5 Generalized Bernoulli Polynomials and Numbers

5.1 Generalized Bernoulli Polynomials

First, we prepare the following lemma.

Lemma 5.1.1

$$\frac{e^{-ix}}{i^{-p}} + \frac{e^{ix}}{i^{p}} = 2cos\left(x - \frac{\pi p}{2}\right)$$

Proof

From $2cos x = e^{ix} + e^{-ix}$,

$$2cos\left(x - \frac{p\pi}{2}\right) = e^{i\left(x - \frac{p\pi}{2}\right)} + e^{-i\left(x - \frac{p\pi}{2}\right)} = e^{ix}\left(e^{\frac{\pi i}{2}}\right)^{-p} + e^{-ix}\left(e^{\frac{\pi i}{2}}\right)^{p}$$

$$= e^{ix} i^{-p} + e^{-ix} i^{p} = \frac{e^{ix}}{i^{p}} + \frac{e^{-ix}}{i^{-p}}$$

Next, we show the following formula which give generalized Bernouilli polynomials.

Formula 5.1.2

$$B_p(x) = -2\Gamma(1+p)\sum_{s=1}^{\infty}\frac{1}{(2\pi s)^p}cos\left(2\pi sx - \frac{\pi p}{2}\right) \quad 0 \leqq x \leqq 1 \qquad (2.p)$$

Proof

Adolf Hurwitz gave the Fourier series as follows.

$$B_m(x) = -m!\sum_{s\neq 0}\frac{e^{2\pi isx}}{(2\pi is)^m} \qquad 0 \leqq x \leqq 1$$

Although m is a natural number here, if this is extended to the real number p , it is as follows.

$$B_p(x) = -\Gamma(1+p)\sum_{s\neq 0}\frac{e^{2\pi isx}}{(2\pi is)^p} = -\Gamma(1+p)\left(\sum_{s=-1}^{-\infty}\frac{e^{2\pi isx}}{(2\pi is)^p} + \sum_{s=1}^{\infty}\frac{e^{2\pi isx}}{(2\pi is)^p}\right)$$

The 1st term in the parentheses of the right side is as follows.

$$\sum_{s=-1}^{-\infty}\frac{e^{2\pi isx}}{(2\pi is)^p} = \sum_{s=1}^{\infty}\frac{e^{-2\pi isx}}{(-2\pi is)^p} = \sum_{s=1}^{\infty}\frac{e^{-2\pi isx}}{(2\pi s)^p(-i)^p}$$

Then,

$$B_p(x) = -\Gamma(1+p)\left\{\sum_{s=1}^{\infty}\frac{e^{-2\pi isx}}{(2\pi s)^p(-i)^p} + \sum_{s=1}^{\infty}\frac{e^{2\pi isx}}{(2\pi s)^p i^p}\right\}$$

$$= -\Gamma(1+p)\sum_{s=1}^{\infty}\frac{1}{(2\pi s)^p}\left(\frac{e^{-2\pi isx}}{i^{-p}} + \frac{e^{2\pi isx}}{i^p}\right)$$

According to Lemma 5.1.1 ,

$$\frac{e^{-ix}}{i^{-p}} + \frac{e^{ix}}{i^{p}} = 2\cos\left(x - \frac{\pi p}{2}\right)$$

Replacing x with $2\pi sx$,

$$\frac{e^{-2\pi isx}}{i^{-p}} + \frac{e^{2\pi isx}}{i^{p}} = 2\cos\left(2\pi sx - \frac{\pi p}{2}\right)$$

Substituting this for the above, we obtain

$$B_p(x) = -2\Gamma(1+p)\sum_{s=1}^{\infty}\frac{1}{(2\pi s)^p}\cos\left(2\pi sx - \frac{\pi p}{2}\right) \tag{2.p}$$

