2 Formulas for Dirichlet Beta

Formulas for Beta obtained in " **1 Dirichlet Beta Generating Functions** " were automorphisms those were expressed by the lower betas. In this chapter, removing the lower betas, we obtain the explicit formulas. In addition, Dirichlet Beta Function is defined as follows.

$$\beta(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^x}$$

2.1 Formulas for Beta at natural number

Formula for Dirichlet Beta at natural number can be obtained from sech family. this is simple and the convergence is fast.

Formula 2.1.1

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \cdots$ are Euler Numbers and n is a natural number, the following expressin holds for $0<x\leqq \pi/2$.

$$\beta(n) = \sum_{r=0}^{\infty}\sum_{s=0}^{n-1} \frac{(2r+1)^s x^s}{s!}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^n} + \frac{x^n}{2}\sum_{r=0}^{\infty}\binom{-n}{2r}\frac{E_{2r}x^{2r}}{(n+2r)!}$$

Especially when $x=1/2,\ 1$,

$$\beta(n) = \sum_{r=0}^{\infty}\sum_{s=0}^{n-1} \frac{(2r+1)^s}{s!\,2^s}\frac{(-1)^r e^{-(r+1/2)}}{(2r+1)^n} + \frac{1}{2^{n+1}}\sum_{r=0}^{\infty}\binom{-n}{2r}\frac{E_{2r}}{(n+2r)!\,2^{2r}}$$

$$\beta(n) = \sum_{r=0}^{\infty}\sum_{s=0}^{n-1} \frac{(2r+1)^s}{s!}\frac{(-1)^r e^{-(2r+1)}}{(2r+1)^n} + \frac{1}{2}\sum_{r=0}^{\infty}\binom{-n}{2r}\frac{E_{2r}}{(n+2r)!}$$

Proof

We obtained the following Dirichlet Beta in Formula 1.1.3 in " **1 Dirichlet Beta Generating Functions ".**

$$\beta(1) = \sum_{r=0}^{\infty}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^1} + \frac{1}{2}\sum_{r=0}^{\infty}\frac{E_{2r}x^{2r+1}}{(2r+1)\,!}$$

$$\beta(2) = \sum_{r=0}^{\infty}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^2} - \frac{1}{2}\sum_{r=0}^{\infty}\frac{E_{2r}x^{2r+2}}{(2r+2)\,!} + \frac{x^1}{1!}\beta(1)$$

$$\beta(3) = \sum_{r=0}^{\infty}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^3} + \frac{1}{2}\sum_{r=0}^{\infty}\frac{E_{2r}x^{2r+3}}{(2r+3)\,!} + \frac{x^1}{1!}\beta(2) - \frac{x^2}{2!}\beta(1)$$

$$\beta(4) = \sum_{r=0}^{\infty}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^4} - \frac{1}{2}\sum_{r=0}^{\infty}\frac{E_{2r}x^{2r+4}}{(2r+4)!} + \frac{x^1}{1!}\beta(3) - \frac{x^2}{2!}\beta(2) + \frac{x^3}{3!}\beta(1)$$

$$\vdots$$

We transform these as follows for simplification.

$$B_n = \beta(n)\ , \quad f_n = \sum_{r=0}^{\infty}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^n} + \frac{(-1)^{n-1}}{2}\sum_{r=0}^{\infty}\frac{E_{2r}x^{n+2r}}{(n+2r)!}$$

Then ,

$$B_1 = f_1$$

$$B_2 = f_2 + \frac{x^1}{1!}B_1$$

$$B_3 = f_3 + \frac{x^1}{1!}B_2 - \frac{x^2}{2!}B_1$$

$$B_4 = f_4 + \frac{x^1}{1!}B_3 - \frac{x^2}{2!}B_2 + \frac{x^3}{3!}B_1$$

$$\vdots$$

Substituting B_k for the below one by one ,

$$B_1 = C_0 x^0 f_1$$

$$B_2 = C_0 x^0 f_2 + C_1 x^1 f_1$$

$$B_3 = C_0 x^0 f_3 + C_1 x^1 f_2 + C_2 x^2 f_1$$

$$B_4 = C_0 x^0 f_4 + C_1 x^1 f_3 + C_2 x^2 f_2 + C_3 x^3 f_1$$

$$\vdots$$

$$B_n = \sum_{s=0}^{n-1} C_s x^s f_{n-s}$$

Where, C_s are as follows.

