1 Generalized Taylor's Theorem

1.1 Repeated integration by parts and its application

1.1.1 Formula of repeated integration by parts

Formula 1.1.1

When $f(x)$ is n times differentiable function on $[a, b]$ and $f(x)$ is

$$g^{<n>}(x) = \int g^{<n-1>}(x)dx + c_n \qquad n=1,2,3,\cdots \qquad c_n \text{ : arbitrary constants,}$$

the following expressions hold.

$$\int_a^b g(x)f(x)dx = \sum_{r=1}^{n}(-1)^{r-1}\left[g^{<r>}(x)f^{(r-1)}(x)\right]_a^b + (-1)^n\int_a^b g^{<n>}(x)f^{(n)}(x)dx \tag{1.1}$$

$$\int_a^b g(x)f(x)dx = \sum_{r=1}^{n}\left[g^{<2r-1>}(x)f^{(2r-2)}(x)\right]_a^b - \sum_{r=1}^{n}\left[g^{<2r>}(x)f^{(2r-1)}(x)\right]_a^b + \int_a^b g^{<2n>}(x)f^{(2n)}(x)dx \tag{1.1'}$$

Proof

$$\begin{aligned}
&\int_a^b g^{<0>}(x)f^{(0)}(x)dx \\
&\quad = \left[g^{<1>}(x)f^{(0)}(x)\right]_a^b - \int_a^b g^{<1>}(x)f^{(1)}(x)dx \\
&\quad = \left[g^{<1>}(x)f(x)\right]_a^b - \left[g^{<2>}(x)f^{(1)}(x)\right]_a^b + \int_a^b g^{<2>}(x)f^{(2)}(x)dx \\
&\quad = \left[g^{<1>}(x)f(x)\right]_a^b - \left[g^{<2>}(x)f^{(1)}(x)\right]_a^b + \left[g^{<3>}(x)f^{(2)}(x)\right]_a^b \\
&\qquad\qquad - \int_a^b g^{<3>}(x)f^{(3)}(x)dx \\
&\quad \vdots \\
&\quad = \sum_{r=1}^{n}(-1)^{r-1}\left[g^{<r>}(x)f^{(r-1)}(x)\right]_a^b + (-1)^n\int_a^b g^{<n>}(x)f^{(n)}(x)dx
\end{aligned}$$

Thus, we obtain (1.1) .

And replacing $\sum_{r=1}^{n}$ with $\sum_{r=1}^{2n}$, we obtain (1.1') .

Another proof

Theorem 16.1.2 (*Super Calculus* **16**) was as follows.

$$\int_a^x \cdots \int_a^x f^{<0>}g^{(0)}dx^n = \sum_{r=0}^{m-1}\binom{-n}{r} f^{<n+r>}g^{(r)} - \sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s} f_a^{<n-r+s>}\, g_a^{(s)}\,\frac{(x-a)^r}{r!}$$

$$+ (-1)^m \sum_{r=1}^{n-1} \sum_{s=0}^{r-1} \sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1} f_a^{\langle m+n-r+s \rangle} g_a^{(m+s)} \frac{(x-a)^r}{r!}$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{r=0}^{n-1} \frac{{}_{n-1}C_r}{m+r} \int_a^x \cdots \int_a^x f^{<m+r>} g^{(m+r)} dx^n$$

Put $n=1$. Then

$$\int_a^x f^{<0>} g^{(0)} dx = \sum_{r=0}^{m-1} (-1)^r \left(f^{<1+r>} g^{(r)} - f_a^{<1+r>} g_a^{(r)} \right) + (-1)^m \int_a^x f^{<m>} g^{(m)} dx$$

Substituting x, m with b, n respectively and replacing f with g, we obtain (1.1).

1.1.2 Series expansion of Exponential Integral

Formula 1.1.2

When $Ei(x) = \int_{-\infty}^x \frac{e^x}{x} dx$, the following expressions hold.

(1) for $x>1$

$$Ei(x) = \frac{e^x}{x} \left\{ 1 + \frac{1!}{x^1} + \frac{2!}{x^2} + \cdots + \frac{(n-1)!}{x^{n-1}} \right\} + n! \int_{-\infty}^x \frac{e^x}{x^{n+1}} dx \tag{2.1}$$

(2) for $x \neq 0$

$$Ei(x) = \frac{e^x}{x} \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\log|x| - \psi(r) - \gamma}{\Gamma(r)} x^r + \gamma \tag{2.2}$$

$$= \log|x| + \frac{e^x}{x} \sum_{r=1}^{\infty} (-1)^r \frac{\psi(r)}{\Gamma(r)} x^r \tag{2.3}$$

Proof

(1) When $g(x) = e^x$,

$$f(x) = \frac{1}{x}, \qquad f^{(r-1)}(x) = (-1)^{r-1} \frac{(r-1)!}{x^r}$$

Substituting these for (1.1) in Formula 1.1.1,

$$\int_{-\infty}^x \frac{e^x}{x} dx = \sum_{r=1}^{n} (-1)^{r-1} \left[e^x (-1)^{r-1} \frac{(r-1)!}{x^r} \right]_{-\infty}^x + \int_{-\infty}^x \left\{ e^x \frac{n!}{x^{n+1}} \right\} dx$$

$$= \frac{e^x}{x} \sum_{r=1}^{n} \frac{(r-1)!}{x^{r-1}} + n! \int_{-\infty}^x \frac{e^x}{x^{n+1}} dx \tag{2.1}$$

(2) When $f(x) = e^x$, $\quad \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$, $\quad \psi(z) = \frac{d}{dz} \log \Gamma(z)$,

$$g(x) = \frac{1}{x}$$

$$g^{<1>}(x) = \log|x|$$

$$g^{<2>}(x) = (\log x)^{<1>} = \frac{\log|x| - \psi(1+1) - \gamma}{\Gamma(1+1)} x^1$$

$$\vdots$$

$$g^{<r>}(x) = (\log x)^{<r-1>} = \frac{\log|x| - \psi(r) - \gamma}{\Gamma(r)} x^{r-1}$$

Substituting these for (1.1) ,

$$\int_{-\infty}^{x} \frac{e^x}{x} dx = \sum_{r=1}^{n} (-1)^{r-1} \left[\frac{\log|x| - \psi(r) - \gamma}{\Gamma(r)} x^{r-1} e^x \right]_{-\infty}^{x}$$

$$+ (-1)^n \int_{-\infty}^{x} \frac{\log|x| - \psi(n) - \gamma}{\Gamma(n)} x^{n-1} e^x dx$$

$$= \frac{e^x}{x} \sum_{r=1}^{n} (-1)^{r-1} \frac{\log|x| - \psi(r) - \gamma}{\Gamma(r)} x^r + (-1)^n \int_{-\infty}^{x} \frac{\log|x| - \psi(n) - \gamma}{\Gamma(n)} x^{n-1} e^x dx$$

Replacing $\psi(n) - \gamma$ with $\sum_{s=1}^{n-1} \frac{1}{s}$,

$$\int_{-\infty}^{x} \frac{e^x}{x} dx = \frac{e^x}{x} \sum_{r=1}^{n} (-1)^{r-1} \frac{\log|x| - \psi(r) - \gamma}{\Gamma(r)} x^r + R_{n+1} \qquad \text{(w1)}$$

$$R_{n+1} = \frac{(-1)^n}{\Gamma(n)} \int_{-\infty}^{x} \log|x| x^{n-1} e^x dx + \frac{(-1)^{n-1}}{\Gamma(n)} \sum_{s=1}^{n-1} \frac{1}{s} \int_{-\infty}^{x} x^{n-1} e^x dx \qquad \text{(wr)}$$

