4 Formulas for Riemann Zeta at even number

Formulas for Zeta at even number obtained in "**1 Zeta Generating Functions**" were automorphisms which were expressed by the lower zetas. In this chapter, removing the lower zetas, we obtain the explicit formulas.

4.1 Formulas of cot x family

Formula 4.1.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers, the following expression holds for $0<x<2\pi$.

$$\zeta(2n) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)(rx)^{2s}}{(2s)!}\frac{\sin rx}{r^{2n+1}} - \frac{|B_{2n}|\,x^{2n}}{2(2n)!}\left\{\left(2^{2n}-2\right)-\frac{\pi\left(2^{2n+1}-2\right)}{x}\right\}$$

Especially when $x=\pi$,

$$\zeta(2n) = \frac{|B_{2n}|\,(2\pi)^{2n}}{2(2n)!}$$

Proof

We obtained the following Riemann Zeta in Formula1.4.3 in **1.4 .**

$$\zeta(2) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin rx}{r^3} - \frac{1}{2}\frac{x^2}{3!} + \frac{\pi}{2}\frac{x^1}{2!}$$

$$\zeta(4) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin rx}{r^5} + \frac{1}{2}\frac{x^4}{5!} - \frac{\pi}{2}\frac{x^3}{4!} + \frac{x^2}{3!}\zeta(2)$$

$$\zeta(6) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin rx}{r^7} - \frac{1}{2}\frac{x^6}{7!} + \frac{\pi}{2}\frac{x^5}{6!} + \frac{x^2}{3!}\zeta(4) - \frac{x^4}{5!}\zeta(2)$$

$$\zeta(8) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin rx}{r^9} + \frac{1}{2}\frac{x^8}{9!} - \frac{\pi}{2}\frac{x^7}{8!} + \frac{x^2}{3!}\zeta(6) - \frac{x^4}{5!}\zeta(4) + \frac{x^6}{7!}\zeta(2)$$

$$\vdots$$

Substituting $\zeta(k)$ for the below one by one, we obtain

$$\zeta(2n) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^s C_s x^{2s}\frac{\sin rx}{r^{2n+1-2s}} - \frac{(-1)^n}{2}\sum_{s=0}^{n-1}C_s\left\{\frac{x^{2n}}{(2n+1-2s)!}-\frac{\pi x^{2n-1}}{(2n-2s)!}\right\}$$

Where, C_s are rational numberes as follows.

$$C_0=-1\ ,\ C_1=\frac{1}{3!}\ ,\ C_2=\frac{1}{5!}-\frac{1}{3!3!}\ ,\ C_3=\frac{1}{7!}-\left(\frac{1}{3!5!}+\frac{1}{5!3!}\right)+\frac{1}{3!3!3!}\ ,$$

$$C_4=\frac{1}{9!}-\left(\frac{1}{3!7!}+\frac{1}{5!5!}+\frac{1}{7!3!}\right)+\left(\frac{1}{3!3!5!}+\frac{1}{3!5!3!}+\frac{1}{5!3!3!}\right)-\frac{1}{3!3!3!3!}$$

$$\vdots$$

And, these are calculable by the following expression.

$$C_s = \frac{2^{2s}-2}{(2s)!}B_{2s} \qquad s=0,1,2,\cdots$$

Substituting this for the above,

$$\zeta(2n) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^s\frac{2^{2s}-2}{(2s)!}B_{2s}x^{2s}\frac{\sin rx}{r^{2n+1-2s}}$$

$$-\frac{(-1)^n}{2}\left\{x^{2n}\sum_{s=0}^{n-1}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n+1-2s)!}-\pi x^{2n-1}\sum_{s=0}^{n-1}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n-2s)!}\right\}$$

Here, following expressions hold.

$$\sum_{s=0}^{n-1}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n+1-2s)!} = -\frac{\left(2^{2n}-2\right)B_{2n}}{(2n)!1!}$$

$$\sum_{s=0}^{n-1}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n-2s)!} = -\frac{\left(2^{2n+1}-2\right)B_{2n}}{(2n)!0!}$$

Then, substituting these for the above,

$$\zeta(2n) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)(rx)^{2s}}{(2s)!}\frac{\sin rx}{r^{2n+1}}$$

$$+\frac{(-1)^n x^{2n}}{2}\left\{\frac{\left(2^{2n}-2\right)B_{2n}}{(2n)!}-\frac{\pi}{x}\frac{\left(2^{2n+1}-2\right)B_{2n}}{(2n)!}\right\}$$

Thus, we obtain the desired expression.

