3 Formulas for Riemann Zeta at odd number

Formulas for Zeta at odd number obtained in " **1 Zeta Generating Functions** " were automorphisms which were expressed by the lower zetas. In this chapter, removing the lower zetas, we obtain the explicit formulas.

3.1 Formulas of cot x family

Formula 3.1.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and $H_s=\sum_{t=1}^{s}1/t$ are Harmonic numbers, the following expressions hold for $0<x<2\pi$.

$$\zeta(2n+1) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}(-1)^s\frac{(2^{2s}-2)B_{2s}(rx)^{2s}}{(2s)!}\frac{\sin rx}{r^{2n+2}}$$

$$+(-1)^n x^{2n}\left\{\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

Especially when $x=\pi$,

$$\zeta(2n+1) =$$

$$(-1)^n\pi^{2n}\left\{\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|\pi^{2r}}{2r}\right\}$$

Proof

We obtained the following Riemann Zeta in Formula1.4.3 in **1.4 .**

$$\zeta(3) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin rx}{r^4}-\frac{x^2}{3!}(\log x-H_3)+\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+2}}{2r(2r+3)!}$$

$$\zeta(5) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin rx}{r^6}+\frac{x^4}{5!}(\log x-H_5)-\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+4}}{2r(2r+5)!}+\frac{x^2}{3!}\zeta(3)$$

$$\zeta(7) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin rx}{r^8}-\frac{x^6}{7!}(\log x-H_7)+\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+6}}{2r(2r+7)!}+\frac{x^2}{3!}\zeta(5)-\frac{x^4}{5!}\zeta(3)$$

$$\zeta(9) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin rx}{r^{10}}+\frac{x^8}{9!}(\log x-H_9)-\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+8}}{2r(2r+9)!}+\frac{x^2}{3!}\zeta(7)-\frac{x^4}{5!}\zeta(5)+\frac{x^6}{7!}\zeta(3)$$

$$\vdots$$

Substituting $\zeta(k)$ for the below one by one, we obtain

$$\zeta(2n+1) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^s C_s x^{2s}\frac{\sin rx}{r^{2n+2-2s}}$$

$$-(-1)^n x^{2n}\sum_{s=0}^{n-1}\frac{C_s}{(2n+1-2s)!}(\log x-H_{2n+1-2s})$$

$$+(-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{C_s}{(2n+1+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}$$

Where, C_s are rational numberes as follows.

$$C_0=-1\ ,\ C_1=\frac{1}{3!}\ ,\ C_2=\frac{1}{5!}-\frac{1}{3!3!}\ ,\ C_3=\frac{1}{7!}-\left(\frac{1}{3!5!}+\frac{1}{5!3!}\right)+\frac{1}{3!3!3!}\ ,$$

$$C_4 = \frac{1}{9!} - \left(\frac{1}{3!7!} + \frac{1}{5!5!} + \frac{1}{7!3!}\right) + \left(\frac{1}{3!3!5!} + \frac{1}{3!5!3!} + \frac{1}{5!3!3!}\right) - \frac{1}{3!3!3!3!}$$

$\vdots$

In fact, these are the coefficients of Taylor series of $\csc x$, and are given by 2^{2s}, factorial and Bernoulli number as follows.

$$C_s = \frac{2^{2s}-2}{(2s)!} B_{2s} \qquad s=0,1,2,\cdots$$

Substituting this for the above,

$$\zeta(2n+1) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^s \frac{(2^{2s}-2)B_{2s}(rx)^{2s}}{(2s)!}\frac{\sin rx}{r^{2n+2}} - (-1)^n x^{2n} \times$$

$$\left\{\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1-2s)!}(\log x - H_{2n+1-2s}) - \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2s}|x^{2s}}{2r}\right\}$$

The inside of the parenthesis of the right side can be transformed as follows.

$$\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1-2s)!}(\log x - H_{2n+1-2s}) - \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2s}|x^{2s}}{2r}$$

$$= \sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}\log x}{(2s)!(2n+1-2s)!} - \sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}$$

$$- \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r} + \frac{(2^{2n}-2)B_{2n}}{(2n)!}\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r}}{2r(1+2r)!}$$

Here, from ***Lemma* 3.4.1** and ***Lemma* 3.4.2c** ,

$$\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1-2s)!} = -\frac{(2^{2n}-2)B_{2n}}{(2n)!}$$

$$\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r}}{2r(2r+1)!} = \log x - 1 + \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin rx}{r^2}$$

Using these,

$$\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1-2s)!}(\log x - H_{2n+1-2s}) - \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2s}|x^{2s}}{2r}$$

$$= -\frac{(2^{2n}-2)B_{2n}}{(2n)!}\log x - \sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}$$

$$- \sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r} + \frac{(2^{2n}-2)B_{2n}}{(2n)!}\left\{\log x - 1 + \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin rx}{r^2}\right\}$$

i.e.

$$\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1-2s)!}(\log x - H_{2n+1-2s}) - \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2s}|x^{2s}}{2r}$$

$$= -\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!} - \sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}$$

$$+ \frac{1}{x}\frac{(2^{2n}-2)B_{2n}}{(2n)!}\sum_{r=1}^{\infty}\frac{\sin rx}{r^2}$$