According to Formula 16.5.1' (*Super Calculus* **16**) ,

$$\int_{-\infty}^{x} e^x x^m dx = e^x \sum_{r=0}^{m} \binom{-1}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \qquad m = 0, 1, 2, \cdots$$

Then

$$\int_{-\infty}^{x} x^{n-1} e^x dx = e^x \sum_{r=0}^{n-1} \binom{-1}{r} \frac{\Gamma(1+n-1)}{\Gamma(1+n-1-r)} x^{n-1-r}$$

$$= e^x \sum_{r=0}^{n-1} (-1)^r \frac{(n-1)!}{(n-1-r)!} x^{n-1-r}$$

$$= (-1)^{n-1} (n-1)!\, e^x \sum_{r=0}^{n-1} (-1)^r \frac{x^r}{r!}$$

Thus, the 2nd term of (wr) is

$$\frac{(-1)^{n-1}}{\Gamma(n)} \sum_{s=1}^{n-1} \frac{1}{s} \int_{-\infty}^{x} x^{n-1} e^x dx = \frac{(-1)^{n-1}}{\Gamma(n)} \sum_{s=1}^{n-1} \frac{1}{s} \cdot (-1)^{n-1} (n-1)!\, e^x \sum_{r=0}^{n-1} (-1)^r \frac{x^r}{r!}$$

$$= \sum_{s=1}^{n-1} \frac{1}{s} \cdot e^x \sum_{r=0}^{n-1} (-1)^r \frac{x^r}{r!}$$

When $n \to \infty$,

$$\lim_{n \to \infty} \frac{(-1)^{n-1}}{\Gamma(n)} \sum_{s=1}^{n-1} \frac{1}{s} \int_{-\infty}^{x} x^{n-1} e^x dx = \sum_{s=1}^{\infty} \frac{1}{s} \cdot e^x \sum_{r=0}^{\infty} (-1)^r \frac{x^r}{r!} = \sum_{s=1}^{\infty} \frac{1}{s} \qquad \text{(wr1)}$$

Next,

$$\int_{-\infty}^{x} e^x x^{n-1} \log|x| dx = \left[\left(e^x x^{n-1} \right)^{<1>} \log|x| \right]_{-\infty}^{x} - \int_{-\infty}^{x} \left(e^x x^{n-1} \right)^{<1>} \frac{1}{|x|} dx$$

$$= \left(e^x x^{n-1} \right)^{<1>} \log|x| - \int_{-\infty}^{x} \left(e^x x^{n-1} \right)^{<1>} \frac{1}{|x|} dx$$

Here,

$$\left(e^x x^{n-1}\right)^{<1>} = (-1)^{n-1}(n-1)!\, e^x \sum_{r=0}^{n-1}(-1)^r \frac{x^r}{r!}$$

Substituting this for the above,

$$\int_{-\infty}^{x} e^x x^{n-1} \log|x|\, dx = \left\{ (-1)^{n-1}(n-1)!\, e^x \sum_{r=0}^{n-1}(-1)^r \frac{x^r}{r!} \right\} \log|x|$$
$$- \int_{-\infty}^{x} \left\{ (-1)^{n-1}(n-1)!\, e^x \sum_{r=0}^{n-1}(-1)^r \frac{x^r}{r!} \right\} \frac{1}{|x|} dx$$

Multipling by $(-1)^n/\Gamma(n)$ both sids ,

$$\frac{(-1)^n}{\Gamma(n)} \int_{-\infty}^{x} e^x x^{n-1} \log|x|\, dx = \left\{ \frac{(-1)^n}{\Gamma(n)} (-1)^{n-1}(n-1)!\, e^x \sum_{r=0}^{n-1}(-1)^r \frac{x^r}{r!} \right\} \log|x|$$
$$- \int_{-\infty}^{x} \left\{ \frac{(-1)^n}{\Gamma(n)} (-1)^{n-1}(n-1)!\, e^x \sum_{r=0}^{n-1}(-1)^r \frac{x^r}{r!} \right\} \frac{1}{|x|} dx$$

Thus, the 1st term of (wr) is

$$\frac{(-1)^n}{\Gamma(n)} \int_{-\infty}^{x} e^x x^{n-1} \log|x|\, dx = -e^x \sum_{r=0}^{n-1}(-1)^r \frac{x^r}{r!} \log|x| + \int_{-\infty}^{x} e^x \sum_{r=0}^{n-1}(-1)^r \frac{x^r}{r!} \frac{1}{|x|} dx$$

When $n \to \infty$,

$$\lim_{n\to\infty} \frac{(-1)^n}{\Gamma(n)} \int_{-\infty}^{x} e^x x^{n-1} \log|x|\, dx = -e^x \sum_{r=0}^{\infty}(-1)^r \frac{x^r}{r!} \log|x| + \int_{-\infty}^{x} e^x \sum_{r=0}^{\infty}(-1)^r \frac{x^r}{r!} \frac{1}{|x|} dx$$
$$= -\log|x| + \int_{-\infty}^{x} \frac{1}{|x|} dx = -\log|x| + [\log|x|]_{-\infty}^{x}$$

i.e.

$$\lim_{n\to\infty} \frac{(-1)^n}{\Gamma(n)} \int_{-\infty}^{x} e^x x^{n-1} \log|x|\, dx = -\log\infty \qquad \text{(wr2)}$$

From (wr1) , (wr2) and (wr)

$$\lim_{n\to\infty} R_{n+1} = \sum_{s=1}^{\infty} \frac{1}{s} - \log\infty = \gamma$$

From this and (w1) , we obtain (2.2) .

Furthermore,

$$\sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^r}{\Gamma(r)} = x e^{-x}$$

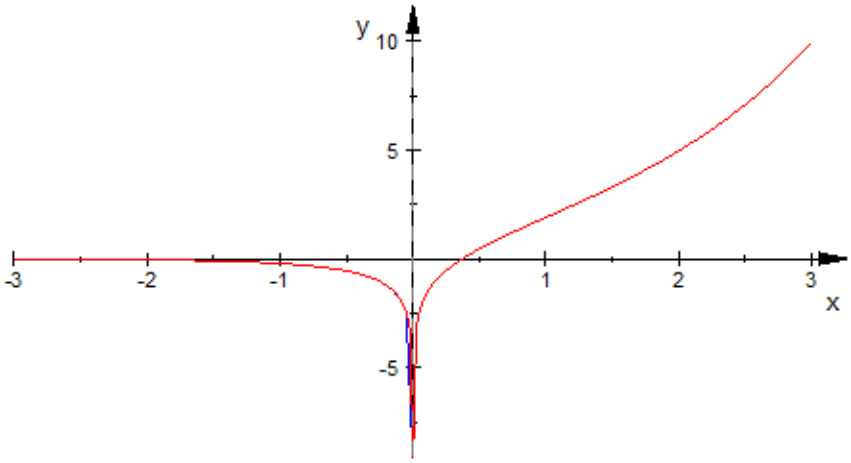
Using this, we obtain (2.3) .