$$C_0 = \frac{1}{0!}, \quad C_1 = \frac{1}{1!}, \quad C_2 = -\frac{1}{2!} + \frac{1}{1!1!}$$

$$C_3 = \frac{1}{3!} - \frac{1}{2!1!} - \frac{1}{1!2!} + \frac{1}{1!1!1!}$$

$$C_4 = -\frac{1}{4!} + \frac{1}{3!1!} + \frac{1}{1!3!} + \frac{1}{2!2!} - \left(\frac{1}{2!1!1!} + \frac{1}{1!2!1!} + \frac{1}{1!1!2!} \right) + \frac{1}{1!1!1!1!}$$

$$\vdots$$

Although these coefficients look complicated, in fact, it is simply calculated with $C_s = 1/s!$.

Then B_n becomes as follows.

$$B_n = \sum_{s=0}^{n-1} \frac{x^s}{s!} f_{n-s}$$

Rewriting this with the previous notations,

$$\beta(n) = \sum_{s=0}^{n-1} \frac{x^s}{s!} \left\{ \sum_{r=0}^{\infty} (-1)^r \frac{e^{-(2r+1)x}}{(2r+1)^{n-s}} + \frac{(-1)^{n-s-1}}{2} \sum_{r=0}^{\infty} \frac{E_{2r} x^{n-s+2r}}{(2r+n-s)!} \right\}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{n-1} (-1)^r \frac{x^s e^{-(2r+1)x}}{s!\,(2r+1)^{n-s}} + \frac{(-1)^{n-s-1}}{2} \sum_{r=0}^{\infty} \sum_{s=0}^{n-1} \frac{1}{s!} \frac{E_{2r} x^{n+2r}}{(n+2r-s)!}$$

i.e.

$$\beta(n) = \sum_{r=0}^{\infty} \sum_{s=0}^{n-1} \frac{(2r+1)^s}{s!} \frac{(-1)^r}{(2r+1)^n e^{2r+1}} - \frac{(-1)^n}{2} \sum_{r=0}^{\infty} \sum_{s=0}^{n-1} \frac{(-1)^s}{s!\,(2r+n-s)!} E_{2r}$$

Here, according to the " 岩波 数学公式 II " p11,

$$\sum_{s=0}^{m}(-1)^s\binom{n}{s}=(-1)^m\binom{n-1}{m}\qquad m\leqq n-1$$

applying this,

$$\sum_{s=0}^{n-1}(-1)^s\binom{n+2r}{s}=(-1)^{n-1}\binom{n+2r-1}{n-1}$$

Furthermore,

$$\binom{p-1+r}{p-1}=(-1)^r\binom{-p}{r}$$

applying this,

$$\binom{n-1+2r}{n-1}=(-1)^{2r}\binom{-n}{2r}$$

Then ,

$$\sum_{s=0}^{n-1}(-1)^s\binom{n+2r}{s}=(-1)^{n-1+2r}\binom{-n}{2r}$$

Substituting this for the above, we obtain

$$\beta(n)=\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{(2r+1)^s x^s}{s!}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^n}+\frac{x^n}{2}\sum_{r=0}^{\infty}\binom{-n}{2r}\frac{E_{2r}x^{2r}}{(n+2r)!}$$

Note

Although this formula is the simplest at $x=1$, the convergence is the fastst around $x=1/2$

Example1

$$\beta(4)=\sum_{r=0}^{\infty}\left\{1+\frac{2r+1}{1!\,2^1}+\frac{(2r+1)^2}{2!\,2^2}+\frac{(2r+1)^3}{3!\,2^3}\right\}\frac{(-1)^r e^{-r-\frac{1}{2}}}{(2r+1)^4}+\frac{1}{2^5}\sum_{r=0}^{\infty}\binom{-4}{2r}\frac{E_{2r}}{(4+2r)!\,2^{2r}}$$

$$\beta(4)=\sum_{r=0}^{\infty}\left\{1+\frac{2r+1}{1!}+\frac{(2r+1)^2}{2!}+\frac{(2r+1)^3}{3!}\right\}\frac{(-1)^r e^{-2r-1}}{(2r+1)^4}+\frac{1}{2}\sum_{r=0}^{\infty}\binom{-4}{2r}\frac{E_{2r}}{(4+2r)!}$$