What on earth is this expression ? Then, if we illustrate this at the time of $p=1,2$, it is as follows.

```
• B := p-> -2*gamma(1+p)*sum(cos(2*PI*s*x-PI*p/2)/(2*PI*s)^p,
  s=1..100)
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$$p \to -2\cdot\Gamma(1+p)\cdot\left(\sum_{s=1}^{100}\frac{\cos\left(2\cdot\pi\cdot s\cdot x - \frac{\pi\cdot p}{2}\right)}{(2\cdot\pi\cdot s)^p}\right)$$

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• plotfunc2d(B(1),B(2), x=0..3, Scaling=Constrained)
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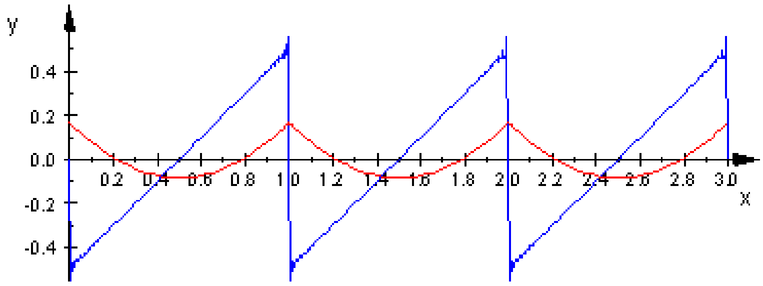
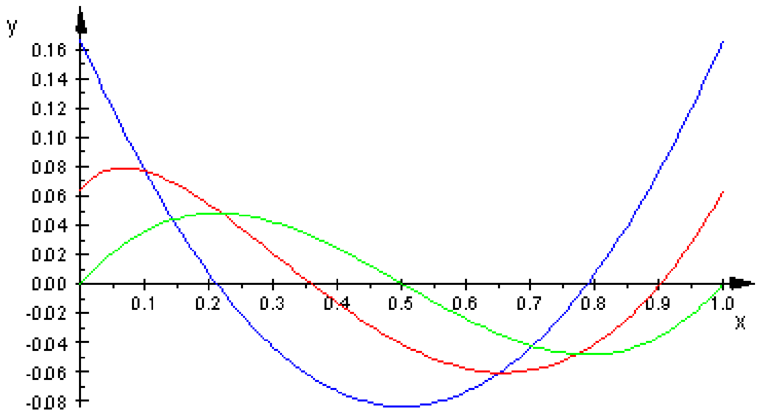


That is, these are the functions $B_1(x-[x])$ & $B_2(x-[x])$ that appeared in " **04** Euler-Maclaurin Summation Formula".

If (2.p) at $p=2.5$ is illustrated with $B_2(x)$ and $B_3(x)$, it is as follows.



Blue is $B_2(x)$, Red is $B_{2.5}(x)$ and Green is $B_3(x)$. $B_{2.5}(x)$ is seen in the middle of $B_2(x)$ and $B_2(x)$ exactly. Thus, $B_{2.5}(x)$ can be regarded as non-integer Bernouilli Polynomial on the domain $[0,1]$.

5.2 Generalized Bernoulli Numbers

Substituting $x=0$ for the generalized Bernoulli polynomial $B_p(x)$ in the previous section,

$$B_p(0) = -2\Gamma(1+p)\sum_{s=1}^{\infty}\frac{1}{(2\pi s)^p}\cos\left(-\frac{\pi p}{2}\right) = -\frac{2\Gamma(1+p)}{(2\pi)^p}\sum_{s=1}^{\infty}\frac{1}{s^p}\cos\frac{\pi p}{2}$$

i.e.

$$B_p(0) = -\frac{2\Gamma(1+p)}{(2\pi)^p}\cos\frac{p\pi}{2}\cdot\zeta(p) \qquad (p\neq 1,-1,-2,-3,\cdots)$$

$p=1$ is a discontinuity of this function, and $p=-1,-2,-3,\cdots$ are singularities of this function. Therefore, at present, $B_p(0)$ is only Bernoulli for non-integer and natural number of two or more. However, these discontinuity and singularities are removable. In Euler's formula

$$B_p = -(-1)^p p\,\zeta(1-p) \qquad p=1,2,3,\cdots$$

if we notice that $(-1)^p$ is required only at $p=1$,

$$B_p = \begin{cases} p\,\zeta(1-p) = -1/2 & p=1 \\ -p\,\zeta(1-p) & p=2,3,4\cdots \end{cases}$$

Extending p to integer,

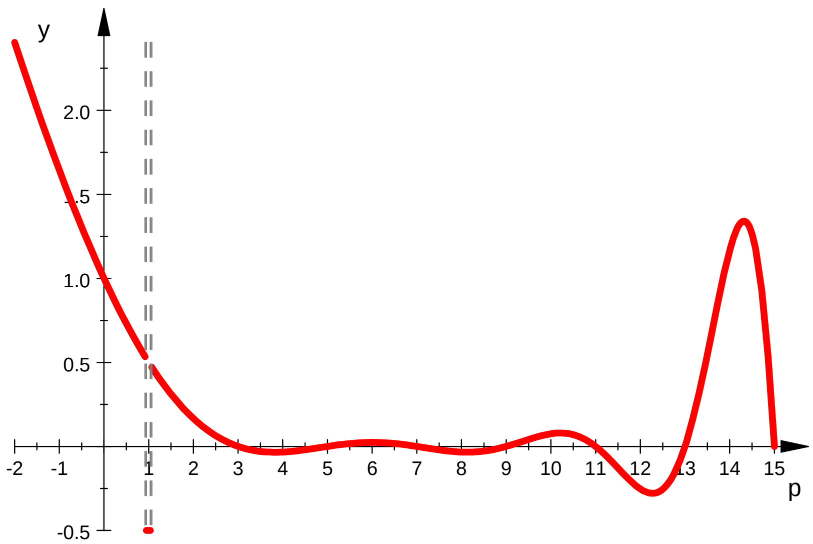
$$B_p = \begin{cases} -1/2 & p=1 \\ -p\,\zeta(1-p) & p \text{ is an integer s.t. } p\neq 1 \end{cases}$$

Thus, we can define the generalized Bernoulli number as follows.

Definition 5.2.1 (Generalized Bernoulli Number)

$$B_p = \begin{cases} -\dfrac{2\Gamma(1+p)}{(2\pi)^p}\cos\dfrac{p\pi}{2}\cdot\zeta(p) & p\neq 1,-1,-2,-3,\cdots \\ -1/2 & p=1 \\ -p\,\zeta(1-p) & p=-1,-2,-3\cdots \end{cases}$$