Substituting this in the above parenthesis

$$\zeta(2n+1) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^s\frac{(2^{2s}-2)B_{2s}(rx)^{2s}}{(2s)!}\frac{\sin rx}{r^{2n+2}} - (-1)^n x^{2n}\times$$

$$\left\{-\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!} - \sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r} + \frac{1}{x}\frac{(2^{2n}-2)B_{2n}}{(2n)!}\sum_{r=1}^{\infty}\frac{\sin rx}{r^2}\right\}$$

$$= -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^s\frac{(2^{2s}-2)B_{2s}(rx)^{2s}}{(2s)!}\frac{\sin rx}{r^{2n+2}} - \frac{(-1)^n}{x}\frac{(2^{2n}-2)B_{2n}x^{2n}}{(2n)!}\sum_{r=1}^{\infty}\frac{\sin rx}{r^2}$$

$$+(-1)^n x^{2n}\left\{\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!} + \sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

Here,

$$\sum_{r=1}^{\infty}(-1)^n\frac{(2^{2n}-2)B_{2n}x^{2n}}{(2n)!}\frac{\sin rx}{r^2} = \sum_{r=1}^{\infty}(-1)^n\frac{(2^{2n}-2)B_{2n}(rx)^{2n}}{(2n)!}\frac{\sin rx}{r^{2n+2}}$$

Thus,

$$\zeta(2n+1) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}(-1)^s\frac{(2^{2s}-2)B_{2s}(rx)^{2s}}{(2s)!}\frac{\sin rx}{r^{2n+2}}$$

$$+(-1)^n x^{2n}\left\{\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!} + \sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

Note

When $x=\pi$, the convergence speed of this formula is the fastest in the Trigonometric functions family.

Example 1

$$\zeta(5) = \pi^4\left\{\frac{269}{21600} + \sum_{r=1}^{\infty}\left(-\frac{1}{(2r+5)!} + \frac{1}{6(2r+3)!} - \frac{7}{360(2r+1)!}\right)\frac{|B_{2r}|\pi^{2r}}{2r}\right\}$$

