08 Factorization and Series Expansion of Zeta Function

8.1 Factorization of Completed Riemann Zeta

Formula 8.1.0 (Hadamard 1893)

When $\zeta(z)$ is Riemann zeta function and the the zeros are z_k $k=1,2,3,\cdots$, the following expression holds.

$$\zeta(z)=\frac{(2\pi/e)^z}{2(z-1)}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}\prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}}$$

Using this formula, we can easily obtain factorization of a completed zeta function.

Formula 8.1.1 (Hadamard product of $\xi(z)$)

Let completed zeta function be as follows.

$$\xi(z)=z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

When non-trivial zeros of $\zeta(z)$ are $z_k=x_k\pm iy_k$ $k=1,2,3,\cdots$ and γ is Euler-Mascheroni constant, $\xi(z)$ is expressed by the Hadamard product as follows.

$$\xi(z)=-e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} \tag{1.0}$$

$$\xi(z)=-e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}} \tag{1.1}$$

Proof

From the Weierstrass expression on the gamma function,

$$\frac{e^{-\gamma z/2}}{\Gamma(1+z/2)}=\prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}}$$

Substituting this for Formula 8.1.0 ,

$$\begin{aligned}\zeta(z)&=\frac{(2\pi/e)^z}{2(z-1)}\frac{e^{-\gamma z/2}}{\Gamma(1+z/2)}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}\\&=\frac{e^{z\log 2\pi}e^{-z}}{z-1}\frac{e^{-\gamma z/2}}{2\Gamma(1+z/2)}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}\\&=\frac{1}{(z-1)z\Gamma(z/2)}e^{\left(\log 2\pi-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}\end{aligned}$$

From this,

$$z(1-z)\Gamma\left(\frac{z}{2}\right)\zeta(z)=-e^{\left(\log 2\pi-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

Multiplying both sides by $\pi^{-\frac{z}{2}} = e^{-\frac{z\log\pi}{2}}$,

$$z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = -e^{\left(\log 2\pi - \frac{\log\pi}{2} - 1 - \frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

i.e.

$$\xi(z) = -e^{\left(\log 2 + \frac{\log\pi}{2} - 1 - \frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} \tag{1.0}$$

This formula means that the completed zeta function is factored by non-trivial zeros of $\zeta(z)$.

Now, let the non-trivial zeros are $z_k = x_k \pm i y_k \quad k=1,2,3,\cdots$. Then,

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} = \prod_{n=1}^{\infty}\left(1-\frac{z}{x_n+iy_n}\right)e^{\frac{z}{x_n+iy_n}}\left(1-\frac{z}{x_n-iy_n}\right)e^{\frac{z}{x_n-iy_n}}$$

$$= \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}}$$

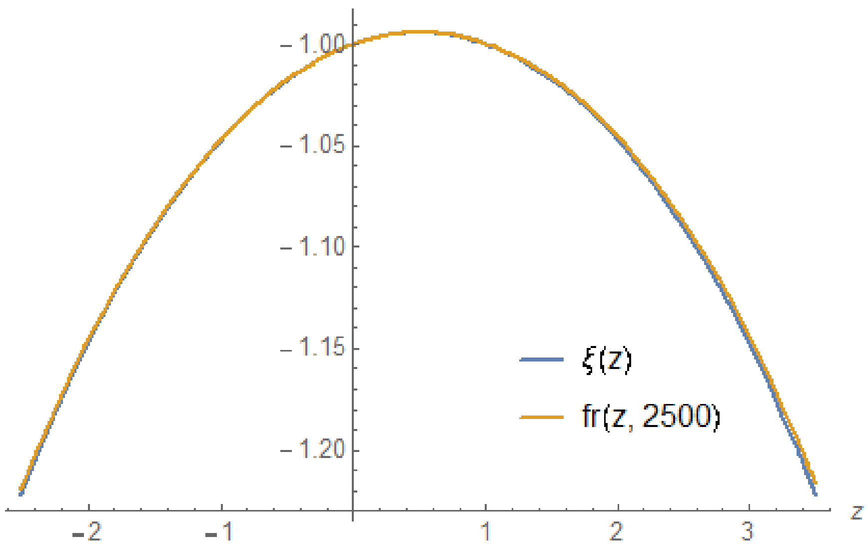
By using this, (1.0) is expressed as follows.

$$\xi(z) = -e^{\left(\log 2 + \frac{\log\pi}{2} - 1 - \frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}} \tag{1.1}$$

If $x_n = 1/2 \quad n=1,2,3,\cdots$, (1.1) becomes

$$\xi(z) = -e^{\left(\log 2 + \frac{\log\pi}{2} - 1 - \frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{z}{1/4+y_n^2}+\frac{z^2}{1/4+y_n^2}\right)e^{\frac{z}{1/4+y_n^2}} \tag{1.1'}$$

Althogh a general formula for generating zeros whose real part is 1/2 is not known, *Mathematica* has a function $y_n = Im\,[\,ZetaZero\,[\,n\,]\,]$ that generates this numerically. When both sides of (1.1') are drawn overlapping using this, it is as follows. The product is calculated to 2,500.



As a special value for (1.1) , an important formula used in the next section is obtained.

Formula 8.1.2 (Special value)

When non-trivial zeros of Riemann zeta function are $x_k \pm i y_k \quad k=1,2,3,\cdots$, the following expression holds.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}} = 1.02336448\cdots \tag{1.2}$$

Proof

Giving $z=1$ to (1.1) ,

$$-1 = -e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}}\prod_{n=1}^{\infty}\left(1-\frac{2x_n}{x_n^2+y_n^2}+\frac{1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}}$$

From this,

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n}{x_n^2+y_n^2}+\frac{1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}}$$

i.e.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}} \quad \left(=1.02336448\cdots\right) \tag{1.2}$$

If $x_n = 1/2 \quad n=1,2,3,\cdots$, (1.2) becomes

$$\prod_{r=1}^{\infty}\left(1-\frac{2\cdot 1/2-1}{(1/2)^2+y_r^2}\right)e^{\frac{2\cdot 1/2}{(1/2)^2+y_r^2}} = e^{\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}} = e^{1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}}$$

From this,

$$\sum_{n=1}^{\infty}\frac{1}{1/4+y_n^2} = 1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2} = 0.0230957\cdots \tag{1.2'}$$

Both sides of (1.2') were computed using known 20000 zeros in an attempt. Both sides was equal up to 4 decimal places.

```
y_n_ := Im[ZetaZero[n]]                      γ := EulerGamma

gl[m_] := Sum[1/(1/4 + y_n^2), {n, 1, m}]    gr := 1 + γ/2 - Log[2] - Log[π]/2

N[gl[20000]]                                 N[gr]
0.0230167                                    0.0230957
```

As a special value for (1.0) , the following formulas is obtained.

