2 Formulas for Riemann Zeta at natural number

Formulas for Zeta at natural number obtained in "**1 Zeta Generating Functions**" were automorphisms which were expressed by the lower zetas. In this chapter, removing the lower zetas, we obtain the explicit formulas.

2.1 Formulas of coth x family

Formulas for Zeta at natural number are obtained from coth x family. These are simple and the convergence speed is fast.

Formula 2.1.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and n is two or more natural number, the following expression holds for $0<x\leqq 2\pi$.

$$\zeta(n)=\frac{x^{n-1}}{(n-1)!}\left(\frac{1}{n-1}-\frac{x}{2n}\right)+\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(rx)^s}{s!}\frac{1}{r^n e^{rx}}-\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

Especially when $x=1,2$,

$$\zeta(n)=\frac{n+1}{2n!(n-1)}+\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{r^s}{s!}\frac{1}{r^n e^r}-\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}}{2r(n-1+2r)!}$$

$$\zeta(n)=\frac{2^{n-1}}{n!(n-1)}+\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(2r)^s}{s!}\frac{1}{r^n e^{2r}}-\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}2^{n-1+2r}}{2r(n-1+2r)!}$$

Proof

We obtained the following Riemann Zeta in Formula 1.1.3 in "**1 Zeta Generating Functions**".

$$0=\log x-\frac{1}{2}\frac{x^1}{1!}+\sum_{r=1}^{\infty}\frac{e^{-xr}}{r^1}+\sum_{r=1}^{\infty}\frac{B_{2r}}{2r(2r)!}x^{2r} \tag{1.1}$$

$$\zeta(2)=-\frac{x^1}{1!}(\log x-H_1)+\frac{1}{2}\frac{x^2}{2!}+\sum_{r=1}^{\infty}\frac{e^{-xr}}{r^2}-\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+1}}{2r(2r+1)!}$$

$$\zeta(3)=\frac{x^2}{2!}(\log x-H_2)-\frac{1}{2}\frac{x^3}{3!}+\sum_{r=1}^{\infty}\frac{e^{-xr}}{r^2}+\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+2}}{2r(2r+2)!}+\frac{x^1}{1!}\zeta(2)$$

$$\zeta(4)=-\frac{x^3}{3!}(\log x-H_3)+\frac{1}{2}\frac{x^4}{4!}+\sum_{r=1}^{\infty}\frac{e^{-xr}}{r^4}-\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+3}}{2r(2r+3)!}-\frac{x^2}{2!}\zeta(2)+\frac{x^1}{1!}\zeta(3)$$

$$\zeta(5)=\frac{x^4}{4!}(\log x-H_4)-\frac{1}{2}\frac{x^5}{5!}+\sum_{r=1}^{\infty}\frac{e^{-xr}}{r^5}+\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+4}}{2r(2r+4)!}+\frac{x^3}{3!}\zeta(2)-\frac{x^2}{2!}\zeta(3)+\frac{x^1}{1!}\zeta(4)$$

$$\vdots$$

For the sake of simplicity, we make the following substitutions.

$$Z_n=\zeta(n)$$

$$f_n=\frac{(-1)^{n-1}x^{n-1}}{(n-1)!}(\log x-H_{n-1})+\frac{(-1)^n}{2}\frac{x^n}{n!}+\sum_{r=1}^{\infty}\frac{e^{-rx}}{r^n}+(-1)^{n-1}\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+n-1}}{2r(2r+n-1)!}$$

Then,

$$Z_2=f_2$$

$$Z_3 = f_3 + \frac{x^1}{1!} Z_2$$

$$Z_4 = f_4 - \frac{x^2}{2!} Z_2 + \frac{x^1}{1!} Z_3$$

$$Z_5 = f_5 + \frac{x^3}{3!} Z_2 - \frac{x^2}{2!} Z_3 + \frac{x^1}{1!} Z_4$$

$$\vdots$$

Substituting Z_k for the below one by one ,

$$Z_2 = \frac{1}{0!} x^0 f_2$$

$$Z_3 = \frac{1}{0!} x^0 f_3 + \frac{1}{1!} x^1 f_2$$

$$Z_4 = \frac{1}{0!} x^0 f_4 + \frac{1}{1!} x^1 f_3 + \left(-\frac{1}{2!} + \frac{1}{1!1!} \right) x^2 f_2$$

$$Z_5 = \frac{1}{0!} x^0 f_5 + \frac{1}{1!} x^1 f_4 + \left(-\frac{1}{2!} + \frac{1}{1!1!} \right) x^2 f_3 + \left(\frac{1}{3!} - \frac{1}{2!1!} - \frac{1}{1!2!} + \frac{1}{1!1!1!} \right) x^3 f_2$$

$$\vdots$$

$$Z_n = \sum_{s=0}^{n-2} C_s x^s f_{n-s}$$

Where, C_s are rational numberes as follows.