(2.1) is an asymptotic expansion and the consistency at large x is good. (2.2) and (2.3) are series expansion and the consistency at small x is good. Calculating Σ in (2.3) to the 18th term and illustrating this with Ei(x), it is as follows. (Blue: Ei(x) , Red: Series.) Both overlap exactly and blue (Ei(x)) is not visible.

```
• E2 := x-> ln(abs(x)) - E^x/x *
            sum((-1)^(r-1)*psi(r)/gamma(r)*x^r, r=1..n)
```

$$x \to \ln(|x|) - \frac{E^x}{x} \cdot \left(\sum_{r=1}^{n} \frac{(-1)^{r-1} \cdot \mathrm{psi}(r)}{\Gamma(r)} \cdot x^r \right)$$

```
• plotfunc2d(Em,Ep,E2(x), x=-3..3,
             Colors=[RGB::Blue,RGB::Blue,RGB::Red])
```



1.1.3 Series expansion of Sine Integral

Formula 1.1.3

When $Si(x)=\int_0^x \frac{\sin x}{x}dx = \int_\infty^x \frac{\sin x}{x}dx + \frac{\pi}{2}$, the following expressions hold.

(1) for $x>1$

$$Si(x)= \frac{\cos x}{x}\sum_{r=1}^{n}(-1)^r\frac{(2r-2)!}{x^{2r-2}} + \frac{\sin x}{x}\sum_{r=1}^{n}(-1)^r\frac{(2r-1)!}{x^{2r-1}} + \frac{\pi}{2} + R_{2n+1} \tag{3.1}$$

$$R_{2n+1} = (-1)^n(2n)!\int_\infty^x \frac{\sin x}{x^{2n+1}}dx \tag{3.1r}$$

(2)

$$Si(x) = \frac{\sin x}{x}\sum_{r=1}^{n}(-1)^{r-1}\frac{\log|x|-\psi(2r-1)-\gamma}{\Gamma(2r-1)}x^{2r-1}$$
$$- \frac{\cos x}{x}\sum_{r=1}^{n}(-1)^{r-1}\frac{\log|x|-\psi(2r)-\gamma}{\Gamma(2r)}x^{2r} + R_{2n+1} \tag{3.2}$$

$$R_{2n+1} = (-1)^n\int_0^x \frac{\log|x|-\psi(2n)-\gamma}{\Gamma(2n)}x^{2n-1}\sin x\,dx \tag{3.2r}$$

$$\lim_{n\to\infty}R_{2n+1} = 0$$

$$Si(x) = \frac{\sin x}{x}\sum_{r=1}^{\infty}(-1)^r\frac{\psi(2r-1)}{\Gamma(2r-1)}x^{2r-1} - \frac{\cos x}{x}\sum_{r=1}^{\infty}(-1)^r\frac{\psi(2r)}{\Gamma(2r)}x^{2r} \tag{3.3}$$

Proof

(1) When $g(x) = \sin x$

$$g^{<2r>}(x) = (-1)^r\sin x$$
$$g^{<2r-1>}(x) = (-1)^r\cos x$$
$$f(x) = \frac{1}{x}, \qquad f^{(r)}(x) = (-1)^r\frac{r!}{x^{r+1}}$$

$$f^{(2r-1)}(x) = (-1)^{2r-1}\frac{(2r-1)!}{x^{2r}} \quad , \quad f^{(2r-2)}(x) = (-1)^{2r-2}\frac{(2r-2)!}{x^{2r-1}}$$

Substituting these for (1.1') ,

$$\int_{\infty}^{x}\frac{\sin x}{x}dx = \sum_{r=1}^{n}\left[(-1)^r\cos x \cdot (-1)^{2r-2}\frac{(2r-2)!}{x^{2r-1}} \right]_{\infty}^{x}$$

$$- \sum_{r=1}^{n}\left[(-1)^r\sin x \cdot (-1)^{2r-1}\frac{(2r-1)!}{x^{2r}} \right]_{\infty}^{x}$$

$$+ \int_{0}^{x}(-1)^n\sin x \ (-1)^{2n}\frac{(2n)!}{x^{2n+1}}dx$$

$$= \cos x\sum_{r=1}^{n}(-1)^r\frac{(2r-2)!}{x^{2r-1}} - \sin x\sum_{r=1}^{n}(-1)^{3r-1}\frac{(2r-1)!}{x^{2r}}$$

$$+ (-1)^n(2n)!\int_{0}^{x}\frac{\sin x}{x^{2n+1}}dx$$

$$\therefore \quad Si(x) = \frac{\cos x}{x}\sum_{r=1}^{n}(-1)^r\frac{(2r-2)!}{x^{2r-2}} + \frac{\sin x}{x}\sum_{r=1}^{n}(-1)^r\frac{(2r-1)!}{x^{2r-1}} + \frac{\pi}{2} + R_{2n+1} \tag{3.1}$$

$$R_{2n+1} = (-1)^n(2n)!\int_{\infty}^{x}\frac{\sin x}{x^{2n+1}}dx \tag{3.1r}$$

(2) When $f(x) = \sin x$

$$f^{(2r-2)}(x) = (-1)^{r-1}\sin x \ , \qquad f^{(2r-1)}(x) = (-1)^{r-1}\cos x$$

$$f^{(2r)}(x) = (-1)^r\sin x$$

$$g(x) = \frac{1}{x}$$

$$g^{<2r-1>}(x) = (\log x)^{<2r-2>} = \frac{\log|x| - \psi(2r-1) - \gamma}{\Gamma(2r-1)}x^{2r-2}$$

$$g^{<2r>}(x) = (\log x)^{<2r-1>} = \frac{\log|x| - \psi(2r) - \gamma}{\Gamma(2r)}x^{2r-1}$$

Substituting these for (1.1') ,

$$\int_{0}^{x}\frac{\sin x}{x}dx = \sum_{r=1}^{n}\left[\frac{\log|x| - \psi(2r-1) - \gamma}{\Gamma(2r-1)}x^{2r-2}\cdot(-1)^{r-1}\sin x \right]_{0}^{x}$$

$$- \sum_{r=1}^{n}\left[\frac{\log|x| - \psi(2r) - \gamma}{\Gamma(2r)}x^{2r-1}\cdot(-1)^{r-1}\cos x \right]_{0}^{x}$$

$$+ \int_{0}^{x}\frac{\log|x| - \psi(2n) - \gamma}{\Gamma(2n)}x^{2n-1}\cdot(-1)^n\sin x\,dx$$

$$= \frac{\sin x}{x}\sum_{r=1}^{n}(-1)^{r-1}\frac{\log|x| - \psi(2r-1) - \gamma}{\Gamma(2r-1)}x^{2r-1}$$

$$- \frac{\cos x}{x}\sum_{r=1}^{n}(-1)^{r-1}\frac{\log|x| - \psi(2r) - \gamma}{\Gamma(2r)}x^{2r} \tag{3.2}$$

$$+ (-1)^n \int_0^x \frac{\log|x| - \psi(2n) - \gamma}{\Gamma(2n)} x^{2n-1} \sin x \, dx \qquad (3.2r)$$

And

$$\lim_{n\to\infty} \frac{\log|x| - \psi(2r) - \gamma}{\Gamma(2n)} = 0 \qquad \Longrightarrow \qquad \lim_{n\to\infty} R_{2n+1} = 0$$

