$ whose real part is 1/2, a part of the right side is expressed as follows.

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)e^{\frac{z}{1/4+y_r^2}}$$

On the other hand, regarding non-trivial zeros $1/2\pm\alpha_s\pm i\beta_s \ (0<\alpha_s<1/2)$ whose real part is not 1/2, a part of the right side is expressed as follows.

$$\prod_{s=1}\left\{1-\frac{(1+2\alpha_s)z}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{z^2}{(1/2+\alpha_s)^2+\beta_s^2}\right\}e^{\frac{(1+2\alpha_s)z}{(1/2+\alpha_s)^2+\beta_s^2}}$$

$$\prod_{s=1}\left\{1-\frac{(1-2\alpha_s)z}{(1/2-\alpha_s)^2+\beta_s^2}+\frac{z^2}{(1/2-\alpha_s)^2+\beta_s^2}\right\}e^{\frac{(1-2\alpha_s)z}{(1/2-\alpha_s)^2+\beta_s^2}}$$

Multiplying these, we obtain the desired expression.

Theorem 8.2.2

Let γ be Euler-Mascheroni constant, non-trivial zeros of Riemann zeta function are $x_n+iy_n \quad n=1,2,3,\cdots$. Among them, zeros whose real part is 1/2 are $1/2\pm iy_r \quad r=1,2,3,\cdots$ and zeros whose real parts is not 1/2 are $1/2\pm\alpha_s\pm i\beta_s \ \left(0<\alpha_s<1/2\right) \quad s=1,2,3,\cdots$. Then the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)=1 \tag{2.2}$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2}=\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1+2\alpha_s}{\left(1/2+\alpha_s\right)^2+\beta_s^2}+\frac{1-2\alpha_s}{\left(1/2-\alpha_s\right)^2+\beta_s^2}\right\} \tag{2.3}$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2}=1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}=0.0230957\cdots \tag{2.4}$$

Proof

Substituting $z=1$ for Formula 8.1.1 (1.1) ,

$$\xi(1)=e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}}\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}}=1 \tag{1.1$_1$}$$

Substituting $z=1$ for Lemma 8.2.1 (2.1) ,

$$\xi(1)=e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}}\prod_{s=1}\left\{1-\frac{2\alpha_s}{\left(1/2+\alpha_s\right)^2+\beta_s^2}\right\}\left\{1+\frac{2\alpha_s}{\left(1/2-\alpha_s\right)^2+\beta_s^2}\right\}$$
$$\times e^{\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\frac{1+2\alpha_s}{\left(1/2+\alpha_s\right)^2+\beta_s^2}+\frac{1-2\alpha_s}{\left(1/2-\alpha_s\right)^2+\beta_s^2}} \tag{2.1$_1$}$$

From these,

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)=\prod_{s=1}\left\{1-\frac{2\alpha_s}{\left(1/2+\alpha_s\right)^2+\beta_s^2}\right\}\left\{1+\frac{2\alpha_s}{\left(1/2-\alpha_s\right)^2+\beta_s^2}\right\}$$

$$e^{\frac{2x_n}{x_n^2+y_n^2}}=e^{\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\frac{1+2\alpha_s}{\left(1/2+\alpha_s\right)^2+\beta_s^2}+\frac{1-2\alpha_s}{\left(1/2-\alpha_s\right)^2+\beta_s^2}}$$

Here, conveniently,

$$\left\{1-\frac{2\alpha_s}{\left(1/2+\alpha_s\right)^2+\beta_s^2}\right\}\left\{1+\frac{2\alpha_s}{\left(1/2-\alpha_s\right)^2+\beta_s^2}\right\}$$
$$=1+\frac{2\alpha_s}{\left(1/2-\alpha_s\right)^2+\beta_s^2}-\frac{2\alpha_s}{\left(1/2+\alpha_s\right)^2+\beta_s^2}-\frac{2\alpha_s}{\left(1/2-\alpha_s\right)^2+\beta_s^2}\frac{2\alpha_s}{\left(1/2+\alpha_s\right)^2+\beta_s^2}$$
$$=1+\frac{2\alpha_s}{\left(1/2-\alpha_s\right)^2+\beta_s^2}\frac{2\alpha_s}{\left(1/2+\alpha_s\right)^2+\beta_s^2}-\frac{2\alpha_s}{\left(1/2-\alpha_s\right)^2+\beta_s^2}\frac{2\alpha_s}{\left(1/2+\alpha_s\right)^2+\beta_s^2}$$
$$=1$$

So,

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)=1 \tag{2.2}$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2}=\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\} \tag{2.3}$$

And, from (1.1_1) and (2.2) ,

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2}=1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}=0.0230957\cdots \tag{2.4}$$

Although the story goes a little aside, using (2.2) , we obtain the following special values.