$$C_2 = -\frac{1}{2!} + \frac{1}{1!1!}$$

$$C_3 = \frac{1}{3!} - \frac{1}{2!1!} - \frac{1}{1!2!} + \frac{1}{1!1!1!}$$

$$C_4 = -\frac{1}{4!} + \frac{1}{3!1!} + \frac{1}{1!3!} + \frac{1}{2!2!} - \left(\frac{1}{2!1!1!} + \frac{1}{1!2!1!} + \frac{1}{1!1!2!} \right) + \frac{1}{1!1!1!1!}$$

$$\begin{aligned} C_5 = &\ \frac{1}{5!} - \left(\frac{1}{4!1!} + \frac{1}{1!4!} + \frac{1}{3!2!} + \frac{1}{2!3!} \right) \\ &+ \left(\frac{1}{3!1!1!} + \frac{1}{1!3!1!} + \frac{1}{1!1!3!} + \frac{1}{2!2!1!} + \frac{1}{2!1!2!} + \frac{1}{1!2!2!} \right) \\ &- \left(\frac{1}{2!1!1!1!} + \frac{1}{1!2!1!1!} + \frac{1}{1!1!2!1!} + \frac{1}{1!1!1!2!} \right) + \frac{1}{1!1!1!1!1!} \end{aligned}$$

$$\vdots$$

Although these are combinations of going mad, when these are actually calculated , it is as follows.

$$C_1 = \frac{1}{1!} \ , C_2 = \frac{1}{2!} \ , C_3 = \frac{1}{3!} \ , \ \cdots \ , C_s = \frac{1}{s!}$$

Then,

$$Z_n = \sum_{s=0}^{n-2} \frac{x^s}{s!} f_{n-s}$$

Returning to the original symbol ,

$$\zeta(n) = \sum_{s=0}^{n-2} \frac{x^s}{s!} \left\{ \frac{(-1)^{n-1-s} x^{n-1-s}}{(n-1-s)!} \left(\log x - H_{n-1-s} \right) + \frac{(-1)^{n-s}}{2} \frac{x^{n-s}}{(n-s)!} \right.$$

$$+\sum_{r=1}^{\infty}\frac{e^{-xr}}{r^{n-s}}+(-1)^{n-1-s}\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+n-1-s}}{2r(2r+n-1-s)!}\Bigg\}$$

Here, the $n-1$ th term must be as follows.

$$\frac{x^{n-1}}{(n-1)!}\left\{\frac{(-1)^0x^0}{0!}(\log x-H_0)+\frac{(-1)^1}{2}\frac{x^1}{1!}+\sum_{r=1}^{\infty}\frac{e^{-xr}}{r^1}+(-1)^0\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r}}{2r(2r)!}\right\}$$

$$=\frac{x^{n-1}}{(n-1)!}\left\{\log x-\frac{1}{2}\frac{x^1}{1!}+\sum_{r=1}^{\infty}\frac{e^{-xr}}{r^1}+\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r}}{2r(2r)!}\right\}$$

However, since the content in the { } of the right side is 0 from (1.1), the $n-1$ th term have to be 0 .

That is, the upper limit of Σ can be $n-1$. Thus,

$$\zeta(n)=\sum_{s=0}^{n-1}\frac{x^s}{s!}\Bigg\{\frac{(-1)^{n-1-s}x^{n-1-s}}{(n-1-s)!}(\log x-H_{n-1-s})+\frac{(-1)^{n-s}}{2}\frac{x^{n-s}}{(n-s)!}$$

$$+\sum_{r=1}^{\infty}\frac{e^{-xr}}{r^{n-s}}+(-1)^{n-1-s}\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+n-1-s}}{2r(2r+n-1-s)!}\Bigg\}$$

$$=(-1)^{n-1}\frac{x^{n-1}}{(n-1)!}\sum_{s=0}^{n-1}(-1)^s\binom{n-1}{s}(\log x-H_{n-s-1})$$

$$+\frac{(-1)^n}{2}\frac{x^n}{n!}\sum_{s=0}^{n-1}(-1)^s\binom{n}{s}+\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(xr)^s}{s!}\frac{1}{r^ne^{xr}}$$

$$+\sum_{r=1}^{\infty}\frac{(-1)^{n-1}}{(n-1+2r)!}\sum_{s=0}^{n-1}(-1)^s\binom{n-1+2r}{s}\frac{B_{2r}x^{2r+n-1}}{2r}$$

Here, according to " 岩波 数学公式 II " p11,

$$\sum_{s=0}^{m}(-1)^s\binom{n}{s}=(-1)^m\binom{n-1}{m}\qquad m\leqq n-1$$

Applying this,

$$\sum_{s=0}^{n-1}(-1)^s\binom{n}{s}=(-1)^{n-1}\binom{n-1}{n-1}=(-1)^{n-1}$$

$$\sum_{s=0}^{n-1}(-1)^s\binom{n-1+2r}{s}=(-1)^{n-1}\binom{n-1+2r-1}{n-1}$$

Furthermore,

$$\binom{p-1+r}{p-1}=(-1)^r\binom{-p}{r}$$

Applying this,

$$\binom{n-1+2r-1}{n-1}=(-1)^{2r-1}\binom{-n}{2r-1}$$

Then,

$$\sum_{s=0}^{n-1}(-1)^s\binom{n-1+2r}{s}=(-1)^{n-1+2r-1}\binom{-n}{2r-1}=(-1)^n\binom{-n}{2r-1}$$

Substituting these (blue) for the above,

$$\zeta(n)=\frac{(-x)^{n-1}}{(n-1)!}\sum_{s=0}^{n-1}(-1)^s\binom{n-1}{s}\left(\log x-H_{n-1-s}\right)-\frac{1}{2}\frac{x^n}{n!}$$

$$+\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(rx)^s}{s!}\frac{1}{r^n e^{rx}}-\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}}{2r}\frac{x^{n-1+2r+}}{(n-1+2r)!}$$