Formula 8.1.3 (Special value)

When non-trivial zeros of Riemann zeta function are $x_k \pm iy_k \quad k=1,2,3,\cdots$, the following expression holds.

$$\prod_{n=1}^{\infty}\left\{1-\frac{1}{(x_n+iy_n)^2}\right\}\left\{1-\frac{1}{(x_n-iy_n)^2}\right\} = \frac{\pi}{3} \tag{1.3}$$

Proof

Giving $z = -1, 1$ to (1.0) respectively,

$$-e^{-\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)}\prod_{k=1}^{\infty}\left(1+\frac{1}{z_k}\right)e^{-\frac{1}{z_k}} = \xi(-1) = -\frac{\pi}{3}$$

$$-e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)}\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right)e^{\frac{1}{z_k}} = \xi(1) = -1$$

Multiplying both sides respectively,

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right)\left(1+\frac{1}{z_k}\right) = \frac{\pi}{3}$$

Let $z_k = x_k \pm iy_k \quad k=1,2,3,\cdots$. Then,

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right) = \prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right)$$

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{z_k}\right) = \prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right)$$

From these,

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right)\left(1+\frac{1}{z_k}\right) = \prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right)$$

$$= \prod_{n=1}^{\infty}\left\{1-\frac{1}{(x_n+iy_n)^2}\right\}\left\{1-\frac{1}{(x_n-iy_n)^2}\right\}$$

Therefore,

$$\prod_{n=1}^{\infty}\left\{1-\frac{1}{(x_n+iy_n)^2}\right\}\left\{1-\frac{1}{(x_n-iy_n)^2}\right\} = \frac{\pi}{3} \tag{1.3}$$

Both sides of (1.3) were computed using known 20000 zeros in an attempt. Both sides was equal up to 3 decimal places.

```
zo_n_ := ZetaZero[n]        zc_n_ := Conjugate[zo_n]

gl[m_] := ∏_{n=1}^{m} (1 - 1/zo_n^2) (1 - 1/zc_n^2)        gr := π/3

N[gl[20 000]]                      N[gr]
1.04703 + 0. i                     1.0472
```

8.2 Non-trivial zeros whose real part is not 1/2

In this section, we will consider how the formulas in the previous section are expressed when non-trivial zeros whose real part is not 1/2 is mixed.

Theorem 8.2.1

Let γ be Euler-Mascheroni constant, non-trivial zeros of Riemann zeta function are $x_n+iy_n \quad n=1,2,3,\cdots$.
Among them, zeros whose real part is 1/2 are $1/2\pm iy_r \quad r=1,2,3,\cdots$, and zeros whose real parts are not 1/2 are $1/2-\alpha_s\pm i\beta_s \quad (0<\alpha_s<1/2) \quad s=1,2,3,\cdots$. Then the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)=1 \tag{2.1}$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2}=1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}=0.0230957\cdots \tag{2.2}$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2}=\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\} \tag{2.3}$$

Proof

According to Formula 8.1.2 , the following expression holds for non-trivial zeros $x_n+iy_n \quad n=1,2,3,\cdots$.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}}=e^{1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}}=1.02336448\cdots \tag{1.2}$$

The known non-trivial zeros are $1/2\pm iy_r \quad r=1,2,3,\cdots$. For these, a part of the left side is as follows.

$$\prod_{r=1}^{\infty}\left(1-\frac{2\cdot 1/2-1}{(1/2)^2+y_r^2}\right)e^{\frac{2\cdot 1/2}{(1/2)^2+y_r^2}}=1\cdot e^{\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}} \tag{1.2$_1$}$$

On the otherhan, for the unknown non-trivial zeros, it will be as follows.

According to Formula 7.4.1 in " **07 Completed Riemann Zeta** ", if $1/2+\alpha_s+i\beta_s$ are zeros of $\zeta(z)$ then $1/2-\alpha_s+i\beta_s$ are also zeros of $\zeta(z)$. And, since $\zeta(z)$ has complex conjugate property, these conjugate numbers $1/2+\alpha_s-i\beta_s$, $1/2-\alpha_s-i\beta_s$ are also zeros of $\zeta(z)$.

As the result, **non-trivial zeros whose real parts are not 1/2 constitutes 1 set from the following four.**

$$1/2+\alpha_s\pm i\beta_s \;,\; 1/2-\alpha_s\pm i\beta_s \qquad (0<\alpha_s<1/2)$$

From these,

$$\left\{1-\frac{2(1/2+\alpha_s)-1}{(1/2+\alpha_s)^2+\beta_s^2}\right\}\left\{1-\frac{2(1/2-\alpha_s)-1}{(1/2-\alpha_s)^2+\beta_s^2}\right\}e^{\frac{2(1/2+\alpha_s)}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{2(1/2-\alpha_s)}{(1/2-\alpha_s)^2+\beta_s^2}}$$

$$=\left\{1-\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}\left\{1+\frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}e^{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}}$$

So, a part of the left side of (1.2) becons as follows.

$$\prod_{s=1}\left\{1-\frac{2(1/2+\alpha_s)-1}{(1/2+\alpha_s)^2+\beta_s^2}\right\}\left\{1-\frac{2(1/2-\alpha_s)-1}{(1/2-\alpha_s)^2+\beta_s^2}\right\}e^{\frac{2(1/2+\alpha_s)}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{2(1/2-\alpha_s)}{(1/2-\alpha_s)^2+\beta_s^2}}$$

$$=\prod_{s=1}\left\{1-\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}\left\{1+\frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}e^{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}}$$