Example2 $\beta(8)$

According to the formula at $x=1/2$ this is calculated. As the result of calculating the series to the 7 th term, the significant 10 digits were obtained.

```
m = 7;
```

$$b[n\_]:=\sum_{r=0}^{m}\sum_{s=0}^{n-1}\frac{(2r+1)^s}{s!\,2^s}\frac{(-1)^r e^{-(r+1/2)}}{(2r+1)^n}+\frac{1}{2^{n+1}}\sum_{r=0}^{m}\frac{\text{Binomial}[-n,2r]\,\text{EulerE}[2r]}{(n+2r)!\,2^{2r}}$$

```
N[b[8], 10]        N[DirichletL [4, 2, 8], 10]
0.9998499902       0.9998499902
```

2.2 Formulas for Beta at even number

Formula 2.2.1

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \cdots$ are Euler Numbers and n is a natural number, the following expressin holds for $0<x \leqq \pi/2$.

$$\beta(2n) = \sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^{2n}} - \frac{(-1)^n}{2}\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{n-1}\binom{2n+2r}{2s}E_{2s}\right\}\frac{|E_{2r}|x^{2n+2r}}{(2n+2r)!} \tag{1.1}$$

Especially when $x=\pi/2$,

$$\beta(2n) = \frac{(-1)^{n-1}}{2}\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{n-1}\binom{2n+2r}{2s}E_{2s}\right\}\frac{|E_{2r}|}{(2n+2r)!}\left(\frac{\pi}{2}\right)^{2n+2r} \tag{1.2}$$

Proof

We obtained the following expressions in Formula 1.2.3 in **1.2** .

$$\beta(2)=\sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^2}+\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|x^{2r+2}}{(2r+2)!}$$

$$\beta(4)=\sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^4}-\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|x^{2r+4}}{(2r+4)!}+\frac{x^2}{2!}\beta(2)$$

$$\beta(6)=\sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^6}+\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|x^{2r+6}}{(2r+6)!}+\frac{x^2}{2!}\beta(4)-\frac{x^4}{4!}\beta(2)$$

$$\beta(8)=\sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^8}-\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|x^{2r+8}}{(2r+8)!}+\frac{x^2}{2!}\beta(6)-\frac{x^4}{4!}\beta(4)+\frac{x^6}{6!}\beta(2)$$

$$\vdots$$

Substituting $\beta(k)$ for the below one by one , we obtain

$$\beta(2n)=\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^rC_s\cos\{(2r+1)x\}\,x^{2s}}{(2r+1)^{2n-2s}} - \frac{(-1)^{n-s}}{2}\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{C_s|E_{2r}|x^{2n+2r}}{(2n-2s+2r)!}$$

Where, C_s are rational numberes as follows.

$$C_0=\frac{1}{0!},\ C_1=\frac{1}{2!},\ C_2=-\frac{1}{4!}+\frac{1}{2!2!},\ C_3=\frac{1}{6!}-\left(\frac{1}{4!2!}+\frac{1}{2!4!}\right)+\frac{1}{2!2!2!},$$

$$C_4=-\frac{1}{8!}+\left(\frac{1}{6!2!}+\frac{1}{4!4!}+\frac{1}{2!6!}\right)-\left(\frac{1}{4!2!2!}+\frac{1}{2!4!2!}+\frac{1}{2!2!4!}\right)+\frac{1}{2!2!2!2!}$$

$$\vdots$$

According to Mr. Sugimoto , these are given by the following expression.

$$C_s = (-1)^s\frac{E_{2s}}{(2s)!} = \frac{|E_{2s}|}{(2s)!}$$

Thus, substituting this for the above, we obtain the desired expressin.

Example1

$$\beta(4)=-\frac{1}{2}\sum_{r=0}^{\infty}\left\{\binom{4+2r}{0}E_0+\binom{4+2r}{2}E_2\right\}\frac{|E_{2r}|}{(4+2r)!}\left(\frac{\pi}{2}\right)^{4+2r}$$

$$\beta(6)=\frac{1}{2}\sum_{r=0}^{\infty}\left\{\binom{6+2r}{0}E_0+\binom{6+2r}{2}E_2+\binom{6+2r}{4}E_4\right\}\frac{|E_{2r}|}{(6+2r)!}\left(\frac{\pi}{2}\right)^{6+2r}$$

