6 Global definition of Riemann Zeta, and generalization of related coefficients

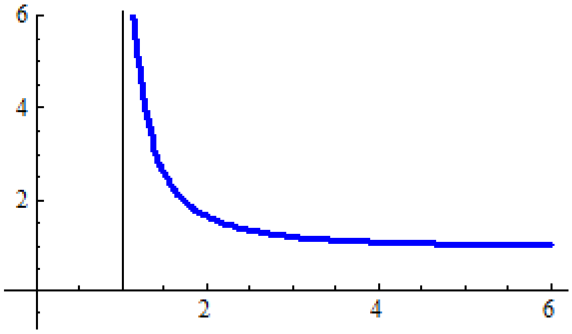
6.1 Patchy definition of Riemann Zeta

Let's review the definition of Riemann Zeta.

6.1.1 The definition by Euler

The very beginning definition of the Riemann Zeta was as follows.

$$\zeta(p) = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \qquad p>1 \tag{1.1}$$

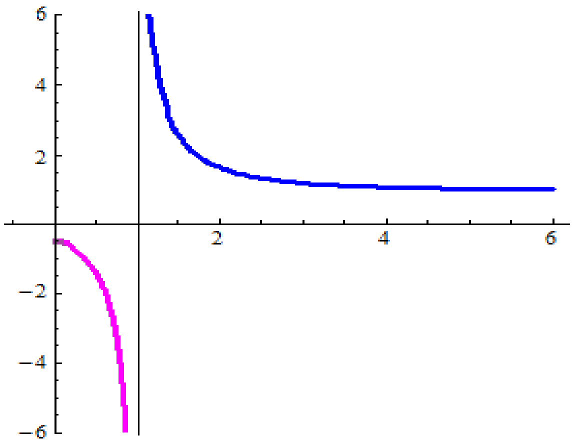


Series on the right-hand side is a generalized harmonic series, and seems to be called p-series. Euler obtained the following even number zeta based on this expression.

$$\zeta(2n) = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

However, this series was divergence at $p \leqq 1$. Therefore, $\zeta(1/2)$, etc. were not able to be treated. Then, Euler derived the following expression.

$$\zeta(p) = \frac{1}{1-2^{1-p}}\left(\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} +- \cdots\right) \qquad p \geqq 0,\ p \neq 1 \tag{1.2}$$



The series in this parenthesis is called Dirichlet Eta. Riemann zeta can be defined on the half-plane Gaussian if using this formula. In addition, the function on $p>1$ may be (1.1) .

As well known, this derived as follows.

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots = \frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+-\cdots + 2\left(\frac{1}{2^p}+\frac{1}{4^p}+\frac{1}{6^p}+\cdots\right)$$

$$= \frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+-\cdots + \frac{2}{2^p}\left(\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots\right)$$

From this,

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots = \frac{1}{1-2^{1-p}}\left(\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+-\cdots\right)$$

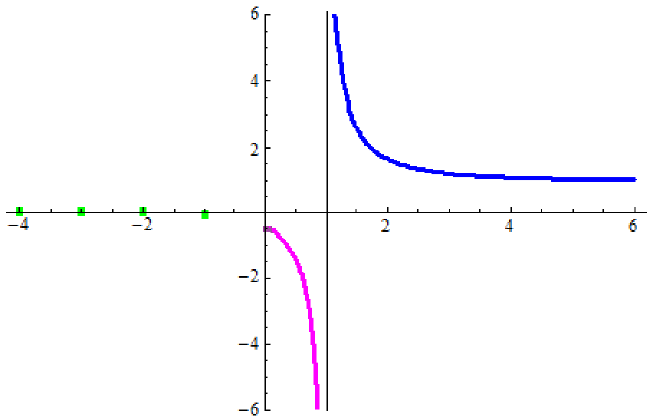
This is a really mysterious equation. When $0 \leqq p < 1$, though the left side is a divergent series, the right side is a convergent series. Substituting $p=0$ for this, Euler obtained the following strange result.

$$\zeta(0) = "1+1+1+1+\cdots" = -\frac{1}{2}$$

Furthermore, Euler found out the following equation.

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \qquad n=1,2,3,4,\cdots \tag{1.3}$$

Where, B_n is Bernouilli number s.t. $\begin{cases} B_0=1,\ B_2=1/6,\ B_4=-1/30,\ \cdots \\ B_1=-1/2,\ B_3=B_5=B_7\ \cdots = 0 \end{cases}$



This is also mysterious equation. Although the left side must be infinite, the right side is a positive or negative rational number.