Here, conveniently,

$$\left\{1-\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}\left\{1+\frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}$$

$$=1+\frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}-\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}-\frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}$$

$$=1+\frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}-\frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}$$

$$=1$$

That is,

$$\prod_{s=1}\left\{1-\frac{2(1/2+\alpha_s)-1}{(1/2+\alpha_s)^2+\beta_s^2}\right\}\left\{1-\frac{2(1/2-\alpha_s)-1}{(1/2-\alpha_s)^2+\beta_s^2}\right\}=1 \tag{w1}$$

Then,

$$\prod_{s=1}\left\{1-\frac{2(1/2+\alpha_s)-1}{(1/2+\alpha_s)^2+\beta_s^2}\right\}\left\{1-\frac{2(1/2-\alpha_s)-1}{(1/2-\alpha_s)^2+\beta_s^2}\right\}e^{\frac{2(1/2+\alpha_s)}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{2(1/2-\alpha_s)}{(1/2-\alpha_s)^2+\beta_s^2}}$$

$$=1\cdot e^{\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}} \tag{1.2$_2$}$$

Since the left side consists of products of (1.2$_1$) and (1.2$_2$) ,

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}}=e^{\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}} \tag{w2}$$

On the other hand,

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)=\prod_{r=1}^{\infty}\left(1-\frac{2\cdot 1/2-1}{(1/2)^2+y_r^2}\right)$$

$$\times\prod_{s=1}\left\{1-\frac{2(1/2+\alpha_s)-1}{(1/2+\alpha_s)^2+\beta_s^2}\right\}\left\{1-\frac{2(1/2-\alpha_s)-1}{(1/2-\alpha_s)^2+\beta_s^2}\right\}$$

Since (w1) is established,

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)=1 \tag{2.1}$$

Therefore, (1.2) and (w2) become as follows respectively.

$$\prod_{n=1}^{\infty} e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2}} = e^{1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}}$$

$$\prod_{n=1}^{\infty} e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2}} = e^{\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}}$$

From these,

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = 1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2} \tag{2.2}$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = \sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} + \sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\} \tag{2.3}$$

Although the story goes a little aside, using (2.1) , we obtain the following special values.

Formula 8.2.2 (Special values)

When non-trivial zeros of Riemann zeta function are $x_k \pm iy_k \quad k=1,2,3,\cdots$, the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right) = 1 \tag{2.4$_+$}$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right) = \frac{\pi}{3} \tag{2.4$_-$}$$

Proof

From the proof of Formula 8.1.3 in the previous section,

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right) = \prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right) = \prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right) \tag{z$_+$}$$

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{z_k}\right) = \prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right) = \prod_{n=1}^{\infty}\left(1+\frac{2x_n+1}{x_n^2+y_n^2}\right) \tag{z$_-$}$$

$$\frac{\pi}{3} = \prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right)\left(1+\frac{1}{z_k}\right) \tag{z$_0$}$$

From Formula 8.2.1 ,

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right) = 1 \tag{2.1}$$

Substituting this for (z_+) ,

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{z_k}\right) = \prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right) = 1 \tag{2.4$_+$}$$

Substituting this and (z_-) for (z_0) sequentially,

$$\frac{\pi}{3} = 1\cdot\prod_{k=1}^{\infty}\left(1+\frac{1}{z_k}\right) = \prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right) \tag{2.4$_-$}$$

(2.4_+) and (2.4_-) were calculated using known non-trivial zeros. It was as follows.

```
zo[n_] := ZetaZero[n]                    zc[n_] := Conjugate[zo[n]]
```

$$g_+[m\_] := \prod_{n=1}^{m}\left(1-\frac{1}{zo_n}\right)\left(1-\frac{1}{zc_n}\right) \qquad g_-[m\_] := \prod_{n=1}^{m}\left(1+\frac{1}{zo_n}\right)\left(1+\frac{1}{zc_n}\right)$$

```
N[g+[1000]]          N[g-[20000]]          N[π / 3]
1. + 0. i            1.04703 + 0. i        1.0472
```

Well, let us return to the subject. By using Theorem 8.2.1 , the very important following theorem is obtained.

Theorem 8.2.3

Let non-trivial zeros of Riemann zeta function are $x_n + iy_n \quad n=1,2,3,\cdots$ and γ be Euler-Mascheroni constant. If the following expression holds, non-trivial zeros whose real parts are not $1/2$ do not exist.

$$\sum_{n=1}^{\infty}\frac{1}{1/4+y_n^2} = 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2} = 0.0230957\cdots \qquad (1.2')$$

Proof

Although non-trivial zeros $1/2 \pm iy_r \quad r=1,2,3,\cdots$ exist in fact, assume non-trivial zeros $1/2+\alpha_s \pm i\beta_s$, $1/2-\alpha_s \pm i\beta_s \quad (0<\alpha_s<1/2)$ exist in addition. Then, the following expression holds from (2.2) , (2.3) .

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} + \sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\} = 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2}$$

Here, the following inequality holds for $0<\alpha_s<1/2$ and arbitrary real number β_s ,

$$\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2} = \frac{1/2-2\alpha_s^2+2\beta_s^2}{\{(1/2+\alpha_s)^2+\beta_s^2\}\{(1/2-\alpha_s)^2+\beta_s^2\}} > 0$$

So,

$$\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\} > 0 \qquad \text{for } 0<\alpha_s<1/2$$

Thus,

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} < 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2}$$

i.e.

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} \neq 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2}$$

As the contrapositive to the above, this theorem holds.

Note

Riemann hypothesis insists that non-trivial zeros whose real parts are not $1/2$ do not exist. Generally, the proof of the non-existence is called devil's proof (probatio diabolica) . This theorem replaces it with a proof of the equation (1.2') .

However, the proof of this equation (1.2') is not easy either. For the purpose, we have to first obtain imaginary part y_r of non-trivial zeros $1/2 \pm i y_r \quad r=1,2,3,\cdots$ as a formula.

