## 01 General Dirichlet Series & Power Series

### 1.1 Definition & Theorems

**Definition 1.1.1 (General Dirichlet Series)**

Let $R$ be a real number set. And let $\sigma, t \in R$ and $\lambda_n \in R$ , $\lambda_n < \lambda_{n+1}$ $\quad n=1,2,3,\cdots$ .
When $s=\sigma+it$ and $a_n$ are complex numbers, we call the following series **General Dirichlet Series**.

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

**Note**

Especially when $\lambda_n = log\ n$ in General Dirichlet Seris, it is called Ordinary Dirichlet Series.
This is discussed in another chapter.

**Theorem 1.1.2**

Let Dirichlet serie $f(s)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$. And assume $f(s)$ is convergent at $s=s_c=\sigma_c+t_c i$ . Then

**1.** $f(s)$ converges uniformly in $|Arg(s-s_c)| \le \theta < \dfrac{\pi}{2}$ .

**2.** $f(s)$ converges for any $s=\sigma+ti \quad s.t. \quad \sigma>\sigma_c$ .

This $\sigma_c$ is called the **line of convergence**. By convention, $\sigma_c=\infty$ if $f(s)$ converges nowhere and $\sigma_c=-\infty$ if $f(s)$ converges everywhere on the complex plane.

**How to calculate $\sigma_c$**

**1.** When $\sum_{k=1}^{n} a_k$ is divergent,

$$\sigma_c = \limsup_{n\to\infty} \frac{log\,|a_1+a_2+\cdots+a_n|}{\lambda_n}$$

**2.** When $\sum_{k=1}^{n} a_k$ is convergent,

$$\sigma_c = \limsup_{n\to\infty} \frac{log\,|a_{n+1}+a_{n+2}+a_{n+3}\cdots|}{\lambda_n}$$

**Absolute Convergence**

A Dirichlet series $f(s)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ is **absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n|\, e^{-\lambda_n s}$ is convergent.
There exist a $\sigma_a\ (\ge \sigma_c)$ such that $f(s)$ converges absolutely for $\sigma>\sigma_a$ and converges non-absolutely for $\sigma<\sigma_a$ . This $\sigma_a$ is called the **line of absolute convergence**.

### How to calculate $\sigma_a$

**1.** When $\sum_{k=1}^{n} |a_k|$ is divergent,

$$\sigma_a = \limsup_{n \to \infty} \frac{\log(|a_1| + |a_2| + \cdots + |a_n|)}{\lambda_n}$$

**2.** When $\sum_{k=1}^{n} |a_k|$ is convergent,

$$\sigma_a = \limsup_{n \to \infty} \frac{\log(|a_{n+1}| + |a_{n+2}| + |a_{n+3}| \cdots)}{\lambda_n}$$

### Uniform Convergence

As seen in Theorem 1.1.2 , Dirichlet serie $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ converges uniformly in a certain domain. If we see it a little in detail, it is as follows.

There exist a $\sigma_u$ $(\sigma_c \le \sigma_u \le \sigma_a)$ such that $f(s)$ converges uniformly for $\sigma > \sigma_u$ and converges non-uniformly for $\sigma < \sigma_u$ . This $\sigma_u$ is called the **line of uniform convergence**.

### How to calculate $\sigma_u$

$$\sigma_u = \limsup_{x \to \infty} \frac{\log T_x}{\log x}$$

where

$$T_x = \sup_{|y|<\infty} \left| \sum_{[x] \le \lambda_n < x} a_n e^{-i \lambda_n y} \right|$$

### Theorem 1.1.3 (Holomorphy)

If Dirichlet serie $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ $(s = \sigma + ti)$ converges for $\sigma > \sigma_c$,

$f(s)$ is **holomophic** at $\sigma > \sigma_c$. And the derivative of $f(s)$ is given as follows.

$$f^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \lambda_n^k a_n e^{-\lambda_n s}$$

### Theorem 1.1.4 (Uniqueness)

Let two Dirichlet series are as follows.

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \quad , \quad g(s) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n s}$$

If both are convergent in a certain domain and $f(s) = g(s)$ holds at there, $a_n = b_n$ for $n = 1, 2, 3, \cdots$ .

### Bibliography


「数論入門」D. B. ザギヤー 著、片山孝次 訳 岩波 1990年, etc.