Formula 8.2.3 (Special values)

When non-trivial zeros of Riemann zeta function are $x_k \pm iy_k \quad k=1,2,3,\cdots$, the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right)=1 \tag{2.5$_+$}$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right)=\frac{\pi}{3} \tag{2.5$_-$}$$

Proof

From the proof of Formula 8.1.3 in the previous section,

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right)=\prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right)=\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right) \tag{z$_+$}$$

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{z_k}\right)=\prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right)=\prod_{n=1}^{\infty}\left(1+\frac{2x_n+1}{x_n^2+y_n^2}\right) \tag{z$_-$}$$

$$\frac{\pi}{3}=\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right)\left(1+\frac{1}{z_k}\right) \tag{z$_0$}$$

From Theorem 8.2.2 ,

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)=1 \tag{2.2}$$

Substituting this for (z_+) ,

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right)=\prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right)=1 \tag{2.5$_+$}$$

Substituting this and (z_-) for (z_0) sequentially,

$$\frac{\pi}{3}=1\cdot\prod_{k=1}^{\infty}\left(1+\frac{1}{z_k}\right)=\prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right) \tag{2.5$_-$}$$

(2.5_+) and (2.5_-) were calculated using known non-trivial zeros. It was as follows.

```
zo[n_] := ZetaZero[n]                          zc[n_] := Conjugate[zo[n]]
```

$$g_+[m\_] := \prod_{n=1}^{m}\left(1-\frac{1}{zo_n}\right)\left(1-\frac{1}{zc_n}\right) \qquad g_-[m\_] := \prod_{n=1}^{m}\left(1+\frac{1}{zo_n}\right)\left(1+\frac{1}{zc_n}\right)$$

```
N[g+[1000]]             N[g-[20000]]           N[π / 3]
1. + 0. i               1.04703 + 0. i         1.0472
```

Well, let us return to the subject. By using Theorem 8.2.2 , the very important following theorem is obtained.

Theorem 8.2.4

Let non-trivial zeros of Riemann zeta function are $x_n+iy_n \quad n=1,2,3,\cdots$ and γ be Euler-Mascheroni constant. If the following expression holds, non-trivial zeros whose real parts is not $1/2$ do not exist.

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2} = 0.0230957\cdots \tag{1.2'}$$

Proof

Although non-trivial zeros $1/2\pm iy_r \quad r=1,2,3,\cdots$ exist in fact, assume non-trivial zeros $1/2+\alpha_s\pm i\beta_s$, $1/2-\alpha_s\pm i\beta_s \quad (0<\alpha_s<1/2)$ exist in addition. Then, the following expression holds from Theorem 8.2.2 (2.3) , (2.4) .

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} + \sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\} = 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2}$$

Here, the following inequality holds for $0<\alpha_s<1/2$ and arbitrary real number β_s ,

$$\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2} = \frac{1/2-2\alpha_s^2+2\beta_s^2}{\{(1/2+\alpha_s)^2+\beta_s^2\}\{(1/2-\alpha_s)^2+\beta_s^2\}} > 0$$

So,

$$\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\} > 0 \qquad \textit{for} \quad 0<\alpha_s<1/2$$

Thus,

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} < 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2}$$

i.e.

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} \neq 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2}$$

As the contrapositive to the above, this theorem holds.

Note

This theorem shows that the equation (1.2') is equivalent to the Riemann hypothesis. However, for the proof of (1.2'), the imaginary part y_r of the non-trivial zeros $1/2\pm iy_r \quad r=1,2,3,\cdots$ have to be obtained as a formula.

Both sides of (1.2') were calculated with the formula manipulation software *Mathematica* using known non-trivial 200000 zeros. Both sides coincided with four decimal places.

```
y_r_ := Im[ZetaZero[r]]          γ := EulerGamma
```

$$gl[m\_] := \sum_{r=1}^{m} \frac{1}{1/4 + y_r^{\,2}} \qquad gr := 1 + \frac{\gamma}{2} - Log[2] - \frac{Log[\pi]}{2}$$

```
N[gl[200 000]]                   N[gr]
0.0230832                        0.0230957
```

8.3 Factorization of $\xi(z)$

Formula 8.1.1 (Hadamard product) is what the completed zeta function $\xi(z)$ is incompletely factored at the non-trivial zeros. However, using Theorem 8.2.2 , the compensation terms disappear and $\xi(z)$ is completely factorized at the non-trivial zeros.

