3 Global Definition of Dirichlet Beta, and Generalized Euler Number

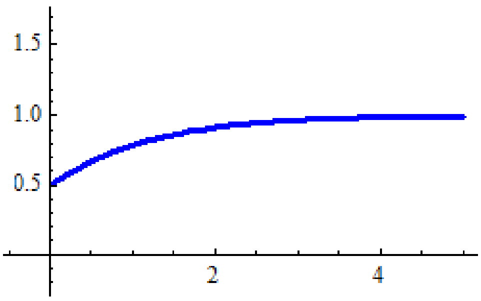
3.1 Patchy Definition of Dirichlet Beta

Let's review the definition of Dirichlet Beta.

3.1.1 The definition by Dirichlet

The very beginning definition of the Dirichlet Beta was as follows.

$$\beta(p) = \frac{1}{1^p} - \frac{1}{3^p} + \frac{1}{5^p} - \frac{1}{7^p} + \cdots \qquad Re(p) \geqq 0 \tag{1.1}$$



From this expression, The following special values were obtaind

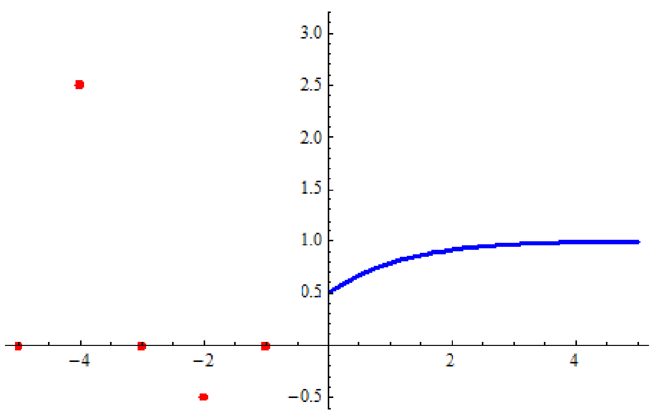
$$\beta(0) = 1 - 1 + 1 - 1 +- \cdots = \frac{1}{2}.$$

$$\beta(2n-1) = \frac{\pi}{4} \frac{|E_{2n-2}|}{(2n-2)!} \left(\frac{\pi}{2}\right)^{2n-2} \qquad n=1,2,3,\cdots.$$

Where, E_n is Euler number s.t. $\begin{cases} E_0=1, \ E_2=-1, \ E_4=5, \ E_6=-61, \ \cdots \\ E_1 = E_3 = E_5 = E_7 \cdots = 0 \end{cases}$

However, this series was divergence at $Re(p) < 0$. Therefore, $\beta(-2)$, etc. were not able to be treated. Then, the following expression was derived.

$$\beta(-n) = \frac{E_n}{2} \qquad n=1,2,3,\cdots \tag{1.2}$$



From this expression, The following special values were obtaind

$$\beta(-1) = \text{"}1 - 3 + 5 - 7 +- \cdots\text{"} = 0$$

$$\beta(-2) = \text{"}1^2 - 3^2 + 5^2 + 7^2 + \cdots\text{"} = -\frac{1}{2}$$

$$\beta(-3) = \text{"}1^3 - 3^3 + 5^3 + 7^3 + \cdots\text{"} = 0$$

$$\beta(-4) = \text{"}1^4 - 3^4 + 5^4 + 7^4 + \cdots\text{"} = \frac{5}{2}$$

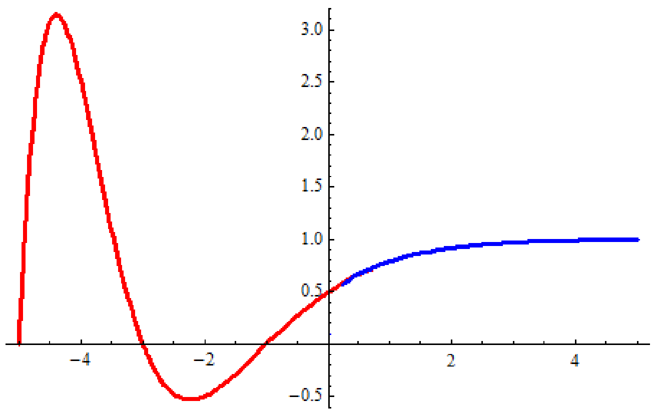
$$\vdots$$

Here, $p = -1, -3, -5, \cdots$ are called the trivial zeros of Dirichlet Beta Function.