Example 2 $\zeta(7)$

According to the formula at $x=\pi$ this is calculated. As the result of calculating the series to the 12 th term, the significant 10 digits were obtained.

```
m = 12;
z[n_] := (-1)^n π^(2 n) ( Sum[(2^(2 s) - 2) BernoulliB[2 s] HarmonicNumber[2 n + 1 - 2 s] / ((2 s)! (2 n + 1 - 2 s)!), {s, 0, n}] +
    Sum[(2^(2 s) - 2) BernoulliB[2 s] / ((2 s)! (2 n + 1 + 2 r - 2 s)!) Abs[BernoulliB[2 r]] π^(2 r) / (2 r), {r, 1, m}, {s, 0, n}] )
{N[z[3], 10], N[Zeta[7], 10]}
{1.008349277 , 1.008349277 }
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Formula 3.1.2

Let $H_s = \sum_{t=1}^{s} 1/t$ be Harmonic number, and let Bernoulli numbers and Euler numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30, \cdots$

$$E_0=1,\ E_2=-1,\quad E_4=5,\qquad E_6=-61,\quad E_8=1385,\ \cdots$$

Then, the following expression holds for $0<x<2\pi$.

$$\zeta(2n+1) = \sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx}{r^{2n+1}}$$

$$-(-1)^n x^{2n}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

Proof

We obtained the following Riemann Zeta in Formula1.4.3 in **1.4 .**

$$\zeta(3) = \sum_{r=1}^{\infty}\frac{\cos rx}{r^3}-\frac{x^2}{2!}(\log x-H_2)+\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+2}}{2r(2r+2)!}$$

$$\zeta(5) = \sum_{r=1}^{\infty}\frac{\cos rx}{r^5}+\frac{x^4}{4!}(\log x-H_4)-\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+4}}{2r(2r+4)!}+\frac{x^2}{2!}\zeta(3)$$

$$\zeta(7) = \sum_{r=1}^{\infty}\frac{\cos rx}{r^7}-\frac{x^6}{6!}(\log x-H_6)+\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+6}}{2r(2r+6)!}+\frac{x^2}{2!}\zeta(5)-\frac{x^4}{4!}\zeta(3)$$

$$\zeta(9) = \sum_{r=1}^{\infty}\frac{\cos rx}{r^9}+\frac{x^8}{8!}(\log x-H_8)-\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+8}}{2r(2r+8)!}+\frac{x^2}{2!}\zeta(7)-\frac{x^4}{4!}\zeta(5)+\frac{x^6}{6!}\zeta(3)$$

$$\vdots$$

Substituting $\zeta(k)$ for the below one by one, we obtain

$$\zeta(2n+1) = \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{C_s x^{2s}}{r^{2n+1-2s}}\cos rx+(-1)^n x^{2n}\sum_{s=0}^{n-1}\frac{(-1)^s C_s}{(2n-2s)!}(\log x-H_{2n-2s})$$

$$-(-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s C_s}{(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}$$

Where, C_s are rational numberes as follows.

$$C_0=\frac{1}{0!}\ ,\ C_1=\frac{1}{2!}\ ,\ C_2=-\frac{1}{4!}+\frac{1}{2!2!}\ ,\ C_3=\frac{1}{6!}-\left(\frac{1}{4!2!}+\frac{1}{2!4!}\right)+\frac{1}{2!2!2!}\ ,$$

$$C_4=-\frac{1}{8!}+\left(\frac{1}{6!2!}+\frac{1}{4!4!}+\frac{1}{2!6!}\right)-\left(\frac{1}{4!2!2!}+\frac{1}{2!4!2!}+\frac{1}{2!2!4!}\right)+\frac{1}{2!2!2!2!}$$

$$\vdots$$

According to Mr. Sugimoto , these are given by the following expression.

$$C_s = (-1)^s\frac{E_{2s}}{(2s)!} = \frac{|E_{2s}|}{(2s)!}\qquad s=0,1,2,\cdots$$

Substituting this for the above,

$$\zeta(2n+1) = \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx}{r^{2n+1}}+(-1)^n x^{2n}\times$$

$$\left\{\sum_{s=0}^{n-1}\frac{E_{2s}(\log x-H_{2n-2s})}{(2s)!(2n-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

The inside of the parenthesis of the right side can be transformed as follows.

$$\sum_{s=0}^{n-1}\frac{E_{2s}(\log x-H_{2n-2s})}{(2s)!(2n-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}$$

$$=\sum_{s=0}^{n-1}\frac{E_{2s}\log x}{(2s)!(2n-2s)!}-\sum_{s=0}^{n-1}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}$$

$$-\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}+\frac{E_{2n}}{(2n)!}\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r}}{2r(2r)!}$$

Here, from ***Lemma* 3.4.1** and ***Lemma* 3.4.2c** ,

$$\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!(2n-2s)!}=-\frac{E_{2n}}{(2n)!}$$

$$\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r}}{2r(2r)!}=\log x+\sum_{r=1}^{\infty}\frac{\cos rx}{r}$$

Using these,

$$\sum_{s=0}^{n-1}\frac{E_{2s}(\log x-H_{2n-2s})}{(2s)!(2n-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}$$

$$=-\frac{E_{2n}}{(2n)!}\log x-\sum_{s=0}^{n-1}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}$$

$$-\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}+\frac{E_{2n}}{(2n)!}\left(\log x+\sum_{r=1}^{\infty}\frac{\cos rx}{r}\right)$$

Since $H_{2n-2n}=0$,

$$\sum_{s=0}^{n-1}\frac{E_{2s}(\log x-H_{2n-2s})}{(2s)!(2n-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}$$

$$=-\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}+\frac{E_{2n}}{(2n)!}\sum_{r=1}^{\infty}\frac{\cos rx}{r}$$

Substituting this in the above parenthesis

$$\zeta(2n+1)=\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx}{r^{2n+1}}+\frac{(-1)^nE_{2n}x^{2n}}{(2n)!}\sum_{r=1}^{\infty}\frac{\cos rx}{r}$$

$$-(-1)^nx^{2n}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

Here,

$$\frac{(-1)^nE_{2n}x^{2n}}{(2n)!}\sum_{r=1}^{\infty}\frac{\cos rx}{r}=\frac{|E_{2n}|(rx)^{2n}}{(2n)!}\sum_{r=1}^{\infty}\frac{\cos rx}{r^{2n+1}}$$

Thus,

$$\zeta(2n+1)=\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx}{r^{2n+1}}$$

$$-(-1)^nx^{2n}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

Example $\zeta(7)$

When $x=1/64$, this is calculated according to the formula. As the result of calculating the series to the 24th term, the significant 10 digits were obtained.

```
m = 24;
z[n_, x_] := Sum[Sum[Abs[EulerE[2 s]] (r x)^(2 s)/(2 s)!, {s, 0, n}] Cos[r x]/r^(2 n + 1), {r, 1, m}] -
    (-1)^n x^(2 n) (Sum[EulerE[2 s] HarmonicNumber[2 n - 2 s]/((2 s)! (2 n - 2 s)!), {s, 0, n}] +
       Sum[Sum[EulerE[2 s]/((2 s)! (2 n + 2 r - 2 s)!), {s, 0, n}] Abs[BernoulliB[2 r]] x^(2 r)/(2 r), {r, 1, m}])
{N[z[3, 1/64], 10], N[Zeta[7], 10]}
{1.008349277 , 1.008349277 }
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In 2D notation:

$$z[n\_, x\_] := \sum_{r=1}^{m}\sum_{s=0}^{n}\frac{\text{Abs}[\text{EulerE}[2\,s]]\,(r\,x)^{2\,s}}{(2\,s)!}\,\frac{\text{Cos}[r\,x]}{r^{2\,n+1}} - (-1)^{n}\,x^{2\,n}\left(\sum_{s=0}^{n}\frac{\text{EulerE}[2\,s]\,\text{HarmonicNumber}[2\,n-2\,s]}{(2\,s)!\,(2\,n-2\,s)!} + \sum_{r=1}^{m}\sum_{s=0}^{n}\frac{\text{EulerE}[2\,s]}{(2\,s)!\,(2\,n+2\,r-2\,s)!}\,\frac{\text{Abs}[\text{BernoulliB}[2\,r]]\,x^{2\,r}}{2\,r}\right)$$

