01 General Dirichlet Series & Power Series

1.1 Definition & Theorems

Definition 1.1.1 (General Dirichlet Series)

Let $R$ be a real number set. And let $\sigma, t \in R$ and $\lambda_n < \lambda_{n+1}$, $n = 1, 2, 3, \ldots$.

When $s = \sigma + it$ and $a_n$ are complex numbers, we call the following series General Dirichlet Series.

\[
\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}
\]

Note

Especially when $\lambda_n = \log n$ in General Dirichlet Series, it is called Ordinary Dirichlet Series.

This is discussed in another chapter.

Theorem 1.1.2

Let Dirichlet serie $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$. And assume $f(s)$ is convergent at $s = s_c = \sigma_c + t_c i$. Then

1. $f(s)$ converges uniformly in $\arg(s - s_c) \leq \theta < \frac{\pi}{2}$.
2. $f(s)$ converges for any $s = \sigma + ti$ s.t. $\sigma > \sigma_c$.

This $\sigma_c$ is called the line of convergence. By convention, $\sigma_c = \infty$ if $f(s)$ converges nowhere and $\sigma_c = -\infty$ if $f(s)$ converges everywhere on the complex plane.

How to calculate $\sigma_c$

1. When $\sum_{k=1}^{n} a_k$ is divergent, 
   \[
   \sigma_c = \lim_{n \to \infty} \sup \frac{\log |a_1 + a_2 + \cdots + a_n|}{\lambda_n}
   \]
2. When $\sum_{k=1}^{n} a_k$ is convergent, 
   \[
   \sigma_c = \lim_{n \to \infty} \sup \frac{\log |a_{n+1} + a_{n+2} + a_{n+3} \cdots|}{\lambda_n}
   \]

Absolute Convergence

A Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n| e^{-\lambda_n s}$ is convergent.

There exist a $\sigma_a (\geq \sigma_c)$ such that $f(s)$ converges absolutely for $\sigma > \sigma_a$ and converges non-absolutely for $\sigma < \sigma_a$. This $\sigma_a$ is called the line of absolute convergence.
How to calculate $\sigma_a$

1. When $\sum_{k=1}^{n} |a_k|$ is divergent,

$$\sigma_a = \limsup_{n \to \infty} \frac{\log \left( |a_1| + |a_2| + \cdots + |a_n| \right)}{\lambda_n}$$

2. When $\sum_{k=1}^{n} |a_k|$ is convergent,

$$\sigma_a = \limsup_{n \to \infty} \frac{\log \left( |a_{n+1}| + |a_{n+2}| + |a_{n+3}| + \cdots \right)}{\lambda_n}$$

Uniform Convergence

As seen in **Theorem 1.1.2**, Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ converges uniformly in a certain domain. If we see it a little in detail, it is as follows.

There exists a $\sigma_u \left( \sigma_c \leq \sigma_u \leq \sigma_a \right)$ such that $f(s)$ converges uniformly for $\sigma > \sigma_u$ and converges non-uniformly for $\sigma < \sigma_u$. This $\sigma_u$ is called the line of uniform convergence.

How to calculate $\sigma_u$

$$\sigma_u = \limsup_{x \to \infty} \frac{\log T_x}{\log x}$$

where

$$T_x = \sup_{|y| < \infty} \left| \sum_{|x| \leq \lambda_n < x} a_n e^{-i \lambda_n y} \right|$$

**Theorem 1.1.3 (Holomorphy)**

If Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ ($s = \sigma + ti$) converges for $\sigma > \sigma_c$, $f(s)$ is **holomorphic** at $\sigma > \sigma_c$. And the derivative of $f(s)$ is given as follows.

$$f^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \lambda_n^k a_n e^{-\lambda_n s}$$

**Theorem 1.1.4 (Uniqueness)**

Let two Dirichlet series are as follows.

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad g(s) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n s}$$

If both are convergent in a certain domain and $f(s) = g(s)$ holds at there, $a_n = b_n$ for $n = 1, 2, 3, \ldots$.

**Bibliography**

「数論入門」D. B. ザギヤー 著、片山考次 訳 岩波 1990年, etc.
1.2 General Dirichlet Series & Power Series

General Dirichlet Series can contain Power Series other than Ordinary Dirichlet Series. In the following section, we will consider this.