3.1.2 Functional Equation

Furthermore, the following functional equation was derived along the lines of the Riemann zeta function.

$$\beta(p) = \left(\frac{2}{\pi}\right)^{1-p} \cos\frac{p\pi}{2}\,\Gamma(1-p)\,\beta(1-p) \qquad p \neq 1, 2, 3, \cdots \tag{1.2'}$$



(1.1) and (1.2') are overlapped at the interval $0 \leqq Re(p) < 1$. (1.2') holds on the whole complex plane except $p = 1, 2, 3, \cdots$. Applying this at $Re(p) < 0$, we can smoothly connect the points of (1.2).
By this equation, the following calculation was enabled.

$$\beta(-1/2) = \text{"}\sqrt{1} - \sqrt{3} + \sqrt{5} - \sqrt{7} +- \cdots\text{"} = 0.2751797\cdots$$

3.1.3 Patchy Definition of Dirichlet Beta

Thus, if Dirichlet Beta is defined based on the above knowledge, it is as follows.

$$\beta(p) = \begin{cases} \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(2r-1)^p} & Re(p) \geqq 0 \\ \displaystyle\left(\frac{2}{\pi}\right)^{1-p} \cos\frac{p\pi}{2}\,\Gamma(1-p) \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(2r-1)^{1-p}} & Re(p) < 0 \end{cases} \tag{1.3}$$

Formula manipulation software easily draws the graph such as **3.1.2**. Because, the function such as (1.3) is implemented in the software. In fact, the graph in this section is drawn by (1.3). Such a graph can never be drawn only by (1.1) or (1.2).

3.1.4 Recursive nature of the functional equation

The definition of $\beta(1-p)$ in the right side of (1.2') is originally (1.1) . However, we can also use (1.2') itself recursively as a definition of $\beta(1-p)$. As the result, (1.2') comes to be hold on the whole complex plane except $p=1,2,3,\cdots$. The cause which makes Dirichlet Beta unclear is just the recursive nature of (1.2'). Then, in the following section, we consider a global function which can define Dirichlet Beta on the whole complex plane without using the functional equation.

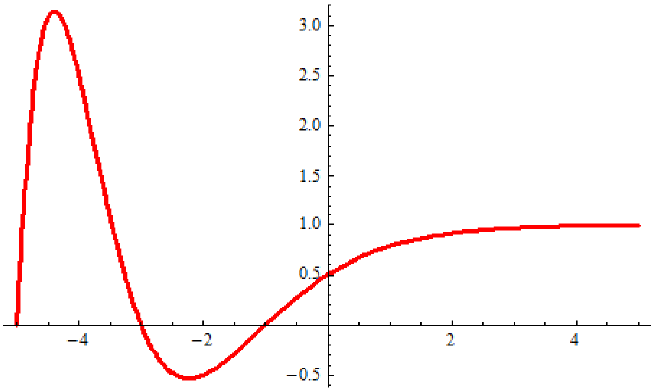
3.2 Global Definition of Dirichlet Beta

3.2.1 Global Definition of Dirichlet Beta

Definition 3.2.1

We difine the Dirichlet Beta Function on the complex plane as follows.

$$\beta(p) = \sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} (2s-1)^{-p} \qquad \textbf{(2.2p)}$$



This formula holds on the whole complex plane. We will make sure of it by examples.

Example1 $\beta(-3) \sim \beta(3)$

```
Table[N[b[p]], {p, -3, 3}]
```

$\{2.45455\times 10^{-91}, -0.5, 2.45455\times 10^{-91}, 0.5, 0.785398, 0.915966, 0.968946\}$

Example2 $\beta(-3.01) \sim \beta(3.01)$

```
Table[b[p], {p, -3.01, 3.01}]
```

{0.0154346, -0.502472, -0.00583466, 0.496072, 0.783461, 0.915146, 0.968629 }

Example3 Non-trivial zero

```
SetPrecision[b[1/2 + 12.9880980123124228 i], 12]
```

$0.\times 10^{-27} + 0.\times 10^{-27}\, i$

3.2.2 Derivation of the definition

(1) Basic sequence

When m, n are natural numbers, the following expression holds. (See **Formula 3.4.1**)

$$\left(m - \frac{1}{2}\right)^n = \sum_{r=0}^{n} {}_nB_r\, {}_mC_r$$

Where,

$$ {}_nB_r = \sum_{s=0}^{r} (-1)^{r-s}\, {}_rC_s \left(s - \frac{1}{2}\right)^n \qquad r=0,1,2,\cdots,n \qquad (2.0)$$

This generates the following values.