8.3 Factorization of $-z\zeta(1-z)$

We would like to expand completed zeta function $\xi(z)$ to Maclaurin series. Although it is possible, there is a restrictions that the convergence radius is 1. This is not interesting. So, in the following section, we will use $-z\zeta(1-z)$ which is a part of $\xi(z)$. Because, the function is expanded to Maclaurin series with convergence radius of infinity. In this section, we factor $-z\zeta(1-z)$ around 0 first.

Formula 8.3.1

When γ is Euler-Mascheroni constant, $\zeta(z)$ is Riemann zeta function and the non-trivial zeros are $x_n+iy_n \quad n=1,2,3,\cdots$, the following expression holds.

$$-z\zeta(1-z) = \frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}} e^{\left(\log 2-1-\frac{\gamma}{2}\right)z} \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right) e^{\frac{2x_n z}{x_n^2+y_n^2}} \tag{3.1}$$

Proof

Let completed zeta function be as follows.

$$\xi(z) = z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Replacing z with $1-z$,

$$\xi(1-z) = z(1-z)\pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) \tag{w_L}$$

As seen in Section 1, $\xi(z)$ was represented by the Hadamard product as follows

$$\xi(z) = -e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

Since functional equation $\xi(z)=\xi(1-z)$ is established,

$$\xi(1-z) = -e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} \tag{w_R}$$

From (w_L) and (w_R),

$$z(1-z)\pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) = -e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

From this,

$$-z\zeta(1-z) = \frac{\frac{1}{2}\pi^{\frac{1-z}{2}}}{\frac{1-z}{2}\Gamma\left(\frac{1-z}{2}\right)} e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

Here,

$$\frac{1}{2}\pi^{\frac{1-z}{2}} = \frac{\sqrt{\pi}}{2}\pi^{-\frac{z}{2}} = \frac{\sqrt{\pi}}{2}e^{-z\log\sqrt{\pi}}$$

$$\frac{1-z}{2}\Gamma\left(\frac{1-z}{2}\right) = \Gamma\left(1+\frac{1-z}{2}\right) = \Gamma\{(3-z)/2\}$$

Substituting these for the above,

$$-z\zeta(1-z) = \frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}}e^{\left(\log 2-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} \tag{3.0}$$

When the non-trivial zeros are $z_k = x_k \pm i y_k \quad k=1,2,3,\cdots$,

$$-z\zeta(1-z) = \frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}}e^{\left(\log 2-1-\frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}} \tag{3.1}$$

If $x_n = 1/2 \quad n=1,2,3,\cdots$

$$-z\zeta(1-z) = \frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}}e^{\left(\log 2-1-\frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{z}{1/4+y_n^2}+\frac{z^2}{1/4+y_n^2}\right)e^{\frac{z}{1/4+y_n^2}} \tag{3.1'}$$

Trivial Zeros

Trivial zeros $z=3,5,7,\cdots$ of $\zeta(1-z)$ are contained in $1/\Gamma\{(3-z)/2\}$. The reason is as follows.
Formula 11.1.1 (1.3_+) in " **11 Series Expansion of Reciprocal of Gamma Function** " **(A la carte)** was as follows.

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{2n-1}\right)e^{\frac{z}{2n-1}} = \frac{\sqrt{\pi}}{\Gamma\{(1-z)/2\}}e^{\left(\frac{\gamma}{2}+\log 2\right)z} \tag{1.3$_+$}$$

The left side becoms

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{2n-1}\right)e^{\frac{z}{2n-1}} = (1-z)e^{z}\prod_{n=1}^{\infty}\left(1-\frac{z}{2n+1}\right)e^{\frac{z}{2n+1}}$$

Dividing both sides by $(1-z)e^{z}$,

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{2n+1}\right)e^{\frac{z}{2n+1}} = \frac{\sqrt{\pi}}{(1-z)\Gamma\{(1-z)/2\}}e^{\left(\frac{\gamma}{2}+\log 2-1\right)z}$$

$$= \frac{\sqrt{\pi}/2}{(1-z)/2\Gamma\{(1-z)/2\}}e^{\left(-\frac{\gamma}{2}+\log 2-1\right)z}\cdot e^{\gamma z}$$

From this,

$$\frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}}e^{\left(\log 2-1-\frac{\gamma}{2}\right)z} = e^{-\gamma z}\prod_{n=1}^{\infty}\left(1-\frac{z}{2n+1}\right)e^{\frac{z}{2n+1}}$$

Since the exponential function $e^{\pm z}$ has no zero, this formula means that $z=3,5,7,\cdots$ are the zeros of $1/\Gamma\{(3-z)/2\}$

8.4 Maclaurin Expansion by Stieltjes Constants

Formula 8.4.1

When $\zeta(z)$ is Riemann zeta function, the following expression holds on whole complex plane.

$$-z\zeta(1-z) = 1 - \sum_{s=1}^{\infty} \frac{s\gamma_{s-1}}{s!} z^s \tag{4.1}$$

Where, γ_s is Stieltjes constant defined by the following expression.

$$\gamma_s = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \frac{(\log k)^s}{k} - \frac{(\log n)^{s+1}}{s+1} \right\}$$

Proof

It is known that the following expression holds on whole complex plane.

$$\zeta(z) = \frac{1}{z-1} + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \gamma_s (z-1)^s \qquad \gamma_s : \textit{Stieltjes constant}$$

Multiplying both sides by $z-1$,

$$(z-1)\zeta(z) = 1 + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \gamma_s (z-1)^{s+1} = 1 + \sum_{s=1}^{\infty} (-1)^{s-1} \frac{s\gamma_{s-1}}{s!} (z-1)^s$$

Replacing z with $1-z$,

$$(1-z-1)\zeta(1-z) = 1 + \sum_{s=1}^{\infty} (-1)^{s-1} \frac{s\gamma_{s-1}}{s!} (1-z-1)^s$$

$$= 1 + \sum_{s=1}^{\infty} (-1)^{2s-1} \frac{s\gamma_{s-1}}{s!} z^s$$

i.e.

$$-z\zeta(1-z) = 1 - \sum_{s=1}^{\infty} \frac{s\gamma_{s-1}}{s!} z^s$$

Note

The function $-z\zeta(1-z)$ does not have singular point on whole complex plane. So, the convergence radius of (4.1) is infinite.