$$\zeta(-1) = "1+2+3+4+\cdots" = -\frac{1}{12}$$

$$\zeta(-2) = "1^2+2^2+3^2+4^2+\cdots" = 0$$

$$\zeta(-3) = "1^3+2^3+3^3+4^3+\cdots" = \frac{1}{120}$$

$$\zeta(-4) = "1^4+2^4+3^4+4^4+\cdots" = 0$$

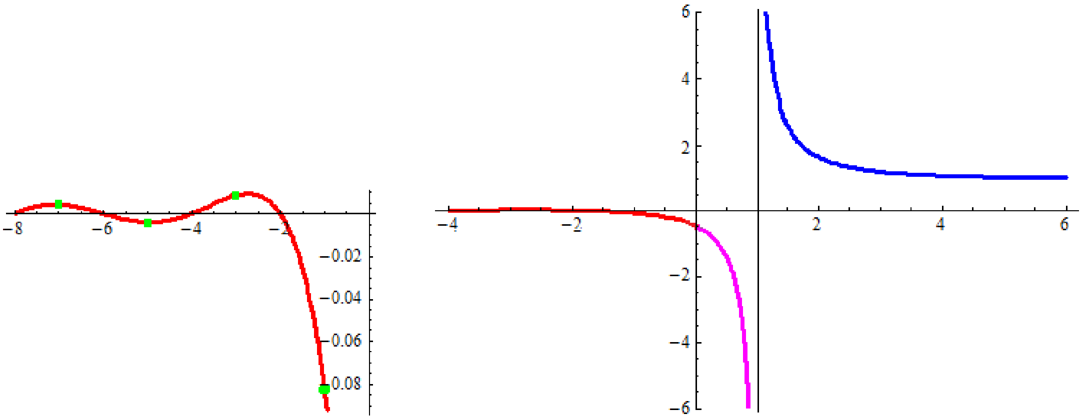
$$\vdots$$

Here, $p=-2,-4,-6,\cdots$ are called the trivial zeros of Riemann Zeta. In addition, (1.3) is obtained also from Euler-Maclaurin summation formula (See " **05 Generalized Bernoulli Polynomials and Numbers** ")

6.1.2 Riemann's functional equation

Riemann discovered the following functional equation.

$$\zeta(p) = \frac{2\Gamma(1-p)}{(2\pi)^{1-p}} \sin\left(\frac{p\pi}{2}\right)\zeta(1-p) \qquad p \neq 0,1 \tag{1.3'}$$



This equation holds on the whole complex plane except $p=0,1$. Applying this at $p<0$, we can smoothly connect the points of (1.3). At that time, $\zeta(1-p)$ of the right side have to be (1.2).

By this equation, the following calculation was enabled.

$$\zeta(-1/2) = \text{"}\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\cdots\text{"} = -0.2078862\cdots$$

6.1.3 Patchy definition of Riemann Zeta

Thus, if Riemann Zeta is defined on the complex plane using the above expressions, it is as follows.

$$\zeta(p) = \begin{cases} \displaystyle\sum_{r=1}^{\infty} \frac{1}{r^p} & Re(p) > 1 \\ \displaystyle\frac{1}{1-2^{1-p}} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^p} & 0 \leqq Re(p) \leqq 1 \\ & p \neq 1 \\ \displaystyle\frac{2\Gamma(1-p)}{(2\pi)^{1-p}} \sin\left(\frac{p\pi}{2}\right) \frac{1}{1-2^p} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^{1-p}} & Re(p) \leqq 0 \\ & p \neq 0 \end{cases} \tag{1.4}$$

or

$$\zeta(p) = \begin{cases} \displaystyle\frac{1}{1-2^{1-p}} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^p} & Re(p) \geqq 0 \\ & p \neq 1 \\ \displaystyle\frac{2\Gamma(1-p)}{(2\pi)^{1-p}} \sin\left(\frac{p\pi}{2}\right) \frac{1}{1-2^p} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^{1-p}} & Re(p) \leqq 0 \\ & p \neq 0 \end{cases} \tag{1.4'}$$

Formula manipulation software easily draws the graph such as the above. Because, the function such as (1.4) or (1.4') is implemented in the software. In fact, the graph in this section is drawn by (1.4). Such a graph can never be drawn only by (1.1) or (1.2).

Moreover, we can never find out the zero of $\zeta(p)$ based on the definition (1.1). It is because the zero does not exist on the plane defined by the expression. .

The definition of Riemann zeta function using the Dirichlet series and the functional equation is so complicated. However, using a double series, we can define Riemann zeta function very simply. Besides, the double series can generalize Bernoulli Number, Euler Numver, tangent number, etc. In the following sections, I mention them.