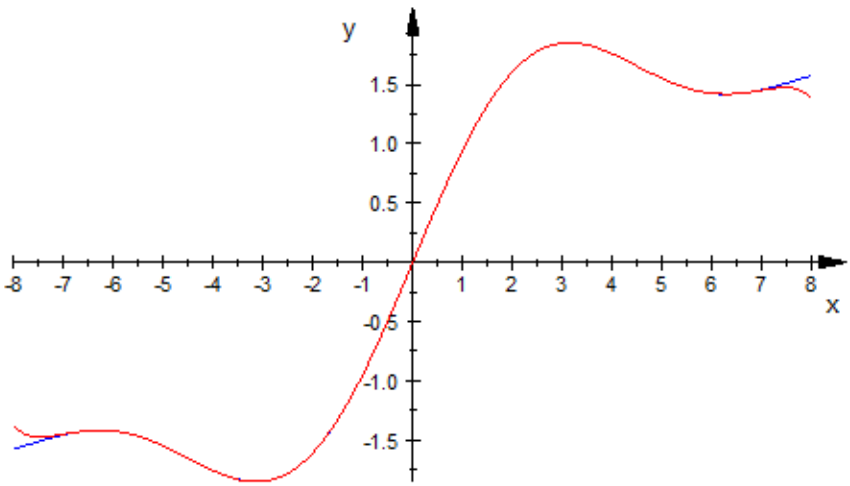
Furthermore

$$\sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^{2r-1}}{\Gamma(2r-1)} = x\cos x \quad , \qquad \sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^{2r}}{\Gamma(2r)} = x\sin x$$

Thus, we obtain (3.3) .

(3.1) is an asymptotic expansion and the consistency at large x is good. (3.2) and (3.3) are series expansion and the consistency at small x is good. Calculating Σ in (3.3) to the 11th term and illustrating this with Si(x), it is as follows. (Blue: Si(x) , Red: Series.)

```
• fS2 := sin(x)/x*sum((-1)^r*psi(2*r-1)
                                /gamma(2*r-1)*x^(2*r-1), r=1..n)
      - cos(x)/x*sum((-1)^r*psi(2*r)
                                /gamma(2*r)*x^(2*r), r=1..n):
• plotfunc2d(Si(x),fS2,x=-8..8)
```



1.1.4 Series expansion of Cosine Integral

Formula 1.1.4

When $Ci(x) = \int_\infty^x \frac{\cos x}{x} dx = \gamma + \log x - \int_0^x \frac{1-\cos x}{x} dx$, the following expressions hold.

(1) for $x>1$

$$Ci(x) = \frac{\sin x}{x} \sum_{r=1}^{n} (-1)^{r-1} \frac{(2r-2)!}{x^{2r-2}} - \frac{\cos x}{x} \sum_{r=1}^{n} (-1)^{r-1} \frac{(2r-1)!}{x^{2r-1}} + R_{2n+1} \qquad (4.1)$$

(2) for $x>0$

$$Ci(x) = \gamma + \frac{\cos x}{x} \sum_{r=1}^{n} (-1)^{r-1} \frac{\log x - \psi(2r-1) - \gamma}{\Gamma(2r-1)} x^{2r-1}$$
$$+ \frac{\sin x}{x} \sum_{r=1}^{n} (-1)^{r-1} \frac{\log x - \psi(2r) - \gamma}{\Gamma(2r)} x^{2r} + R_{2n+1} \qquad (4.2)$$

$$R_{2n+1} = (-1)^n \int_0^x \frac{\log x - \psi(2n) - \gamma}{\Gamma(2n)} x^{2n-1} \cos x \, dx \tag{4.2r}$$

$$\lim_{n \to \infty} R_{2n+1} = 0$$

$$Ci(x) = \log x + \frac{\cos x}{x} \sum_{r=1}^{\infty} (-1)^r \frac{\psi(2r-1)}{\Gamma(2r-1)} x^{2r-1}$$

$$+ \frac{\sin x}{x} \sum_{r=1}^{\infty} (-1)^r \frac{\psi(2r)}{\Gamma(2r)} x^{2r} \tag{4.3}$$

Proof

(1) When $g(x) = \cos x$

$$g^{<2r>}(x) = (-1)^r \cos x$$

$$g^{<2r-1>}(x) = (-1)^{r-1} \sin x$$

$$f(x) = \frac{1}{x}, \qquad f^{(r)}(x) = (-1)^r \frac{r!}{x^{r+1}}$$

$$f^{(2r-1)}(x) = (-1)^{2r-1} \frac{(2r-1)!}{x^{2r}}, \qquad f^{(2r-2)}(x) = (-1)^{2r-2} \frac{(2r-2)!}{x^{2r-1}}$$

Substituting these for (1.1') ,

$$\int_\infty^x \frac{\cos x}{x} dx = \sum_{r=1}^{n} \left[(-1)^{r-1} \sin x \cdot (-1)^{2r-2} \frac{(2r-2)!}{x^{2r-1}} \right]_\infty^x$$

$$- \sum_{r=1}^{n} \left[(-1)^r \cos x \cdot (-1)^{2r-1} \frac{(2r-1)!}{x^{2r}} \right]_\infty^x$$

$$+ \int_\infty^x (-1)^n \cos x \cdot (-1)^{2n} \frac{(2n)!}{x^{2n+1}} dx$$

$$= \frac{\sin x}{x} \sum_{r=1}^{n} (-1)^{r-1} \frac{(2r-2)!}{x^{2r-2}} - \frac{\cos x}{x} \sum_{r=1}^{n} (-1)^{r-1} \frac{(2r-1)!}{x^{2r-1}}$$

$$+ (-1)^n (2n)! \int_\infty^x \frac{\cos x}{x^{2n+1}} dx \tag{4.1}$$

(2) When $f(x) = 1 - \cos x$

$$f^{(2r-2)}(x) = (-1)^r \cos x, \qquad f^{(2r-1)}(x) = (-1)^{r-1} \sin x$$

$$f^{(2r)}(x) = (-1)^{r-1} \cos x$$

$$g(x) = \frac{1}{x}$$

$$g^{<2r-1>}(x) = (\log x)^{<2r-2>} = \frac{\log x - \psi(2r-1) - \gamma}{\Gamma(2r-1)} x^{2r-2}$$

$$g^{<2r>}(x) = (\log x)^{<2r-1>} = \frac{\log x - \psi(2r) - \gamma}{\Gamma(2r)} x^{2r-1}$$