Theorem 8.3.1 (Factorization of $\xi(z)$)

Let Riemann zeta function be $\zeta(z)$, the non-trivial zeros are $z_n = x_n \pm i y_n \quad n=1,2,3,\cdots$ and completed zeta function be as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Then, $\xi(z)$ is factorized as follows.

$$\xi(z) = \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right) \tag{3.1}$$

Proof

From Formula 8.1.1 (1.1) ,

$$\xi(z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)\cdot e^{\sum_{n=1}^{\infty}\frac{2x_n z}{x_n^2+y_n^2}}$$

On the other hand, from Theorem 8.2.2 (2.4) ,

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = 1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}$$

From this,

$$e^{\sum_{n=1}^{\infty}\frac{2x_n z}{x_n^2+y_n^2}} = e^{\left(1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}\right)z}$$

Substituting this for the right side of the above,

$$\xi(z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)\cdot e^{\left(1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}\right)z}$$

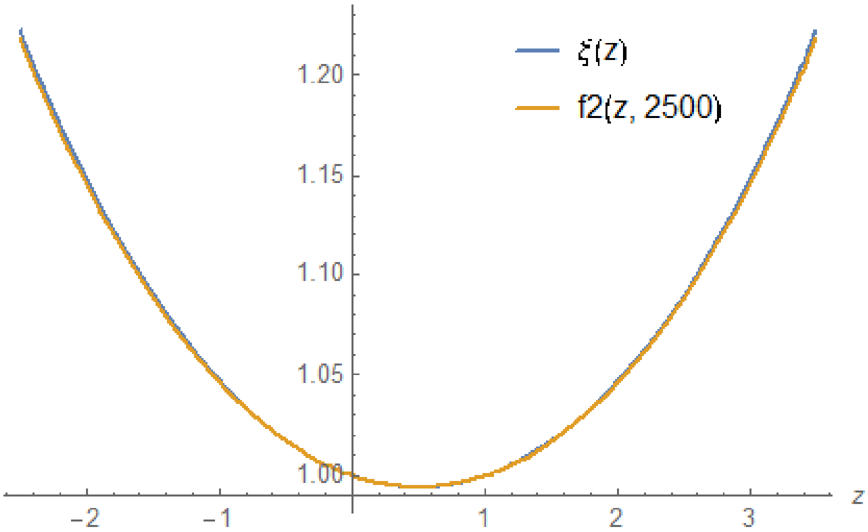
$$= \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right) \tag{3.1}$$

In addition, this formula is known.

If $x_n = 1/2 \quad n=1,2,3,\cdots$, (3.1) becomes

$$\xi(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{1/4+y_n^2}+\frac{z^2}{1/4+y_n^2}\right) \tag{3.1'}$$

When both sides of (3.1') are drawn overlapping using known non-trivial 2,500 zeros, it is as follows..
This is exactly the same as the one in Formula 8.1.1 (1.1') .



ξ(z)
f2(z, 2500)
1.20
1.15
1.10
1.05
1.00
−2
−1
1
2
3
z

8.4 Factorization of $\Xi(z)$

By replacing z with $z+1/2$ in Theorem 8.3.1, completed zeta function $\Xi(z)$ that is an even function is obtained.

Theorem 8.4.1 (Factorization of $\Xi(z)$)

Let Riemann zeta function be $\zeta(z)$, the non-trivial zeros are $z_n = x_n \pm i y_n$ $n=1,2,3,\cdots$ and completed zeta function be as follows.