When p is horizontal axis and B_p is vertical axis, this is illustrated as follows.



According to the definition, $B_{-4.9}\sim B_{4.9}$ are shown in two tables as follows.

n	$B_{-4.n}$	$B_{-3.n}$	$B_{-2.n}$	$B_{-1.n}$	$B_{-0.n}$
9	4.9915289558	4.0556394217	3.1597013355	2.3239544010	1.5747717916
8	4.8965940882	3.9639624506	3.0730296140	2.2446565601	1.5057836944
7	4.8019226839	3.8726975025	2.9869782519	2.1662498955	1.4380021297
6	4.7075272044	3.7818625772	2.9015716057	2.0887644945	1.3714593994
5	4.6134206098	3.6914762876	2.8168346682	2.0122308858	1.3061876743
4	4.5196163712	3.6015578683	2.7327930617	1.9366800023	1.2422189111
3	4.4261284844	3.5121271822	2.6494730278	1.8621431391	1.1795847635
2	4.3329714820	3.4232047271	2.5669014161	1.7886519078	1.1183164882
1	4.2401604462	3.3348116400	2.4851056694	1.7162381868	1.0584448465
0	4.1477110205	3.2469697011	2.4041138063	1.6449340668	1

Note) Blues are removable singularities.

n	$B_{0.n}$	$B_{1.n}$	$B_{2.n}$	$B_{3.n}$	$B_{4.n}$
0	1	-0.5	0.1666666666	0	-0.0333333333
1	0.9430114019	0.4589508448	0.1427610484	-0.0084614493	-0.0316898571
2	0.8875076831	0.4195995367	0.1205345707	-0.0156134795	-0.0294502827
3	0.8335165336	0.3819569885	0.0999673907	-0.0215139555	-0.0266973159
4	0.7810645792	0.3460316451	0.0810359704	-0.0262242815	-0.0235142614
5	0.7301772544	0.3118293374	0.0637130047	-0.0298092507	-0.0199845510
6	0.6808786703	0.2793531391	0.0479673653	-0.0323368645	-0.0161912437
7	0.6331914800	0.2486032259	0.0337640613	-0.0338781212	-0.0122164984
8	0.5871367399	0.2195767398	0.0210642174	-0.0345067720	-0.0081410226
9	0.5427337678	0.1922676575	0.0098250724	-0.0342990445	-0.0040434999

Note) Red is the removable discontinuity.

Although Definition 5.2.1 used Riemann Zeta Function, I found out the definition by a double series later. (See " 06 Global definition of Riemann Zeta, and generalization of related coefficients "). When it is published here, it is as follows.

Definition 5.2.2 (Generalized Bernoulli Number)

$$B_p = \begin{cases} -\dfrac{1}{2} & p = 1 \\ \dfrac{p}{2^p - 1} \displaystyle\sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^{p-1} & p \neq 1 \end{cases}$$

5.3 Generalized Bernoulli's Formula

Formula 5.3.1

When $p \neq -1$,

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1} \sum_{r=0}^{p} \binom{p+1}{r} B_r n^{p+1-r} \tag{1.1}$$