Substituting these for (1.1') ,

$$\int_0^x \frac{1-\cos x}{x}dx = \sum_{r=1}^{n}\left[\frac{\log x-\psi(2r-1)-\gamma}{\Gamma(2r-1)}x^{2r-2}\cdot(-1)^r\cos x\right]_0^x$$

$$-\sum_{r=1}^{n}\left[\frac{\log x-\psi(2r)-\gamma}{\Gamma(2r)}x^{2r-1}\cdot(-1)^{r-1}\sin x\right]_0^x$$

$$+\int_0^x \frac{\log x-\psi(2n)-\gamma}{\Gamma(2n)}x^{2n-1}\cdot(-1)^{n-1}\cos x\,dx$$

$$=\cos x\sum_{r=1}^{n}(-1)^r\frac{\log x-\psi(2r-1)-\gamma}{\Gamma(2r-1)}x^{2r-2}-\cos 0\cdot(-1)^1\frac{\log x-\psi(1)-\gamma}{\Gamma(1)}0^0$$

$$-\sin x\sum_{r=1}^{n}(-1)^{r-1}\frac{\log x-\psi(2r)-\gamma}{\Gamma(2r)}x^{2r-1}$$

$$+(-1)^{n-1}\int_0^x \frac{\log x-\psi(2n)-\gamma}{\Gamma(2n)}x^{2n-1}\cos x\,dx$$

$$=-\frac{\cos x}{x}\sum_{r=1}^{n}(-1)^{r-1}\frac{\log x-\psi(2r-1)-\gamma}{\Gamma(2r-1)}x^{2r-1}+\log x$$

$$-\frac{\sin x}{x}\sum_{r=1}^{n}(-1)^{r-1}\frac{\log x-\psi(2r)-\gamma}{\Gamma(2r)}x^{2r}$$

$$-(-1)^n\int_0^x \frac{\log x-\psi(2n)-\gamma}{\Gamma(2n)}x^{2n-1}\cos x\,dx$$

Therefore,

$$Ci(x)=\gamma+\frac{\cos x}{x}\sum_{r=1}^{n}(-1)^{r-1}\frac{\log x-\psi(2r-1)-\gamma}{\Gamma(2r-1)}x^{2r-1}$$

$$+\frac{\sin x}{x}\sum_{r=1}^{n}(-1)^{r-1}\frac{\log x-\psi(2r)-\gamma}{\Gamma(2r)}x^{2r}+R_{2n+1} \tag{4.2}$$

$$R_{2n+1}=(-1)^n\int_0^x \frac{\log x-\psi(2n)-\gamma}{\Gamma(2n)}x^{2n-1}\cos x\,dx \tag{4.2r}$$

And

$$\lim_{n\to\infty}\frac{\log x-\psi(2n)-\gamma}{\Gamma(2n)}=0 \quad\Longrightarrow\quad \lim_{n\to\infty}R_{2n+1}=0$$

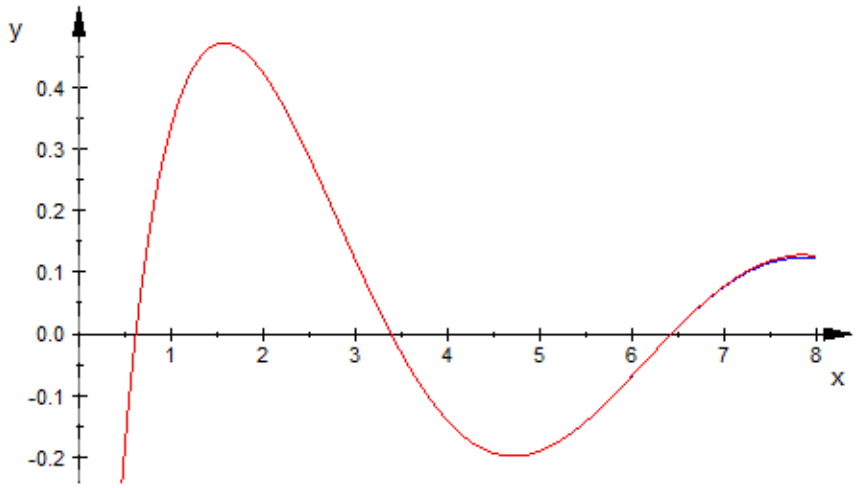
Furthermore

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{x^{2r-1}}{\Gamma(2r-1)}=x\cos x \quad , \quad \sum_{r=1}^{\infty}(-1)^{r-1}\frac{x^{2r}}{\Gamma(2r)}=x\sin x$$

Using these, we obtain (4.3) .

(4.1) is an asymptotic expansion and the consistency at large x is good. (4.2) and (4.3) are series expansion and the consistency at small x is good. Calculating Σ in (4.3) to the 12th term and illustrating this with Ci(x), it is as follows. (Blue: Ci(x) , Red: Series.)

```
• fC2 := ln(x) + cos(x)/x*sum((-1)^r*psi(2*r-1)
                                  /gamma(2*r-1)*x^(2*r-1), r=1..n)
          + sin(x)/x*sum((-1)^r*psi(2*r)
                                  /gamma(2*r)*x^(2*r), r=1..n):
• plotfunc2d(Ci(x),fC2,x=0..8)
```


y
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
1
2
3
4
5
6
7
8
x

1.2 Generalized Taylor's Theorem

1.2.1 Lagrange form of the remainder

Lemma 1.2.1

When $f(x)$ is n times differentiable function on $[a, b]$ and $f(x)$ is

$$g^{<n>}(x) = \int g^{<n-1>}(x)dx + c_n \qquad n=1,2,3,\cdots \qquad c_n \text{ : arbitrary constants}$$

the following expressions hold.

$$f(b)-f(a) = \sum_{r=1}^{n}(-1)^{r-1}\left[g^{<r>}(x)\, f^{(r)}(x)\right]_a^b + R_{n+1} \tag{1.1}$$

$$R_{n+1} = (-1)^n \int_a^b g^{<n>}(x) f^{(n+1)}(x)dx \tag{1.1s}$$

$$= (-1)^n \left[g^{<n+1>}(x)\right]_a^b f^{(n+1)}(\xi) \qquad a<\xi<b \tag{1.1r}$$

Proof

Formula 1.1.1 was as follows.

$$\int_a^b g(x)f(x)dx = \sum_{r=1}^{n}(-1)^{r-1}\left[g^{<r>}(x)f^{(r-1)}(x)\right]_a^b + R_{n+1}$$

$$R_{n+1} = (-1)^n \int_a^b g^{<n>}(x)f^{(n)}(x)dx$$

Replacing $f(x)$ with $f^{(1)}(x)$ and substituting $g^{<0>}(x)=1$,

$$\int_a^b f^{(1)}(x)dx = \sum_{r=1}^{n}(-1)^{r-1}\left[g^{<r>}(x)f^{(r)}(x)\right]_a^b + R_{n+1}$$

$$R_{n+1} = (-1)^n \int_a^b g^{<n>}(x)f^{(n+1)}(x)dx$$

Thus, we obtain (1.1), (1.1s) immediately.

Next, we prove (1.1r) . Let

$$\int_a^b f^{(1)}(x)dx = f(b) - f(a) = \left[g^{<1>}(x)f^{(1)}(x)\right]_a^b + \left[g^{<2>}(x)\right]_a^b K$$

transposing $f(a)$,

$$f(b) = f(a) + \left[g^{<1>}(x)f^{(1)}(x)\right]_a^b + \left[g^{<2>}(x)\right]_a^b K$$

And we difine $F(x)$ as follows.

$$F(x) = -f(b) + f(x) + g^{<1>}(b)f^{(1)}(b) - g^{<1>}(x)f^{(1)}(x) + \left\{g^{<2>}(b) - g^{<2>}(x)\right\}K$$

Then,

$$F(b) = -f(b) + f(b) + g^{<1>}(b)f^{(1)}(b) - g^{<1>}(b)f^{(1)}(b) + \left\{g^{<2>}(b) - g^{<2>}(b)\right\}K = 0$$

$$F(a) = -f(b) + f(a) + g^{<1>}(b)f^{(1)}(b) - g^{<1>}(a)f^{(1)}(a) + \left\{g^{<2>}(b) - g^{<2>}(b)\right\}K$$

$$= -f(b)+f(a)+\left[g^{<1>}(x)f^{(1)}(x)\right]_a^b + \left[g^{<2>}(x)\right]_a^b K$$
$$= -f(b)+f(b) = 0$$