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right)$$

Then, $\Xi(z)$ is factorized as follows.

$$\Xi(z) = \Xi(0)\prod_{n=1}^{\infty}\left\{1-\frac{2(x_n-1/2)z}{(x_n-1/2)^2+y_n^2}+\frac{z^2}{(x_n-1/2)^2+y_n^2}\right\} \tag{4.1}$$

$$\text{Where, } \Xi(0) = \prod_{n=1}^{\infty}\frac{(x_n-1/2)^2+y_n^2}{x_n^2+y_n^2} = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.99424155\cdots \tag{4.1$_0$}$$

Proof

From Theorem 8.3.1 ,

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

$$\xi(z) = \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right) \tag{3.1}$$

Replacing z with $1/2+z$ in the first expression,

$$\xi\left(\frac{1}{2}+z\right) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) =: \Xi(z)$$

Substituting $z=0$ for this,

$$\Xi(0) = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.99424155\cdots$$

Replacing z with $1/2+z$ in (3.1) ,

$$\xi\left(\frac{1}{2}+z\right) = \prod_{n=1}^{\infty}\left\{1-\frac{2x_n}{x_n^2+y_n^2}\left(\frac{1}{2}+z\right)+\frac{1}{x_n^2+y_n^2}\left(\frac{1}{2}+z\right)^2\right\}$$

$$= \prod_{n=1}^{\infty}\left\{1-\frac{x_n}{x_n^2+y_n^2}+\frac{1/4}{x_n^2+y_n^2}-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right\}$$

$$= \prod_{n=1}^{\infty}\left\{\frac{x_n^2-x_n+1/4+y_n^2}{x_n^2+y_n^2}-\frac{(2x_n-1)z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right\}$$

i.e.

$$\xi\left(\frac{1}{2}+z\right) = \prod_{n=1}^{\infty}\frac{(x_n-1/2)^2+y_n^2}{x_n^2+y_n^2}\left\{1-\frac{2(x_n-1/2)z}{(x_n-1/2)^2+y_n^2}+\frac{z^2}{(x_n-1/2)^2+y_n^2}\right\}$$

Since $\xi(1/2+z) = \Xi(z)$,

$$\Xi(z) = \prod_{n=1}^{\infty}\frac{(x_n-1/2)^2+y_n^2}{x_n^2+y_n^2}\left\{1-\frac{2(x_n-1/2)z}{(x_n-1/2)^2+y_n^2}+\frac{z^2}{(x_n-1/2)^2+y_n^2}\right\}$$

Substituting $z=0$ for this,

$$\Xi(0) = \prod_{n=1}^{\infty}\frac{(x_n-1/2)^2+y_n^2}{x_n^2+y_n^2}$$

Substituting this for the right side of $\Xi(z)$, we obtain (4.1) ,

Lemma 8.4.2

Among Theorem 8.4.1 (4.1) , the product $\Xi_h(z)$ of the factor whose real part x_n is 1/2 is expressed as follows.

$$\Xi_h(z) = \Xi_h(0)\prod_{r=1}\left(1+\frac{z^2}{y_r^2}\right) \tag{4.2}$$

Where, $$\Xi_h(0) = \prod_{r=1}^{\infty}\frac{y_r^2}{1/4+y_r^2} \tag{4.2$_0$}$$

Proof

Replacing a part of x_n with $1/2$ in Theorem 8.4.1 , we obtain the desired expression immediately.

Note

It is the Riemann hypothesis that $\Xi(z) = \Xi_h(z)$ must be.

Lemma 8.4.3

Assume that the factor whose real part is not 1/2 exists among Theorem 8.4.1 (4.1) . Then, when two real numbers are α_s, β_s $s.t.$ $0<\alpha_s<1/2$ & $|\beta_s| > \sqrt{1/8}$, the product $\Xi_\alpha(z)$ of these factors is expressed as follows.

$$\Xi_\alpha(z) = \Xi_\alpha(0)\prod_{s=1}\left\{1+\frac{2\left(\beta_s^2-\alpha_s^2\right)z^2}{\left(\alpha_s^2+\beta_s^2\right)^2}+\frac{z^4}{\left(\alpha_s^2+\beta_s^2\right)^2}\right\} \tag{4.3}$$

Where, $$\Xi_\alpha(0) = \prod_{s=1}\frac{\alpha_s^2+\beta_s^2}{(1/2+\alpha_s)^2+\beta_s^2}\frac{\alpha_s^2+\beta_s^2}{(1/2-\alpha_s)^2+\beta_s^2} < 1 \tag{4.3$_0$}$$

Proof

If such a product of factors exists, according to Theorem 7.4.1 in " **07 Completed Riemann Zeta** ", the one set have to consist of the following four.

$$1/2+\alpha_s \pm i\beta_s \quad , \quad 1/2-\alpha_s \pm i\beta_s \qquad (0 < \alpha_s < 1/2)$$

