5 Formulas for Riemann Zeta at complex number

We can extend "**2 Formulas for Riemann Zeta at natural number** " to the complex number easily.

5.1 Formulas of coth x family

Formula 5.1.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and p is a real number such that $p \ne 1, 0, -1, -2, \cdots$ and $\Gamma(p,x)=\int_x^\infty t^{p-1}e^{-t}dt$ is incomplete gamma function, the following expression holds for $x=u+vi \quad s.t. \quad 0< |x| \leqq 2\pi \ , u \geqq 0$.

$$\zeta(p) = \frac{x^{p-1}}{\Gamma(p)}\left(\frac{1}{p-1}-\frac{x}{2p}\right)+\sum_{r=1}^{\infty}\frac{\Gamma(p,xr)}{\Gamma(p)r^p}-\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}$$

Especially when $x=1$,

$$\zeta(p)=\frac{p+1}{2(p-1)\Gamma(p+1)}+\sum_{r=1}^{\infty}\frac{\Gamma(p,r)}{\Gamma(p)r^p}-\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{B_{2r}}{2r\Gamma(p+2r)}$$

Proof

We obtained the following Riemann Zeta in Formula 2.1.1 in **"2 Formulas for Riemann Zeta at natural number".**

$$\zeta(n) = \frac{x^{n-1}}{(n-1)!}\left(\frac{1}{n-1}-\frac{x}{2n}\right)+\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(rx)^s}{s!}\frac{1}{r^n e^{rx}}-\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

On the other hand, for the relation between the sum of exponential and the incomplete gamma function, the following formula is known.

$$\sum_{s=0}^{n-1}\frac{x^s}{s!} = \frac{\Gamma(n,x)}{\Gamma(n)}e^x$$

Substituting this for the above,

$$\zeta(n) = \frac{x^{n-1}}{(n-1)!}\left(\frac{1}{n-1}-\frac{x}{2n}\right)+\sum_{r=1}^{\infty}\frac{\Gamma(n,xr)}{\Gamma(n)r^n}-\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

Extending natural number n to complex number p and replacing the factorial with the gamma function, we obtain

$$\zeta(p) = \frac{x^{p-1}}{\Gamma(p)}\left(\frac{1}{p-1}-\frac{x}{2p}\right)+\sum_{r=1}^{\infty}\frac{\Gamma(p,xr)}{\Gamma(p)r^p}-\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}$$

Example $\zeta(1/2+14.134725141734693790457251983\,i)$

When $x=1+i$, this is calculated according to the formula. As the result of calculating the series to the 55 th term, the significant 23 digits were obtained.

```
m = 55 ;
z[p_, x_] := x^(p-1)/Gamma[p] (1/(p-1) - x/(2 p)) + Sum[Gamma[p, x r]/(Gamma[p] r^p), {r, 1, m}]
   - Sum[Binomial[-p, 2 r - 1] BernoulliB[2 r] x^(p-1+2 r)/(2 r Gamma[p + 2 r]), {r, 1, m}]

N[z[1 / 2 + 14.134725141734693790457251983 i, 1 + i], 20]
0. × 10^-22 + 0. × 10^-22 i
```

```
N[Zeta[1/2 + 14.134725141734693790457251983 i], 20]
0. × 10^-26 + 0. × 10^-26 i
```

In a similar way, we obtain the following formula from Formula 2.1.1' (**2.1**).

Formula 5.1.1'

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and p is a real number such that $p \neq 1, 0, -1, -2, \cdots$ and $\Gamma(p,x)=\int_x^\infty t^{p-1}e^{-t}dt$ is incomplete gamma function, the following expression holds for $x=u+vi\ \ s.t.\ \ 0<|x| \leqq 2\pi\ ,\ u \geqq 0$.

$$\zeta(p)=\frac{x^{p-1}}{\Gamma(p)}\left\{\frac{1}{p-1}+\frac{(p-1)x}{2p}-\log x\right\}+\sum_{r=1}^{\infty}\frac{\Gamma(p-1,xr)}{\Gamma(p-1)r^p}-\sum_{r=1}^{\infty}\binom{1-p}{2r}\frac{B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}$$

Especially when $x=1$,

$$\zeta(p)=\frac{p^2+1}{2(p-1)\Gamma(p+1)}+\sum_{r=1}^{\infty}\frac{\Gamma(p-1,r)}{\Gamma(p-1)r^p}-\sum_{r=1}^{\infty}\binom{1-p}{2r}\frac{B_{2r}}{2r\Gamma(p+2r)}$$