8.5 Maclaurin Expansion by Hadamard Product

As seen in Section 3, when non-trivial zeros of $\zeta(z)$ are $z_k = x_k \pm i y_k \quad k=1,2,3,\cdots$, the function $-z\zeta(1-z)$ was factored asfollows.

$$-z\zeta(1-z) = \frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}} e^{\left(\log 2 - 1 - \frac{\gamma}{2}\right)z} \prod_{r=1}^{\infty} \left(1 - \frac{2x_r z}{x_r^2+y_r^2} + \frac{z^2}{x_r^2+y_r^2}\right) e^{\frac{2x_r z}{x_r^2+y_r^2}} \tag{3.1}$$

Each function which constitutes this is expanded to Maclaurin series as follows, respectively.

Maclaurin expansion of the reciprocal of gamma function

According to Formula 12.3.3 in " **12 Series Expansion of Gamma Function & the Reciprocal** " (**A la carte**) , when $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials,

$$\frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}} = 1 + \sum_{s=1}^{\infty} (-1)^s \frac{\beta_s}{(2s)!!} z^s \tag{5.g}$$

$$= 1 + \frac{z}{2!!}\psi_0\left(\frac{3}{2}\right) + \frac{z^2}{4!!}\left\{\psi_0^2\left(\frac{3}{2}\right) - \psi_1\left(\frac{3}{2}\right)\right\}$$

$$+ \frac{z^3}{6!!}\left\{\psi_0^3\left(\frac{3}{2}\right) - 3\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right) + \psi_2\left(\frac{3}{2}\right)\right\} + \cdots$$

where,

$$\beta_s = \sum_{k=1}^{s} (-1)^k B_{s,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \cdots, \psi_{s-1}\left(\frac{3}{2}\right)\right) \qquad s=1,2,3,\cdots$$

Maclaurin expansion of the reciprocal of exponential function

Exponential function of (3.1) is expanded to Maclaurin series as follows.

$$e^{\left(\log 2 - 1 - \frac{\gamma}{2}\right)z} = \sum_{s=1}^{\infty} \frac{z^s}{s!}\left(\log 2 - 1 - \frac{\gamma}{2}\right)^s \tag{5.e}$$

$$= 1 + \frac{z^1}{1!}\left(\log 2 - 1 - \frac{\gamma}{2}\right)^1 + \frac{z^2}{2!}\left(\log 2 - 1 - \frac{\gamma}{2}\right)^2 + \frac{z^3}{3!}\left(\log 2 - 1 - \frac{\gamma}{2}\right)^3 + \cdots$$

Maclaurin expansion of non-trivial zeros

According to Formula 3.6.1 in " **03 Vieta's Formulas in Infinite-degree Equation** " (**Infinite-degree Equation**) , the non-trivial zeros of (3.1) is expanded to Maclaurin series as follows.

$$\prod_{r=1}^{\infty}\left(1 - \frac{2x_r z}{x_r^2+y_r^2} + \frac{z^2}{x_r^2+y_r^2}\right) e^{\frac{2x_r z}{x_r^2+y_r^2}} = 1 - 0z - \left\{\frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^2 - \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2}\right\}z^2$$

$$-\left\{\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^3 - \sum_{r_1=1}^{\infty}\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2} + \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1(x_{r_1}+x_{r_2})}{(x_{r_1}^2+y_{r_1}^2)(x_{r_2}^2+y_{r_2}^2)}\right\}z^3$$

$$- \cdots \tag{5.z}$$

Maclaurin series of $-z\zeta(1-z)$

Maclaurin series of $-z\zeta(1-z)$ consists of the product of (5.g), (5.e) and (5.z). That is,

$$-z\zeta(1-z)=\left(1+\sum_{s=1}^{\infty}(-1)^s\frac{\beta_s}{(2s)!!}z^s\right)\times\sum_{s=1}^{\infty}\frac{z^s}{s!}\left(\log 2-1-\frac{\gamma}{2}\right)^s$$

$$\times\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)e^{\frac{2x_r z}{x_r^2+y_r^2}}$$

According to Formula 1.1.2 in " **01 Power of Infinite Series** " (**Infinite-degree Equation**), The product of the three series of (3.1) is expressed as follows. (Where, $a_0=b_0=c_0=1$)

$$\left(\sum_{r=0}^{\infty}a_r z^r\right)\left(\sum_{r=0}^{\infty}b_r z^r\right)\left(\sum_{r=0}^{\infty}c_r z^r\right)=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}a_{r-s}b_{s-t}c_t z^r$$

$$=1+z^1(a_1+b_1+c_1)$$
$$+z^2(a_2+b_2+c_2+a_1b_1+b_1c_1+c_1a_1)$$
$$+z^3(a_3+b_3+c_3+a_2b_1+a_2c_1+b_2c_1+b_2a_1+c_2a_1+c_2b_1+a_1b_1c_1)$$
$$+$$
$$\vdots$$

So, put

$$a_1=\frac{1}{2!!}\psi_0\left(\frac{3}{2}\right),\quad a_2=\frac{1}{4!!}\left\{\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right\}$$
$$,\ a_3=\frac{1}{6!!}\left\{\psi_0^3\left(\frac{3}{2}\right)-3\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right)+\psi_2\left(\frac{3}{2}\right)\right\}$$

$$b_1=\frac{1}{1!}\left(\log 2-1-\frac{\gamma}{2}\right)^1,\quad b_2=\frac{1}{2!}\left(\log 2-1-\frac{\gamma}{2}\right)^2,\quad b_3=\frac{1}{3!}\left(\log 2-1-\frac{\gamma}{2}\right)^3$$

$$c_1=0,\quad c_2=-\frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2+\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}\ ,$$

$$c_3=-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1(x_{r1}+x_{r2})}{(x_{r1}^2+y_{r1}^2)(x_{r2}^2+y_{r2}^2)}$$

Then,

$$-z\zeta(1-z)=1+z^1(a_1+b_1+c_1)$$
$$+z^2(a_2+b_2+c_2+a_1b_1+b_1c_1+c_1a_1)$$
$$+z^3(a_3+b_3+c_3+a_2b_1+a_2c_1+b_2c_1+b_2a_1+c_2a_1+c_2b_1+a_1b_1c_1)$$
$$+$$
$$\vdots$$

Since $c_1=0$

$$-z\zeta(1-z) = 1 + z^1(a_1+b_1)$$
$$+ z^2(a_2+b_2+c_2+a_1b_1)$$
$$+ z^3(a_3+b_3+c_3+a_2b_1+b_2a_1+c_2a_1+c_2b_1) \tag{5.0}$$
$$+$$
$$\vdots$$

Coefficients of the 1st, 2nd, 3rd degree

Comparing Formula 8.4.1 and (5.0) , we obtain the following formula.