$n\backslash r$	1	2	3	4	5	$\cdots$
1:	1	0	0	0	0	$\cdots$
2:	0	2	0	0	0	$\cdots$
3:	$\frac{1}{4}$	3	6	0	0	$\cdots$
4:	0	5	24	24	0	$\cdots$
5:	$\frac{1}{16}$	$\frac{15}{2}$	75	180	120	$\cdots$
$\vdots$	$\vdots$					

Since the alternate series is required in order to obtain Dirichlet Beta, we transform (2.0) as follows.

$$ {}_nB_r = \sum_{s=1}^{r} (-1)^{s-1} {}_rC_s \left(s - \frac{1}{2} \right)^n \qquad r=1,2,3,\cdots \tag{2.1}$$

Since the start of the subscript was changed into 1, this generates the following value.

$n\backslash r$	1	2	3	4	5	6	$\cdots$
1:	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\cdots$
2:	$\frac{1}{4}$	$-\frac{7}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\cdots$
3:	$\frac{1}{8}$	$-\frac{25}{8}$	$\frac{47}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$\cdots$
4:	$\frac{1}{16}$	$-\frac{79}{16}$	$\frac{385}{16}$	$-\frac{383}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\cdots$
5:	$\frac{1}{32}$	$-\frac{241}{32}$	$\frac{2399}{32}$	$-\frac{5761}{32}$	$\frac{3839}{32}$	$-\frac{1}{32}$	$\cdots$
$\vdots$	$\vdots$						

(2) Euler number

Multiplying (2.1) by 2^{n-r} and summing up them, we obtain Euler number.

$$E_n = \sum_{r=1}^{\infty} 2^{n-r} \sum_{s=1}^{r} (-1)^{s-1} {}_rC_s \left(s - \frac{1}{2} \right)^n \qquad n=1,2,3,\cdots$$

i.e.

$$E_n = \sum_{r=1}^{\infty} \frac{1}{2^r} \sum_{s=1}^{r} (-1)^{s-1} {}_rC_s (2s-1)^n \qquad n=1,2,3,\cdots \tag{2.2n}$$

$n \backslash r$	1	2	3	4	5	6	$\cdots$	Total
1:	$\frac{1}{2}\cdot 2^{0}$	$-\frac{2^{-1}}{2}$	$-\frac{2^{-2}}{2}$	$-\frac{2^{-3}}{2}$	$-\frac{2^{-4}}{2}$	$-\frac{2^{-5}}{2}$	$\cdots$	0
2:	$\frac{1}{4}\cdot 2^{1}$	$-\frac{7}{4}\cdot 2^{0}$	$\frac{2^{-1}}{4}$	$\frac{2^{-2}}{4}$	$\frac{2^{-3}}{4}$	$\frac{2^{-4}}{4}$	$\cdots$	-1
3:	$\frac{1}{8}\cdot 2^{2}$	$-\frac{25}{8}\cdot 2^{1}$	$\frac{47}{8}\cdot 2^{0}$	$-\frac{2^{-1}}{8}$	$-\frac{2^{-2}}{8}$	$-\frac{2^{-3}}{8}$	$\cdots$	0
4:	$\frac{1}{16}\cdot 2^{3}$	$-\frac{79}{16}\cdot 2^{2}$	$\frac{385}{16}\cdot 2^{1}$	$-\frac{383}{16}\cdot 2^{0}$	$\frac{2^{-1}}{16}$	$\frac{2^{-2}}{16}$	$\cdots$	5
5:	$\frac{1}{32}\cdot 2^{4}$	$-\frac{241}{32}\cdot 2^{3}$	$\frac{2399}{32}\cdot 2^{2}$	$-\frac{5761}{32}\cdot 2^{1}$	$\frac{3839}{32}\cdot 2^{0}$	$-\frac{2^{-1}}{32}$	$\cdots$	0
$\vdots$	$\vdots$							

Here, what should be paid attention is that the upper limit of outside Σ has to be ∞ by all means.