Formula 4.1.2

Let Bernoulli numbers and Euler numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30, \cdots$

$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385, \cdots$

Then, the following expression holds for $0<x<2\pi$.

$$\zeta(2n)=\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx}{r^{2n}}-\frac{|E_{2n}|x^{2n}}{2(2n)!}+\frac{\pi}{2}\frac{2^{2n}\left(2^{2n}-1\right)|B_{2n}|x^{2n-1}}{(2n)!}$$

Especially when $x=\pi$,

$$\zeta(2n) = -\frac{1}{2^{2n}-2}\left\{\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(\pi r)^{2s}}{(2s)!}\frac{(-1)^r}{r^{2n}}-\frac{|E_{2n}|\pi^{2n}}{2(2n)!}\right\}$$

Proof

We obtained the following Riemann Zeta in Formula1.4.3 in **1.4 .**

$$\zeta(2) = \sum_{r=1}^{\infty}\frac{\cos rx}{r^2}-\frac{1}{2}\frac{x^2}{2!}+\frac{\pi}{2}\frac{x^1}{1!}$$

$$\zeta(4) = \sum_{r=1}^{\infty}\frac{\cos rx}{r^4}+\frac{1}{2}\frac{x^4}{4!}-\frac{\pi}{2}\frac{x^3}{3!}+\frac{x^2}{2!}\zeta(2)$$

$$\zeta(6) = \sum_{r=1}^{\infty}\frac{\cos rx}{r^6}-\frac{1}{2}\frac{x^6}{6!}+\frac{\pi}{2}\frac{x^5}{5!}+\frac{x^2}{2!}\zeta(4)-\frac{x^4}{4!}\zeta(2)$$

$$\zeta(8) = \sum_{r=1}^{\infty}\frac{\cos rx}{r^8}+\frac{1}{2}\frac{x^8}{8!}-\frac{\pi}{2}\frac{x^7}{7!}+\frac{x^2}{2!}\zeta(6)-\frac{x^4}{4!}\zeta(4)+\frac{x^6}{6!}\zeta(2)$$

$\vdots$

Substituting $\zeta(k)$ for the below one by one, we obtain

$$\zeta(2n)=\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}C_s x^{2s}\frac{\cos rx}{r^{2n-2s}}+\frac{(-1)^n}{2}x^{2n}\sum_{s=0}^{n-1}\frac{(-1)^s C_s}{(2n-2s)!}-(-1)^n\frac{\pi x^{2n-1}}{2}\sum_{s=0}^{n-1}\frac{(-1)^s C_s}{(2n-1-2s)!}$$

Where, C_s are rational numberes as follows.

$$C_0=\frac{1}{0!}\ ,\ C_1=\frac{1}{2!}\ ,\ C_2=-\frac{1}{4!}+\frac{1}{2!2!}\ ,\ C_3=\frac{1}{6!}-\left(\frac{1}{4!2!}+\frac{1}{2!4!}\right)+\frac{1}{2!2!2!}\ ,$$

$$C_4=-\frac{1}{8!}+\left(\frac{1}{6!2!}+\frac{1}{4!4!}+\frac{1}{2!6!}\right)-\left(\frac{1}{4!2!2!}+\frac{1}{2!4!2!}+\frac{1}{2!2!4!}\right)+\frac{1}{2!2!2!2!}$$

$\vdots$

And, these are calculable by the following expression.

$$C_s=(-1)^s\frac{E_{2s}}{(2s)!}=\frac{|E_{2s}|}{(2s)!}$$

Substituting this for the above,

$$\zeta(2n)=\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|x^{2s}}{(2s)!}\frac{\cos rx}{r^{2n-2s}}+\frac{(-1)^n}{2}x^{2n}\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!(2n-2s)!}$$
$$-(-1)^n\frac{\pi x^{2n-1}}{2}\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!(2n-1-2s)!}$$

Here, following expressions hold.