$$= \frac{1}{p+1} \left\{ B_{p+1}(n) - B_{p+1}(0) \right\} \tag{1.1'}$$

Proof

$$\{S_{n-1}\} = 0^p,\ 1^p,\ 2^p,\ \cdots,\ (n-1)^p \qquad p \neq -1$$

From this,

$$f(x) = x^p$$

$$\int_0^n f(x)\,dx = \frac{1}{p+1}\left(n^{p+1} - 0^{p+1}\right)$$

$$f^{(r-1)}(x) = \frac{\Gamma(1+p)}{\Gamma(1+p-r+1)} x^{p-r+1} \quad (r=1 \sim p+1)$$

On the other hand, replacing m with p in Formula 4.2.1 (See **04** Euler-Maclaurin Summation Formula),

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x)\,dx + \sum_{r=1}^{p} \frac{B_r}{r!}\left\{ f^{(r-1)}(b) - f^{(r-1)}(a) \right\} + R_p$$

$$R_p = \frac{(-1)^{1+p}}{\Gamma(1+p)} \int_a^b B_p(x - \lfloor x \rfloor) f^{(p)}(x)\,dx$$

Substituting the above expressions for this ,

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1}\left(n^{p+1} - 0^{p+1}\right) + \sum_{r=1}^{p} \frac{B_r}{r!} \frac{\Gamma(1+p)}{\Gamma(1+p-r+1)} \left(n^{p-r+1} - 0^{p-r+1}\right) + R_p$$

$$= \sum_{r=0}^{p} \frac{B_r}{r!} \frac{\Gamma(1+p)}{\Gamma(1+p-r+1)} \left(n^{p+1-r} - 0^{p+1-r}\right) + R_p$$

$$= \frac{1}{1+p} \sum_{r=0}^{p} \frac{\Gamma(2+p)}{r!\,\Gamma(2+p-r)} B_r \left(n^{p+1-r} - 0^{p+1-r}\right) + R_p$$

$$R_p = \frac{(-1)^{1+p}}{\Gamma(1+p)} \int_0^n B_p(x - \lfloor x \rfloor) \frac{\Gamma(1+p)}{\Gamma(1)} x^0\,dx$$

i.e.

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1} \sum_{r=0}^{p} \binom{p+1}{r} B_r \left(n^{p+1-r} - 0^{p+1-r}\right) + R_p$$

$$R_p = (-1)^{p+1} \int_0^n B_p(x - \lfloor x \rfloor)\,dx$$

As seen in **5.1** ,

$$B_p(x - \lfloor x \rfloor) = B_p(x) = -2\Gamma(1+p) \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^p} \cos\left(2\pi s x - \frac{\pi p}{2}\right)$$

Substituting this for R_p ,

$$R_p = (-1)^p 2\Gamma(1+p) \int_0^n \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^p} cos\left(2\pi sx - \frac{\pi p}{2} \right) dx$$

$$= (-1)^p 2\Gamma(1+p) \sum_{s=1}^{\infty} \frac{1}{(2\pi s)^p} \int_0^n cos\left(2\pi sx - \frac{\pi p}{2} \right) dx$$

When $s = 1, 2, 3, \cdots$,

$$\int_0^n cos\left(2\pi sx - \frac{\pi p}{2} \right) dx = \frac{1}{2\pi s} \left[sin\left(2\pi sx - \frac{\pi p}{2} \right) \right]_0^n$$

$$= \frac{1}{2\pi s} \left\{ sin\left(2\pi sn - \frac{\pi p}{2} \right) - sin\left(-\frac{\pi p}{2} \right) \right\}$$

$$= 0$$

That is, $R_p = 0$. Therefore,

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1} \sum_{r=0}^{p} \binom{p+1}{r} B_r \left(n^{p+1-r} - 0^{p+1-r} \right) \tag{1.w}$$