(3) Dirichlet Beta Function

Finally, substituting (2.2n) for

$$\beta(-n) = \frac{E_n}{2} \qquad n=1,2,3,\cdots \tag{1.2}$$

we obtain Dirichlet Beta at negative integer.

$$\beta(-n) = \sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} {}_rC_s (2s-1)^n \qquad n=1,2,3,\cdots \tag{2.2$_{-n}$}$$

We can extend the natural number n to the complex number p and reverse the sighn of p easily. Thus,

$$\beta(p) = \sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} (2s-1)^{-p} \tag{2.2p}$$

Note

When $Re(p) \geqq 0$, **Definition3.2.1** reduce to the usual definition

$$\beta(p) = \sum_{s=1}^{\infty} (-1)^{s-1} (2s-1)^{-p} \tag{2.3p}$$

Because, when $p \geqq 0$

$$\sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} (2s-1)^{-p} \geqq 0$$

Then, (2.2p) is nonnegative term series. Therefore, the swapping of rows and columns is possible. That is,

$$\beta(p) = \sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} (2s-1)^{-p}$$

$$= \frac{1}{2} \sum_{s=1}^{\infty} (-1)^{s-1} (2s-1)^{-p} \sum_{r=1}^{\infty} \frac{1}{2^r} \binom{r}{s}$$

Here, (See " **06 Global definition of Riemann Zeta** ".)

$$\sum_{r=s}^{\infty}\frac{{}_rC_s}{2^r} = 2 \quad for\ \ s=1,2,3,\cdots$$

Using this, we obtain

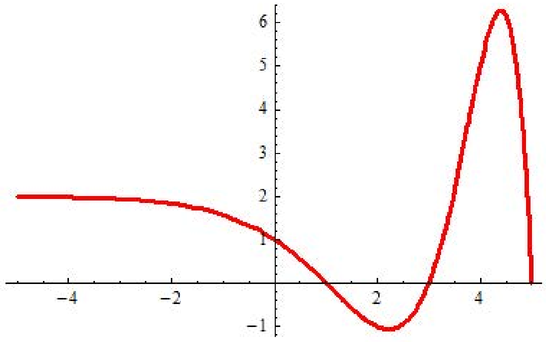
$$\beta(p) = \sum_{s=1}^{\infty}(-1)^{s-1}(2s-1)^{-p}$$

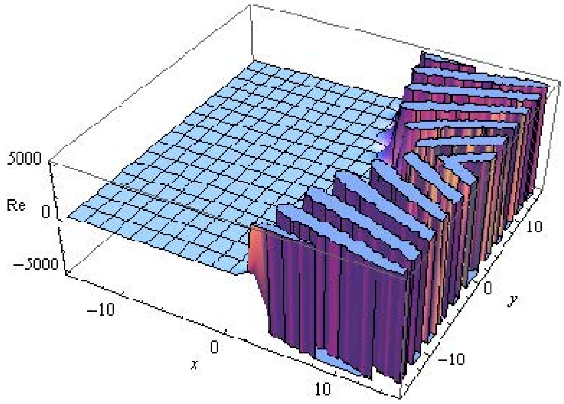
3.3 Generalized Euler Number

Euler Number (2.2n) in the previous section can be generalized (into complex number) immediately.

Definition 3.3.1 (Generalized Euler Number)

$$E_p = \sum_{r=1}^{\infty} \frac{1}{2^r} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} (2s-1)^p \tag{3.1}$$





Example1 $E(-3) \sim E(4)$

```
Table[N[eu[p]], {p, -3, 4}]
{1.93789 , 1.83193 , 1.5708 , 1., 4.90909×10^-91, -1., 4.90909×10^-91, 5.}
```

Example2 $E(-3.01) \sim E(3.01)$

```
Table[eu[p], {p, -3.01, 3.01}]
{1.93852 , 1.83356 , 1.57464 , 1.00781 , 0.0116553 , -0.99487 , -0.0303694 }
```

Example3 Non-trivial zero

```
SetPrecision[eu[-1 / 2 + 12.9880980123124228 i], 12]
0. ×10^-27 + 0. ×10^-27 i
```

Functional equation

The following functional equation holds.

$$E_p = \left(\frac{2}{\pi}\right)^{1+p} \cos\frac{p\pi}{2} \Gamma(1+p) E_{-1-p} \qquad p \neq -1, -2, -3, \cdots$$

Notation of Dirichlet Beta by Generalized Euler Number

Using generalized Euler Number, we can notete Dirichlet Beta very easily.