$$\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!(2n-2s)!}=-\frac{E_{2n}}{(2n)!}$$

$$\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!(2n-1-2s)!}=\frac{2^{2n}\left(2^{2n}-1\right)B_{2n}}{(2n)!}$$

Then, substituting these for the above,

$$\zeta(2n)=\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx}{r^{2n}}-\frac{|E_{2n}|x^{2n}}{2(2n)!}+\frac{\pi}{2}\frac{2^{2n}\left(2^{2n}-1\right)|B_{2n}|x^{2n-1}}{(2n)!}$$

Especially when $x=\pi$,

$$\zeta(2n)=\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(\pi r)^{2s}}{(2s)!}\frac{\cos\pi r}{r^{2n}}-\frac{|E_{2n}|\pi^{2n}}{2(2n)!}+\frac{(2\pi)^{2n}|B_{2n}|}{2(2n)!}\left(2^{2n}-1\right)$$

i.e.

$$\zeta(2n)=\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(\pi r)^{2s}}{(2s)!}\frac{\cos\pi r}{r^{2n}}-\frac{|E_{2n}|\pi^{2n}}{2(2n)!}+\left(2^{2n}-1\right)\zeta(2n)$$

From this,

$$\zeta(2n)=-\frac{1}{2^{2n}-2}\left\{\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(\pi r)^{2s}}{(2s)!}\frac{(-1)^r}{r^{2n}}-\frac{|E_{2n}|\pi^{2n}}{2(2n)!}\right\}$$

Example $\zeta(6)$

According to the formula at $x=\pi$ this is calculated. As the result of calculating the series to the 18400 th term, the significant 10 digits were obtained.

```
m = 18400 ;
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$$z[n\_] := -\frac{1}{2^{2n}-2}\left(\sum_{r=1}^{m}\sum_{s=0}^{n-1}\frac{\text{Abs}[\text{EulerE}[2\,s]]\,(\pi\,r)^{2s}}{(2\,s)\,!}\,\frac{(-1)^{r}}{r^{2n}}-\frac{\text{Abs}[\text{EulerE}[2\,n]]\,\pi^{2n}}{2\,(2\,n)\,!}\right)$$

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N[z[3], 10]        N[Zeta[6], 10]
1.017343062        1.017343062
```

By-products

$$\sum_{s=0}^{n-1}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!\,(2n-2s)!} = -\frac{\left(2^{2n+1}-2\right)B_{2n}}{(2n)!\,0!}$$

$$\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!\,(2n-1-2s)!} = \frac{2^{2n}\left(2^{2n}-1\right)B_{2n}}{(2n)!}$$

4.2 Formulas of tan x family

Formula 4.2.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers, the following expression holds for $0<x\leq\pi$.

$$\zeta(2n)=\frac{2^{2n}}{2^{2n}-2}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)(rx)^{2s-1}}{(2s)!}\frac{(-1)^r\sin rx}{r^{2n}}+\frac{|B_{2n}|(2x)^{2n}}{2(2n)!}$$

Especially when $x=\pi$,

$$\zeta(2n)=\frac{|B_{2n}|\,(2\pi)^{2n}}{2(2n)!}$$

Proof

We obtained the following Dirichlet Eta in Formula1.5.3 in **1.5 .**

$$\eta(2)=\frac{1}{x}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^3}+\frac{1}{2}\frac{x^2}{3!}$$

$$\eta(4)=\frac{1}{x}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^5}-\frac{1}{2}\frac{x^4}{5!}+\frac{x^2}{3!}\eta(2)$$

$$\eta(6)=\frac{1}{x}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^7}+\frac{1}{2}\frac{x^6}{7!}+\frac{x^2}{3!}\eta(4)-\frac{x^4}{5!}\eta(2)$$

$$\eta(8)=\frac{1}{x}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^9}-\frac{1}{2}\frac{x^8}{9!}+\frac{x^2}{3!}\eta(6)-\frac{x^4}{5!}\eta(4)+\frac{x^6}{7!}\eta(2)$$

$$\vdots$$

Substituting $\eta(k)$ for the below one by one , we obtain

$$\eta(2n)=-\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^s C_s\,(rx)^{2s}(-1)^{r-1}\frac{\sin rx}{r^{2n+1}}+\frac{(-1)^n x^{2n}}{2}\sum_{s=0}^{n-1}\frac{C_s}{(2n+1-2s)!}$$