## 1.2 General Dirichlet Series & Power Series

General Dirichlet Series can contain Power Series other than Ordinary Dirichlet Series. In the following section, we will consider this.

### 1.2.1 Relation between Power Series and General Dirichlet Series

Relations between power series $\sum_{n=1}^{\infty} c_n (z-z_0)^n$ and general Dirichlet series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ are as follows.

$$c_n = a_n \quad , \quad n = \lambda_n \quad , \quad z - z_0 = e^{-s} \tag{2.1}$$

Especially, when the power series starts from $n=0$ ,

$$c_{n-1} = a_n \quad , \quad n-1 = \lambda_n \quad , \quad z - z_0 = e^{-s} \tag{2.0}$$

**Calculation**

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n = \sum_{n=1}^{\infty} c_{n-1} (z-z_0)^{n-1} = \sum_{n=1}^{\infty} a_n (e^{-s})^{\lambda_n} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \sum_{n=1}^{\infty} c_{n-1} (e^{-s})^{\lambda_n} = \sum_{n=1}^{\infty} c_{n-1} (z-z_0)^{n-1} = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

**Note**

When a function $f(z)$ is expressed with a power series, Dirichlet series equivalent to this does not express $f(s)$ . Since $z - z_0 = e^{-s}$, it is $f(z) = f(e^{-s} + z_0)$ . That is, Dirichlet series equivalent to the power series express the composition $f$ .

### 1.2.2 Circle of convergence & Line of convergence

As seen above, the relation between the variable $z - z_0$ of power series and the variable $s$ of Dirichlet series was as follows..

$$z - z_0 = e^{-s}$$

That is,

$$s = -log(z-z_0) = -log|z-z_0| - i\,arg(z-z_0)$$

Here, let $s = \sigma + it$ , $z = x + iy$ . Then

$$\sigma + it = -log|(x-x_0) + i(y-y_0)| - i\,arg\{(x-x_0) + i(y-y_0)\}$$

Although coordinates $x-y$ are transferred to coordinates $\sigma - t$ by the conversion, the coordinates $\sigma - t$ are similar to pola coordinates $r - \theta$ . The difference among both is only the existence of $log$ in the real part.

From this,

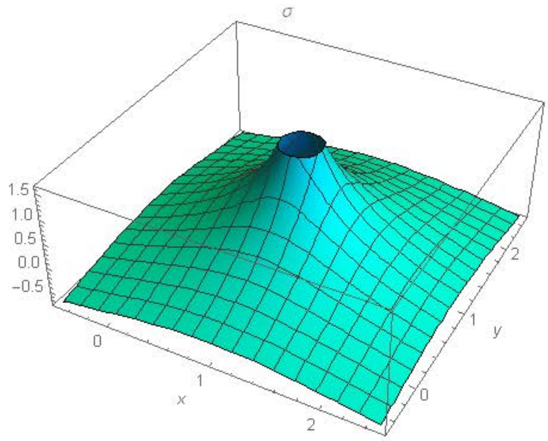
$$\sigma = -log\sqrt{(x-x_0)^2 + (y-y_0)^2} \tag{2.σ}$$

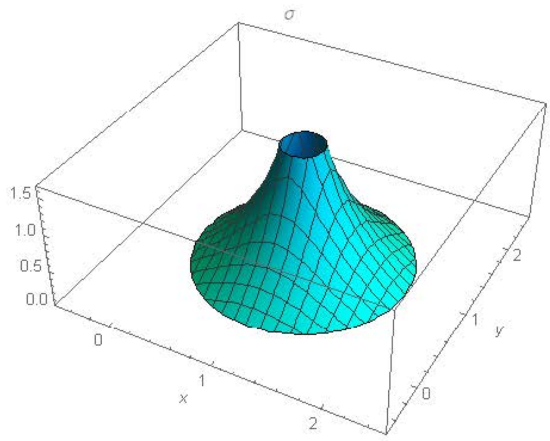
$$t = -arg\{(x-x_0) + i(y-y_0)\} \tag{2.t}$$

When $z_0 = x_0 + iy_0 = 1 + i1$ , the real part (2.$\sigma$) is drawn on the left figure. All the horizontal sections of the real part (2.$\sigma$) are circles centered on the point $(1,1)$ . Radius of each circle is given by

$$\sqrt{(x-1)^2 + (y-1)^2}$$

For example, if the radius is $1$ , $\sigma = -log\,1 = 0$ from (2.$\sigma$) . If the left figure is horizontally cut at this height, it is the right figure.