Formula 3.3.2

$$\beta(p) = \frac{E_{-p}}{2} \tag{3.2}$$

Where, $E_{-p} = \sum_{r=1}^{\infty} \frac{1}{2^r} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} (2s-1)^{-p}$

3.4 Expression with binomial coefficients of a power of half-integer

Formula 3.4.1

When m, n are natural numbers, the following expression holds.

$$\left(n-\frac{1}{2}\right)^m = \sum_{r=0}^{m} {}_mB_r\, {}_nC_r$$

Where,

$${}_mB_r = \sum_{s=0}^{r} (-1)^{r-s}\, {}_rC_s \left(s-\frac{1}{2}\right)^m \qquad r=0,1,2,\cdots,m$$

Proof

$(n-1/2)^2$ and the difference sequence are as follows.

$$\left(n-\frac{1}{2}\right)^2 : \frac{1}{4} \quad \frac{1}{4} \quad \frac{9}{4} \quad \frac{25}{4} \quad \frac{49}{4} \quad \frac{81}{4} \quad \frac{121}{4} \quad \cdots$$

$$d_1: \quad 0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad \cdots$$

$$d_2: \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad \cdots$$

On the other hand, if ${}_nC_0$ of Pascal's triangle is multiplied by 2 , it is as follows.

$$2\,{}_nC_0 : \quad 2 \quad 2 \quad 2 \quad 2 \quad \cdots$$

Comparing both, we obtain the followings.

$$d_2(n) = 2\,{}_nC_0$$

$$d_1(n) = 0 + 2\sum_{r=1}^{n-1} {}_rC_0 = 0 + 2\,{}_nC_1 \qquad \left(\because \sum_{r=1}^{n-1} {}_rC_0 = {}_nC_1\right)$$

$$\left(n-\frac{1}{2}\right)^2 = \frac{1}{4} + \sum_{r=1}^{n-1}\left(0 + 2\,{}_rC_1\right)$$

$$= \frac{1}{4} + 0\sum_{r=1}^{n-1} 1 + 2\sum_{r=1}^{n-1} {}_rC_1$$

Here, $1 = \binom{n}{0}$, $\sum_{r=1}^{n} 1 = \binom{n}{1}$, $\sum_{r=1}^{n-1} {}_rC_1 = \binom{n}{2}$. Then

$$\left(n-\frac{1}{2}\right)^2 = \frac{1}{4}\binom{n}{0} + 0\binom{n}{1} + 2\binom{n}{2} \tag{2.n}$$

Coefficients (blue and magenta) of the right hand side are corresponding with the first terms of $(n-1/2)^2$ and the difference sequence. And the first terms of the difference sequences are given as follows.

$$d_1:\ 0 = \left(\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2,\ 2 = \left(\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2,\ 4 = \left(\frac{5}{2}\right)^s - \left(\frac{3}{2}\right)^2,\ \cdots$$

$$d_2:\ 2 = \left(\frac{3}{2}\right)^2 - 2\cdot\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^3,\ 2 = \left(\frac{5}{2}\right)^2 - 2\cdot\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2,\ \cdots$$

That is, the r-th coefficient ${}_2B_r$ of the right side of (2.n) can be expressed as follows.

$$ {}_2B_r = \sum_{s=0}^{r} (-1)^{r-s} {}_rC_s \left(s - \frac{1}{2} \right)^2 $$

Next, $(n-1/2)^3$ and the differences sequence are as follows.