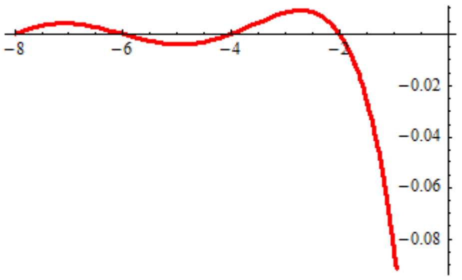
6.2 Global definition of Riemann Zeta

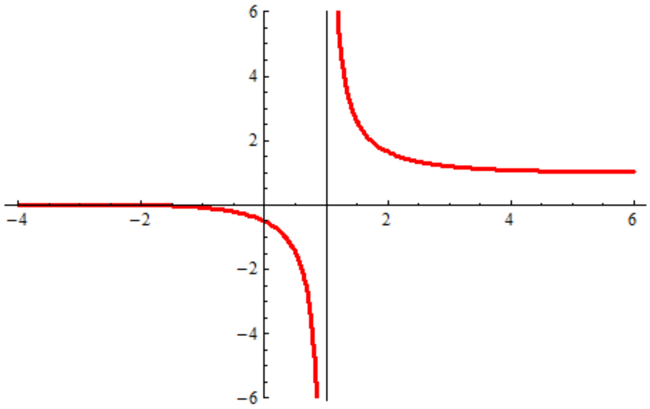
6.2.1 Global definition of Riemann Zeta

Definition 6.2.1

We difine the Riemann Zeta Function on the complex plane as follows.

$$\zeta(p) = \frac{1}{1-2^{1-p}} \sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^{-p} \qquad p \neq 1 \tag{2.4p}$$





It is said that it is proved by Hasse around 1930 that this formula holds on the whole complex plane except $p=1$. We will make sure of it by examples.

Example1 $\zeta(-6) \sim \zeta(-1)$

```
Table[z[p], {p, -6, -1}]
```

$\left\{0, -\frac{1}{252}, 0, \frac{1}{120}, 0, -\frac{1}{12}\right\}$

Example2 $\zeta(-1/2), \zeta(0), \zeta(3)$

```
N[{z[-1 / 2], z[0], z[3]}]
{-0.207886 , -0.5 , 1.20206 }
```

Example3 Non-trivial zero

```
SetPrecision[z[1 / 2 + 14.134725141734693790457251983 i], 10]
```

$0. \times 10^{-35} + 0. \times 10^{-35}\, i$

6.2.2 Derivation of the definition

So necessary for the next section, I derive the definition along the lines of Mr.Sugimoto .

(1) Basic sequence

When m, n are natural numbers, the following expression holds. (See " **07 New Formula for Power Sum** ")

$$m^n = \sum_{r=0}^{n} {}_nB_r\, {}_mC_r$$

Where,

$${}_nB_r = \sum_{s=1}^{r} (-1)^{r-s} \binom{r}{s} s^n \qquad r = 0, 1, 2, \cdots, n \tag{2.0}$$

This generates the following values.

$n \backslash r$	1	2	3	4	5	⋯
1:	1	0	0	0	0	⋯
2:	1	2	0	0	0	⋯
3:	1	6	6	0	0	⋯
4:	1	14	36	24	0	⋯
5:	1	30	150	240	120	⋯
⋮	⋮					

Here, what should be paid attention is that all values are 0 for $r > n$. This is a property resulting from a general binomial coefficient. Thereby, at any time, we can replace the upper limit r by ∞ and can change n to the real number. This well behaved property is maintained all the time hereafter.

Since the alternate series is required in order to obtain Riemann Zeta, we transform (2.0) as follows.

$$ {}_nB_r = \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^n \qquad r=1,2,3,\cdots,n \tag{2.1} $$

$n \backslash r$	1	2	3	4	5	⋯
1:	1	0	0	0	0	⋯
2:	1	-2	0	0	0	⋯
3:	1	-6	6	0	0	⋯
4:	1	-14	36	-24	0	⋯
5:	1	-30	150	-240	120	⋯
⋮	⋮					

(2) Tangent number

Multiplying (2.1) by 2^{n-r} and summing up them, we obtain a tangent number.

$$ T_n = \sum_{r=1}^{n} 2^{n-r} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^n \qquad n=1,2,3,\cdots \tag{2.2n} $$

$n \backslash r$	1	2	3	4	5	⋯	Total
1:	$1 \cdot 2^0$	0	0	0	0	⋯	1
2:	$1 \cdot 2^1$	$-2 \cdot 2^0$	0	0	0	⋯	0
3:	$1 \cdot 2^2$	$-6 \cdot 2^1$	$6 \cdot 2^0$	0	0	⋯	-2
4:	$1 \cdot 2^3$	$-14 \cdot 2^2$	$36 \cdot 2^1$	$-24 \cdot 2^0$	0	⋯	0
5:	$1 \cdot 2^4$	$-30 \cdot 2^3$	$150 \cdot 2^2$	$-240 \cdot 2^1$	$120 \cdot 2^0$	⋯	16
⋮	⋮						

In addition, although original tangent number is a non-negative integer, in this chapter, we call the sighed one also tangent number. There is the following relation between T_{2n-1} and the original tangent number T^*_{2n-1} .

$$T^*_{2n-1} = (-1)^{n-1} T_{2n-1} \qquad n=1,2,3,\cdots$$

(3) Bernoulli number

Multiplying (2.2n) by $\dfrac{n+1}{2^{n+1}\left(2^{n+1}-1\right)}$, we obtain a Bernoulli number B_{n+1} .