When the radius is $1$ , the power series converges in the circle. This means that the corresponding Dirichlet series should converge at the upper than the circle i.e. $\sigma > 0$. The height $\boldsymbol{\sigma_c = -log\,R}$ which provides the convergence circle with the radius $R$ is the **line of convergence**.

So far, we have seen the relationship between circle of convergence and line of convergence in the figure. It is as follows analytically.

using $z - z_0 = e^{-s}$

$$e^{-\sigma_c} \equiv R > \left|z - z_0\right| = \left|e^{-s}\right| = \left|e^{-\sigma - it}\right| = e^{-\sigma}$$

i.e.

$$e^{-\sigma} < e^{-\sigma_c} = R$$

Taking the logarithm of both sides, we obtain

$$\sigma > \sigma_c = -logR \tag{2.2}$$

**c.f.**

According to the definition in **1.1** , the convergence line $\sigma_c$ is calculated as follows.

When $\displaystyle\sum_{n=1}^{\infty} c_n \left( or \sum_{n=0}^{\infty} c_n \right)$ is convergent, $\displaystyle\sum_{k=1}^{n} c_k$ is also convergent. Then

$$\sigma_c = \lim_{n\to\infty} sup \frac{log\left|a_{n+1} + a_{n+2} + a_{n+3} \cdots\right|}{\lambda_n} = \lim_{n\to\infty} sup \frac{log\left|c_{n+1} + c_{n+2} + c_{n+3} \cdots\right|}{n}$$

The calculation by this formula is difficult in many cases. It is better to use the above (2.2) .

**Byproduct**

From the above and $R = \lim_{n\to\infty} inf\left|c_n\right|^{-1/n}$, $\sigma_c = -logR$ , we obtain the following

$$\lim_{n\to\infty} sup\left|c_{n+1} + c_{n+2} + c_{n+3} \cdots\right|^{1/n} = \lim_{n\to\infty} sup\left|c_n\right|^{1/n} \ (= 1/R) \tag{2.3}$$

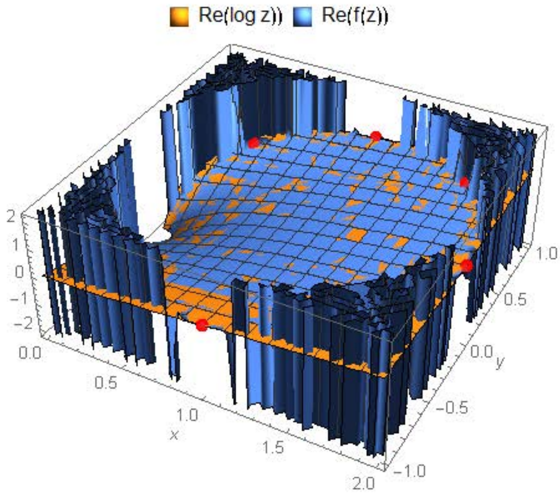
### 1.3 Dirichlet series expression of primary functions

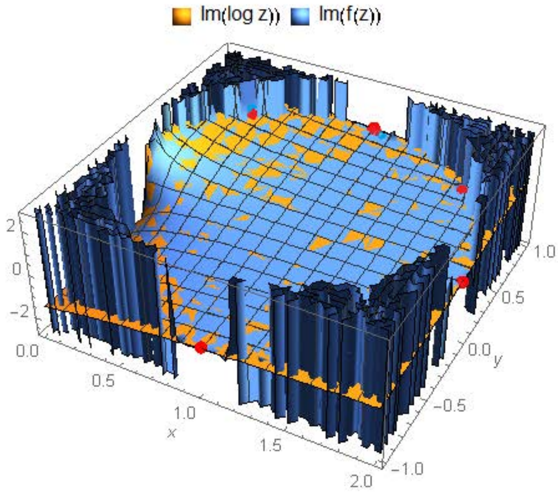
### 1.3.1 Dirichlet series expression of logarithmic function

Function $\log z$ is expanded in power series as follows.

$$\log z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n = f(z) \tag{1.f}$$

When $z = x+iy$ , the real part and the imaginary part of both sides are illustrated respectively as follows. The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, we can see that the radius of convergence of the right-hand side is 1. Further, for comparison with latter figures, appropriate 5 points on the circumference is marked in red.