Example2 $\beta(6)$

According to (1.2) this is calculated. As the result of calculating the 6000 terms, the significant 4 digits were obtained.

```
m = 6000;
```

$$b[n\_] := \frac{(-1)^{n-1}}{2}\sum_{r=0}^{m}\left|\sum_{s=0}^{n-1}\text{Binomial}[2n+2r,2s]\,\text{EulerE}[2s]\right|\frac{\text{Abs}[\text{EulerE}[2r]]}{(2n+2r)!}\left(\frac{\pi}{2}\right)^{2n+2r}$$

```
N[b[3]]          N[DirichletL[4, 2, 6]]
0.998602         0.998685
```

As seen in Example2, the convergence of (1.2) is vary slow. Then let us focus on the following expression.

$$\beta(2)=\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|}{(2r+2)!}\left(\frac{\pi}{2}\right)^{2r+2}=Catalan\quad(=0.915965\cdots)$$

If (1.2) is transformed as follows using this, the convergence is faster.

Formula 2.2.1'

$$\beta(2n)=\frac{(-1)^{n-1}}{2}\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{n-2}\binom{2n+2r}{2s}E_{2s}\right\}\frac{|E_{2r}|}{(2n+2r)!}\left(\frac{\pi}{2}\right)^{2n+2r}$$

$$+(-1)^{n-1}\frac{E_{2n-2}}{(2n-2)!}\left(\frac{\pi}{2}\right)^{2n-2}Catalan\qquad(=0.915965\cdots)\qquad(1.2')$$

Example2' $\beta(6)$

The same calculation as Example2 was performed by (1.2'). As the result of calculating the 70 terms, the significant 6 digits were obtained.

```
m = 70;
```

$$b[n\_] := \frac{(-1)^{n-1}}{2}\sum_{r=0}^{m}\left|\sum_{s=0}^{n-2}\text{Binomial}[2n+2r,2s]\,\text{EulerE}[2s]\right|\frac{\text{Abs}[\text{EulerE}[2r]]}{(2n+2r)!}\left(\frac{\pi}{2}\right)^{2n+2r}+(-1)^{n-1}\frac{\text{EulerE}[2n-2]}{(2n-2)!}\left(\frac{\pi}{2}\right)^{2n-2}\text{Catalan}$$

```
N[b[3]]          N[DirichletL[4, 2, 6]]
0.998685         0.998685
```

Formula 2.2.2

Let Bernoulli numbers and Euler numbers are as follows.

$$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$$
$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$$

Then, the following expression holds for $0<x\leq\pi/2$.

$$\beta(2n) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^{2n+1}}$$
$$+(-1)^n\frac{x^{2n}}{2}\sum_{r=1}^{\infty}\left\{\sum_{s=0}^{n-1}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!\,(2n+1+2r-2s)!}\right\}\frac{|E_{2r}|\,x^{2r}}{2r}$$

Proof

We obtained the following expressions in Formula 1.2.3 in **1.2** .

$$\beta(2) = \frac{1}{x}\sum_{r=0}^{\infty}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^3}+\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|\,x^{2r+2}}{(2r+3)!}$$

$$\beta(4) = \frac{1}{x}\sum_{r=0}^{\infty}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^5}-\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|\,x^{2r+4}}{(2r+5)!}+\frac{x^2}{3!}\beta(2)$$

$$\beta(6) = \frac{1}{x}\sum_{r=0}^{\infty}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^7}+\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|\,x^{2r+6}}{(2r+7)!}+\frac{x^2}{3!}\beta(4)-\frac{x^4}{5!}\beta(2)$$

$$\beta(8) = \frac{1}{x}\sum_{r=0}^{\infty}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^9}-\frac{1}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|\,x^{2r+8}}{(2r+9)!}+\frac{x^3}{3!}\beta(6)-\frac{x^5}{5!}\beta(4)+\frac{x^3}{3!}\beta(2)$$

$\vdots$

Substituting $\beta(k)$ for the below one by one , we obtain

$$\beta(2n) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^s C_s x^{2s}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^{2n+1-2s}}$$
$$+(-1)^n\frac{x^{2n}}{2}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{C_s}{(2n+1+2r-2s)!}\frac{|E_{2r}|\,x^{2r}}{2r}$$

Where, C_s are rational numberes as follows.