So, replacing a part of x_n , y_n with $1/2\pm\alpha_s$, β_s respectively in Theorem 8.4.1 ,

$$\Xi_\alpha(z) = \Xi_\alpha(0)\prod_{s=1}^{\infty}\left\{1-\frac{2(+\alpha_s)z}{(+\alpha_s)^2+\beta_s^2}+\frac{z^2}{(+\alpha_s)^2+\beta_s^2}\right\}\left\{1-\frac{2(-\alpha_s)z}{(-\alpha_s)^2+\beta_s^2}+\frac{z^2}{(-\alpha_s)^2+\beta_s^2}\right\}$$

$$= \Xi_s(0)\prod_{s=1}^{\infty}\left(1-\frac{2\alpha_s z}{\alpha_s^2+\beta_s^2}+\frac{z^2}{\alpha_s^2+\beta_s^2}\right)\left(1+\frac{2\alpha_s z}{\alpha_s^2+\beta_s^2}+\frac{z^2}{\alpha_s^2+\beta_s^2}\right)$$

$$= \Xi_s(0)\prod_{s=1}^{\infty}\left\{1+\frac{2\left(\beta_s^2-\alpha_s^2\right)z^2}{\left(\alpha_s^2+\beta_s^2\right)^2}+\frac{z^4}{\left(\alpha_s^2+\beta_s^2\right)^2}\right\} \tag{4.3}$$

$$\Xi_s(0) = \prod_{s=1}^{\infty}\frac{(+\alpha_s)^2+\beta_s^2}{(1/2+\alpha_s)^2+\beta_s^2}\,\frac{(-\alpha_s)^2+\beta_s^2}{(1/2-\alpha_s)^2+\beta_s^2}$$

$$= \prod_{s=1}^{\infty}\frac{\alpha_s^2+\beta_s^2}{(1/2+\alpha_s)^2+\beta_s^2}\,\frac{\alpha_s^2+\beta_s^2}{(1/2-\alpha_s)^2+\beta_s^2} \tag{4.3$_0$}$$

Last,

$$\left\{(1/2+\alpha_s)^2+\beta_s^2\right\}\left\{(1/2-\alpha_s)^2+\beta_s^2\right\}$$

$$= (1/2+\alpha_s)^2(1/2-\alpha_s)^2 + (1/2+\alpha_s)^2\beta_s^2 + (1/2-\alpha_s)^2\beta_s^2 + \beta_s^4$$

$$= \frac{1}{16}+\frac{\beta_s^2-\alpha_s^2}{2} + 2\alpha_s^2\beta_s^2+\alpha_s^4+\beta_s^4$$

i.e.

$$\left\{(1/2+\alpha_s)^2+\beta_s^2\right\}\left\{(1/2-\alpha_s)^2+\beta_s^2\right\} = \frac{1}{16}+\frac{\beta_s^2-\alpha_s^2}{2}+\left(\alpha_s^2+\beta_s^2\right)^2$$

Here, if $0 < \alpha_s < 1/2$ & $|\beta_s| > \sqrt{1/8}$, the following inequality holds.

$$\beta_s^2 > \alpha_s^2 - \frac{1}{8}$$

From this,

$$\frac{1}{16}+\frac{\beta_s^2-\alpha_s^2}{2} > 0$$

Therefore,

$$\left\{(1/2+\alpha_s)^2+\beta_s^2\right\}\left\{(1/2-\alpha_s)^2+\beta_s^2\right\} > \left(\alpha_s^2+\beta_s^2\right)^2$$

Since, both sides are positive,

$$\frac{1}{(1/2+\alpha_s)^2+\beta_s^2}\,\frac{1}{(1/2-\alpha_s)^2+\beta_s^2} < \frac{1}{\left(\alpha_s^2+\beta_s^2\right)^2}$$

From this,

$$\frac{\alpha_s^2+\beta_s^2}{(1/2+\alpha_s)^2+\beta_s^2}\,\frac{\alpha_s^2+\beta_s^2}{(1/2-\alpha_s)^2+\beta_s^2} < 1$$

If there are a plurality of such sets,

$$\prod_{s=1} \frac{\alpha_s^2+\beta_s^2}{(1/2+\alpha_s)^2+\beta_s^2} \frac{\alpha_s^2+\beta_s^2}{(1/2-\alpha_s)^2+\beta_s^2} < 1$$

Note

The conditional expression $|\beta_s| > \sqrt{1/8}$ is valid. It is because the zero of $\zeta(z)$ does not exist in the domain $0 < \alpha_s < 1/2$ & $|\beta_s| \le \sqrt{1/8}$.