That is, $F(a) = F(b)$. Then according Rolle's theorem , at least one $a < \xi < b$ exists such that

$$F'(\xi) = f^{(1)}(\xi) - g^{<0>}(\xi)f^{(1)}(\xi) - g^{<1>}(\xi)f^{(2)}(\xi) - g^{<1>}(\xi)K = 0$$

From this,

$$K = \frac{1}{g^{<1>}(\xi)}\left\{f^{(1)}(\xi) - g^{<0>}(\xi)f^{(1)}(\xi) - g^{<1>}(\xi)f^{(2)}(\xi)\right\}$$
$$= -f^{(2)}(\xi) \qquad (\because \quad g^{<0>}(\xi)=1)$$

Therefore,

$$f(b) = f(a)+\left[g^{<1>}(x)f^{(1)}(x)\right]_a^b - \left[g^{<2>}(x)\right]_a^b f^{(2)}(\xi)$$

Next, we ask for K such that

$$f(b) = f(a)+\left[g^{<1>}(x)f^{(1)}(x)\right]_a^b - \left[g^{<2>}(x)f^{(2)}(x)\right]_a^b + \left[g^{<3>}(x)\right]_a^b K$$

Let us difine $F(x)$ as follows.

$$F(x) = -f(b)+f(x) + g^{<1>}(b)f^{(1)}(b) - g^{<1>}(x)f^{(1)}(x)$$
$$- g^{<2>}(b)f^{(2)}(b)+g^{<2>}(x)f^{(2)}(x)+\left\{g^{<3>}(b) - g^{<3>}(x)\right\}K$$

Then,

$$F(b) = 0$$
$$F(a) = -f(b)+f(a) + g^{<1>}(b)f^{(1)}(b) - g^{<1>}(a)f^{(1)}(a)$$
$$- g^{<2>}(b)f^{(2)}(b)+g^{<2>}(a)f^{(2)}(a)+\left\{g^{<3>}(b) - g^{<3>}(a)\right\}K$$
$$= -f(b)+f(b) = 0$$

That is, $F(a) = F(b)$. Then according Rolle's theorem , at least one $a < \xi < b$ exists such that

$$F'(\xi) = f^{(1)}(\xi) - g^{<0>}(\xi)f^{(1)}(\xi)+g^{<2>}(\xi)f^{(3)}(\xi) - g^{<2>}(\xi)K = 0$$

From this ,

$$K = \frac{1}{g^{<2>}(\xi)}\left\{f^{(1)}(\xi) - g^{<0>}(\xi)f^{(1)}(\xi)+g^{<2>}(\xi)f^{(3)}(\xi)\right\}$$
$$= f^{(3)}(\xi) \qquad (\because \quad g^{<0>}(\xi)=1)$$

Therefore,

$$f(b) = f(a)+\left[g^{<1>}(x)f^{(1)}(x)\right]_a^b - \left[g^{<2>}(x)f^{(2)}(x)\right]_a^b + \left[g^{<3>}(x)\right]_a^b f^{(3)}(\xi)$$

Hereafter, by induction, we obtain (1.1r) .

1.2.2 Generalized Taylor's Theorem

Theorem 1.2.2

When $f(x)$ is $n+1$ times differentiable function on $[a,b]$, ξ exists such that $a < \xi < b$, the following expressions hold.

$$f(b)-f(a) = \sum_{r=1}^{n} \frac{(b-a)^r}{r!} f^{(r)}(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \qquad \textbf{(2.1)}$$

$$f(b)-f(a)=\sum_{r=1}^{n}(-1)^{r-1}\frac{(b-a)^r}{r!}f^{(r)}(b)+(-1)^n\frac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi) \qquad \textbf{(2.2)}$$

Proof

In Lemma 1.2.1, let

$$g^{<r>}(x)=\int_b^x\cdots\int_b^x 1dx^r \quad =\frac{(x-b)^r}{r!} \quad r=0,1,2,\cdots n+1.$$

Then

$$\begin{aligned} f(b)-f(a) &= \sum_{r=1}^{n}(-1)^{r-1}\left[\frac{(x-b)^r}{r!}f^{(r)}(x)\right]_a^b+(-1)^n\left[\frac{(x-b)^{n+1}}{(n+1)!}\right]_a^b f^{(n+1)}(\xi) \\ &= \sum_{r=1}^{n}(-1)^{r-1}\left\{-\frac{(a-b)^r}{r!}f^{(r)}(a)\right\}-(-1)^n\frac{(a-b)^{n+1}}{(n+1)!}f^{(n+1)}(\xi) \\ &= \sum_{r=1}^{n}\frac{(b-a)^r}{r!}f^{(r)}(a)+\frac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi) \qquad (2.1) \end{aligned}$$

On the contrary, let

$$g^{<r>}(x)=\int_a^x\cdots\int_a^x 1dx^r \quad =\frac{(x-a)^r}{r!} \quad r=0,1,2,\cdots n+1$$

Then

$$\begin{aligned} f(b)-f(a) &= \sum_{r=1}^{n}(-1)^{r-1}\left[\frac{(x-a)^r}{r!}f^{(r)}(x)\right]_a^b+(-1)^n\left[\frac{(x-a)^{n+1}}{(n+1)!}\right]_a^b f^{(n+1)}(\xi) \\ &= \sum_{r=1}^{n}(-1)^{r-1}\frac{(b-a)^r}{r!}f^{(r)}(b)+(-1)^n\frac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi) \qquad (2.2) \end{aligned}$$

Note

(2.1) is a power series and (2.1) is a series of functions. That is, Generalized Taylor's Theorem consists of a power series and a series of functions.

1.2.3 Taylor Expansion of Logarithmic Function

Formula 1.2.3

$$\log x=\sum_{r=1}^{n}\frac{(-1)^{r-1}}{r}(x-1)^r+\frac{(-1)^n}{n+1}\left(\frac{x-1}{\xi}\right)^{n+1} \qquad 1<\xi<x \qquad (3.1)$$

$$=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r}(x-1)^r \qquad 0<x\leqq 2 \qquad (3.1')$$

$$\log x=\sum_{r=1}^{n}\frac{1}{r}\left(\frac{x-1}{x}\right)^r+\frac{1}{n+1}\left(\frac{x-1}{\xi}\right)^{n+1} \qquad 1<\xi<x \qquad (3.2)$$

$$=\sum_{r=1}^{\infty}\frac{1}{r}\left(\frac{x-1}{x}\right)^r \qquad x\geqq\frac{1}{2} \qquad (3.2')$$

Proof

Let

$$f(x)=\log x \quad , \quad (\log x)^{(r)} = (-1)^{r-1}\frac{(r-1)!}{x^r}$$

Then according to (2.1) of Theorem1.2.2 ,

$$\log b - \log a = \sum_{r=1}^{n}\frac{(b-a)^r}{r!}(-1)^{r-1}\frac{(r-1)!}{a^r} + \frac{(b-a)^{n+1}}{(n+1)!}(-1)^n\frac{n!}{\xi^{n+1}}$$

$$= \sum_{r=1}^{n}\frac{(-1)^{r-1}}{r}\left(\frac{b-a}{a}\right)^r + (-1)^n\frac{1}{n+1}\left(\frac{b-a}{\xi}\right)^{n+1} \quad a<\xi<b$$

Substituting $b=x$, $a=1$ for this,

$$\log x = \sum_{r=1}^{n}\frac{(-1)^{r-1}}{r}(x-1)^r + \frac{(-1)^n}{n+1}\left(\frac{x-1}{\xi}\right)^{n+1} \qquad 1<\xi<x \tag{3.1}$$

When $n\to\infty$, the 1st term of the right side is converged for $0<x\leqq 2$. Then (3.1') follows.