$$C_0 = -1\ ,\ C_1 = \frac{1}{3!}\ ,\ C_2 = \frac{1}{5!}-\frac{1}{3!3!}\ ,\ C_3 = \frac{1}{7!}-\left(\frac{1}{3!5!}+\frac{1}{5!3!}\right)+\frac{1}{3!3!3!}\ ,$$
$$C_4 = \frac{1}{9!}-\left(\frac{1}{3!7!}+\frac{1}{5!5!}+\frac{1}{7!3!}\right)+\left(\frac{1}{3!3!5!}+\frac{1}{3!5!3!}+\frac{1}{5!3!3!}\right)-\frac{1}{3!3!3!3!}$$

$\vdots$

In fact, these are the coefficients of Taylor series of csc x, and are given by 2^{2s} , factorial and Bernoulli number as follows.

$$C_s = \frac{2^{2s}-2}{(2s)!}B_{2s} \qquad s=0,1,2,\cdots$$

Thus, substituting this for the above, we obtain the desired expressin.

Note

When $x=1/4$, the convergence speed of this formula is the fastest in this chapter.

2.3 Formulas for Beta at odd number

Formula 2.3.1

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \cdots$ are Euler Numbers and n is a natural number, the following expressin holds for $0 < x \leqq \pi/2$.

$$\beta(2n-1) = \sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^{2n-1}}$$

Especially when $x=\pi/2$,

$$\beta(2n-1) = \frac{\pi}{4}\frac{|E_{2n-2}|}{(2n-2)!}\left(\frac{\pi}{2}\right)^{2n-2}$$

Proof

We obtained the following expressions in Formula 1.2.3 in **1.2** .

$$\beta(1) = \sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^1} \tag{1.1}$$

$$\beta(3) = \sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^3}+\frac{x^2}{2!}\beta(1)$$

$$\beta(5) = \sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^5}+\frac{x^2}{2!}\beta(3)-\frac{x^4}{4!}\beta(1)$$

$$\beta(7) = \sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^7}+\frac{x^2}{2!}\beta(5)-\frac{x^4}{4!}\beta(3)+\frac{x^6}{6!}\beta(1)$$

$$\vdots$$

Substituting $\beta(k)$ for the below one by one , we obtain

$$\beta(2n-1) = \sum_{r=0}^{\infty}\sum_{s=0}^{n-1}C_s\,x^{2s}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^{2n-1-2s}}$$

Where, C_s is the same as the coefficient in the Formula 2.2.1 and is given by

$$C_s = (-1)^s\frac{E_{2s}}{(2s)!} = \frac{|E_{2s}|}{(2s)!}$$

Then ,

$$\beta(2n-1) = \sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^{2n-1}}$$

Furthermore, this is rewritten as follows.

$$\beta(2n-1)=\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{n-2}\frac{|E_{2s}|\{(2r+1)x\}^{2s}}{(2s)!}+\frac{|E_{2n-2}|\{(2r+1)x\}^{2n-2}}{(2n-2)!}\right\}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^{2n-1}}$$

i.e.

$$\beta(2n-1) = \sum_{r=0}^{\infty}\sum_{s=0}^{n-2}\frac{|E_{2s}|\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^{2n-1}}$$

$$+\frac{|E_{2n-2}|x^{2n-2}}{(2n-2)!}\sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^1}$$

Here, from (1.1) and Madhava series ,

$$\sum_{r=0}^{\infty}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^1} = \beta(1) = \frac{\pi}{4}$$

Using this,

$$\beta(2n-1)=\sum_{r=0}^{\infty}\sum_{s=0}^{n-2}\frac{|E_{2s}|\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^{2n-1}}+\frac{\pi}{4}\frac{|E_{2n-2}|x^{2n-2}}{(2n-2)!}$$

Therefore, when $x=\pi/2$,

$$\beta(2n-1) = \frac{\pi}{4}\frac{|E_{2n-2}|}{(2n-2)!}\left(\frac{\pi}{2}\right)^{2n-2}$$

Formula 2.3.2

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers , the following expression holds for $0<x\leqq\pi/2$.

$$\beta(2n-1) = -\frac{1}{x}\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^sB_{2s}\left(2^{2s}-2\right)\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^{2n}}$$

Proof

We obtained the following expressions in Formula1.2.3 in **1.2** .