Example $\zeta(1/2-14.134725141734693790457251983i)$

When $x=1-i$, this is calculated according to the formula. As the result of calculatingthe series to the 51 th term, the significant 22 digits were obtained.

```
m = 51;
z[p_, x_] := x^(p-1)/Gamma[p] (1/(p-1) + (p-1) x/(2 p) - Log[x]) + Sum[Gamma[p-1, x r]/(Gamma[p-1] r^p), {r, 1, m}]
              - Sum[Binomial[1-p, 2 r] BernoulliB[2 r] x^(p-1+2 r)/(2 r Gamma[p + 2 r]), {r, 1, m}]

N[z[1/2 - 14.134725141734693790457251983 i, 1 - i], 20]
0. × 10^-21 + 0. × 10^-21 i

N[Zeta[1/2 - 14.134725141734693790457251983 i], 20]
0. × 10^-26 + 0. × 10^-26 i
```

5.2 Formula of tanh x family

Formula 5.2.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and p is a real number such that $p\neq 1,0,-1,-2,\cdots$ and $\Gamma(p,x)=\int_x^\infty t^{p-1}e^{-t}dt$ is incomplete gamma function, the following expression holds for $x=u+vi \quad s.t.\quad 0<|x|\leqq\pi\ ,\ u\geqq 0$.

$$\zeta(p)=\frac{2^{p-1}}{2^{p-1}-1}\left\{\frac{x^p}{2\Gamma(p+1)}-\sum_{r=1}^{\infty}\frac{\Gamma(p,xr)}{\Gamma(p)}\frac{(-1)^r}{r^p}+\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{(2^{2r}-1)B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}\right\}$$

Especially when $x=1$,

$$\zeta(p)=\frac{2^{p-1}}{2^{p-1}-1}\left\{\frac{1}{2\Gamma(p+1)}-\sum_{r=1}^{\infty}\frac{\Gamma(p,r)}{\Gamma(p)}\frac{(-1)^r}{r^p}+\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{(2^{2r}-1)B_{2r}}{2r\Gamma(p+2r)}\right\}$$

Proof

We obtained the following Riemann Zeta in Formula2.2.1 in **2.2 .**

$$\zeta(n)=\frac{2^{n-1}}{2^{n-1}-1}\left\{\frac{x^n}{2n!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(xr)^s}{s!}\frac{(-1)^r}{r^n e^{xr}}+\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{(2^{2r}-1)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

Substituting

$$\sum_{s=0}^{n-1}\frac{(xr)^s}{s!}=\frac{\Gamma(n,xr)}{\Gamma(n)}e^{xr}$$

for this,

$$\zeta(n)=\frac{2^{n-1}}{2^{n-1}-1}\left\{\frac{x^n}{2n!}-\sum_{r=1}^{\infty}\frac{\Gamma(n,xr)}{\Gamma(n)}\frac{(-1)^r}{r^n}+\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{(2^{2r}-1)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

Extending natural number n to complex number p and replacing the factorial with the gamma function, we obtain the desired expression.

Example $\zeta(0.7)$

When $x=1/2+i$, this is calculated according to the formula. As the result of calculating the series to the 37 th term, the significant 10 digits were obtained.

```
m = 37 ;
z[p_, x_]:= 2^(p-1)/(2^(p-1) - 1) ( x^p/(2 Gamma[p + 1]) - Sum[Gamma[p, x r]/Gamma[p] (-1)^r/r^p, {r, 1, m}] +
    Sum[Binomial[-p, 2 r - 1] ((2^(2 r) - 1) BernoulliB[2 r] x^(p-1+2 r))/(2 r Gamma[p + 2 r]), {r, 1, m}] )
```

```
N[z[0.7, 1 / 2 + i]];            N[Zeta[0.7]];
SetPrecision [%, 10]             SetPrecision [%, 10

-2.778388446  + 0. × 10^-11 i    -2.778388446
```

5.3 Formulas of csch x family

Formula 5.3.1

When $B_0=1, B_2=1/6, B_4=-1/30, B_6=1/42, \cdots$ are Bernoulli numbers and p is a real number such that $p \neq 1, 0, -1, -2, \cdots$ and $\Gamma(p,x)=\int_x^{\infty} t^{p-1}e^{-t}dt$ is incomplete gamma function, the following expression holds for $x=u+vi \;\; s.t. \;\; 0<|x| \leqq \pi , u \geqq 0$.

$$\zeta(p) = \frac{2^p}{2^p-1}\left\{\frac{x^{p-1}}{2(p-1)\Gamma(p)} + \sum_{r=1}^{\infty}\frac{\Gamma\{p,x(2r-1)\}}{\Gamma(p)(2r-1)^p} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{(2^{2r}-2)B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}\right\}$$

Especially when $x=1$,

$$\zeta(p) = \frac{2^p}{2^p-1}\left\{\frac{1}{2(p-1)\Gamma(p)} + \sum_{r=1}^{\infty}\frac{\Gamma(p,2r-1)}{\Gamma(p)(2r-1)^p} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{(2^{2r}-2)B_{2r}}{2r\Gamma(p+2r)}\right\}$$

Proof

We obtained the following Riemann Zeta in Formula2.3.1 in **2.3** .