Next, from (2.2) of Theorem1.2.2 ,

$$\log b - \log a = \sum_{r=1}^{n}(-1)^{2(r-1)}\frac{(b-a)^r}{r!}\frac{(r-1)!}{b^r} + (-1)^{2n}\frac{(b-a)^{n+1}}{(n+1)!}\frac{n!}{\xi^{n+1}}$$

$$= \sum_{r=1}^{n}\frac{1}{r}\left(\frac{b-a}{b}\right)^r + \frac{1}{n+1}\left(\frac{b-a}{\xi}\right)^{n+1} \quad a<\xi<b$$

Substituting $b=x$, $a=1$ for this,

$$\log x = \sum_{r=1}^{n}\frac{1}{r}\left(\frac{x-1}{x}\right)^r + \frac{1}{n+1}\left(\frac{x-1}{\xi}\right)^{n+1} \qquad 1<\xi<x \tag{3.2}$$

When $n\to\infty$, the 1st term of the right side is converged for $x\geqq 1/2$. Then (3.2') follows.

Note

(3.2') is obtained also by replacing x with $1/x$ in (3.1'). Moreover, it can also be obtained easily as follows.

$$\log\frac{1}{1-y} = y + \frac{y^2}{2} + \frac{y^3}{3} + \cdots \quad |y|<1$$

Using this,

$$\log x = \log\left(\frac{1}{\frac{1}{x}}\right) = \log\left\{\frac{1}{1-\left(\frac{x-1}{x}\right)}\right\} = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots$$

Example

$$\log 3 = \frac{1}{1}\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \cdots$$

1.2.4 Generalized Binomial Theorem

Theorem 1.2.4

$$(1+x)^\alpha = \sum_{r=0}^{n}\binom{\alpha}{r}x^r + \binom{\alpha}{n+1}x^{n+1}(1+\xi)^{\alpha-(n+1)} \qquad 0<\xi<x \tag{4.1}$$

$$= \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r \qquad |x|<1 \qquad \textbf{(4.1')}$$

$$= x^{\alpha} \sum_{r=0}^{\infty} \binom{\alpha}{r} \frac{1}{x^r} \qquad \left\{ = \sum_{r=0}^{\infty} \binom{\alpha}{\alpha-r} x^{\alpha-r} \right\} \qquad |x|>1 \qquad \textbf{(4.2')}$$

$$(1+x)^{\alpha} = \sum_{r=0}^{\infty} (-1)^r \binom{-\alpha}{r} \left(\frac{x}{1+x} \right)^r \qquad x > -\frac{1}{2} \qquad \textbf{(4.3')}$$

$$= \sum_{r=0}^{\infty} \binom{\alpha-1+r}{\alpha-1} \left(\frac{x}{1+x} \right)^r \qquad x > -\frac{1}{2} \qquad (4.3'')$$

Proof

Let

$$f(x) = (1+x)^{\alpha} \qquad (\alpha : \text{real number})$$

$$\{(1+x)^{\alpha}\}^{(r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} (1+x)^{\alpha-r}$$

Then, according to (2.1) of Theorem1.2.2 ,

$$(1+b)^{\alpha} - (1+a)^{\alpha} = \sum_{r=1}^{n} \frac{(b-a)^r}{r!} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} (1+a)^{\alpha-r}$$

$$+ \frac{(b-a)^{n+1}}{(n+1)!} \frac{\Gamma(1+\alpha)}{\Gamma\{1+\alpha-(n+1)\}} (1+\xi)^{\alpha-(n+1)}$$

$$= \sum_{r=1}^{n} \binom{\alpha}{r} (b-a)^r (1+a)^{\alpha-r} + \binom{\alpha}{n+1} (b-a)^{n+1} (1+\xi)^{\alpha-(n+1)}$$

Substituting $b=x$, $a=0$ for this,

$$(1+x)^{\alpha} = \sum_{r=0}^{n} \binom{\alpha}{r} x^r + \binom{\alpha}{n+1} x^{n+1} (1+\xi)^{\alpha-(n+1)} \qquad 0<\xi<x \qquad (4.1)$$

Since the radius of convergence of the 1st term is 1,

$$(1+x)^{\alpha} = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r \qquad |x|<1 \qquad (4.1')$$

Replacing x with $1/x$,

$$\left(1+\frac{1}{x}\right)^{\alpha} = \sum_{r=0}^{\infty} \binom{\alpha}{r} \frac{1}{x^r} \qquad \frac{1}{|x|}<1$$

Multipling by x^{α} both sids ,

$$(1+x)^{\alpha} = x^{\alpha} \sum_{r=0}^{\infty} \binom{\alpha}{r} \frac{1}{x^r} \qquad \left\{ = \sum_{r=0}^{\infty} \binom{\alpha}{\alpha-r} x^{\alpha-r} \right\} \qquad |x|>1 \qquad (4.2')$$

Next, from (2.2) of Theorem1.2.2 ,

$$(1+b)^{\alpha} - (1+a)^{\alpha} = \sum_{r=1}^{n} (-1)^{r-1} \frac{(b-a)^r}{r!} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} (1+b)^{\alpha-r}$$

$$+ (-1)^n \frac{(b-a)^{n+1}}{(n+1)!} \frac{\Gamma(1+\alpha)}{\Gamma\{1+\alpha-(n+1)\}} (1+\xi)^{\alpha-(n+1)}$$

$$= (1+b)^{\alpha}\sum_{r=1}^{n}(-1)^{r-1}\binom{\alpha}{r}\left(\frac{b-a}{1+b}\right)^{r}$$

$$+ (-1)^{n}\binom{\alpha}{n+1}(b-a)^{n+1}(1+\xi)^{\alpha-(n+1)} \qquad a<\xi<b$$

Substituting $b=x$, $a=0$ for this,

$$(1+x)^{\alpha} = 1+(1+x)^{\alpha}\sum_{r=1}^{n}(-1)^{r-1}\binom{\alpha}{r}\left(\frac{x}{1+x}\right)^{r}+(-1)^{n}\binom{\alpha}{n+1}x^{n+1}(1+\xi)^{\alpha-(n+1)}$$

Let

$$a_n = (-1)^{r-1}\binom{\alpha}{r}\left(\frac{x}{1+x}\right)^{r}.$$

Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n}\binom{\alpha}{n+1}\left(\frac{x}{1+x}\right)^{n+1}}{(-1)^{n-1}\binom{\alpha}{n}\left(\frac{x}{1+x}\right)^{n}}\right| = \left|\frac{-(n+1)x}{(\alpha-n+1)(1+x)}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{-\left(1+\frac{1}{n}\right)x}{\left(\frac{\alpha}{n}-1+\frac{1}{n}\right)(1+x)}\right| = \left|\frac{x}{1+x}\right|$$

When $x>-\frac{1}{2}$, $-1<\frac{x}{1+x}<1$ $\quad\therefore\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

Therefore Σ convergis in the right side as follows.