3.2 Formulas of tan x family

Formula 3.2.1

Let Bernoulli numbers and Tangent numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30, \cdots$

$T_1=1,\ T_3=2,\ T_5=16,\ T_7=272,\ T_9=7936, \cdots$

Then, the following expression holds for $0<x\leqq\pi$.

$$\zeta(2n+1) = \frac{2^{2n}}{2^{2n}-1}\left\{\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)(rx)^{2s}}{(2s)!}\frac{(-1)^r \sin rx}{r^{2n+2}} - (-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!\,(2n+1+2r-2s)!}\frac{\left(2^{2r}-1\right)|B_{2r}|\,x^{2r}}{2r}\right\}$$

Especially when $x=\pi$,

$$\zeta(2n+1) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2^{2n}-1}\sum_{r=1}^{\infty}\left\{\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!\,(2n+1+2r-2s)!}\right\}T_{2r-1}\left(\frac{\pi}{2}\right)^{2r}$$

Proof

We obtained the following Riemann Zeta in Formula1.5.3 in **1.5 .**

$$\eta(1) = \frac{1}{x}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^2} + \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+1)!}x^{2r}$$

$$\eta(3) = \frac{1}{x}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^4} - \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+3)!}x^{2r+2} + \frac{x^2}{3!}\eta(1)$$

$$\eta(5) = \frac{1}{x}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^6} + \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+5)!}x^{2r+4} + \frac{x^2}{3!}\eta(3) - \frac{x^4}{5!}\eta(1)$$

$$\eta(7) = \frac{1}{x}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^8} - \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+7)!}x^{2r+6} + \frac{x^2}{3!}\eta(5) - \frac{x^4}{5!}\eta(3) + \frac{x^6}{7!}\eta(1)$$

$$\vdots$$

Substituting $\eta(k)$ for the below one by one, we obtain

$$\eta(2n+1) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}(-1)^s C_s x^{2s}\frac{(-1)^{r-1}\sin rx}{r^{2n+2-2s}} - (-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{C_s}{(2n+1+2r-2s)!}\frac{\left(2^{2r}-1\right)|B_{2r}|\,x^{2r}}{2r}$$

Where, C_s is the same as the coefficient in Formula 3.1.1 and is given by $C_s=\dfrac{2^{2s}-2}{(2s)!}B_{2s}$. Then ,

$$\eta(2n+1) = \frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)(rx)^{2s}}{(2s)!}\frac{(-1)^r \sin rx}{r^{2n+2}}$$

$$- (-1)^n x^{2n} \sum_{r=1}^{\infty} \sum_{s=0}^{n} \frac{(2^{2s}-2)B_{2s}}{(2s)!\,(2n+1+2r-2s)!} \frac{2^{2r}(2^{2r}-1)|B_{2r}|}{2r} \left(\frac{\pi}{2}\right)^{2r}$$

And using $\zeta(2n+1) = \dfrac{2^{2n}}{2^{2n}-1}\eta(2n+1)$, $T_{2r-1} = \dfrac{2^{2r}(2^{2r}-1)|B_{2r}|}{2r}$, we obtain the desired expression.

Example1

$$\zeta(5) = -\frac{16\pi^4}{15} \sum_{r=1}^{\infty} \left\{ -\frac{1}{(2r+5)!} + \frac{1}{6(2r+3)!} - \frac{7}{360(2r+1)!} \right\} \frac{(2^{2r}-1)|B_{2r}|\,\pi^{2r}}{2r}$$

Example2 $\zeta(7)$

According to the formula at $x=\pi$ this is calculated. As the result of calculating the 3000 terms, the significant 4 digits were obtained.

```
m = 3000;
z[n_] := (-1)^(n-1) (2 π)^(2 n)/(2^(2 n) - 1) Sum[Sum[(2^(2 s) - 2) BernoulliB[2 s]/((2 s)! (2 n + 1 + 2 r - 2 s)!) (2^(2 r) - 1) Abs[BernoulliB[2 r]] π^(2 r)/(2 r), {s, 0, n}], {r, 1, m}]

{N[z[3]], N[Zeta[7]]}
{1.00802 , 1.00835 }
```

As seen in Example2, the convergence of the formula at $x=\pi$ is vary slow. Then, from **Lemma 3.4.2t**,

$$\sum_{r=1}^{\infty} \frac{T_{2r-1}}{(2r+1)!} \left(\frac{\pi}{2}\right)^{2r} = \sum_{r=1}^{\infty} \frac{(2^{2r}-1)|B_{2r}|\,\pi^{2r}}{2r(2r+1)!} = \log 2$$

If the formula at $x=\pi$ is transformed as follows using this, the convergence is faster.