$$ \left(n - \frac{1}{2} \right)^3 : \quad -\frac{1}{8} \quad \frac{1}{8} \quad \frac{27}{8} \quad \frac{125}{8} \quad \frac{343}{8} \quad \frac{729}{8} \quad \frac{1331}{8} \quad \frac{2197}{8} \quad \cdots $$

$$ d_1 : \quad \frac{1}{4} \quad \frac{13}{4} \quad \frac{49}{4} \quad \frac{109}{4} \quad \frac{193}{4} \quad \frac{301}{4} \quad \frac{433}{4} \quad \cdots $$

$$ d_2 : \quad 3 \quad 9 \quad 15 \quad 21 \quad 27 \quad 33 \quad \cdots $$

$$ d_3 : \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad \cdots $$

On the other hand, if ${}_nC_0$ of Pascal's triangle is multiplied by 6 , it is as follows.

$$ 6\,{}_nC_0 : \quad 6 \quad 6 \quad 6 \quad 6 \quad \cdots $$

Comparing both, we obtain the followings.

$$ d_3(n) = 6\,{}_nC_0 $$

$$ d_2(n) = 3 + 6\sum_{r=1}^{n-1} {}_rC_0 = 3 + 6\,{}_nC_1 \qquad \left(\because \sum_{r=1}^{n-1} {}_rC_0 = {}_nC_1 \right) $$

$$ d_1(n) = \frac{13}{4} + \sum_{r=1}^{n-1} \left(3 + 6\,{}_rC_1 \right) $$

$$ = \frac{1}{4} + 3 + \sum_{r=1}^{n-1} \left(3 + 6\,{}_rC_1 \right) $$

$$ = \frac{1}{4} + 3\sum_{r=1}^{n} 1 + 6\sum_{r=1}^{n-1} {}_rC_1 $$

Here, $\sum_{r=1}^{n} 1 = \binom{n}{1}$, $\sum_{r=1}^{n-1} {}_rC_1 = \binom{n}{2}$. Then

$$ d_1(n) = \frac{1}{4} + 3\binom{n}{1} + 6\binom{n}{2} $$

Therefore,

$$ \left(n - \frac{1}{2} \right)^3 = -\frac{1}{8} + \sum_{r=1}^{n-1} \left\{ \frac{1}{4} + 3\binom{r}{1} + 6\binom{r}{2} \right\} $$

$$ = -\frac{1}{8} + \frac{1}{4}\sum_{r=1}^{n-1} 1 + 3\sum_{r=1}^{n-1} \binom{r}{1} + 6\sum_{r=1}^{n-1} \binom{r}{2} $$

Here,

$$ 1 = \binom{n}{0} , \quad \sum_{r=1}^{n} 1 = \binom{n}{1} , \quad \sum_{r=1}^{n-1} \binom{r}{1} = \binom{n}{2} , \quad \sum_{r=1}^{n-1} \binom{r}{2} = \binom{n}{3} $$

Substituting these for the above, we obtain

$$\left(n-\frac{1}{2}\right)^3 = -\frac{1}{8}\binom{n}{0} + \frac{1}{4}\binom{n}{1} + 3\binom{n}{2} + 6\binom{n}{3} \tag{3.n}$$

Coefficients (blue and magenta) of the right hand side are corresponding with the first terms of $(n-1/2)^3$ and the difference sequence. And the first terms of the difference sequences are given as follows.

$$d_1:\ \frac{1}{4} = \left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3,\ \frac{13}{4} = \left(\frac{3}{2}\right)^3 - \left(\frac{1}{2}\right)^3,\ \frac{49}{4} = \left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^3,\ \cdots$$

$$d_2:\ 3 = \left(\frac{3}{2}\right)^3 - 2\cdot\left(\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^3,\ 9 = \left(\frac{5}{2}\right)^3 - 2\cdot\left(\frac{3}{2}\right)^3 + \left(\frac{1}{2}\right)^3,\ \cdots$$

$$d_3:\ 6 = \left(\frac{5}{2}\right)^3 - 3\cdot\left(\frac{3}{2}\right)^3 + 3\cdot\left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3,\ \cdots$$

That is, the r-th coefficient ${}_3B_r$ of the right side of (3.n) can be expressed as follows.

$${}_3B_r = \sum_{s=0}^{r} (-1)^{r-s}\,{}_rC_s\left(s-\frac{1}{2}\right)^3$$

Hereafter by induction, we obtain the desired expression.

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