Where, C_s is the same as the coefficient in Formula 4.1.1 and is given by $C_s=\frac{2^{2s}-2}{(2s)!}B_{2s}$. Then ,

$$\eta(2n)=\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s\left(2^{2s}-2\right)B_{2s}(rx)^{2s-1}}{(2s)!}\frac{(-1)^r\sin rx}{r^{2n}}$$

$$+\frac{(-1)^n x^{2n}}{2}\sum_{s=0}^{n-1}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!\,(2n+1-2s)!}$$

Here,

$$\sum_{s=0}^{n-1}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!\,(2n+1-2s)!}=-\frac{\left(2^{2n}-2\right)B_{2n}}{(2n)!\,1!}$$

Substituting this for the above,

$$\eta(2n)=\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)(rx)^{2s-1}}{(2s)!}\frac{(-1)^r\sin rx}{r^{2n}}+\frac{\left(2^{2n}-2\right)|B_{2n}|\,x^{2n}}{2(2n)!}$$

applying $\zeta(2n) = \frac{2^{2n-1}}{2^{2n-1}-1}\eta(2n) = \frac{2^{2n}}{2^{2n}-2}\eta(2n)$ to this,

$$\zeta(2n) = \frac{2^{2n}}{2^{2n}-2}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)(rx)^{2s-1}}{(2s)!}\frac{(-1)^r \sin rx}{r^{2n}} + \frac{2^{2n}}{2^{2n}-2}\frac{\left(2^{2n}-2\right)|B_{2n}|x^{2n}}{2(2n)!}$$

i.e.

$$\zeta(2n) = \frac{2^{2n}}{2^{2n}-2}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)(rx)^{2s-1}}{(2s)!}\frac{(-1)^r \sin rx}{r^{2n}} + \frac{|B_{2n}|(2x)^{2n}}{2(2n)!}$$

Example $\zeta(6)$

When x=1/64, this is calculated according to the formula. As the result of calculating the series to the 32th term , the significant 10 digits were obtained.

```
m = 32 ;
z[n_, x_] := 2^(2 n)/(2^(2 n) - 2) Sum[(-1)^s BernoulliB[2 s] (2^(2 s) - 2) (r x)^(2 s-1)/(2 s)! (-1)^r Sin[r x]/r^(2 n), {r, 1, m}, {s, 0, n-1}]
             + Abs[EulerE[2 n]] (2 x)^(2 n)/(2 (2 n) !)

N[z[3, 1/64], 10]        N[Zeta[6], 10]
1.017343062              1.017343062
```

Formula 4.2.2

Let Bernoulli numbers and Euler numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30, \cdots$

$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385, \cdots$

Then, the following expression holds for $0 < x \leqq \pi$

$$\zeta(2n) = -\frac{2^{2n}}{2^{2n}-2}\left\{\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{(-1)^r \cos rx}{r^{2n}} - \frac{|E_{2n}|x^{2n}}{2(2n)!}\right\}$$

Especially when $x=\pi$,

$$\zeta(2n) = -\frac{1}{2^{2n}-2}\left\{\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(\pi r)^{2s}}{(2s)!}\frac{(-1)^r}{r^{2n}} - \frac{|E_{2n}|\pi^{2n}}{2(2n)!}\right\}$$

Proof

We obtained the following Dirichlet Eta in Formula1.5.3 in **1.5 .**

$$\eta(2) = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^2} + \frac{1}{2}\frac{x^2}{2!}$$

$$\eta(4) = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^4} - \frac{1}{2}\frac{x^4}{4!} + \frac{x^2}{2!}\eta(2)$$

$$\eta(6) = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^6} + \frac{1}{2}\frac{x^6}{6!} + \frac{x^2}{2!}\eta(4) - \frac{x^4}{4!}\eta(2)$$

$$\eta(8) = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^8} - \frac{1}{2}\frac{x^8}{8!} + \frac{x^2}{2!}\eta(6) - \frac{x^4}{4!}\eta(4) + \frac{x^6}{6!}\eta(2)$$

$$\vdots$$

Substituting $\eta(k)$ for the below one by one , we obtain

$$\eta(2n) = \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}C_s x^{2s}(-1)^{r-1}\frac{\cos rx}{r^{2n-2s}} - \frac{(-1)^n x^{2n}}{2}\sum_{s=0}^{n-1}\frac{(-1)^s C_s}{(2n-2s)!}$$