$$(1+x)^{\alpha} = 1 + (1+x)^{\alpha}\sum_{r=1}^{\infty}(-1)^{r-1}\binom{\alpha}{r}\left(\frac{x}{1+x}\right)^{r} \qquad x>-\frac{1}{2}$$

Solving this for x,

$$(1+x)^{\alpha} = \frac{1}{1-\sum_{r=1}^{\infty}(-1)^{r-1}\binom{\alpha}{r}\left(\frac{x}{1+x}\right)^{r}} \qquad x>-\frac{1}{2}$$

Further, replacing α with $-\alpha$,

$$(1+x)^{\alpha} = \sum_{r=0}^{\infty}(-1)^{r}\binom{-\alpha}{r}\left(\frac{x}{1+x}\right)^{r} \qquad x>-\frac{1}{2} \qquad (4.3')$$

or

$$(1+x)^{\alpha} = \sum_{r=0}^{\infty}\binom{\alpha-1+r}{\alpha-1}\left(\frac{x}{1+x}\right)^{r} \qquad x>-\frac{1}{2} \qquad (4.3'')$$

Example

$$(1+x)^{\pi} = 1 + \binom{\pi}{1}x^{1}+\binom{\pi}{2}x^{2}+\binom{\pi}{3}x^{3}+\cdots \qquad |x|<1$$

$$= x^{\pi}\left\{1 + \binom{\pi}{1}\frac{1}{x^{1}}+\binom{\pi}{2}\frac{1}{x^{2}}+\binom{\pi}{3}\frac{1}{x^{3}}+\cdots\right\} \qquad x>1$$

$$= 1 + \binom{-\pi}{1}\left(-\frac{x}{1+x}\right)^1 + \binom{-\pi}{2}\left(-\frac{x}{1+x}\right)^2 + \cdots \qquad x > -\frac{1}{2}$$

Note

(4.1') is usual binomial series. (4.2') is the series in the opposite direction. (See *Super Calculus* **3.2.2**) (4.3') is a series of functions.

1.2.5 Taylor Expansion of Power Function

Replacing x with $x-1$ in (4.1'), (4.2'), (4.3') , we obtain the following immediately.

$$x^\alpha = \sum_{r=0}^{\infty}\binom{\alpha}{r}(x-1)^r \qquad 0<x<2 \tag{5.1}$$

$$x^\alpha = (x-1)^\alpha \sum_{r=0}^{\infty}\binom{\alpha}{r}\frac{1}{(x-1)^r} \qquad x>2 \tag{5.2}$$

$$x^\alpha = \sum_{r=0}^{\infty}\binom{-\alpha}{r}\left(\frac{1}{x}-1\right)^r \qquad x>\frac{1}{2} \tag{5.3}$$

Example

$$x^e = 1 + \binom{e}{1}(x-1)^1 + \binom{e}{2}(x-1)^2 + \binom{e}{3}(x-1)^3 + \cdots \qquad 0<x<2$$

$$= (x-1)^e\left\{1 + \binom{e}{1}\frac{1}{(x-1)^1} + \binom{e}{2}\frac{1}{(x-1)^2} + \binom{e}{3}\frac{1}{(x-1)^3} + \cdots\right\} \qquad x>2$$

$$= 1 + \binom{-e}{1}\left(\frac{1}{x}-1\right)^1 + \binom{-e}{2}\left(\frac{1}{x}-1\right)^2 + \binom{-e}{3}\left(\frac{1}{x}-1\right)^3 + \cdots \qquad x>\frac{1}{2}$$

As a special case of (5.3) , the following interesting expressions are obtained though they are nonsense.

Taylor Expansion of x

$$x = \sum_{r=0}^{\infty}\left(\frac{x-1}{x}\right)^r \qquad x>\frac{1}{2} \tag{5.4}$$

$$x = \sum_{r=1}^{\infty}\left(\frac{x}{1+x}\right)^r \qquad x>-\frac{1}{2} \tag{5.5}$$

Proof

Substituting $\alpha=1$ for (5.3) , we obtain (5.4). And replacing x with $x+1$, we obtain (5.5). Moreover, (5.4) can also be obtained easily as follows.

$$x = \frac{1}{1-\frac{x-1}{x}} = 1 + \frac{x-1}{x} + \left(\frac{x-1}{x}\right)^2 + \left(\frac{x-1}{x}\right)^3 + \cdots \qquad x>\frac{1}{2}$$

Examples

$$2 = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots \qquad \left(= \frac{1}{1-1/2}\right)$$

$$= \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \cdots \qquad \left(= \frac{2}{3}\frac{1}{1-2/3}\right)$$

$$3 = 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \cdots \qquad \left(= \frac{1}{1-2/3}\right)$$

$$= \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots \qquad \left(= \frac{3}{4}\frac{1}{1-3/4}\right)$$

1.2.6 Taylor Expansion of Fractional Function

Replacing α with $-\alpha$ in (5.1), (5.3) , we obtain the following immediately.

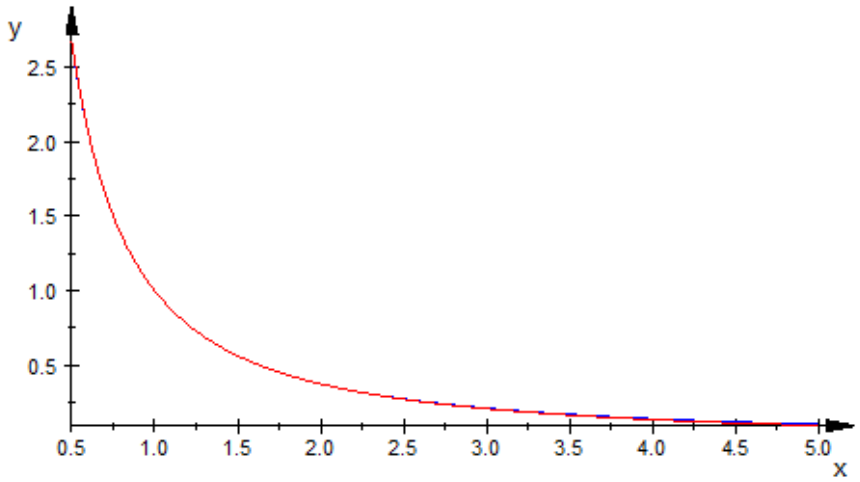
$$\frac{1}{x^\alpha} = \sum_{r=0}^{\infty} \binom{-\alpha}{r} (x-1)^r \qquad 0 < x < 2 \tag{6.1}$$

$$\frac{1}{x^\alpha} = \sum_{r=0}^{\infty} \binom{\alpha}{r} \left(\frac{1}{x}-1\right)^r \qquad x > \frac{1}{2} \tag{6.2}$$

Example

$$\frac{1}{x^{\sqrt{2}}} = 1 + \binom{\sqrt{2}}{1}\left(\frac{1}{x}-1\right)^1 + \binom{\sqrt{2}}{2}\left(\frac{1}{x}-1\right)^2 + \binom{\sqrt{2}}{3}\left(\frac{1}{x}-1\right)^3 + \cdots \qquad x > \frac{1}{2}$$

Calculating the right -side to the 5th term and illustrating this with the left side, it is as follows. (Blue: Left , Red: Right.) Both overlap exactly and blue (Left) is not visible.



2011.06.15

K. Kono

Alien's Mathematics