Formula 8.5.1

When γ is Euler-Mascheroni constant, γ_s is Stieltjes constant, $\psi_n(z)$ is the polygamma function and non-trivial zeros of Riemann zeta function are x_n+iy_n $n=1,2,3,\cdots$, the following expressions hold.

$$-\gamma_0 = \frac{1}{2!!}\psi_0\left(\frac{3}{2}\right) + \log 2 - 1 - \frac{\gamma}{2} \tag{5.1$_1$}$$

$$-\gamma_1 = \frac{\gamma_0^2}{2} - \frac{1}{4!!}\psi_1\left(\frac{3}{2}\right) - \frac{1}{2}\sum_{r1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2 + \sum_{r1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2} \tag{5.1$_2$}$$

$$-\frac{\gamma_2}{2} = \frac{\gamma_0^3}{3} + \gamma_0\gamma_1 + \frac{1}{6!!}\psi_2\left(\frac{3}{2}\right)$$
$$-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r1}^2+y_{r1}^2}\right)^3 + \sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1(x_{r1}+x_{r2})}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$
(5.1$_3$)

Proof

From Formula 8.4.1 ,

$$-z\zeta(1-z) = 1 - \frac{\gamma_0}{0!}z^1 - \frac{\gamma_1}{1!}z^2 - \frac{\gamma_2}{2!}z^3 - \frac{\gamma_3}{3!}z^4 - \cdots$$

Comparing these coefficients with the coefficients of (5.0) ,

$$-\gamma_0 = a_1+b_1 \tag{1}$$
$$-\gamma_1 = a_2+b_2+a_1b_1 + c_2 \tag{2}$$
$$-\frac{\gamma_2}{2} = a_3+b_3+a_2b_1+b_2a_1+c_2(a_1+b_1) + c_3 \tag{3w}$$

From (2) ,

$$c_2 = -\gamma_1 - (a_2+b_2) - a_1b_1$$

Substituting this and (1) for (3w) ,

$$-\frac{\gamma_2}{2} = a_3+b_3+a_2b_1+b_2a_1+\{-\gamma_1 - (a_2+b_2) - a_1b_1\}(-\gamma_0) + c_3$$

i.e.

$$-\frac{\gamma_2}{2} = a_3+b_3+a_2b_1+b_2a_1+\gamma_0\gamma_1+\gamma_0(a_2+b_2)+\gamma_0 a_1b_1+c_3 \tag{3}$$

Substituting the above $a_1 \sim c_3$ for (1), (2) and (3) respectively,

Coefficient of the 1st degree

$$-\gamma_0 = a_1+b_1 = \frac{1}{2^1 1!}\psi_0\left(\frac{3}{2}\right) + \log 2 - 1 - \frac{\gamma}{2}$$

i.e.

$$-\gamma_0 = \frac{1}{2!!}\psi_0\left(\frac{3}{2}\right) + \log 2 - 1 - \frac{\gamma}{2} \tag{5.1$_1$}$$

Coefficient of the 2nd degree

$$\begin{aligned}-\gamma_1 &= a_2+b_2+c_2+a_1b_1 \\ &= \frac{1}{4!!}\left\{\psi_0^2\left(\frac{3}{2}\right) - \psi_1\left(\frac{3}{2}\right)\right\} + \frac{1}{2!}\left(\log 2 - 1 - \frac{\gamma}{2}\right)^2 \\ &\quad - \frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2 + \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2} \\ &\quad + \frac{1}{2!!}\psi_0\left(\frac{3}{2}\right)\left(\log 2 - 1 - \frac{\gamma}{2}\right)^1\end{aligned}$$

i.e.

$$\begin{aligned}-\gamma_1 = &\frac{1}{4!!}\left\{\psi_0^2\left(\frac{3}{2}\right) - \psi_1\left(\frac{3}{2}\right)\right\} + \frac{1}{2}\left(\log 2 - 1 - \frac{\gamma}{2}\right)^2 + \frac{1}{2}\psi_0\left(\frac{3}{2}\right)\left(\log 2 - 1 - \frac{\gamma}{2}\right) \\ &- \frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2 + \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}\end{aligned}$$

Here, from (1) ,

$$\log 2 - 1 - \frac{\gamma}{2} = -\gamma_0 - \frac{1}{2}\psi_0\left(\frac{3}{2}\right) \tag{5.1$_1$'}$$

Using this for the 2nd and the 3rd term of the right hand,

$$\begin{aligned}&\frac{1}{2}\left(\log 2 - 1 - \frac{\gamma}{2}\right)^2 + \frac{1}{2}\psi_0\left(\frac{3}{2}\right)\left(\log 2 - 1 - \frac{\gamma}{2}\right) \\ &\quad = \frac{1}{2}\left\{\gamma_0 + \frac{1}{2}\psi_0\left(\frac{3}{2}\right)\right\}^2 - \frac{1}{2}\psi_0\left(\frac{3}{2}\right)\left\{\gamma_0 + \frac{1}{2}\psi_0\left(\frac{3}{2}\right)\right\} \\ &\quad = \frac{\gamma_0^2}{2} - \frac{1}{8}\psi_0^2\left(\frac{3}{2}\right)\end{aligned}$$