$$\beta(1) = \frac{1}{x}\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^2}$$

$$\beta(3) = \frac{1}{x}\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^4}+\frac{x^2}{3!}\beta(1)$$

$$\beta(5) = \frac{1}{x}\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^6}+\frac{x^2}{3!}\beta(3)-\frac{x^4}{5!}\beta(1)$$

$$\beta(7) = \frac{1}{x}\sum_{r=0}^{\infty}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^8}+\frac{x^2}{3!}\beta(5)-\frac{x^4}{5!}\beta(3)+\frac{x^6}{7!}\beta(1)$$

$$\vdots$$

Substituting $\beta(k)$ for the below one by one , we obtain

$$\beta(2n-1) = -\frac{1}{x}\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}(-1)^sC_s\{(2r+1)x\}^{2s}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^{2n}}$$

Where, C_s is the same as the coefficient in Formula 2.2.2 and is given by $C_s=\dfrac{2^{2s}-2}{(2s)!}B_{2s}$

Then ,

$$\beta(2n-1) = -\frac{1}{x}\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^sB_{2s}\left(2^{2s}-2\right)\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^{2n}}$$

2.4 Formulas for Beta at complex number

We can extend Formula 2.1.1 to the complex number easily.

Formula 2.4.1

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \cdots$ are Euler Numbers and p is a complex number such that $p \neq 1,0,-1,-2,\cdots$ and $\Gamma(p,x)=\int_x^{\infty} t^{p-1}e^{-t}dt$ is incomplete gamma function, the following expression holds for $x=u+vi$ s.t. $0<|x| \leqq \pi/2$, $u \geqq 0$.

$$\beta(p) = \sum_{r=0}^{\infty} \frac{\Gamma\{p,(2r+1)x\}}{\Gamma(p)} \frac{(-1)^r}{(2r+1)^p} + \frac{x^p}{2}\sum_{r=0}^{\infty}\binom{-p}{2r}\frac{E_{2r}x^{2r}}{\Gamma(p+1+2r)}$$

Especially when $x=1$,

$$\beta(p) = \sum_{r=0}^{\infty} \frac{\Gamma(p,2r+1)}{\Gamma(p)} \frac{(-1)^r}{(2r+1)^p} + \frac{1}{2}\sum_{r=0}^{\infty}\binom{-p}{2r}\frac{E_{2r}}{\Gamma(p+1+2r)}$$

Proof

We obtained the following Dirichlet Beta in Formula 2.1.1 .

$$\beta(n) = \sum_{r=0}^{\infty}\sum_{s=0}^{n-1} \frac{(2r+1)^s x^s}{s!} \frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^n} + \frac{x^n}{2}\sum_{r=0}^{\infty}\binom{-n}{2r}\frac{E_{2r}x^{2r}}{(n+2r)!}$$

On the other hand, for the relation between the sum of exponential and the incomplete gamma function, the following formula is known.

$$\sum_{s=0}^{n-1}\frac{x^s}{s!} = \frac{\Gamma(n,x)}{\Gamma(n)}e^x$$

Substituting this for the above,

$$\beta(n) = \sum_{r=0}^{\infty} \frac{\Gamma\{n,(2r+1)x\}}{\Gamma(n)} \frac{(-1)^r}{(2r+1)^n} + \frac{x^n}{2}\sum_{r=0}^{\infty}\binom{-n}{2r}\frac{E_{2r}x^{2r}}{(n+2r)!}$$

Extending natural number n to complex number p and replacing the factorial with the gamma function, we obtain

$$\beta(p) = \sum_{r=0}^{\infty} \frac{\Gamma\{p,(2r+1)x\}}{\Gamma(p)} \frac{(-1)^r}{(2r+1)^p} + \frac{x^p}{2}\sum_{r=0}^{\infty}\binom{-p}{2r}\frac{E_{2r}x^{2r}}{\Gamma(p+1+2r)}$$

Example $\beta(1/2+12.98809801231i)$

Whn $x=1+i$, this is calculated. As the result of calculating the series to the 143 th term, the significant 11 digits were obtained.

```
m = 143;
b[p_, x_] := Sum[Gamma[p, (2 r + 1) x]/Gamma[p] (-1)^r/(2 r + 1)^p, {r, 0, m}] + x^p/2 Sum[Binomial[-p, 2 r] EulerE[2 r] x^(2 r)/Gamma[p + 1 + 2 r], {r, 0, m}]

N[b[1/2 + 12.98809801231 i, 1 + i]]
1.53477 × 10^-12 - 5.79803 × 10^-12 i

N[DirichletL[4, 2, 1/2 + 12.98809801231 i]]
1.19633 × 10^-12 - 4.99564 × 10^-12 i
```

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Alien's Mathematics