$$B_{n+1} = \frac{n+1}{2^{n+1}\left(2^{n+1}-1\right)} T_n = \frac{n+1}{2^{n+1}\left(2^{n+1}-1\right)} \sum_{r=1}^{n} 2^{n-r} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^n$$

i.e.

$$B_{n+1} = \frac{n+1}{2^{n+1}-1} \sum_{r=1}^{n} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^n \qquad n=1,2,3,\cdots \tag{2.3$_{n1}$}$$

n	1	2	3	4	5	6	7	$\cdots$
T_n	1	0	2	0	16	0	272	$\cdots$
B_{n+1}	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	$\cdots$

(4) Riemann Zeta

Finally, substituting (2.3_{n1}) for the well known

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \qquad n=1,2,3,\cdots$$

we obtain Riemann Zeta at negative integer.

$$\zeta(-n) = \frac{1}{1-2^{n+1}} \sum_{r=1}^{n} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^n \qquad n=1,2,3,\cdots \tag{2.4$_{-n}$}$$

Since all values are 0 for $r>n$. replacing the upper limit r by ∞ we can extend the natural number n to the complex number p .

$$\zeta(-p) = \frac{1}{1-2^{1+p}} \sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^p \qquad p \neq -1 \tag{2.4$_{-p}$}$$

Since p is a complex number, we can reverse the sighn.

$$\zeta(p) = \frac{1}{1-2^{1-p}} \sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^{-p} \qquad p \neq 1 \tag{2.4$_{p}$}$$

6.3 Generalization of related coefficients

6.3.1 Generalized Stirling Number of the 2nd kind

Dividing (2.1) of the previous section by $r!$, we obtain Stirling Number of the 2nd kind.

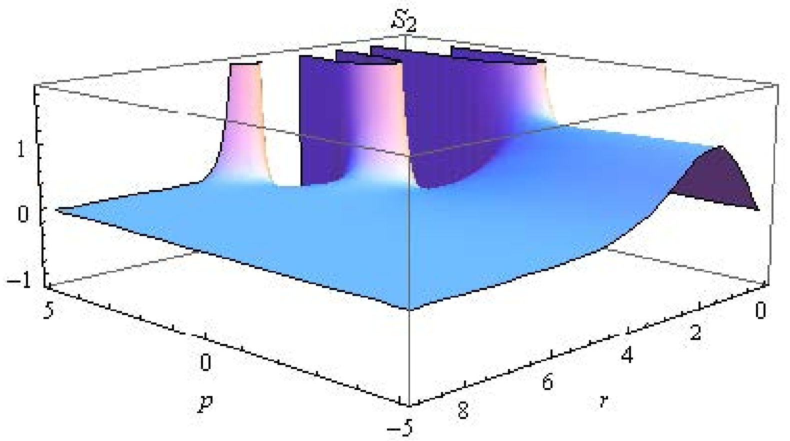
$$S_2(n,r) = \frac{1}{r!}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}s^n \qquad r=1,2,3,\cdots,n$$

$n\backslash r$	1	2	3	4	5	
1:	1	0	0	0	0	⋯
2:	1	-1	0	0	0	⋯
3:	1	-3	2	0	0	⋯
4:	1	-7	6	-1	0	⋯
5:	1	-15	25	-10	1	⋯
⋮	⋮					

Since all values are 0 for $r>n$, we can replace the upper limit r by ∞ . Then, since n does not need to be a natural number any longer, it is extensible to a more global number.

Definition 6.3.1 (Generalized Stirling Number of the 2nd kind)

$$S_2(p,r) = \frac{1}{r!}\sum_{s=1}^{\infty}(-1)^{s-1}\binom{r}{s}s^p \qquad r=1,2,3,\cdots \qquad (3.1)$$



Example $S_2(3,1)\sim S_2(3,5)$ and $S_2(3.01,1)\sim S_2(3.01,5)$

```
Table[S2[3, r], {r, 1, 5}]
{1, -3, 1, 0, 0}
Table[S2[3.01, r], {r, 1, 5}]
{1., -3.02782 , 1.02189 , -0.00142627 , -0.00010664 }
```

Although original Stirling Number is a non-negative integer, in this chapter, we call the sighed one also Stirling Number. There is the following relation between $S_2(p,r)$ and the original Stirling Number $S_2^*(p,r)$.

$$S_2^*(p,r) = (-1)^{r+1}S_2(p,r) \qquad r=0,1,2,\cdots \qquad (3.1')$$

The following formula is known for a natural number n .

$${}_nS_2 = 2^{n-1} - 1 \ , \ {}_nS_3 = \frac{1}{6}\left(3^n - 3\cdot 2^n + 3\right)$$

(3.1') satisfies the following formula also for a complex number p .

$$S_2^*(p,2) = 2^{p-1} - 1 \ , \ S_2^*(p,3) = \frac{1}{6}\left(3^p - 3\cdot 2^p + 3\right)$$