Substituting this for the right hand of the above,

$$\begin{aligned}-\gamma_1 = &\frac{1}{8}\psi_0^2\left(\frac{3}{2}\right) - \frac{1}{8}\psi_1\left(\frac{3}{2}\right) + \frac{\gamma_0^2}{2} - \frac{1}{8}\psi_0^2\left(\frac{3}{2}\right) \\ &- \frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2 + \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}\end{aligned}$$

$$= \frac{\gamma_0^2}{2} - \frac{1}{4!!}\psi_1\left(\frac{3}{2}\right) - \frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^2 + \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2} \tag{5.1$_2$}$$

Coefficient of the 3rd degree

$$-\frac{\gamma_2}{2} = a_3+b_3+a_2b_1+b_2a_1+\gamma_0\gamma_1+\gamma_0(a_2+b_2)+\gamma_0 a_1b_1+c_3 \tag{3}$$

Substituting the above $a_1 \sim c_3$ for this respectively,

$$\begin{aligned}-\frac{\gamma_2}{2} = {} & \frac{1}{3!}\left(\log 2-1-\frac{\gamma}{2}\right)^3+\frac{1}{2!}\left\{\frac{1}{2}\psi_0\left(\frac{3}{2}\right)+\gamma_0\right\}\left(\log 2-1-\frac{\gamma}{2}\right)^2 \\ & +\left\{\frac{1}{4!!}\left(\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right)+\frac{\gamma_0}{2}\psi_0\left(\frac{3}{2}\right)\right\}\left(\log 2-1-\frac{\gamma}{2}\right) \\ & +\frac{1}{6!!}\left\{\psi_0^3\left(\frac{3}{2}\right)-3\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right)+\psi_2\left(\frac{3}{2}\right)\right\} \\ & +\frac{\gamma_0}{4!!}\left\{\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right\}+\gamma_0\gamma_1 \\ & -\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^3+\sum_{r_1=1}^{\infty}\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1(x_{r_1}+x_{r_2})}{(x_{r_1}^2+y_{r_1}^2)(x_{r_2}^2+y_{r_2}^2)}\end{aligned}$$

Here, using **(5.1$_1$')** for the 2nd and the 3rd term of the right hand,

$$\begin{aligned}& \frac{1}{3!}\left(\log 2-1-\frac{\gamma}{2}\right)^3+\frac{1}{2!}\left\{\frac{1}{2}\psi_0\left(\frac{3}{2}\right)+\gamma_0\right\}\left(\log 2-1-\frac{\gamma}{2}\right)^2 \\ & \qquad +\left\{\frac{1}{4!!}\left(\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right)+\frac{\gamma_0}{2}\psi_0\left(\frac{3}{2}\right)\right\}\left(\log 2-1-\frac{\gamma}{2}\right) \\ & = -\frac{1}{3!}\left\{\gamma_0+\frac{1}{2}\psi_0\left(\frac{3}{2}\right)\right\}^3+\frac{1}{2!}\left\{\frac{1}{2}\psi_0\left(\frac{3}{2}\right)+\gamma_0\right\}\left\{\gamma_0+\frac{1}{2}\psi_0\left(\frac{3}{2}\right)\right\}^2 \\ & \qquad -\left\{\frac{1}{4!!}\left(\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right)+\frac{\gamma_0}{2}\psi_0\left(\frac{3}{2}\right)\right\}\left\{\gamma_0+\frac{1}{2}\psi_0\left(\frac{3}{2}\right)\right\} \\ & = \frac{1}{3}\gamma_0^3+\frac{\gamma_0^2}{2}\psi_0\left(\frac{3}{2}\right)+\frac{\gamma_0}{4}\psi_0^2\left(\frac{3}{2}\right)+\frac{1}{24}\psi_0^3\left(\frac{3}{2}\right) \\ & \qquad -\frac{\gamma_0}{8}\psi_0^2\left(\frac{3}{2}\right)+\frac{\gamma_0}{8}\psi_1\left(\frac{3}{2}\right)-\frac{1}{16}\psi_0^3\left(\frac{3}{2}\right)+\frac{1}{16}\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right) \\ & \qquad -\frac{\gamma_0^2}{2}\psi_0\left(\frac{3}{2}\right)-\frac{\gamma_0}{4}\psi_0^2\left(\frac{3}{2}\right) \\ & = \frac{\gamma_0^3}{3}-\frac{\gamma_0}{8}\psi_0^2\left(\frac{3}{2}\right)-\frac{1}{48}\psi_0^3\left(\frac{3}{2}\right)+\frac{\gamma_0}{8}\psi_1\left(\frac{3}{2}\right)+\frac{1}{16}\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right)\end{aligned}$$

Substituting this for the right hand of the above,

$$-\frac{\gamma_2}{2} = \frac{\gamma_0^3}{3}-\frac{\gamma_0}{8}\psi_0^2\left(\frac{3}{2}\right)-\frac{1}{48}\psi_0^3\left(\frac{3}{2}\right)+\frac{\gamma_0}{8}\psi_1\left(\frac{3}{2}\right)+\frac{1}{16}\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right)$$

$$+\frac{1}{48}\psi_0^3\left(\frac{3}{2}\right)-\frac{1}{16}\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right)+\frac{1}{48}\psi_2\left(\frac{3}{2}\right)$$

$$+\frac{\gamma_0}{8}\psi_0^2\left(\frac{3}{2}\right)-\frac{\gamma_0}{8}\psi_1\left(\frac{3}{2}\right)+\gamma_0\gamma_1$$

$$-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$

$$=\frac{\gamma_0^3}{3}+\gamma_0\gamma_1+\frac{1}{6!!}\psi_2\left(\frac{3}{2}\right)$$

$$-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$

(5.1_3)

8.6 Proposition equivalent to the Riemann Hypothesis

From Formula 8.5.1 in the previous section, a proposition equivalent to the Riemann hypothesis is obtained

Proposition 8.6.1

When γ_s is Stieltjes constant, $\psi_n(z)$ is the polygamma function and non-trivial zeros of Riemann zeta function are $1/2+iy_r \quad r=1,2,3,\cdots$, the following expressions hold.