Next, when $n>1$,

$$\sum_{s=0}^{n-1}(-1)^s\binom{n-1}{s}\log x=0$$

Furthermore, according to my work " **04 Higher Integral** " **4.6.2** ,

$$\sum_{s=0}^{n-1}(-1)^s\binom{n}{s}H_{n-s}=\frac{(-1)^{n-1}}{n} \qquad \{H_r=\psi(1+r)+\gamma\}$$

Usint this,

$$\sum_{s=0}^{n-1}(-1)^s\binom{n-1}{s}H_{n-1-s}=\frac{(-1)^{n-2}}{n-1}=-\frac{(-1)^{n-1}}{n-1}$$

Substituting these (blue) for the above, we obtain

$$\zeta(n)=\frac{x^{n-1}}{(n-1)!(n-1)}-\frac{x^n}{2n!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(xr)^s}{s!}\frac{1}{r^n e^{xr}}-\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

i.e.

$$\zeta(n)=\frac{x^{n-1}}{(n-1)!}\left(\frac{1}{n-1}-\frac{x}{2n}\right)+\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(xr)^s}{s!}\frac{1}{r^n e^{xr}}-\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

Example 1

$$\zeta(3)=\frac{x^2}{2!2}-\frac{x^3}{2\cdot 3!}+\sum_{r=1}^{\infty}\left\{1+\frac{(xr)^1}{1!}+\frac{(xr)^2}{2!}\right\}\frac{1}{r^3e^{xr}}-\sum_{r=1}^{\infty}\binom{-3}{2r-1}\frac{B_{2r}x^{2+2r}}{2r(2+2r)!} \qquad 0<x\le 2\pi$$

$$\zeta(5)=\frac{6}{2\cdot 5!4}+\sum_{r=1}^{\infty}\left(1+\frac{r^1}{1!}+\frac{r^2}{2!}+\frac{r^3}{3!}+\frac{r^4}{4!}\right)\frac{1}{r^5e^r}-\sum_{r=1}^{\infty}\binom{-5}{2r-1}\frac{B_{2r}}{2r(4+2r)!}$$

$$\zeta(5)=\frac{2^4}{5!4}+\sum_{r=1}^{\infty}\left\{1+\frac{2r}{1!}+\frac{(2r)^2}{2!}+\frac{(2r)^3}{3!}+\frac{(2r)^4}{4!}\right\}\frac{1}{r^5e^{2r}}-\sum_{r=1}^{\infty}\binom{-5}{2r-1}\frac{B_{2r}2^{4+2r}}{2r(4+2r)!}$$

Example 2 $\zeta(7)$

According to the formula at x=2 this is calculated. As the result of calculating the series to the 8 th term, the significant 10 digits were obtained.

```
m = 8;
z[n_] := 2^(n-1)/(n! (n - 1)) + Sum[Sum[(2 r)^s/s! 1/(r^n e^(2 r)), {s, 0, n-1}], {r, 1, m}]
         - Sum[Binomial[-n, 2 r - 1] BernoulliB[2 r] 2^(n-1+2 r)/(2 r (n - 1 + 2 r)!), {r, 1, m}]
N[z[7], 10]        N[Zeta[7], 10]
1.008349277        1.008349277
```

Note

In this formula, the convergence is the fastst around x=2.

Formula 2.1.1'

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and n is two or more natural number, the following expression holds for $0<x\leqq 2\pi$.

$$\zeta(n)=\frac{x^{n-1}}{(n-1)!}\left\{\frac{1}{n-1}+\frac{(n-1)x}{2n}-\log x\right\}$$

$$+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(xr)^s}{s!}\frac{1}{r^n e^{xr}}-\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

Especially when $x=1$,

$$\zeta(n)=\frac{n^2+1}{2n!(n-1)}+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{r^s}{s!}\frac{1}{r^n e^r}-\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{B_{2r}}{2r(n-1+2r)!}$$

Proof

We start with the middle of the previous proof .

$$\zeta(n)=\sum_{s=0}^{n-2}\frac{x^s}{s!}\left\{\frac{(-1)^{n-1-s}x^{n-1-s}}{(n-1-s)!}\left(\log x-H_{n-1-s}\right)+\frac{(-1)^{n-s}}{2}\frac{x^{n-s}}{(n-s)!}\right.$$

$$\left.+\sum_{r=1}^{\infty}\frac{e^{-xr}}{r^{n-s}}+(-1)^{n-1-s}\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+n-1-s}}{2r(2r+n-1-s)!}\right\}$$

$$=(-1)^{n-1}\frac{x^{n-1}}{(n-1)!}\sum_{s=0}^{n-2}(-1)^s\binom{n-1}{s}\left(\log x-H_{n-1-s}\right)$$

$$+\frac{(-1)^n}{2}\frac{x^n}{n!}\sum_{s=0}^{n-2}(-1)^s\binom{n}{s}+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(xr)^s}{s!}\frac{1}{r^n e^{xr}}$$

$$+\sum_{r=1}^{\infty}\frac{(-1)^{n-1}}{(n-1+2r)!}\sum_{s=0}^{n-2}(-1)^s\binom{n-1+2r}{s}\frac{B_{2r}x^{n-1-2r}}{2r}$$