Formula 3.2.1'

$$\zeta(2n+1) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2^{2n}-1} \left\{ \sum_{r=1}^{\infty}\sum_{s=0}^{n-1} \frac{(2^{2s}-2)B_{2s}}{(2s)!\,(2n+1+2r-2s)!} \frac{(2^{2r}-1)|B_{2r}|\,\pi^{2r}}{2r} + \frac{(2^{2n}-2)B_{2n}}{(2n)!}\log 2 \right\}$$

Example2' $\zeta(7)$

The same calculation as Example2 was performed by this formula . As the result of calculating the 55 terms, the significant 6 digits were obtained.

```
m = 55;
z[n_] := (-1)^(n-1) (2 π)^(2 n)/(2^(2 n) - 1) (Sum[Sum[(2^(2 s) - 2) BernoulliB[2 s]/((2 s)! (2 n + 1 + 2 r - 2 s)!) (2^(2 r) - 1) Abs[BernoulliB[2 r]] π^(2 r)/(2 r), {s, 0, n-1}], {r, 1, m}]
    + (2^(2 n) - 2) BernoulliB[2 n]/(2 n)! Log[2])

{N[z[3]], N[Zeta[7]]}
{1.00835, 1.00835}
```

Formula 3.2.2

Let Bernoulli numbers and Euler numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$

$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$

Then, the following expression holds for $0<x\leqq\pi$.

$$\zeta(2n+1) = -\frac{2^{2n}}{2^{2n}-1}\left\{\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{(-1)^r\cos rx}{r^{2n+1}}\right.$$
$$\left.-(-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r}}{2r}\right\}$$

Proof

We obtained the following Riemann Zeta in Formula1.5.3 in **1.5 .**

$$\eta(1) = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^1}+\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r)!}x^{2r}$$

$$\eta(3) = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^3}-\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+2)!}x^{2r+2}+\frac{x^2}{2!}\eta(1)$$

$$\eta(5) = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^5}+\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+4)!}x^{2r+4}+\frac{x^2}{2!}\eta(3)-\frac{x^4}{4!}\eta(1)$$

$$\eta(7) = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^7}-\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+6)!}x^{2r+6}+\frac{x^2}{2!}\eta(5)-\frac{x^4}{4!}\eta(3)+\frac{x^6}{6!}\eta(1)$$

$$\vdots$$

Substituting $\eta(k)$ for the below one by one, we obtain

$$\eta(2n+1)=\sum_{r=1}^{\infty}\sum_{s=0}^{n}C_s x^{2s}\frac{(-1)^{r-1}\cos rx}{r^{2n+1-2s}}+(-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n}(-1)^s C_s\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r}}{2r(2r+2n-2s)!}$$

Where, C_s is the same as the coefficient in the Formula 3.1.2 and is given by

$$C_s = (-1)^s\frac{E_{2s}}{(2s)!} = \frac{|E_{2s}|}{(2s)!}$$

Then,

$$\eta(2n+1) = -\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{(-1)^r\cos rx}{r^{2n+1}}$$
$$+(-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r}}{2r}$$

And using $\zeta(2n+1) = \dfrac{2^{2n}}{2^{2n}-1}\eta(2n+1)$, we obtain the desired expression.

3.3 Formulas of csc x family

Formula 3.3.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and $H_s=\sum_{t=1}^{s}1/t$ are Harmonic numbers, the following expressions hold for $0<x\leqq\pi$.

$$\zeta(2n+1) = -\frac{2^{2n+1}}{2^{2n+1}-1}\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s\left(2^{2s}-2\right)B_{2s}\{(2r-1)x\}^{2s}}{(2s)!}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n+2}}$$

$$+\frac{(-1)^n(2x)^{2n}}{2^{2n+1}-1}\left\{\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}\left(2^{2r}-2\right)|B_{2r}|}{(2s)!(2n+1+2r-2s)!}\frac{x^{2r}}{2r}\right\}$$

Especially when $x=\pi$,

$$\zeta(2n+1) = \frac{(-1)^n(2\pi)^{2n}}{2^{2n+1}-1}\left\{\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}\left(2^{2r}-2\right)|B_{2r}|}{(2s)!(2n+1+2r-2s)!}\frac{\pi^{2r}}{2r}\right\}$$

Proof

Originally, this should be derived from Formula1.6.3 in **1.6 .** However, since it is complicated, we derive this from the previous two sections. From Formula3.1.1 and Formula3.2.1 ,

$$\zeta(2n+1) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}(-1)^s\frac{\left(2^{2s}-2\right)B_{2s}(rx)^{2s}}{(2s)!}\frac{\sin rx}{r^{2n+2}}$$

$$+(-1)^n x^{2n}\left\{\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

$$\eta(2n+1) = \frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(-1)^sB_{2s}\left(2^{2s}-2\right)(rx)^{2s}}{(2s)!}\frac{(-1)^r\sin rx}{r^{2n+2}}$$

$$-(-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r}}{2r}$$

Adding both,

$$\zeta(2n+1)+\eta(2n+1) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s\left(2^{2s}-2\right)B_{2s}(rx)^{2s}}{(2s)!}\frac{\sin rx-(-1)^r\sin rx}{r^{2n+2}}$$

$$+(-1)^n x^{2n}\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}$$

$$-(-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{\left(2^{2r}-2\right)|B_{2r}|x^{2r}}{2r}$$

Here,

$$\zeta(2n+1)+\eta(2n+1) = \left(1+\frac{2^{2n}-1}{2^{2n}}\right)\zeta(2n+1) = \frac{2^{2n+1}-1}{2^{2n}}\zeta(2n+1)$$

$$\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s\left(2^{2s}-2\right)B_{2s}(rx)^{2s}}{(2s)!}\frac{\sin rx-(-1)^r\sin rx}{r^{2n+2}}$$

$$= 2\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s\left(2^{2s}-2\right)B_{2s}\{(2r-1)x\}^{2s}}{(2s)!}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n+2}}$$