Example $S_2^*(1+i,2)$ and $S_2^*(1+i,3)$

```
N[{S2x[1 + i, 2]          ,  2^(1+i-1) - 1}]
{-0.230761 + 0.638961 i  ,  -0.230761 + 0.638961 i }
N[{S2x[1 + i, 3]          ,  (3^(1+i) - 3 × 2^(1+i) + 3) / 6}]
{-0.0418227 - 0.193673 i  ,  -0.0418227 - 0.193673 i }
```

6.3.2 Generalized Tangent Number

From (2.2n) in the previous section,

$$T_n = \sum_{r=1}^{n} 2^{n-r} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^n \qquad n=1,2,3,\cdots$$

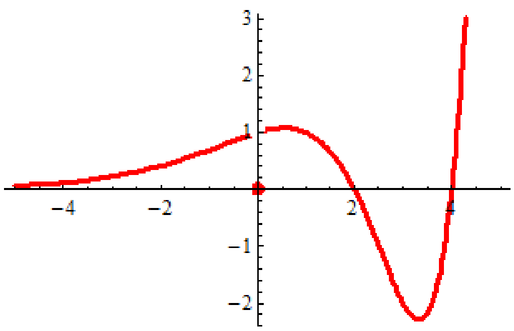
Since all values are 0 for $r>n$. we can replace the upper limit r by ∞ and extend the natural number n to the complex number p .

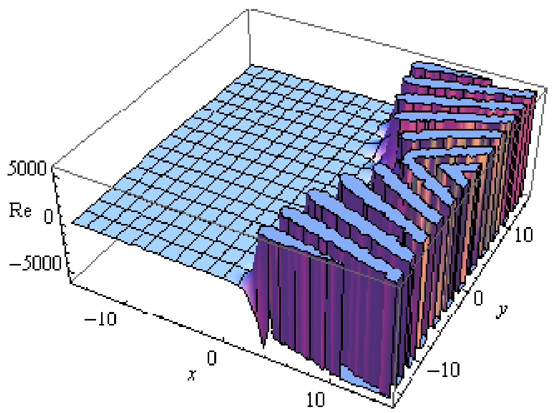
$$T_p = \sum_{r=1}^{\infty} 2^{p-r} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^p \qquad p \neq 0 \tag{3.2}$$

Adding the point $p=0$ to this, we define as follows.

Definition 6.3.2 (Generalized Tangent Number)

$$T_p = \begin{cases} 0 & p=0 \\ \sum_{r=1}^{\infty} 2^{p-r} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^p & p \neq 0 \end{cases} \tag{3.2}$$





Example1 $T_1 \sim T_5$ **and** $T_{1.01} \sim T_{5.01}$

```
Table[t[p], {p, 1, 5}]
{1, 0, -2, 0, 16}
Table[t[p], {p, 1.01, 5.01}]
{0.996272 , -0.0171201 , -2.01559 , 0.0800307 , 16.1997 }
```

Example2 Non-trivial zero

```
SetPrecision[t[-1 / 2 + 14.134725141734693790457251983 i], 10]
0. × 10^-35 + 0. × 10^-35 i
```

6.3.3 Generalized Bernoulli Number

From (2.3_{n1}) in the previous section,

$$B_{n+1} = \frac{n+1}{2^{n+1}-1} \sum_{r=1}^{n} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^n \qquad n=1,2,3,\cdots$$

Since all values are 0 for $r>n$. we can replace the upper limit r by ∞ and extend the natural number n to the complex number p .

$$B_{p+1} = \frac{p+1}{2^{p+1}\left(2^{p+1}-1\right)} \sum_{r=1}^{\infty} 2^{p-r} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^p \qquad p \ne -1$$

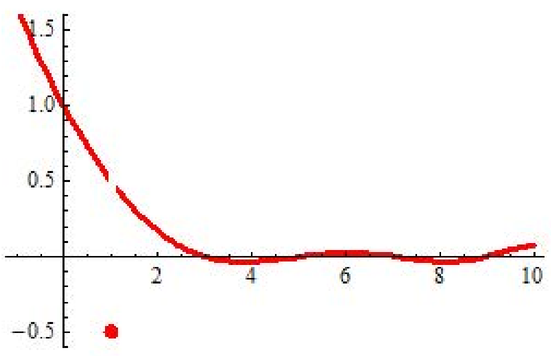
Replacing p with $p-1$,

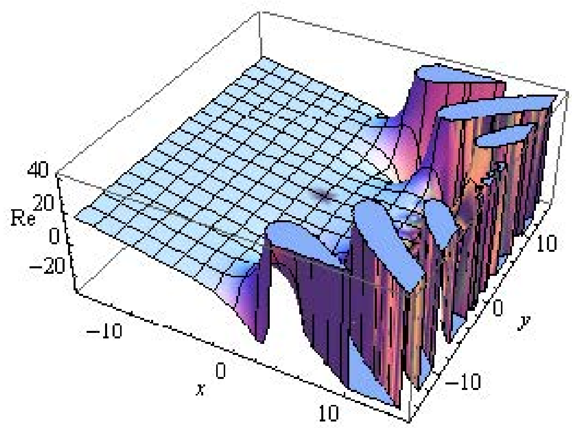
$$B_p = \frac{p}{2^p-1} \sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^{p-1} \qquad p \ne 1 \tag{3.3p}$$

Adding two points $p=0,1$ to this, we define as follows.