$$= \frac{1}{p+1} \sum_{r=0}^{p} \binom{p+1}{r} B_r n^{p+1-r} \tag{1.1}$$

Furthermore

$$\binom{p+1}{p+1} B_{m+1} \left(n^0 - 0^0 \right) = 0$$

Then, (1.w) is rewritten as follows.

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1} \sum_{r=0}^{p+1} \binom{p+1}{r} B_r \left(n^{p+1-r} - 0^{p+1-r} \right)$$

Expressing this with the generalized Bernoulli polynomial,

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1} \left\{ B_{p+1}(n) - B_{p+1}(0) \right\} \tag{1.1'}$$

Approximate expression

Formula 5.3.1 holds only as a concept, and regrettably, we cannot use this for actual calculation. If it is why, obtaining $\sum^{p}$ at non-integer p is difficult and the generalized Bernoulli polynomial on the domain wider than $[0,1]$ is unknown.

Then, let us consider the approximate expression of Formula 5.3.1. From (1.1') ,

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1} B_{p+1}(n) - \frac{1}{p+1} B_{p+1}(0)$$

Here, for $B_{p+1}(0)$, we may use the following generalized Bernouilli number (See **5.2**).

$$B_p = -\frac{2\Gamma(1+p)}{(2\pi)^p} cos\frac{p\pi}{2} \cdot \zeta(p) \qquad p \neq 1, -1, -2, -3, \cdots$$

For $B_{p+1}(n)$, let us approximate this with the following expression.

$$B_{p+1}(n) \doteqdot \sum_{r=0}^{m}\binom{p+1}{r}B_r n^{p+1-r}$$

Where, $|p|\uparrow+1 \leqq m < \infty$. We can adopt the large m , and the accuracy improves as the result. However, m that is larger than $n-1$ is meaningless, and infinite m is impossible. Thus, we obtain the following approximation.

Formula 5.3.2

When $p \neq 0,-1,-2,-3,\cdots$

$$\sum_{k=0}^{n-1}k^p \doteqdot \frac{1}{p+1}\left\{\sum_{r=0}^{m}\binom{p+1}{r}B_r n^{p+1-r} + \frac{2\Gamma(p+2)}{(2\pi)^{p+1}}\cos\frac{(p+1)\pi}{2}\zeta(p+1)\right\}$$

Where $|p|\uparrow+1 \leqq m < \infty$

Example: $\sum_{k=0}^{100}k^{0.1}$

When $m = 0.1\uparrow+1 = 2$,

$$\sum_{k=0}^{101-1}k^{0.1} \doteqdot \frac{1}{0.1+1}\left\{\sum_{r=0}^{2}\binom{0.1+1}{r}B_r x^{0.1+1-r} + \frac{2\Gamma(0.1+2)}{(2\pi)^{0.1+1}}\cos\frac{(0.1+1)\pi}{2}\zeta(0.1+1)\right\}$$

$$= 144.45654994\cdots$$

This all digits (8 digits after the decimal point) are significant digits. This result is completely consistent with the example1 of Formula 4.6.1. (See **4.6**) This is no wonder. Formula 5.3.2 is rewritten by expression similar to the Formula 4.6.1.

Formula 5.3.2'