Thus,

$$\zeta(2n+1) = -\frac{2^{2n+1}}{2^{2n+1}-1}\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s(2^{2s}-2)B_{2s}\{(2r-1)x\}^{2s}}{(2s)!}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n+2}}$$
$$+\frac{(-1)^n(2x)^{2n}}{2^{2n+1}-1}\left\{\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}(2^{2r}-2)|B_{2r}|}{(2s)!(2n+1+2r-2s)!}\frac{x^{2r}}{2r}\right\}$$

Note

As seen in the proof, these are the weighted averages of Formula3.1.1 and Formula3.2.1 and are not original formulas. These are only complicated, and there is not much value.

Formula 3.3.2

Let $H_s=\sum_{t=1}^{s}1/t$ be Harmonic number, and let Bernoulli numbers and Euler numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$

$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$

Then, the following expression holds for $0<x\leqq\pi$.

$$\zeta(2n+1) = \frac{2^{2n+1}}{2^{2n+1}-1}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|\{(2r-1)x\}^{2s}}{(2s)!}\frac{\cos\{(2r-1)x\}}{(2r-1)^{2n+1}}$$
$$-\frac{(-1)^n(2x)^{2n}}{2^{2n+1}-1}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}(2^{2r}-2)|B_{2r}|}{(2s)!(2n+2r-2s)!}\frac{x^{2r}}{2r}\right\}$$

Especially when $x=\pi/2$,

$$\zeta(2n+1) = \frac{(-1)^{n-1}\pi^{2n}}{2^{2n+1}-1}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}(2^{2r}-2)|B_{2r}|}{(2s)!(2n+2r-2s)!}\frac{1}{2r}\left(\frac{\pi}{2}\right)^{2r}\right\}$$

Proof

From Formula3.1.2 and Formula3.2.2 ,

$$\zeta(2n+1) = \sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx}{r^{2n+1}}$$
$$-(-1)^n x^{2n}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

$$\eta(2n+1) = -\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{(-1)^r\cos rx}{r^{2n+1}}$$
$$+(-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{(2^{2r}-1)|B_{2r}|x^{2r}}{2r}$$

Adding both,

$$\zeta(2n+1)+\eta(2n+1) = \sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx-(-1)^r\cos rx}{r^{2n+1}}$$
$$-(-1)^n x^{2n}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{(2^{2r}-2)|B_{2r}|x^{2r}}{2r}\right\}$$

Here,

$$\zeta(2n+1)+\eta(2n+1)=\frac{2^{2n+1}-1}{2^{2n}}\zeta(2n+1)$$

$$\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx-(-1)^r\cos rx}{r^{2n+1}}=2\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|\{(2r-1)x\}^{2s}}{(2s)!}\frac{\cos\{(2r-1)x\}}{(2r-1)^{2n+1}}$$

Thus,

$$\zeta(2n+1)=\frac{2^{2n+1}}{2^{2n+1}-1}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|\{(2r-1)x\}^{2s}}{(2s)!}\frac{\cos\{(2r-1)x\}}{(2r-1)^{2n+1}}$$
$$-\frac{(-1)^n(2x)^{2n}}{2^{2n+1}-1}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}(2^{2r}-2)|B_{2r}|}{(2s)!(2n+2r-2s)!}\frac{x^{2r}}{2r}\right\}$$

Note

These are also the weighted averages of Formula3.1.2 and Formula3.2.2 . However, the formula at $x=\pi/2$ is original and is worthy.

Example1

$$\zeta(5)=\frac{\pi^4}{31}\left\{\frac{83}{288}+\sum_{r=1}^{\infty}\left(\frac{1}{(4+2r)!}-\frac{1}{2(2+2r)!}+\frac{5}{24(2r)!}\right)\frac{(2^{2r}-2)|B_{2r}|}{2r}\left(\frac{\pi}{2}\right)^{2r}\right\}$$

Example 2 $\zeta(7)$

According to the formula at $x=\pi/2$ this is calculated. As the result of calculating the series to the 13 th term, the significant 10 digits were obtained.

```
m = 13;
z[n_] := ((-1)^(n-1) π^(2 n))/(2^(2 n+1) - 1) (Sum[(EulerE[2 s] HarmonicNumber[2 n - 2 s])/((2 s)! (2 n - 2 s)!), {s, 0, n}] -
    Sum[(EulerE[2 s] (2^(2 r) - 2) Abs[BernoulliB[2 r]])/((2 s)! (2 n + 2 r - 2 s)!) 1/(2 r) (π/2)^(2 r), {r, 1, m}, {s, 0, n}])

{N[z[3], 10], N[Zeta[7], 10]}
{1.008349277 , 1.008349277 }
```

3.4 Lemmas

Lemma 3.4.1

Let Bernoulli numbers and Euler numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30, \cdots$

$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385, \cdots$

Then, the following expressions hold.