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = \gamma_0 - \frac{1}{2}\log\pi + \frac{1}{2}\psi_0\left(\frac{3}{2}\right) \tag{6.1$_1$}$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^2 = \gamma_0^2 + 2\gamma_0 + 2\gamma_1 - \log\pi + \psi_0\left(\frac{3}{2}\right) - \frac{1}{4}\psi_1\left(\frac{3}{2}\right) \tag{6.1$_2$}$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^3 = \gamma_0^3 + 3\gamma_0^2 + 6\gamma_0 + 6\gamma_1 + 3\gamma_0\gamma_1 + \frac{3}{2}\gamma_2 - 3\log\pi$$
$$+3\psi_0\left(\frac{3}{2}\right) - \frac{3}{4}\psi_1\left(\frac{3}{2}\right) + \frac{1}{16}\psi_2\left(\frac{3}{2}\right) \tag{6.1$_3$}$$

Proof

If Riemann hypothesis is true, the following expression holds from Thorem 8.2.3 ,

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = 1 + \frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2} \tag{1.2'}$$

On the other hand, from the previous section,

$$\log 2 - 1 - \frac{\gamma}{2} = -\gamma_0 - \frac{1}{2}\psi_0\left(\frac{3}{2}\right) \tag{5.1$_1$'}$$

Substituting the latter for the former,

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = \gamma_0 - \frac{\log\pi}{2} + \frac{1}{2}\psi_0\left(\frac{3}{2}\right) \tag{6.1$_1$}$$

Next, substituting $x_{r1} = 1/2$ for (5.1$_2$) in Formula 8.5.1 ,

$$-\gamma_1 = \frac{\gamma_0^2}{2} - \frac{1}{4!!}\psi_1\left(\frac{3}{2}\right) - \frac{1}{2}\sum_{r1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2 + \sum_{r1=1}^{\infty}\frac{1}{1/4+y_{r1}^2}$$

Substituting (1.2') for this and arranging it,

$$\sum_{r1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2 = \gamma_0^2 + 2\gamma_1 - \frac{1}{4}\psi_1\left(\frac{3}{2}\right) + 2\left(1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2}\right)$$

Further, using (5.1$_1$') ,

$$\sum_{r1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2 = \gamma_0^2 + 2\gamma_0 + 2\gamma_1 - \log\pi + \psi_0\left(\frac{3}{2}\right) - \frac{1}{4}\psi_1\left(\frac{3}{2}\right) \tag{6.1$_2$}$$

According to Corollary 3.6.1 in " **03 Vieta's Formulas in Infinite-degree Equation** " (**Infinite-degree Equation**) , if $x_{r1} = 1/2$,

$$-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1(x_{r1}+x_{r2})}{(x_{r1}^2+y_{r1}^2)(x_{r2}^2+y_{r2}^2)}$$

$$=-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2$$

Substituting this for (5.1_3) in Formula 8.5.1 ,

$$-\frac{\gamma_2}{2}=\frac{\gamma_0^3}{3}+\gamma_0\gamma_1+\frac{1}{6!!}\psi_2\left(\frac{3}{2}\right)-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2$$

Further, substituting (6.1_2) for this,

$$-\frac{\gamma_2}{2}=\frac{\gamma_0^3}{3}+\gamma_0\gamma_1+\frac{1}{6!!}\psi_2\left(\frac{3}{2}\right)-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3$$
$$+\gamma_0^2+2\gamma_0+2\gamma_1-\log\pi+\psi_0\left(\frac{3}{2}\right)-\frac{1}{4}\psi_1\left(\frac{3}{2}\right)$$

From this,

$$\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3=\gamma_0^3+3\gamma_0^2+6\gamma_0+6\gamma_1+3\gamma_0\gamma_1+\frac{3}{2}\gamma_2-3\log\pi$$
$$+3\psi_0\left(\frac{3}{2}\right)-\frac{3}{4}\psi_1\left(\frac{3}{2}\right)+\frac{1}{16}\psi_2\left(\frac{3}{2}\right)\qquad(6.1_3)$$

Numerical Calculation

(6.1_2) & (6.1_3) were calculated with the formula manipulation software *Mathematica* , it was as follows

```
y_n_ := Im[ZetaZero[n]]   γ_s_ := StieltjesGamma[s]   ψ_k_[p_] := PolyGamma[k, p]
```

2nd degree

```
f2[m_] := Sum[(1/(1/4 + y_r^2))^2, {r, 1, m}]
g2 := γ_0^2 + 2 γ_0 + 2 γ_1 - Log[π] + ψ_0[3/2] - 1/4 ψ_1[3/2]
N[{f2[1095]    , g2}]
{0.0000371006 , 0.0000371006}
```

3rd degree

```
f3[m_] := Sum[(1/(1/4 + y_r^2))^3, {r, 1, m}]
g3 := γ_0^3 + 3 γ_0^2 + 6 γ_0 + 6 γ_1 + 3 γ_0 γ_1 + 3/2 γ_2 - 3 Log[π]
          + 3 ψ_0[3/2] - 3/4 ψ_1[3/2] + 1/16 ψ_2[3/2]

SetPrecision[f3[15000], 25]          SetPrecision[g3, 25]
1.436778602886913134053 32×10^-7     1.4367786028869177×10^-7
```

(6.1_1) is the same as the following expression in Theorem 8.2.3 .

$$\sum_{r=1}^{\infty} \frac{1}{1/4+y_r^2} = 1+\frac{\gamma}{2} - \log 2 - \frac{\log \pi}{2} = 0.0230957\cdots \tag{1.2'}$$

So, (6.1_1) is equivalent to the Riemann hypothesis according Theorem 8.2.3 .

Since (6.1_2) & (6.1_3) are derived using (1.2'), these are also equivalent to the Riemann hypothesis. However, (6.1_2) & (6.1_3) are considerably faster than (6.1_1) at convergence speed. (6.1_3) was calculated up to 15000 terms. As the result, 15 significant figures were obtained. If this is calculated on a high-speed computer, both sides will approach endlessly. But even so, it only serves as circumstantial evidence for the Riemann hypothesis.

2017.12.01

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Alien's Mathematics