Here, according to " 岩波 数学公式 II " p11,

$$\sum_{s=0}^{m}(-1)^s\binom{n}{s}=(-1)^m\binom{n-1}{m}\qquad m\leqq n-1$$

Applying this,

$$\sum_{s=0}^{n-2}(-1)^s\binom{n}{s}=(-1)^{n-2}\binom{n-1}{n-2}=(-1)^n\binom{n-1}{1}=(-1)^n(n-1)$$

$$\sum_{s=0}^{n-2}(-1)^s\binom{n-1+2r}{s}=(-1)^{n-2}\binom{n-1+2r-1}{n-2}=(-1)^n\binom{n-1+2r-1}{n-2}$$

Furthermore.

$$\binom{p-1+r}{p-1}=(-1)^r\binom{-p}{r}$$

Applying this,

$$\binom{n-1+2r-1}{n-2}=\binom{n-1-1+2r}{n-1-1}=(-1)^{2r}\binom{-(n-1)}{2r}=\binom{1-n}{2r}$$

Then,

$$\sum_{s=0}^{n-2}(-1)^s\binom{n-1+2r}{s}=(-1)^n\binom{n-1+2r-1}{n-2}=(-1)^n\binom{1-n}{2r}$$

Substituting these (blue) for the above,

$$\zeta(n) = (-1)^{n-1}\frac{x^{n-1}}{(n-1)!}\sum_{s=0}^{n-2}(-1)^s\binom{n-1}{s}\left(\log x - H_{n-1-s}\right) + \frac{(n-1)}{2}\frac{x^n}{n!}$$

$$+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(xr)^s}{s!}\frac{1}{r^n e^{xr}} + \sum_{r=1}^{\infty}\frac{(-1)^{n-1}}{(n-1+2r)!}(-1)^n\binom{1-n}{2r}\frac{B_{2r}x^{n-1-2r}}{2r}$$

Next, when $n>1$,

$$\sum_{s=0}^{n-2}(-1)^s\binom{n-1}{s} = (-1)^n$$

Then,

$$\sum_{s=0}^{n-2}(-1)^s\binom{n-1}{s}\log x = (-1)^n \log x$$

Furthermore, according to my work " **04 Higher Integral** " **4.6.2** ,

$$\sum_{s=0}^{n-1}(-1)^s\binom{n}{s}H_{n-s} = \frac{(-1)^{n-1}}{n} \qquad \{H_r = \psi(1+r)+\gamma\}$$

Usint this,

$$\sum_{s=0}^{n-2}(-1)^s\binom{n-1}{s}H_{n-1-s} = \frac{(-1)^{n-2}}{n-1} = \frac{(-1)^n}{n-1}$$

Then,

$$(-1)^{n-1}\frac{x^{n-1}}{(n-1)!}\sum_{s=0}^{n-2}(-1)^s\binom{n-1}{s}\left(\log x - H_{n-1-s}\right)$$

$$= (-1)^{n-1}\frac{x^{n-1}}{(n-1)!}\left\{(-1)^n\log x - \frac{(-1)^n}{n-1}\right\}$$

$$= \frac{x^{n-1}}{(n-1)!}\left(\frac{1}{n-1} - \log x\right)$$

Thus,

$$\zeta(n) = \frac{x^{n-1}}{(n-1)!}\left(\frac{1}{n-1} - \log x\right) + \frac{(n-1)x^n}{2n!}$$

$$+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(xr)^s}{s!}\frac{1}{r^n e^{xr}} - \sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

i.e.

$$\zeta(n) = \frac{x^{n-1}}{(n-1)!}\left\{\frac{1}{n-1} + \frac{(n-1)x}{2n} - \log x\right\}$$

$$+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(xr)^s}{s!}\frac{1}{r^n e^{xr}} - \sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

Example 1

$$\zeta(3) = \frac{x^2}{2!}\left(\frac{1}{2} - \log x\right) + \frac{2x^3}{2\cdot 3!} + \sum_{r=1}^{\infty}\left(1+\frac{xr}{1!}\right)\frac{1}{r^3 e^{xr}} - \sum_{r=1}^{\infty}\binom{-2}{2r}\frac{B_{2r}x^{2+2r}}{2r(2+2r)!} \qquad 0<x\le 2\pi$$

$$\zeta(5)=\frac{5^2+1}{2\cdot 5!\,4}+\sum_{r=1}^{\infty}\left(1+\frac{r^1}{1!}+\frac{r^2}{2!}+\frac{r^3}{3!}\right)\frac{1}{r^5 e^r}-\sum_{r=1}^{\infty}\binom{-4}{2r}\frac{B_{2r}}{2r(4+2r)!}$$

Example 2 $\zeta(7)$

When X=2, this is calculated according to the formula. As the result of calculating the series to the 7th term, the significant 10 digits were obtained.

```
m = 7;
```

$$z[n\_, x\_] := \frac{x^{n-1}}{(n-1)!}\left(\frac{1}{n-1}+\frac{(n-1)\,x}{2\,n}-\mathrm{Log}[x]\right)+\sum_{r=1}^{m}\sum_{s=0}^{n-2}\frac{(x\,r)^s}{s!}\,\frac{1}{r^n\,e^{x\,r}}-\sum_{r=1}^{m}\frac{\mathrm{Binomial}[1-n,\,2\,r]\,\mathrm{BernoulliB}[2\,r]\,x^{n-1+2\,r}}{2\,r\,(n-1+2\,r)!}$$

```
N[z[7, 2], 10]        N[Zeta[7], 10]
1.008349277           1.008349277
```

2.2 Formula of tanh x family

Formula for Zeta at natural number is obtained from tanh x family. This is simple and the convergence speed is fast.