Definition 6.3.3 (Generalized Bernoulli Number)

$$B_p = \begin{cases} -\dfrac{1}{2} & p=1 \\ \dfrac{p}{2^p-1} \displaystyle\sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^{p-1} & p \ne 1 \end{cases} \tag{3.3}$$





Example1 $B_2 \sim B_6$ **and** $B_{2.01} \sim B_{6.01}$

```
Table[b[p],{p,2,6}]   Table[b[p], {p, 2.01, 6.01}]
```

$\left\{\frac{1}{6},\ 0,\ -\frac{1}{30},\ 0,\ \frac{1}{42}\right\}$ {0.1642, -0.000906643, -0.0331981, 0.000398534, 0.0238119}

Example2 $B_{-4.0} \sim B_{-4.9}$

```
Table[SetPrecision[b[p], 11], {p, -4.0, -4.9, -0.1}]
```

{4.1477110206, 4.2401604462, 4.3329714820, 4.4261284844, 4.5196163712,
4.6134206098, 4.7075272044, 4.8019226839, 4.8965940883, 4.9915289558}

Example3 **Non-trivial zero**

```
SetPrecision[b[1 / 2 + 14.134725141734693790457251983 i], 10]
```

$0.\times 10^{-34} + 0.\times 10^{-34}\, i$

cf.

The values of Example2 are exactly consistent with the values of the table of 5.2 in " **5 Generalized Bernoulli Polynomials and Numbers** ".

Functional equation

The following functional equation holds.

$$B_{1-p} = \frac{1-p}{p}\,\frac{2\Gamma(1-p)}{(2\pi)^{1-p}}\sin\frac{p\pi}{2}\cdot B_p \qquad p \neq 0,1,2,3,\cdots$$

6.3.4 Notation of Riemann Zeta by the various coefficients

Using these various coefficients, we can define Riemann Zeta Function as follows.

Formula 6.3.4

$$\zeta(p) = \frac{1}{1-2^{1-p}}\sum_{r=1}^{\infty}\frac{r!}{2^{r+1}}S_2(-p,r) \qquad (3.4)$$

$$\zeta(p) = \frac{1}{1-2^{1-p}}\,\frac{T_{-p}}{2^{1-p}} \qquad p \neq 1 \qquad (3.5)$$

$$\zeta(p) = -\frac{B_{1-p}}{1-p} \qquad p \neq 1 \qquad (3.6)$$

6.4 Dirichlet Eeta Function

Looking back at the results of the previous sections, the element of all the coefficients was the following series.

$$\sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^{-p}$$

here, the global definition of $\zeta(p)$ and the related expression were

$$\zeta(p) = \frac{1}{1-2^{1-p}} \sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^{-p} \qquad p \neq 1$$

$$\zeta(p) = \frac{1}{1-2^{1-p}} \eta(p)$$

Comparing these, we notice the following immediately.

6.4.1 Global definition of Dirichlet Eeta Function

Definition 6.4.1

We difine the Dirichlet Eeta Function on the complex plane as follows.

$$\eta(p) = \sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} s^{-p} \tag{4.1}$$

This definition results in the usual definition of $\eta(p)$ in a half-plane. Some preparation is required in order to show this.

Lemma

$$\sum_{r=s}^{\infty} \frac{{}_rC_s}{2^r} = 2 \qquad for\ \ s=1,2,3,\cdots$$

Proof

$$\frac{{}_1C_1}{2^1} + \frac{{}_2C_1}{2^2} + \frac{{}_3C_1}{2^3} + \frac{{}_4C_1}{2^4} + \cdots$$

$$= \frac{{}_0C_0}{2^1} + \frac{{}_1C_0}{2^2} + \frac{{}_2C_0}{2^3} + \frac{{}_3C_0}{2^4} + \cdots \qquad = \frac{1}{2^1}\left(1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots\right)$$

$$+ \frac{{}_0C_0}{2^2} + \frac{{}_1C_0}{2^3} + \frac{{}_2C_0}{2^4} + \cdots \qquad + \frac{1}{2^2}\left(1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots\right)$$

$$+ \frac{{}_0C_0}{2^3} + \frac{{}_1C_0}{2^4} + \frac{{}_2C_0}{2^5} + \cdots \qquad + \frac{1}{2^3}\left(1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots\right)$$