1.2.1 Relation between Power Series and General Dirichlet Series

Relations between power series
\[ \sum_{n=1}^{\infty} c_n (z - z_0)^n \] and general Dirichlet series
\[ \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \] are as follows.

\[ c_n = a_n, \quad n = \lambda_n, \quad z - z_0 = e^{-s} \tag{2.1} \]

Especially, when the power series starts from \( n = 0 \),

\[ c_{n-1} = a_n, \quad n - 1 = \lambda_n, \quad z - z_0 = e^{-s} \tag{2.0} \]

Calculation

\[ \sum_{n=0}^{\infty} c_n (z - z_0)^n = \sum_{n=1}^{\infty} c_{n-1} (z - z_0)^{n-1} = \sum_{n=1}^{\infty} a_n (e^{-s})^{\lambda_n} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \]

\[ \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \sum_{n=1}^{\infty} c_{n-1} (e^{-s})^{\lambda_n} = \sum_{n=1}^{\infty} c_n (z - z_0)^{n-1} = \sum_{n=0}^{\infty} c_n (z - z_0)^n \]

Note

When a function \( f(z) \) is expressed with a power series, Dirichlet series equivalent to this does not express \( f(s) \). Since \( z - z_0 = e^{-s} \), it is \( f(z) = f(e^{-s} + z_0) \). That is, Dirichlet series equivalent to the power series express the composition \( f \).

1.2.2 Circle of convergence & Line of convergence

As seen above, the relation between the variable \( z - z_0 \) of power series and the variable \( s \) of Dirichlet series was as follows..

\[ z - z_0 = e^{-s} \]

That is,

\[ s = -\log(z - z_0) = -\log |z - z_0| + i \arg(z - z_0) \]

Here, let \( s = \sigma + it \), \( z = x + iy \). Then

\[ \sigma + it = -\log |(x - x_0) + i (y - y_0)| + i \arg\{(x - x_0) + i (y - y_0)\} \]

Although coordinates \( x - y \) are transferred to coordinates \( \sigma - t \) by the conversion, the coordinates \( \sigma - t \) are similar to pola coordinates \( r - \theta \). The difference among both is only the existence of \( \log \) in the real part.

From this,

\[ \sigma = -\log \sqrt{(x - x_0)^2 + (y - y_0)^2} \tag{2.σ} \]
\[ t = -\arg\{(x - x_0) + i (y - y_0)\} \tag{2.t} \]

When \( z_0 = x_0 + iy_0 = 1 + i1 \), the real part (2.σ) is drawn on the left figure. All the horizontal sections of the real part (2.σ) are circles centered on the point \((1,1)\). Radius of each circle is given by

\[ \sqrt{(x - 1)^2 + (y - 1)^2} \]
For example, if the radius is $1$, $\sigma = -\log 1 = 0$ from $\{2, \sigma\}$. If the left figure is horizontally cut at this height, it is the right figure.

When the radius is $1$, the power series converges in the circle. This means that the corresponding Dirichlet series should converge at the upper than the circle i.e. $\sigma > 0$. The height $\sigma_c = -\log R$ which provides the convergence circle with the radius $R$ is the line of convergence.

So far, we have seen the relationship between circle of convergence and line of convergence in the figure. It is as follows analytically.

using $z - z_0 = e^{-s}$

$$e^{-\sigma} = R > |z - z_0| = |e^{-s}| = |e^{-\sigma - it}| = e^{-\sigma}$$

i.e.

$$e^{-\sigma} < e^{-\alpha} = R$$

Taking the logarithm of both sides, we obtain

$$\sigma > \sigma_c = -\log R$$

(2.2)

c.f.

According to the definition in (1.1), the convergence line $\sigma_c$ is calculated as follows.

When $\sum_{n=1}^{\infty} c_n$ (or $\sum_{n=0}^{\infty} c_n$) is convergent, $\sum_{k=1}^{n} c_k$ is also convergent. Then

$$\sigma_c = \lim_{n \to \infty} \sup \log \left| \frac{a_{n+1} + a_{n+2} + a_{n+3} \ldots}{\lambda_n} \right| = \lim_{n \to \infty} \sup \log \left| \frac{c_{n+1} + c_{n+2} + c_{n+3} \ldots}{n} \right|$$

The calculation by this formula is difficult in many cases. It is better to use the above (2.2).