When $p \neq 0,-1,-2,-3,\cdots$

$$\sum_{k=0}^{n-1}k^p \doteqdot \frac{1}{p+1}\sum_{r=0}^{m}\binom{p+1}{r}B_r n^{p+1-r} + \zeta(-p)$$

Where $|p|\uparrow+1 \leqq m < \infty$

Proof

From (1.3') in Formula 5.4.1 (later) ,

$$\zeta(1-p) = \frac{2\Gamma(p)}{(2\pi)^p}\cos\frac{p\pi}{2}\cdot\zeta(p) \qquad p \neq 1,0,-1,-2,\cdots$$

Replacing p with $p+1$,

$$\zeta(-p) = \frac{2\Gamma(p+1)}{(2\pi)^{p+1}}\cos\frac{(p+1)\pi}{2}\cdot\zeta(p+1) \qquad p \neq 0,-1,-2,\cdots$$

Substituting this for Formula 5.3.2 ,

$$\sum_{k=0}^{n-1}k^p \doteqdot \frac{1}{p+1}\left\{\sum_{r=0}^{m}\binom{p+1}{r}B_r n^{p+1-r} + (p+1)\zeta(-p)\right\}$$

$$= \frac{1}{p+1}\sum_{r=0}^{m}\binom{p+1}{r}B_r n^{p+1-r} + \zeta(-p)$$

Formula 4.6.1. ("**04 Euler-Maclaurin Summation Formula**") was as follows.

$$\sum_{k=1}^{n-1} k^p = \zeta(-p) + \frac{1}{p+1}\sum_{r=0}^{m}\binom{p+1}{r}B_r n^{p+1-r} + R_m$$

$$R_m = \frac{1}{mB(m,-p)}\int_n^\infty \frac{B_m(x-\lfloor x\rfloor)}{x^{-p+m}}dx$$

The difference between this and Formula 5.3.2' is only the existence of a remainder term.

Thus, the following is drawn from the above-mentioned.

$$B_{p+1}(0) = -(p+1)\zeta(-p)$$

$$B_{p+1}(n) = \sum_{r=0}^{m}\binom{p+1}{r}B_r n^{p+1-r} + \frac{p+1}{mB(m,-p)}\int_n^\infty \frac{B_m(x-\lfloor x\rfloor)}{x^{-p+m}}dx$$

Note

The 2nd term of this right side is complicated. If the conciseness is required, using the Hurwitz zeta function, we can express as follows. However, doing so far is close to a foul.

$$B_{p+1}(n) = -(p+1)\zeta(-p,n)$$

5.4 Generalized Bernoulli Number & Zeta Function

Formula 5.4.1

When $\zeta(p), B_p$ denote the Riemann zeta function and the generalized Bernoulli number ,

$$\zeta(p) = -\frac{B_{1-p}}{1-p} \qquad p \ne 1, 0 \tag{1.1}$$

$$\zeta(1-p) = -\frac{B_p}{p} \qquad p \ne 0, 1 \tag{1.1'}$$

$$B_p = \frac{p}{1-p}\frac{2\Gamma(p)}{(2\pi)^p}\cos\frac{p\pi}{2}\cdot B_{1-p} \qquad p \ne 1, 0, -1, -2, \cdots \tag{1.2}$$

$$B_{1-p} = \frac{1-p}{p}\frac{2\Gamma(1-p)}{(2\pi)^{1-p}}\sin\frac{p\pi}{2}\cdot B_p \qquad p \ne 0, 1, 2, 3, \cdots \tag{1.2'}$$

$$\zeta(p) = \frac{2\Gamma(1-p)}{(2\pi)^{1-p}}\sin\frac{p\pi}{2}\cdot\zeta(1-p) \qquad p \ne 0, 1, 2, 3, \cdots \tag{1.3}$$

$$\zeta(1-p) = \frac{2\Gamma(p)}{(2\pi)^p}\cos\frac{p\pi}{2}\cdot\zeta(p) \qquad p \ne 1, 0, -1, -2, \cdots \tag{1.3'}$$

Proof

From Formula 5.3.1 ,

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1}\left\{\sum_{r=0}^{p+1}\binom{p+1}{r}B_r n^{p+1-r} - B_{p+1}(0)\right\}$$