Formula 2.2.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and n is two or more natural number, the following expression holds for $0<x\leqq\pi$.

$$\zeta(n)=\frac{2^{n-1}}{2^{n-1}-1}\left\{\frac{x^n}{2n!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(xr)^s}{s!}\frac{(-1)^r}{r^n e^{xr}}+\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-1\right)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

Especially when $x=1$,

$$\zeta(n)=\frac{2^{n-1}}{2^{n-1}-1}\left\{\frac{1}{2n!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{r^s}{s!}\frac{(-1)^r}{r^n e^r}+\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-1\right)B_{2r}}{2r(n-1+2r)!}\right\}$$

Proof

We obtained the following Dirichlet Eeta in Formula1.2.3 in **1.2 .**

$$\eta(1)=\frac{1}{2}\frac{x^1}{1!}-\sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^1}-\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)B_{2r}x^{2r}}{2r(2r)!}$$

$$\eta(2)=-\frac{1}{2}\frac{x^2}{2!}-\sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^2}+\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)B_{2r}x^{2r+1}}{2r(2r+1)!}+\frac{x^1}{1!}\eta(1)$$

$$\eta(3)=\frac{1}{2}\frac{x^3}{3!}-\sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^3}-\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)B_{2r}x^{2r+2}}{2r(2r+2)!}-\frac{x^2}{2!}\eta(1)+\frac{x^1}{1!}\eta(2)$$

$$\eta(4)=-\frac{1}{2}\frac{x^4}{4!}-\sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^4}+\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)B_{2r}x^{2r+3}}{2r(2r+3)!}+\frac{x^3}{3!}\eta(1)-\frac{x^2}{2!}\eta(2)+\frac{x^1}{1!}\eta(3)$$

$$\vdots$$

$$\eta(n)=-\frac{(-1)^n}{2}\frac{x^n}{n!}-\sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^n}+(-1)^n\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)B_{2r}x^{2r+n-1}}{2r(2r+n-1)!}-\sum_{s=1}^{n-1}\frac{(-1)^s x^s}{s!}\eta(n-s)$$

Substituting $\eta(k)$ for the below one by one in a similar way as the proof of Formula 2.1.1 , we obtain

$$\eta(n)=\frac{C_n x^n}{2}-\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{x^s}{s!\,r^{n-s}}\frac{(-1)^r}{e^{rx}}+(-1)^n\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s}{s!\,(n-1+2r-s)!}\frac{\left(2^{2r}-1\right)B_{2r}x^{2r+n-1}}{2r}$$

Where, C_s is the same as the coefficient in the Formula 2.1.1 and is given by $C_s=\dfrac{1}{s!}$. Then ,

$$\eta(n)=\frac{x^n}{2n!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{x^s}{s!\,r^{n-s}}\frac{(-1)^r}{e^{rx}}+(-1)^n\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s}{s!\,(n-1+2r-s)!}\frac{\left(2^{2r}-1\right)B_{2r}x^{2r+n-1}}{2r}$$

$$=\frac{x^n}{2n!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{x^s}{s!\,r^{n-s}}\frac{(-1)^r}{e^{rx}}+(-1)^n\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^s\binom{n-1+2r}{s}\frac{\left(2^{2r}-1\right)B_{2r}x^{2r+n-1}}{2r(n-1+2r)!}$$

Here, as seen in the proof of Formula 2.1.1 ,

$$\sum_{s=0}^{n-1}(-1)^s\binom{n-1+2r}{s}=(-1)^n\binom{-n}{2r-1}$$

Applying this,

$$\eta(n) = \frac{x^n}{2n\,!} - \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(xr)^s}{s!}\frac{(-1)^r}{r^n e^{xr}} + \sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{(2^{2r}-1)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

And using $\zeta(n) = \dfrac{2^{n-1}}{2^{n-1}-1}\eta(n)$, we obtain the desired exression.

Example 1

$$\zeta(3) = \frac{2^2}{2^2-1}\left\{\frac{x^3}{2\cdot 3!} - \sum_{r=1}^{\infty}\left(1+\frac{xr}{1!}+\frac{x^2r^2}{2!}\right)\frac{(-1)^r}{r^3e^{xr}} + \sum_{r=1}^{\infty}\binom{-3}{2r-1}\frac{(2^{2r}-1)B_{2r}x^{2+2r}}{2r(2+2r)!}\right\}$$

$$\zeta(5) = \frac{2^4}{2^4-1}\left\{\frac{1}{2\cdot 5!} - \sum_{r=1}^{\infty}\left(1+\frac{r^1}{1!}+\frac{r^2}{2!}+\frac{r^3}{3!}+\frac{r^4}{4!}\right)\frac{(-1)^r}{r^5e^r} + \sum_{r=1}^{\infty}\binom{-5}{2r-1}\frac{(2^{2r}-1)B_{2r}}{2r(4+2r)!}\right\}$$