Where, C_s is the same as the coefficient in the Formula 4.1.2 and is given by

$$C_s = (-1)^s\frac{E_{2s}}{(2s)!} = \frac{|E_{2s}|}{(2s)!}$$

Then ,

$$\eta(2n) = -\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|\,(rx)^{2s}}{(2s)!}\frac{(-1)^r\cos rx}{r^{2n}} - \frac{(-1)^n x^{2n}}{2}\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!\,(2n-2s)!}$$

Here,

$$\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!\,(2n-2s)!} = -\frac{E_{2n}}{(2n)!}$$

Substituting this for the above,

$$\eta(2n) = -\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|\,(rx)^{2s}}{(2s)!}\frac{(-1)^r\cos rx}{r^{2n}} + \frac{|E_{2n}|\,x^{2n}}{2(2n)!}$$

And applying $\zeta(2n) = \dfrac{2^{2n}}{2^{2n}-2}\eta(2n)$ to this, we obtain the desired expression.

4.3 Formulas of csc x family

Formula 4.3.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers , the following expression holds for $0<x\leqq\pi$.

$$\zeta(2n) = -\frac{2^{2n}}{2^{2n}-1}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)\{(2r-1)x\}^{2s-1}}{(2s)!}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n}} + \frac{\pi}{2}\frac{2^{2n}|B_{2n}|x^{2n-1}}{(2n)!}$$

Especially when $x=\pi$,

$$\zeta(2n) = \frac{|B_{2n}|(2\pi)^{2n}}{2(2n)!}$$

Proof

We obtained the following Dirichlet Lambda in Formula1.6.3 in **1.6** .

$$\lambda(2) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^3} + \frac{x^1}{2!}\frac{\pi}{4}$$

$$\lambda(4) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^5} - \frac{x^3}{4!}\frac{\pi}{4} + \frac{x^2}{3!}\lambda(2)$$

$$\lambda(6) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^7} + \frac{x^5}{6!}\frac{\pi}{4} + \frac{x^2}{3!}\lambda(4) - \frac{x^4}{5!}\lambda(2)$$

$$\lambda(8) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^9} - \frac{x^7}{8!}\frac{\pi}{4} + \frac{x^2}{3!}\lambda(6) - \frac{x^4}{5!}\lambda(4) + \frac{x^6}{7!}\lambda(2)$$

$$\vdots$$

Substituting $\lambda(k)$ for the below one by one , we obtain

$$\lambda(2n) = -\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s C_s x^{2s-1}}{(2r-1)^{2n+1-2s}}\sin\{(2r-1)x\} + (-1)^n\frac{\pi x^{2n-1}}{4}\sum_{s=0}^{n-1}\frac{C_s}{(2n-2s)!}$$

Where, C_s is the same as the coefficient in Formula 4.1.1 and is given by $C_s = \dfrac{2^{2s}-2}{(2s)!}B_{2s}$. Then ,

$$\lambda(2n) = -\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s\left(2^{2s}-2\right)B_{2s}\{(2r-1)x\}^{2s}}{(2s)!}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n}} + (-1)^n\frac{\pi x^{2n-1}}{4}\sum_{s=0}^{n-1}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n-2s)!}$$

Here,

$$\sum_{s=0}^{n-1}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n-2s)!} = -\frac{\left(2^{2n+1}-2\right)B_{2n}}{(2n)!0!}$$

Substituting this for the above,

$$\lambda(2n) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)\{(2r-1)x\}^{2s}}{(2s)!}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n}} + \frac{\pi}{4}\frac{\left(2^{2n+1}-2\right)|B_{2n}|x^{2n-1}}{(2n)!}$$

applying $\zeta(2n) = \frac{2^{2n}}{2^{2n}-1}\lambda(2n) = \frac{2^{2n+1}}{2^{2n+1}-2}\lambda(2n)$ to this,

$$\zeta(2n) = -\frac{2^{2n}}{2^{2n}-1}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)\{(2r-1)x\}^{2s-1}}{(2s)!}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n}} + \frac{\pi}{2}\frac{2^{2n}|B_{2n}|x^{2n-1}}{(2n)!}$$