$$\sum_{s=0}^{n} \frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1-2s)!} = 0$$

$$\sum_{s=0}^{n} \frac{E_{2s}}{(2s)!(2n-2s)!} = 0$$

Proof

$$x\,csch\,x = 1 - \sum_{s=1}^{\infty} \frac{(2^{2s}-2)B_{2s}}{(2s)!} x^{2s} \qquad |x|<\pi$$

i.e.

$$\frac{2x}{e^x - e^{-x}} = -\frac{(2^0-2)B_0}{0!} - \frac{(2^2-2)B_2}{2!}x^2 - \frac{(2^4-2)B_4}{4!}x^4 - \cdots$$

Here, let $C_{2s} = -(2^{2s}-2)B_{2s}$. Then

$$\frac{2x}{e^x - e^{-x}} = \frac{C_0}{0!} + \frac{C_2}{2!}x^2 + \frac{C_4}{4!}x^4 + \frac{C_6}{6!}x^6 + \cdots$$

$$\frac{e^x - e^{-x}}{2x} = \frac{1}{1!} + \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \frac{1}{7!}x^6 + \frac{1}{9!}x^8 + \cdots$$

The Cauchy product is

$$1 = C_0 + \left(\frac{C_0}{0!3!} + \frac{C_2}{2!1!} \right)x^2 + \left(\frac{C_0}{0!5!} + \frac{C_2}{2!3!} + \frac{C_4}{4!1!} \right)x^4 +$$

In order to hold this for arbitrary x , the followings are necessary.

$$C_0 = 1,\ \frac{C_0}{0!3!} + \frac{C_2}{2!1!} = 0,\ \frac{C_0}{0!5!} + \frac{C_2}{2!3!} + \frac{C_4}{4!1!} = 0, \cdots$$

That is,

$$\sum_{s=0}^{n} \frac{C_{2s}}{(2s)!(2n+1-2s)!} = \sum_{s=0}^{n} \frac{-(2^{2s}-2)B_{2s}}{(2s)!(2n+1-2s)!} = 0$$

Next,

$$sech\,x = \sum_{s=0}^{\infty} \frac{E_{2s}}{(2s)!} x^{2s} \qquad |x| < \frac{\pi}{2}$$

i.e.

$$\frac{2}{e^x + e^{-x}} = \frac{E_0}{0!} + \frac{E_1}{1!}x^1 + \frac{E_2}{2!}x^2 + \frac{E_3}{3!}x^3 + \frac{E_4}{4!}x^4 + \cdots$$

$$\frac{e^x + e^{-x}}{2} = \frac{1}{0!} + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

The Cauchy product is

$$1 = E_0 + \left(\frac{E_0}{0!1!} + \frac{E_1}{1!0!} \right)x^1 + \left(\frac{E_0}{0!2!} + \frac{E_1}{1!1!} + \frac{E_2}{2!0!} \right)x^2 + \cdots$$

In order to hold this for arbitrary x , the followings are necessary.

$$E_0=1\,,\quad \frac{E_0}{0!1!}+\frac{E_1}{1!0!}=0\,,\quad \frac{E_0}{0!2!}+\frac{E_1}{1!1!}+\frac{E_2}{2!0!}=0\,,\cdots$$

That is,

$$\sum_{s=0}^{n}\frac{E_s}{s!(n-s)!}=0$$

Since $E_1=E_3=E_5\cdots=0$, we obtain

$$\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n-2s)!}=0$$

Lemma 3.4.2c

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers, the following expressions hold for $0<x<2\pi$.

$$\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r}}{2r(2r)!}=\sum_{r=1}^{\infty}\frac{\cos rx}{r}+\log x$$

$$\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r}}{2r(2r+1)!}=\frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin rx}{r^2}+\log x-1$$

Especially, when $x=\pi$,

$$\sum_{r=1}^{\infty}\frac{|B_{2r}|\pi^{2r}}{2r(2r)!}=\log\frac{\pi}{2}$$

$$\sum_{r=1}^{\infty}\frac{|B_{2r}|\pi^{2r}}{2r(2r+1)!}=\log\pi-1$$

Proof

The 1st order integrals of Taylor series and Fourier series of $\cot x$ were as follows. (See " **5 Termwise Higher Integral** ")

$$\int_{\frac{\pi}{2}}^{x}\cot x\,dx=\log x-\sum_{r=1}^{\infty}\frac{2^{2r}|B_{2r}|}{2r(2r)!}x^{2r}\qquad 0<x<\pi$$

$$\int_{\frac{\pi}{2}}^{x}\cot x\,dx=\frac{1}{2^0}\sum_{r=1}^{\infty}\frac{1}{r^1}\sin\left(2rx-\frac{1\pi}{2}\right)-\log 2\qquad 0<x<\pi$$

From these,

$$\sum_{r=1}^{\infty}\frac{2^{2r}|B_{2r}|}{2r(2r)!}x^{2r}=\sum_{r=1}^{\infty}\frac{1}{r^1}\cos(2rx)+\log 2x$$

Replacing x with $x/2$,

$$\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r}}{2r(2r)!}=\sum_{r=1}^{\infty}\frac{\cos rx}{r}+\log x$$

Next, the 2nd order integrals of Taylor series and Fourier series of $\cot x$ were as follows.