Replacing p with $-p$,

$$\sum_{k=0}^{n-1} k^{-p} = \frac{1}{-p+1}\left\{\sum_{r=0}^{-p+1}\binom{-p+1}{r}B_r n^{-p+1-r} - B_{-p+1}(0)\right\} \qquad p>1$$

Here, let $n \to \infty$.

Since $p+r-1 > 0$ from $p>1$, $r \geqq 0$, $\lim_{n\to\infty} n^{-p+1-r} = 0$. Then,

$$\lim_{n\to\infty}\sum_{r=0}^{-p+1}\binom{-p+1}{r}B_r n^{-p+1-r} = 0$$

That is, for $p>1$,

$$\sum_{k=0}^{\infty} k^{-p} = \frac{-B_{1-p}(0)}{1-p} = \zeta(p)$$

And analytically continuing the domain to $p \ne 1, 0$ from $p>1$, we obtain (1.1) .

Replacing p with $1-p$ in (1.1) , we obtain (1.1') . This corresponds to Euler's expression

$$\zeta(1-m) = -\frac{(-1)^m B_m}{m} \qquad m = 1, 2, 3, \cdots$$

Next, from Definition 5.2.1 ,

$$B_p = -\frac{2\Gamma(1+p)}{(2\pi)^p}\cos\frac{p\pi}{2}\cdot\zeta(p) \qquad p \ne 1, -1, -2, -3, \cdots$$

Substituting (1.1) for this, we obtain (1.2) . And replacing p with $1-p$ in (1.2) , we obtain (1.2') . These are the functional equations for the generalized Bernoulli number.

Last, solving (1.1), (1.1') of B_{1-p} , B_p and substituting these for (1.2'), (1.2) , we obtain (1.3), (1.3') . These are the functional equations for the well known Riemann zeta function.

Example 1

Substituting Generalized Bernoulli Numbers on the tables in **5.2** for (1.1) ,

$$\zeta(1.5) = -\frac{B_{1-1.5}}{1-1.5} = -\frac{B_{-0.5}}{-0.5} = -\frac{-1.3061876743}{0.5} = 2.612375348$$

$$\zeta(0.1) = -\frac{B_{1-0.1}}{1-0.1} = -\frac{B_{0.9}}{0.9} = -\frac{0.5427337678}{0.9} = -0.603037519$$

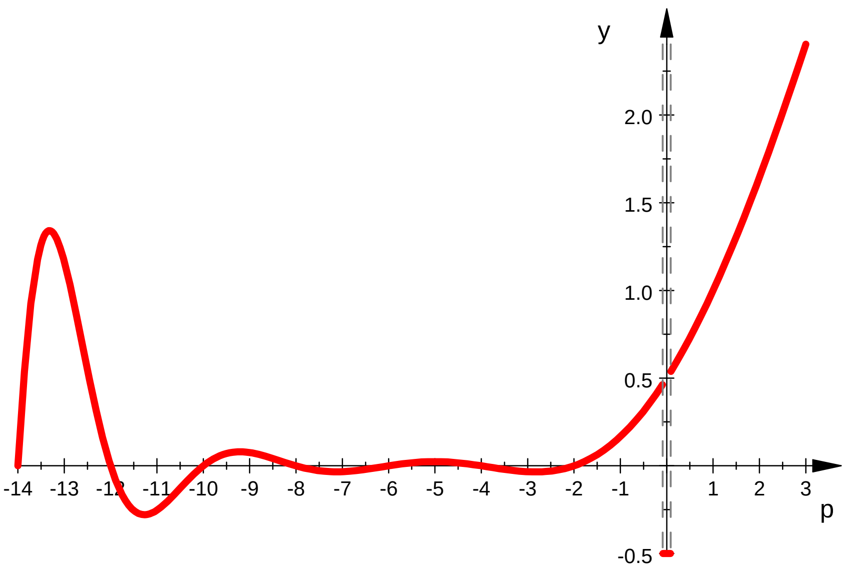
Example 2

Substituting Generalized Bernoulli Numbers on the tables in **5.2** for (1.1') ,

$$\zeta(-0.9) = \zeta(1-1.9) = -\frac{B_{1.9}}{1.9} = -\frac{0.1922676575}{1.9} = -0.101193503$$

$$\zeta(-2.1) = \zeta(1-3.1) = -\frac{B_{3.1}}{3.1} = -\frac{-0.0084614493}{3.1} = 0.002729499$$

Finally, let us illustrate B_{1-p} . As for the expression, we use $B_{1-p} = (p-1)\zeta(p)$. $B_1\ (=-1/2)$ at the time of $p=0$ is a removable discontinuity, and it is just under the origin though it is not easy to see. Moreover, this figure and the previous figure are the line symmetry at both sides of $p=1/2$. This is the result of the reflection relation between $\zeta(p)$ and $\zeta(1-p)$ shown by (1.3) .



2011.08.25

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Alien's Mathematics