Formula 4.3.2

Let Bernoulli numbers and Euler numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30, \cdots$

$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385, \cdots$

Then, the following expression holds for $0 < x \leqq \pi$

$$\zeta(2n) = \frac{2^{2n}}{2^{2n}-1}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|\{(2r-1)x\}^{2s}}{(2s)!}\frac{\cos\{(2r-1)x\}}{(2r-1)^{2n}} + \frac{\pi}{4}\frac{2^{4n}|B_{2n}|x^{2n-1}}{(2n)!}$$

Especially when $x = \pi/2$,

$$\zeta(2n) = \frac{|B_{2n}|(2\pi)^{2n}}{2(2n)!}$$

Proof

We obtained the following Dirichlet Lambda in Formula1.6.3 in **1.6 .**

$$\lambda(2) = \sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^2} + \frac{x^1}{1!}\frac{\pi}{4}$$

$$\lambda(4) = \sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^4} - \frac{x^3}{3!}\frac{\pi}{4} + \frac{x^2}{2!}\lambda(2)$$

$$\lambda(6) = \sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^6} + \frac{x^5}{5!}\frac{\pi}{4} + \frac{x^2}{2!}\lambda(4) - \frac{x^4}{4!}\lambda(2)$$

$$\lambda(8) = \sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^8} - \frac{x^7}{7!}\frac{\pi}{4} + \frac{x^2}{2!}\lambda(6) - \frac{x^4}{4!}\lambda(4) + \frac{x^6}{6!}\lambda(2)$$

$$\vdots$$

Substituting $\lambda(k)$ for the below one by one , we obtain

$$\lambda(2n) = \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{C_s x^{2s}}{(2r-1)^{2n-2s}}\cos\{(2r-1)x\} - (-1)^n\frac{\pi x^{2n-1}}{4}\sum_{s=0}^{n-1}C_s\frac{(-1)^s}{(2n-1-2s)!}$$

Where, C_s is the same as the coefficient in the Formula4.1.2 and is given by

$$C_s = (-1)^s \frac{E_{2s}}{(2s)!} = \frac{|E_{2s}|}{(2s)!}$$

Then ,

$$\lambda(2n) = \sum_{r=1}^{\infty}\sum_{s=0}^{n-1} \frac{|E_{2s}|\,x^{2s}}{(2s)!\,(2r-1)^{2n-2s}} \cos\{(2r-1)x\} - (-1)^n \frac{\pi x^{2n-1}}{4} \sum_{s=0}^{n-1} \frac{E_{2s}}{(2s)!\,(2n-1-2s)!}$$

Here,

$$\sum_{s=0}^{n-1} \frac{E_{2s}}{(2s)!\,(2n-1-2s)!} = \frac{2^{2n}\left(2^{2n}-1\right)B_{2n}}{(2n)!}$$

Substituting this for the above,

$$\lambda(2n) = \sum_{r=1}^{\infty}\sum_{s=0}^{n-1} \frac{|E_{2s}|\,x^{2s}}{(2s)!\,(2r-1)^{2n-2s}} \cos\{(2r-1)x\} - (-1)^n \frac{\pi}{4} \frac{2^{2n}\left(2^{2n}-1\right)B_{2n}\,x^{2n-1}}{(2n)!}$$

And applying $\zeta(2n) = \dfrac{2^{2n}}{2^{2n}-1}\lambda(2n)$ to this, we obtain the desired expression.

Example $\zeta(6)$

When x=1/64, this is calculated according to the formula. As the result of calculating the series to the 25th term, the significant 10 digits were obtained.

```
m = 25 ;
```

$$z[n\_, x\_] := \frac{2^{2n}}{2^{2n}-1} \sum_{r=1}^{m}\sum_{s=0}^{n-1} \frac{\text{Abs}[\text{EulerE}[2s]]\,((2r-1)x)^{2s}}{(2s)!} \frac{\text{Cos}[(2r-1)x]}{(2r-1)^{2n}} + \frac{\pi}{4} \frac{2^{4n}\,\text{Abs}[\text{BernoulliB}[2n]]\,x^{2n-1}}{(2n)!}$$

```
N[z[3, 1/64], 10]        N[Zeta[6], 10]
1.017343062              1.017343062
```

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Alien's Mathematics