Example 2 $\zeta(7)$

When x=3/2, this is calculated according to the formula. As the result of calculating the series to the 10th term, the significant 10 digits were obtained.

```
m = 7;
```

$$z[n\_, x\_] := \frac{x^{n-1}}{(n-1)\,!}\left(\frac{1}{n-1} + \frac{(n-1)\,x}{2\,n} - \mathrm{Log}[x]\right) + \sum_{r=1}^{m}\sum_{s=0}^{n-2}\frac{(x\,r)^s}{s\,!}\,\frac{1}{r^n\,e^{x\,r}} - \sum_{r=1}^{m}\frac{\mathrm{Binomial}[1-n,\,2\,r]\,\mathrm{BernoulliB}[2\,r]\,x^{n-1+2\,r}}{2\,r\,(n-1+2\,r)\,!}$$

```
N[z[7, 2], 10]        N[Zeta[7], 10]
1.008349277           1.008349277
```

Note

In this formula, the convergence is the faststest around $x=3/2$.

2.3 Formulas of csch x family

Formulas for Zeta at natural number are obtained from csch x family. These are simple and the convergence speed is the fastest.

Formula 2.3.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and n is two or more natural number, the following expression holds for $0<x\leqq\pi$.

$$\zeta(n)=\frac{2^n}{2^n-1}\left\{\frac{x^{n-1}}{2(n-1)!\,(n-1)}+\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{\{x(2r-1)\}^s}{s\,!}\frac{e^{-x(2r-1)}}{(2r-1)^n}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-2\right)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

Especially when $x=1$,

$$\zeta(n)=\frac{2^n}{2^n-1}\left\{\frac{1}{2(n-1)!\,(n-1)}+\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(2r-1)^s}{s\,!}\frac{e^{-(2r-1)}}{(2r-1)^n}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-2\right)B_{2r}}{2r(n-1+2r)!}\right\}$$

Proof

We obtained the following Dirichlet Eeta in Formula1.3.3 in **1.3** .

$$0=\frac{1}{2}\frac{x^0}{0!}\left(\log\frac{x}{2}-H_0\right)+\sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^1}-\frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r}}{2r(2r)\,!}\qquad(1.1)$$

$$\lambda(2)=-\frac{1}{2}\frac{x^1}{1!}\left(\log\frac{x}{2}-H_1\right)+\sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^2}+\frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+1}}{2r(2r+1)\,!}$$

$$\lambda(3)=\frac{1}{2}\frac{x^2}{2!}\left(\log\frac{x}{2}-H_2\right)+\sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^3}-\frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+2}}{2r(2r+2)\,!}+\frac{x^1}{1!}\lambda(2)$$

$$\lambda(4)=-\frac{1}{2}\frac{x^3}{3!}\left(\log\frac{x}{2}-H_3\right)+\sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^4}+\frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+3}}{2r(2r+3)\,!}-\frac{x^2}{2!}\lambda(2)+\frac{x^1}{1!}\lambda(3)$$

$\vdots$

Substituting $\lambda(k)$ for the below one by one in a similar way as the proof of Formula 2.1.1 , we obtain

$$\lambda(n)=\sum_{s=0}^{n-2}\frac{x^s}{s\,!}\left\{\frac{(-1)^{n-1-s}x^{n-1-s}}{2(n-1-s)\,!}\left(\log\frac{x}{2}-H_{n-1-s}\right)+\sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^{n-s}}-\frac{(-1)^{n-1-s}}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+n-1-s}}{2r(2r+n-1-s)!}\right\}$$

Here, the $n-1$ th term must be as follows.

$$\frac{x^{n-1}}{(n-1)\,!}\left\{\frac{(-1)^0x^0}{2\cdot0!}\left(\log\frac{x}{2}-H_0\right)+\sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^1}-\frac{(-1)^0}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r}}{2r(2r)!}\right\}$$

However, since the content in the { } of the right side is 0 from (1.1) , the $n-1$ th term have to be 0 . That is, the upper limit of Σ can be $n-1$. Thus,

$$\lambda(n) = \sum_{s=0}^{n-1} \frac{x^s}{s!} \left\{ \frac{(-1)^{n-1-s} x^{n-1-s}}{2(n-1-s)!} \left(\log\frac{x}{2} - H_{n-1-s} \right) + \sum_{r=1}^{\infty} \frac{e^{-(2r-1)x}}{(2r-1)^{n-s}} - \frac{(-1)^{n-1-s}}{2} \sum_{r=1}^{\infty} \frac{(2^{2r}-2)B_{2r} x^{2r+n-1-s}}{2r(2r+n-1-s)!} \right\}$$

$$= \frac{(-1)^{n-1}}{2} \frac{x^{n-1}}{(n-1)!} \sum_{s=0}^{n-1} (-1)^s \binom{n-1}{s} \left(\log\frac{x}{2} - H_{n-1-s} \right)$$

$$+ \sum_{r=1}^{\infty} \sum_{s=0}^{n-1} \frac{\{x(2r-1)\}^s}{s!} \frac{e^{-x(2r-1)}}{(2r-1)^n}$$

$$- \frac{1}{2} \sum_{r=1}^{\infty} \frac{(-1)^{n-1}}{(n-1+2r)!} \sum_{s=0}^{n-1} (-1)^s \binom{n-1+2r}{s} \frac{(2^{2r}-2)B_{2r} x^{n-1+2r}}{2r}$$