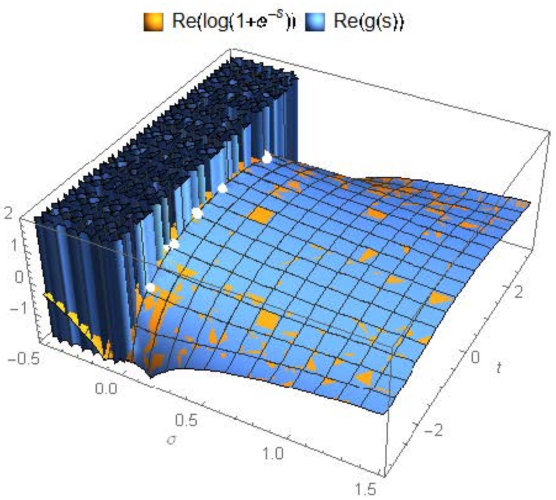



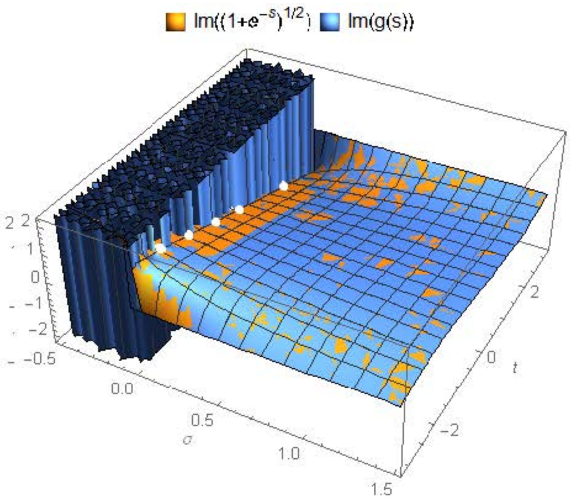

Substituting $z-1 = e^{-s}$ for (1.f) ,

$$\log\left(1+e^{-s}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{-ns} = g(s) \tag{1.g}$$

Let $a_n = \dfrac{(-1)^{n-1}}{n}$ , $\lambda_n = n$ . Then the right-hand side becomes $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ , which is general Dirichlet series. Where, we should note that the left-hand side is not $\log s$ but $\log(1+e^{-s})$ .

When $s = \sigma+it$ , the real part and the imaginary part of both sides are illustrated respectively as follows.




The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, we can see that the line of convergence of the right-hand side is 0. Further, we can see that 5 points in the former figures are transferred from the circle of convergence to the line of convergence by $s = -log(z-1)$ . ( White points.)

In addition, the line of convergence $\sigma_c$ is calculated as follows using **1.2.2** (2.2) .

$$\sigma_c = -logR = -log\,1 = 0$$

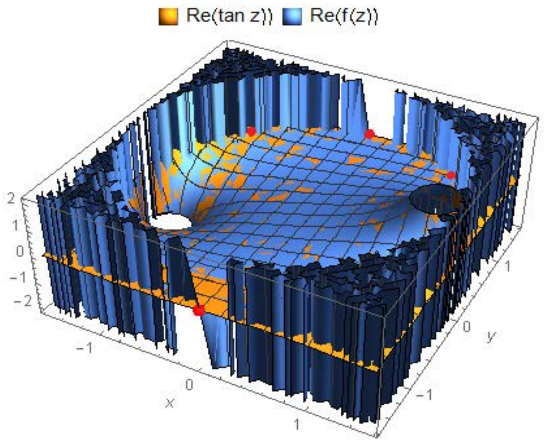
### 1.3.2 Dirichlet series expression of tangent function

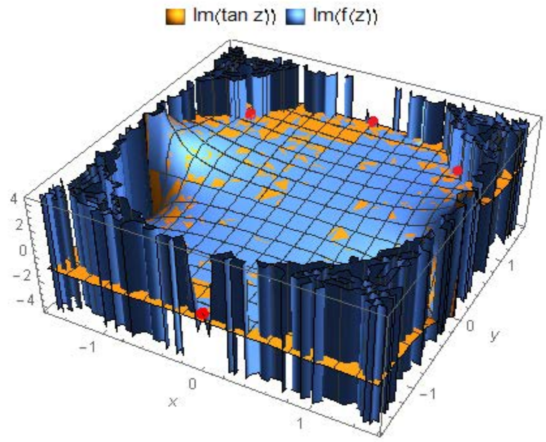
Function $tan\,z$ is expanded in power series as follows. Where, $B_k$ is Bernouilli number.