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{x^{n-1}}{2(n-1)!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{\{x(2r-1)\}^s}{s\,!}\frac{e^{-x(2r-1)}}{(2r-1)^n} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{(2^{2r}-2)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

On the other hand,

$$\sum_{s=0}^{n-1}\frac{x^s}{s\,!} = \frac{\Gamma(n,x)}{\Gamma(n)}e^x$$

Then,

$$\sum_{s=0}^{n-1}\frac{\{x(2r-1)\}^s}{s\,!} = \frac{\Gamma\{n,x(2r-1)\}}{\Gamma(n)}e^{x(2r-1)}$$

Substituting this for the above,

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{x^{n-1}}{2(n-1)!\,(n-1)} + \sum_{r=1}^{\infty}\frac{\Gamma\{n,x(2r-1)\}}{\Gamma(n)(2r-1)^n} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{(2^{2r}-2)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

Extending natural number n to complex number p and replacing the factorial with the gamma function, we obtain the desired expression.

In a similar way, we obtain the following formula from Formula 2.3.1' (**2.3**).

Formula 5.3.1'

When $B_0=1, B_2=1/6, B_4=-1/30, B_6=1/42, \cdots$ are Bernoulli numbers and p is a real number such that $p \neq 1, 0, -1, -2, \cdots$ and $\Gamma(p,x) = \int_x^{\infty} t^{p-1} e^{-t} dt$ is incomplete gamma function,

the following expression holds for $x = u + vi$ s.t. $0 < |x| \leqq \pi$, $u \geqq 0$.

$$\zeta(p) = \frac{2^p}{2^p - 1} \left\{ \frac{x^{p-1}}{2\Gamma(p)} \left(\frac{1}{p-1} - \log\frac{x}{2} \right) + \sum_{r=1}^{\infty} \frac{\Gamma\{p-1, x(2r-1)\}}{\Gamma(p-1)(2r-1)^p} + \frac{1}{2} \sum_{r=1}^{\infty} \binom{1-p}{2r} \frac{\left(2^{2r}-2\right) B_{2r} x^{p-1+2r}}{2r\Gamma(p+2r)} \right\}$$

Especially when $x = 1, 2$,

$$\zeta(p) = \frac{2^p}{2^p - 1} \left\{ \frac{1}{2\Gamma(p)} \left(\frac{1}{p-1} + \log 2 \right) + \sum_{r=1}^{\infty} \frac{\Gamma(p-1, 2r-1)}{\Gamma(p-1)(2r-1)^p} + \frac{1}{2} \sum_{r=1}^{\infty} \binom{1-p}{2r} \frac{\left(2^{2r}-2\right) B_{2r}}{2r\Gamma(p+2r)} \right\}$$

$$\zeta(p) = \frac{2^p}{2^p - 1} \left\{ \frac{2^{p-2}}{(p-1)\Gamma(p)} + \sum_{r=1}^{\infty} \frac{\Gamma(p-1, 4r-2)}{\Gamma(p-1)(2r-1)^p} + \frac{1}{2} \sum_{r=1}^{\infty} \binom{1-p}{2r} \frac{\left(2^{2r}-2\right) B_{2r} 2^{p-1+2r}}{2r\Gamma(p+2r)} \right\}$$

Example $\zeta(i)$

According to the formula at $x = 2$ this is calculated. As the result of calculating the series to the 18 th term, the significant 10 digits were obtained.

```
m = 18 ;
z[p_] := 2^p/(2^p - 1) ( 2^(p-2)/((p - 1) Gamma[p]) + Sum[Gamma[p - 1, 4 r - 2]/(Gamma[p - 1] (2 r - 1)^p), {r, 1, m}]
         + 1/2 Sum[Binomial[1 - p, 2 r] ((2^(2 r) - 2) BernoulliB[2 r] 2^(p-1+2 r))/(2 r Gamma[p + 2 r]), {r, 1, m}] )

N[z[i], 10]
0.0033002237 - 0.4181554491 i

N[Zeta[i], 10]
0.0033002237 - 0.4181554491 i
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Alien's Mathematics