Here, as seen in the proof of Formula 2.1.1 ,

$$\sum_{s=0}^{n-1} (-1)^s \binom{n-1}{s} \log\frac{x}{2} = 0$$

$$\sum_{s=0}^{n-1} (-1)^s \binom{n-1}{s} H_{n-1-s} = -\frac{(-1)^{n-1}}{n-1}$$

$$\sum_{s=0}^{n-1} (-1)^s \binom{n-1+2r}{s} = (-1)^n \binom{-n}{2r-1}$$

Substituting these for the above,

$$\lambda(n) = \frac{x^{n-1}}{2(n-1)!\,(n-1)} + \sum_{r=1}^{\infty} \sum_{s=0}^{n-1} \frac{\{x(2r-1)\}^s}{s!} \frac{e^{-x(2r-1)}}{(2r-1)^n} + \frac{1}{2} \sum_{r=1}^{\infty} \binom{-n}{2r-1} \frac{(2^{2r}-2)B_{2r} x^{n-1+2r}}{2r(n-1+2r)!}$$

And using $\zeta(n) = \frac{2^n}{2^n-1}\lambda(n)$, we obtain the desired exression.

Note

In this formula, the convergence is the fastst around x=1. We had better not think other than this.

Example 1

$$\zeta(3) = \frac{8}{7} \left\{ \frac{1}{8} + \sum_{r=1}^{\infty} \left(1 + \frac{2r-1}{1!} + \frac{(2r-1)^2}{2!} \right) \frac{e^{-(2r-1)}}{(2r-1)^3} + \frac{1}{2} \sum_{r=1}^{\infty} \binom{-3}{2r-1} \frac{(2^{2r}-2)B_{2r}}{2r(2+2r)!} \right\}$$

Example 2 $\zeta(7)$

According to the formula at x=1 this is calculated. As the result of calculating the series to the 6 th term, the significant 10 digits were obtained.

```
m = 6;
z[n_] := 2^n/(2^n - 1) ( 1/(2 (n - 1)! (n - 1)) + Sum[Sum[(2 r - 1)^s/s! e^-(2 r-1)/(2 r - 1)^n, {s, 0, n-1}], {r, 1, m}]
   + 1/2 Sum[Binomial[-n, 2 r - 1] (2^(2 r) - 2) BernoulliB[2 r]/(2 r (n - 1 + 2 r)!), {r, 1, m}] )
```

```
N[z[7], 10]          N[Zeta[7], 10]
1.008349277          1.008349277
```

Formula 2.3.1'

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and n is two or more natural number, the following expression holds for $0<x\leqq\pi$.

$$\zeta(n)=\frac{2^n}{2^n-1}\left\{\frac{x^{n-1}}{2(n-1)!}\left(\frac{1}{n-1}-\log\frac{x}{2}\right)+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{\{x(2r-1)\}^s}{s!}\frac{e^{-x(2r-1)}}{(2r-1)^n}\right.$$
$$\left.+\frac{1}{2}\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{\left(2^{2r}-2\right)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

Especially when $x=1,2$,

$$\zeta(n)=\frac{2^n}{2^n-1}\left\{\frac{1}{2(n-1)!}\left(\frac{1}{n-1}+\log 2\right)+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(2r-1)^s}{s!}\frac{e^{-(2r-1)}}{(2r-1)^n}\right.$$
$$\left.+\frac{1}{2}\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{\left(2^{2r}-2\right)B_{2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n)=\frac{2^n}{2^n-1}\left\{\frac{2^{n-2}}{(n-1)!\,(n-1)}+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(4r-2)^s}{s!}\frac{e^{-(4r-2)}}{(2r-1)^n}\right.$$
$$\left.+\frac{1}{2}\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{\left(2^{2r}-2\right)B_{2r}2^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

Proof

We start with the middle of the previous proof .

$$\lambda(n)=\sum_{s=0}^{n-2}\frac{x^s}{s!}\left\{\frac{(-1)^{n-1-s}x^{n-1-s}}{2(n-1-s)!}\left(\log\frac{x}{2}-H_{n-1-s}\right)+\sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^{n-s}}\right.$$
$$\left.-\frac{(-1)^{n-1-s}}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+n-1-s}}{2r(2r+n-1-s)!}\right\}$$

$$=\frac{(-1)^{n-1}}{2}\frac{x^{n-1}}{(n-1)!}\sum_{s=0}^{n-2}(-1)^s\binom{n-1}{s}\left(\log\frac{x}{2}-H_{n-1-s}\right)$$
$$+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{\{x(2r-1)\}^s}{s!}\frac{e^{-x(2r-1)}}{(2r-1)^n}$$
$$-\frac{1}{2}\sum_{r=1}^{\infty}\frac{(-1)^{n-1}}{(n-1+2r)!}\sum_{s=0}^{n-2}(-1)^s\binom{n-1+2r}{s}\frac{\left(2^{2r}-2\right)B_{2r}x^{n-1+2r}}{2r}$$