$$tan\,z = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n}\left(2^{2n}-1\right)}{2n\,(2n-1)\,!}B_{2n}z^{2n-1} = f(z) \tag{2.f}$$

When $z = x+iy$ , the real part and the imaginary part of both sides are illustrated respectively as follows. The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, the radius of convergence of the right-hand side looks like $\pi/2$

Further, for comparison with latter figures, appropriate 4 points on the circumference is marked in red.

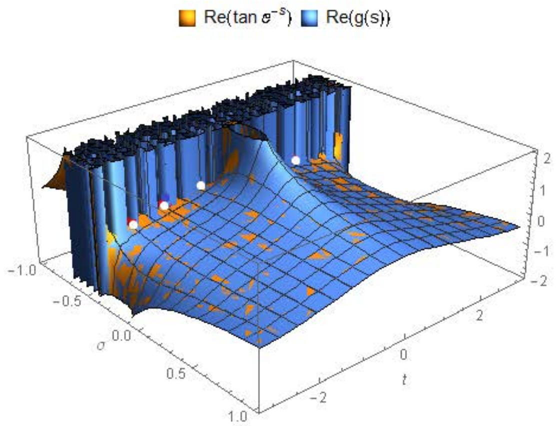


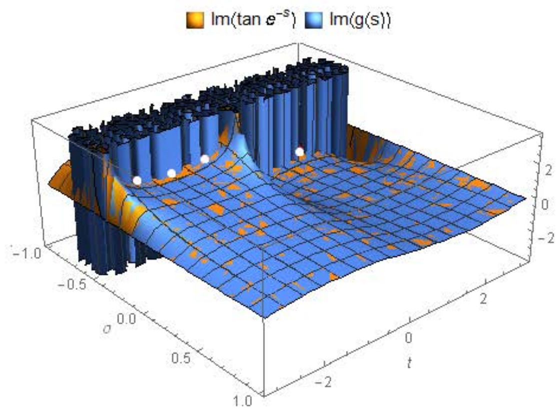




Substituting $z = e^{-s}$ for (2.f) ,

$$tan\,e^{-s} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n}\left(2^{2n}-1\right)}{2n\,(2n-1)\,!}B_{2n}e^{-(2n-1)s} = g(s) \tag{2.g}$$

Let $a_n = (-1)^{n-1}\frac{2^{2n}\left(2^{2n}-1\right)}{2n\,(2n-1)\,!}B_{2n}$ , $\lambda_n = 2n-1$ , Then the right-hand side becomes $\sum_{n=1}^{\infty}a_n e^{-\lambda_n s}$ , which is general Dirichlet series. Where, we should note that the left-hand side is not $tan\,s$ but $tan\,e^{-s}$ . When $s = \sigma+it$ , the real part and the imaginary part of both sides are illustrated respectively as follows.



The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, we can see that the line of convergence of the right-hand side is near $-0.5$ . Further, we can see that 4 points in the former figures are transferred from the circle of convergence to the line of convergence by $s = -\log z$ . ( White points.)
In addition, the line of convergence $\sigma_c$ is calculated as follows using **1.2.2** (2.2) .

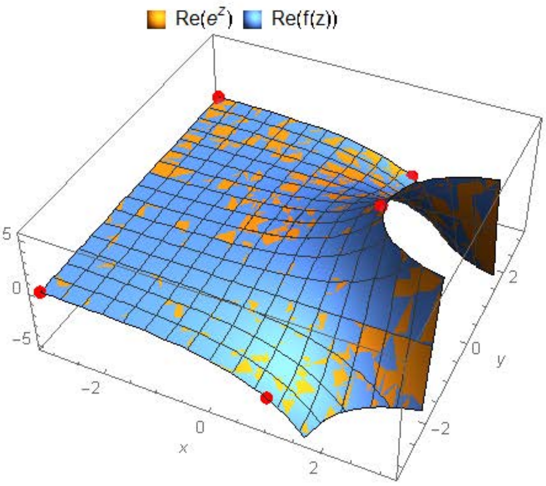
$$\sigma_c = -\log R = -\log\frac{\pi}{2} = -0.451852\cdots$$