$$\vdots$$

$$= \left(\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) \cdot 2$$

$$= \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^3} + \cdots = 2$$

Next,

$$
\begin{aligned}
&\frac{{}_{2}C_{2}}{2^{2}}+\frac{{}_{3}C_{2}}{2^{3}}+\frac{{}_{4}C_{2}}{2^{4}}+\frac{{}_{5}C_{2}}{2^{5}}+\cdots \\
&= \frac{{}_{1}C_{1}}{2^{2}}+\frac{{}_{2}C_{1}}{2^{3}}+\frac{{}_{3}C_{1}}{2^{4}}+\frac{{}_{4}C_{1}}{2^{5}}+\cdots \\
&\qquad +\frac{{}_{1}C_{1}}{2^{3}}+\frac{{}_{2}C_{1}}{2^{4}}+\frac{{}_{3}C_{1}}{2^{5}}+\cdots \\
&\qquad\qquad +\frac{{}_{1}C_{1}}{2^{4}}+\frac{{}_{2}C_{1}}{2^{5}}+\frac{{}_{3}C_{1}}{2^{6}}+\cdots \\
&\qquad\qquad\qquad \vdots
\end{aligned}
$$

Here,

$$
\begin{aligned}
\frac{{}_{1}C_{1}}{2^{2}}+\frac{{}_{2}C_{1}}{2^{3}}+\frac{{}_{3}C_{1}}{2^{4}}+\cdots &= \frac{1}{2^{1}}\left(\frac{{}_{1}C_{1}}{2^{1}}+\frac{{}_{2}C_{1}}{2^{2}}+\frac{{}_{3}C_{1}}{2^{3}}+\cdots\right) = \frac{2}{2^{1}} = \frac{1}{2^{0}} \\
\frac{{}_{1}C_{1}}{2^{3}}+\frac{{}_{2}C_{1}}{2^{4}}+\frac{{}_{3}C_{1}}{2^{5}}+\cdots &= \frac{1}{2^{2}}\left(\frac{{}_{1}C_{1}}{2^{1}}+\frac{{}_{2}C_{1}}{2^{2}}+\frac{{}_{3}C_{1}}{2^{3}}+\cdots\right) = \frac{2}{2^{2}} = \frac{1}{2^{1}} \\
\frac{{}_{1}C_{1}}{2^{4}}+\frac{{}_{2}C_{1}}{2^{5}}+\frac{{}_{3}C_{1}}{2^{6}}+\cdots &= \frac{1}{2^{3}}\left(\frac{{}_{1}C_{1}}{2^{1}}+\frac{{}_{2}C_{1}}{2^{2}}+\frac{{}_{3}C_{1}}{2^{3}}+\cdots\right) = \frac{2}{2^{3}} = \frac{1}{2^{2}} \\
\vdots \qquad\qquad & \qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

Then,

$$
\frac{{}_{2}C_{2}}{2^{2}}+\frac{{}_{3}C_{2}}{2^{3}}+\frac{{}_{4}C_{2}}{2^{4}}+\frac{{}_{5}C_{2}}{2^{5}}+\cdots = \frac{1}{2^{0}}+\frac{1}{2^{1}}+\frac{1}{2^{3}} = 2
$$

Next,

$$
\begin{aligned}
&\frac{{}_{3}C_{3}}{2^{3}}+\frac{{}_{4}C_{3}}{2^{4}}+\frac{{}_{5}C_{3}}{2^{5}}+\frac{{}_{6}C_{3}}{2^{6}}+\cdots \\
&= \frac{{}_{2}C_{2}}{2^{3}}+\frac{{}_{3}C_{2}}{2^{4}}+\frac{{}_{4}C_{2}}{2^{5}}+\frac{{}_{5}C_{2}}{2^{6}}+\cdots \\
&\qquad +\frac{{}_{2}C_{2}}{2^{4}}+\frac{{}_{3}C_{2}}{2^{5}}+\frac{{}_{4}C_{2}}{2^{6}}+\cdots \\
&\qquad\qquad +\frac{{}_{2}C_{2}}{2^{5}}+\frac{{}_{3}C_{2}}{2^{6}}+\frac{{}_{4}C_{2}}{2^{7}}+\cdots \\
&\qquad\qquad\qquad +\frac{{}_{2}C_{2}}{2^{6}}+\frac{{}_{3}C_{2}}{2^{7}}+\frac{{}_{4}C_{2}}{2^{8}}+\cdots \\
&\qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

Here,

$$
\frac{{}_{2}C_{2}}{2^{3}}+\frac{{}_{3}C_{2}}{2^{4}}+\frac{{}_{4}C_{2}}{2^{5}}+\cdots = \frac{1}{2^{1}}\left(\frac{{}_{2}C_{2}}{2^{2}}+\frac{{}_{3}C_{2}}{2^{3}}+\frac{{}_{4}C_{2}}{2^{4}}+\cdots\right) = \frac{2}{2^{1}} = \frac{1}{2^{0}}
$$

$$\frac{{}_2C_2}{2^4}+\frac{{}_3C_2}{2^5}+\frac{{}_4C_2}{2^6}+\cdots = \frac{1}{2^2}\left(\frac{{}_2C_2}{2^2}+\frac{{}_3C_2}{2^3}+\frac{{}_4C_2}{2^4}+\cdots\right) = \frac{2}{2^2} = \frac{1}{2^1}$$

$$\frac{{}_2C_2}{2^5}+\frac{{}_3C_2}{2^6}+\frac{{}_4C_2}{2^7}+\cdots = \frac{1}{2^3}\left(\frac{{}_2C_2}{2^2}+\frac{{}_3C_2}{2^3}+\frac{{}_4C_2}{2^4}+\cdots\right) = \frac{2}{2^3} = \frac{1}{2^2}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Then,

$$\frac{{}_3C_3}{2^3}+\frac{{}_4C_3}{2^4}+\frac{{}_5C_3}{2^5}+\frac{{}_6C_3}{2^6}+\cdots = \frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^3} = 2$$