Here, as seen in the proof of Formula 2.1.1' ,

$$\sum_{s=0}^{n-2}(-1)^s\binom{n-1+2r}{s}=(-1)^n\binom{1-n}{2r}$$

$$\sum_{s=0}^{n-2}(-1)^s\binom{n-1}{s}\log\frac{x}{2} = (-1)^n\log\frac{x}{2}$$

$$\sum_{s=0}^{n-2}(-1)^s\binom{n-1}{s}H_{n-1-s} = \frac{(-1)^n}{n-1}$$

Substituting these for the above,

$$\lambda(n) = \frac{x^{n-1}}{2(n-1)!}\left(\frac{1}{n-1}-\log\frac{x}{2}\right)+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{\{x(2r-1)\}^s}{s!}\frac{e^{-x(2r-1)}}{(2r-1)^n}$$
$$+\frac{1}{2}\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{(2^{2r}-2)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

And using $\zeta(n) = \dfrac{2^n}{2^n-1}\lambda(n)$, we obtain the desired exression.

Note

When x=1, the convergence speed of this formula is the fastest in this chapter.

Example 1

$$\zeta(3)=\frac{8}{7}\left\{\frac{1}{4}\left(\frac{1}{2}+\log 2\right)+\sum_{r=1}^{\infty}\left(1+\frac{2r-1}{1!}\right)\frac{e^{-(2r-1)}}{(2r-1)^3}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-2}{2r}\frac{(2^{2r}-2)B_{2r}}{2r(2+2r)!}\right\}$$

$$\zeta(3)=\frac{8}{7}\left\{\frac{1}{2}+\sum_{r=1}^{\infty}\left(1+\frac{4r-2}{1!}\right)\frac{e^{-(4r-2)}}{(2r-1)^3}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-2}{2r}\frac{(2^{2r}-2)B_{2r}2^{2+2r}}{2r(2+2r)!}\right\}$$

Example 2 $\zeta(7)$

According to the formula at x=1 this is calculated. As the result of calculating the series to the 6 th term, the significant 10 digits were obtained.

```
m = 6;
```

$$z[n\_] := \frac{2^n}{2^n-1}\left(\frac{1}{2\,(n-1)!}\left(\frac{1}{n-1}+\text{Log}[2]\right)+\sum_{r=1}^{m}\sum_{s=0}^{n-2}\frac{(2r-1)^s}{s!}\frac{e^{-(2r-1)}}{(2r-1)^n}\right.$$
$$\left.+\frac{1}{2}\sum_{r=1}^{m}\text{Binomial}[1-n,2r]\frac{(2^{2r}-2)\,\text{BernoulliB}[2r]}{2r\,(n-1+2r)!}\right)$$

```
N[z[7], 10]        N[Zeta[7], 10]
1.008349277        1.008349277
```

2.4 Formulas of trigonometric function family

The following formulas for zeta at natural number are obtained also from each family of cot x, tan and csc x.

(1) For a complex number $x=u+vi \quad s.t. \quad 0<|x| \leqq 2\pi \ , v \geqq 0$

$$\zeta(n) = \frac{(-ix)^{n-1}}{(n-1)!}\left(\frac{1}{n-1}+\frac{ix}{2n}\right) + \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-irx)^s}{s!}\frac{e^{irx}}{r^n} + (-ix)^{n-1}\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{|B_{2r}|x^{2r}}{2r(n-1+2r)!}$$

$$\zeta(n)= \frac{(-ix)^{n-1}}{(n-1)!}\left\{\frac{1}{n-1}-\frac{(n-1)ix}{2n}-\log x+\frac{i\pi}{2}\right\} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(-irx)^s}{s!}\frac{e^{irx}}{r^n} + (-ix)^{n-1}\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{|B_{2r}|x^{2r}}{2r(n-1+2r)!}$$

(2) For a complex number $x=u+vi \quad s.t. \quad 0<|x| \leqq \pi \ , v \geqq 0$

$$\zeta(n) = \frac{2^{n-1}}{2^{n-1}-1}\left\{\frac{(-ix)^n}{2n!} - \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-irx)^s}{s!}\frac{(-1)^r e^{irx}}{r^n} - (-ix)^{n-1}\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{(-ix)^{n-1}}{2(n-1)!(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{\{-(2r-1)ix\}^s}{s!}\frac{e^{(2r-1)ix}}{(2r-1)^n} - \frac{(-ix)^{n-1}}{2}\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-2\right)|B_{2r}|x^{2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{(-ix)^{n-1}}{2(n-1)!}\left(\frac{1}{n-1}-\log\frac{x}{2}+\frac{i\pi}{2}\right) + \sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{\{-(2r-1)ix\}^s}{s!}\frac{e^{(2r-1)ix}}{(2r-1)^n} - \frac{(-ix)^{n-1}}{2}\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{\left(2^{2r}-2\right)|B_{2r}|x^{2r}}{2r(n-1+2r)!}\right\}$$

These were drawn from Formula 1.4.4, 1.5.4 and 1.6.4 in " **1 Zeta Generating Functions** " in a way similar to previous 3 sections.

However, these are the same as the formulas in previous 3 sections. That is, By replacing x with ix , these reduce to the previous 3 formulas. Conversely, by replacing x with $-ix$, the formulas in previous 3 sections reduce to these. Therefore, these were described only as reference.

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Alien's Mathematics