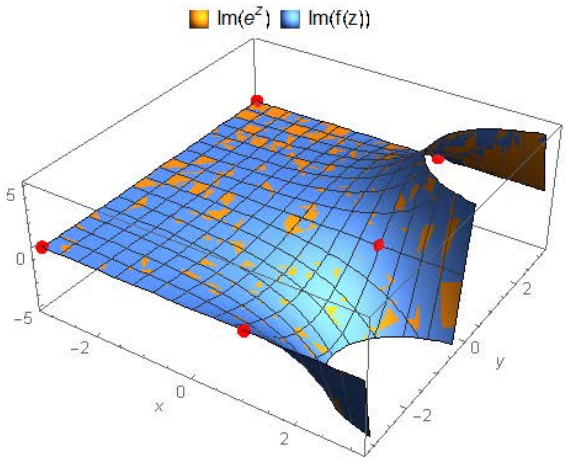
### 1.3.3 Dirichlet series expression of exponential function

Function $e^z$ is expanded in power series as follows.

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} = f(z) \tag{3.f}$$

When $z = x+iy$ , the real part and the imaginary part of both sides are illustrated respectively as follows.
The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue.



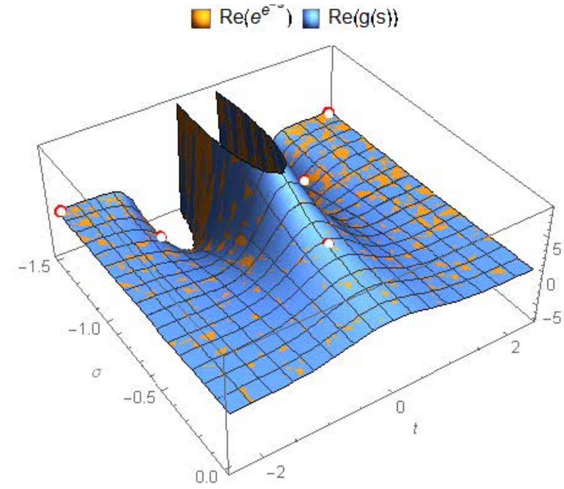

In both figures, something like a circle of convergence is found nowhere. Both sides are completely overlap. For comparison with latter figures, appropriate 5 points is marked in red.

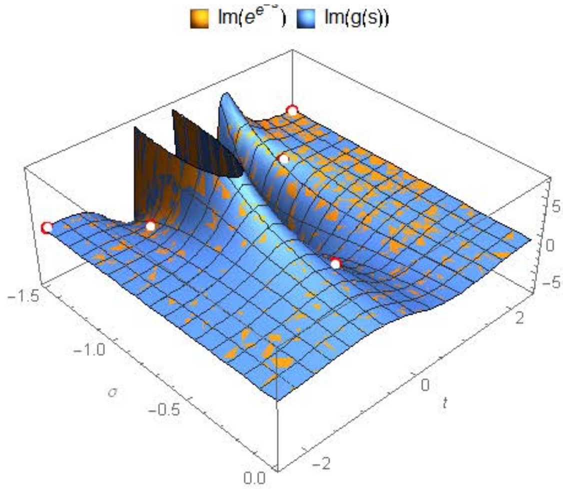
Substituting $z = e^{-s}$ for (3.f) ,

$$e^{e^{-s}} = \sum_{n=1}^{\infty}\frac{e^{-(n-1)s}}{(n-1)!} = g(s) \tag{3.g}$$

Let $a_n = \dfrac{1}{(n-1)!}$ , $\lambda_n = n-1$ , Then the right-hand side becomes $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ , which is general Dirichlet seies. Where, we should note that the left-hand side is not $e^s$ but $e^{e^{-s}}$ .

When $s = \sigma + it$ , the real part and the imaginary part of both sides are illustrated respectively on the following page.
The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, something like a line of convergence is found nowhere.
Further, we can see that 5 points in the former figures are transferred to white points by $s = -\log z$ .




In addition, the line of convergence $\sigma_c$ is calculated as follows using **1.2.2** (2.2) .

$$\sigma_c = -\log R = -\log \infty = -\infty$$

2016.08.05

Kano. Kono

**Alien's Mathematics**