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\cot x\,dx^2=x(\log x-1)-\sum_{r=1}^{\infty}\frac{2^{2r}|B_{2r}|}{2r\,(2r+1)!}x^{2r+1}$$
$$-\left\{\frac{\pi}{2}\left(\log\frac{\pi}{2}-1\right)-\sum_{r=1}^{\infty}\frac{2^{2r}|B_{2r}|}{2r(2r+1)!}\left(\frac{\pi}{2}\right)^{2r+1}\right\}$$

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\cot x\,dx^2 = \frac{1}{2^1}\sum_{r=1}^{\infty}\frac{1}{r^2}\sin\left(2rx-\frac{2\pi}{2}\right) - \left(x-\frac{\pi}{2}\right)\log 2$$

From these,

$$x(\log x-1)-\sum_{r=1}^{\infty}\frac{2^{2r}|B_{2r}|}{2r(2r+1)!}x^{2r+1}-\left\{\frac{\pi}{2}\left(\log\frac{\pi}{2}-1\right)-\sum_{r=1}^{\infty}\frac{2^{2r}|B_{2r}|}{2r(2r+1)!}\left(\frac{\pi}{2}\right)^{2r+1}\right\}$$
$$= -\frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{r^2}\sin(2rx) - \left(x-\frac{\pi}{2}\right)\log 2$$

Giving $x=0$ to this,

$$-\left\{\frac{\pi}{2}\left(\log\frac{\pi}{2}-1\right)-\sum_{r=1}^{\infty}\frac{2^{2r}|B_{2r}|}{2r(2r+1)!}\left(\frac{\pi}{2}\right)^{2r+1}\right\} = \frac{\pi}{2}\log 2$$

Therefore,

$$x(\log x-1)-\sum_{r=1}^{\infty}\frac{2^{2r}|B_{2r}|}{2r(2r+1)!}x^{2r+1} = -\frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{r^2}\sin(2rx) - x\log 2$$

From this,

$$\sum_{r=1}^{\infty}\frac{2^{2r}|B_{2r}|x^{2r}}{2r(2r+1)!} = \log(2x) - 1 + \frac{1}{2x}\sum_{r=1}^{\infty}\frac{1}{r^2}\sin(2rx)$$

Replacing x with $x/2$,

$$\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r}}{2r(2r+1)!} = \log x - 1 + \frac{1}{x}\sum_{r=1}^{\infty}\frac{\sin(rx)}{r^2}$$

Lemma 3.4.2t

When $B_0=1, B_2=1/6, B_4=-1/30, B_6=1/42, \cdots$ are Bernoulli numbers, the following expressions hold for $x \leqq |\pi|$.

$$\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r}}{2r(2r)!} = \sum_{r=1}^{\infty}\frac{(-1)^r}{r}\cos rx + \log 2$$

$$\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r}}{2r(2r+1)!} = \frac{1}{x}\sum_{r=1}^{\infty}\frac{(-1)^r}{r^2}\sin rx + \log 2$$

Especially, when $x=\pi$,

$$\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|\pi^{2r}}{2r(2r+1)!} = \log 2$$

Proof

The 1st order integrals of Taylor series and Fourier series of $\tan x$ were as follows. (See " **5 Termwise Higher Integral** ")

$$\int_0^x \tan x\,dx = \sum_{r=1}^{\infty}\frac{2^{2r}\left(2^{2r}-1\right)|B_{2r}|x^{2r}}{2r(2r)!} \qquad |x|<\frac{\pi}{2}$$

$$\int_0^x \tan x\,dx = \frac{1}{2^0}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^1}\sin\left(2rx-\frac{1\pi}{2}\right) + \log 2 \qquad |x|<\frac{\pi}{2}$$

From these,

$$\sum_{r=1}^{\infty}\frac{2^{2r}\left(2^{2r}-1\right)|B_{2r}|x^{2r}}{2r(2r)!} = -\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r}\cos(2rx) + \log 2$$

Replacing x with $x/2$,

$$\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)\left|B_{2r}\right|x^{2r}}{2r(2r)!}=\sum_{r=1}^{\infty}\frac{(-1)^{r}}{r}\cos rx+\log 2$$

Next, the 2nd order integrals of Taylor series and Fourier series of $\tan x$ were as follows.

$$\int_0^x\int_0^x \tan x\,dx^2=\sum_{r=1}^{\infty}\frac{2^{2r}\left(2^{2r}-1\right)\left|B_{2r}\right|}{2r(2r+1)!}x^{2r+1}$$

$$\int_0^x\int_0^x \tan x\,dx^2=\frac{1}{2^1}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^2}\sin\left(2rx-\frac{2\pi}{2}\right)+x\log 2$$

From these,

$$\sum_{r=1}^{\infty}\frac{2^{2r}\left(2^{2r}-1\right)\left|B_{2r}\right|x^{2r}}{2r(2r+1)!}=-\frac{1}{2x}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^2}\sin(2rx)+\log 2$$

Replacing x with $x/2$,

$$\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)\left|B_{2r}\right|x^{2r}}{2r(2r+1)!}=\frac{1}{x}\sum_{r=1}^{\infty}\frac{(-1)^{r}}{r^2}\sin rx+\log 2$$

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