Theorem 8.4.4

When Riemann zeta function is $\zeta(z)$ and the non-trivial zeros sre $z_n = x_n \pm iy_n$ $n=1,2,3,\cdots$,
If the following expression holds, non-trivial zeros whose real parts is not $1/2$ do not exist.

$$\prod_{r=1}^{\infty} \frac{y_r^2}{1/4+y_r^2} = -\frac{1}{4\pi^{1/4}} \Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.99424155\cdots \tag{4.4$_0$}$$

Proof

Although non-trivial zeros $1/2 \pm iy_r$ $r=1,2,3,\cdots$ exist in fact, assume non-trivial zeros $1/2+\alpha_s \pm i\beta_s$, $1/2-\alpha_s \pm i\beta_s$ $(0 < \alpha_s < 1/2$ & $|\beta_s| > \sqrt{1/8})$ exist in addition. Then, the following expression holds from Theorem 8.4.1 , Lemma 8.4.2 and Lemma 8.4.3 .

$$\Xi(z) = \Xi_h(z)\Xi_\alpha(z)$$

$$= \Xi(0) \prod_{r=1}\left(1+\frac{z^2}{y_r^2}\right) \cdot \prod_{s=1}\left\{1+\frac{2\left(\beta_s^2-\alpha_s^2\right)z^2}{\left(\alpha_s^2+\beta_s^2\right)^2}+\frac{z^4}{\left(\alpha_s^2+\beta_s^2\right)^2}\right\} \tag{4.1'}$$

$$\Xi(0) = \Xi_h(0)\Xi_\alpha(0)$$

$$= \prod_{r=1}^{\infty} \frac{y_r^2}{1/4+y_r^2} \cdot \prod_{s=1} \frac{\alpha_s^2+\beta_s^2}{(1/2+\alpha_s)^2+\beta_s^2} \frac{\alpha_s^2+\beta_s^2}{(1/2-\alpha_s)^2+\beta_s^2} = -\frac{1}{4\pi^{1/4}} \Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)$$

$$= 0.99424155\cdots \tag{4.1$_0$'}$$

And, according to Lemma 8.4.3 , when $0 < \alpha_s < 1/2$ & $|\beta_s| > \sqrt{1/8}$,

$$\prod_{s=1} \frac{\alpha_s^2+\beta_s^2}{(1/2+\alpha_s)^2+\beta_s^2} \frac{\alpha_s^2+\beta_s^2}{(1/2-\alpha_s)^2+\beta_s^2} < 1 \tag{4.3$_0$}$$

Then, form (4.1$_0$') ,

$$\prod_{r=1}^{\infty} \frac{y_r^2}{1/4+y_r^2} > -\frac{1}{4\pi^{1/4}} \Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.99424155\cdots$$

i.e.

$$\prod_{r=1}^{\infty} \frac{y_r^2}{1/4+y_r^2} \ne -\frac{1}{4\pi^{1/4}} \Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)$$

As the contrapositive to the above, this theorem holds.

Both sides of (4.4_0) were calculated with the formula manipulation software *Mathematica* using known non-trivial 100000 zeros. Both sides coincided with five decimal places.

```
y_r_ := Im[ZetaZero[r]]
Ξ0[m_] := Product[y_r^2 / (1/4 + y_r^2), {r, 1, m}]        Ξ[0] := -1/(4 π^(1/4)) Gamma[1/4] Zeta[1/2]
N[Ξ0[100000]]                                            N[Ξ[0]]
0.994247                                                 0.994242
```

cf.

If the square root of (4.4_0) is taken, it is as follows. This is also equivalent to the Riemann hypothesis.

$$\prod_{r=1}^{\infty} \frac{y_r}{\sqrt{1/4+y_r^2}} = \frac{1}{2\pi^{1/8}} \sqrt{-\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)} = 0.99711662\cdots \tag{4.5}$$

Each factor on the left side is an imaginary part when non-trivial zero $z_r = 1/2 + i\,y_r$ is converted to polar coordinates. That is,

$$\prod_{r=1}^{\infty} \sin\theta_r = \frac{1}{2\pi^{1/8}} \sqrt{-\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)} = 0.99711662\cdots \qquad (4.5\theta)$$

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2018.03.18 Added Sec.3

2018.04.02 Added Sec.4 (Renewed)

2024.04.04 Updated Sec.1

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Alien's Mathematics