Byproduct

From the above and $R = \lim_{n \to \infty} \inf |c_n|^{-1/n}$, $\sigma_c = -\log R$, we obtain the following

$$\limsup_{n \to \infty} |c_{n+1} + c_{n+2} + c_{n+3} \ldots|^{1/n} = \limsup_{n \to \infty} |c_n|^{1/n} (= 1/R)$$

(2.3)
1.3 Dirichlet series expression of primary functions

1.3.1 Dirichlet series expression of logarithmic function

Function $\log z$ is expanded in power series as follows.

$$\log z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n = f(z)$$  \hfill (1.f)

When $z = x + iy$, the real part and the imaginary part of both sides are illustrated respectively as follows. The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, we can see that the radius of convergence of the right-hand side is 1. Further, for comparison with latter figures, appropriate 5 points on the circumference is marked in red.

$$\log(1+e^{-s}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{-ns} = g(s)$$  \hfill (1.g)

Let $a_n = \frac{(-1)^{n-1}}{n}$, $\lambda_n = n$. Then the right-hand side becomes $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, which is general Dirichlet series. Where, we should note that the left-hand side is not $\log s$ but $\log(1+e^{-s})$.

When $s = \sigma + it$, the real part and the imaginary part of both sides are illustrated respectively as follows.

The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, we can see that the line of convergence of the right-hand side...
is 0. Further, we can see that 5 points in the former figures are transferred from the circle of convergence to the line of convergence by \( s = -\log(z - 1) \). (White points.)

In addition, the line of convergence \( \sigma_c \) is calculated as follows using 1.2.2 (2.2).

\[
\sigma_c = -\log R = -\log 1 = 0
\]

### 1.3.2 Dirichlet series expression of tangent function

Function \( \tan z \) is expanded in power series as follows. Where, \( B_k \) is Bernoulli number.

\[
\tan z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n} \binom{2^n - 1}{n}}{2n \ (2n-1) !} z^{2n-1} = f(z)
\]

(2.f)

When \( z = x + iy \), the real part and the imaginary part of both sides are illustrated respectively as follows.

The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, the radius of convergence of the right-hand side looks like \( \pi/2 \). Further, for comparison with latter figures, appropriate 4 points on the circumference is marked in red.

Substituting \( z = e^{-s} \) for (2.f),

\[
\tan e^{-s} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n} \binom{2^n - 1}{n}}{2n \ (2n-1) !} B_{2n} e^{-(2n-1)s} = g(s)
\]

(2.g)

Let \( a_n = (-1)^{n-1} \frac{2^{2n} \binom{2^n - 1}{n}}{2n \ (2n-1) !} B_{2n} \), \( \lambda_n = 2n - 1 \), Then the right-hand side becomes \( \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \), which is general Dirichlet series. Where, we should note that the left-hand side is not \( \tan s \) but \( \tan e^{-s} \).

When \( s = \sigma + it \), the real part and the imaginary part of both sides are illustrated respectively as follows.
The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, we can see that the line of convergence of the right-hand side is near $-0.5$. Further, we can see that 4 points in the former figures are transferred from the circle of convergence to the line of convergence by $s = -\log z$. (White points.)

In addition, the line of convergence $\sigma_c$ is calculated as follows using 1.2.2 (2.2).

$$\sigma_c = -\log R = -\log \frac{\pi}{2} = -0.451852\ldots$$

1.3.3 Dirichlet series expression of exponential function

Function $e^z$ is expanded in power series as follows.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = f(z) \quad (3.f)$$

When $z = x + iy$, the real part and the imaginary part of both sides are illustrated respectively as follows. The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue.

In both figures, something like a circle of convergence is found nowhere. Both sides are completely overlap. For comparison with latter figures, appropriate 5 points is marked in red.

Substituting $z = e^{-s}$ for (3.f),

$$e^{e^{-s}} = \sum_{n=1}^{\infty} \frac{e^{-(n-1)s}}{(n-1)!} = g(s) \quad (3.g)$$

Let $a_n = \frac{1}{(n-1)!}$, $\lambda_n = n - 1$. Then the right-hand side becomes $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, which is general Dirichlet series. Where, we should note that the left-hand side is not $e^s$ but $e^{e^{-s}}$.

When $s = \sigma + it$, the real part and the imaginary part of both sides are illustrated respectively on the following page.

The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, something like a line of convergence is found nowhere. Further, we can see that 5 points in the former figures are transferred to white points by $s = -\log z$. 


In addition, the line of convergence $\sigma_c$ is calculated as follows using 1.2.2 \( (2.2) \) .

$$\sigma_c = -\log R = -\log \infty = -\infty$$