Hereafter, by induction, we obtain the disired expression.

Theorem 6.4.2

When $\eta(p)$ is Dirichlet Eta function, if $Re(p) \geqq 0$, the following expression holds.

$$\eta(p) = \sum_{r=1}^{\infty}\frac{1}{2^{r+1}}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}s^{-p} = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^p}$$

Proof

$$\eta(p) = \sum_{r=1}^{\infty}\frac{1}{2^{r+1}}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}s^{-p}$$

$$= \frac{1}{2^2}\left\{\binom{1}{1}1^{-p}\right\}$$

$$+ \frac{1}{2^3}\left\{\binom{2}{1}1^{-p} - \binom{2}{2}2^{-p}\right\}$$

$$+ \frac{1}{2^4}\left\{\binom{3}{1}1^{-p} - \binom{3}{2}2^{-p} + \binom{3}{3}3^{-p}\right\}$$

$$+ \frac{1}{2^5}\left\{\binom{4}{1}1^{-p} - \binom{4}{2}2^{-p} + \binom{4}{3}3^{-p} - \binom{4}{4}4^{-p}\right\}$$

$$\vdots$$

$$= \frac{1^{-p}}{2}\left\{\frac{1}{2^1}\binom{1}{1}+\frac{1}{2^2}\binom{2}{1}+\frac{1}{2^3}\binom{3}{1}+\frac{1}{2^4}\binom{4}{1}+\cdots\right\}$$

$$- \frac{2^{-p}}{2}\left\{\frac{1}{2^2}\binom{2}{2}+\frac{1}{2^3}\binom{3}{2}+\frac{1}{2^4}\binom{4}{2}+\frac{1}{2^5}\binom{5}{2}+\cdots\right\}$$

$$+ \frac{3^{-p}}{2}\left\{\frac{1}{2^3}\binom{3}{3}+\frac{1}{2^4}\binom{4}{3}+\frac{1}{2^5}\binom{5}{3}+\frac{1}{2^6}\binom{6}{3}+\cdots\right\}$$

$$- \frac{4^{-p}}{2}\left\{\frac{1}{2^4}\binom{4}{4}+\frac{1}{2^5}\binom{5}{4}+\frac{1}{2^6}\binom{6}{4}+\frac{1}{2^7}\binom{7}{4}+\cdots\right\}$$

⋮

$$= \sum_{s=1}^{\infty} (-1)^{s-1} \frac{s^{-p}}{2} \sum_{r=s}^{\infty} \frac{1}{2^r} \binom{r}{s}$$ (The orders of the alternating series are unchanged.)

According to the above Lemma ,

$$\sum_{r=s}^{\infty} \frac{{}_rC_s}{2^r} = 2 \qquad for \;\; s=1,2,3,\cdots$$

Using this

$$\eta(p) = \sum_{s=1}^{\infty} (-1)^{s-1} s^{-p}$$

When $p=0$

$$\eta(0) = \sum_{s=1}^{\infty} (-1)^{s-1} s^{-0} = 1 - 1 + 1 - 1 + - \cdots = \frac{1}{2}$$

When $Re(p)>0$, let $p=x+iy$. Then

$$\eta(x+iy) = \sum_{s=1}^{\infty} (-1)^{s-1} s^{-x-iy} \qquad x>0$$

Using exponential function,

$$\eta(x+iy) = \sum_{s=1}^{\infty} (-1)^{s-1} e^{-x\log s - iy\log s}$$

Using trigonometric functions,

$$\eta(x+iy) = \sum_{s=1}^{\infty} (-1)^{s-1} e^{-x\log s} \{ \cos(y\log s) - i\sin(y\log s) \}$$

When s is a natural number and y is a real number ,

$$0 < |\cos(y\log s) - i\sin(y\log s)| < 2$$

Therefore, this series diverges at $x<0$ and converges at $x>0$. Q.E.D

6.4.3 Notation of Dirichlet Eeta by the various coefficients

Using these various coefficients, we can define Dirichlet Eeta Function as follows.

Formula 6.4.3

$$\eta(p) = \sum_{r=1}^{\infty} \frac{r!}{2^{r+1}} S_2(-p,r) \tag{4.2}$$

$$\eta(p) = \frac{T_{-p}}{2^{1-p}} \tag{4.3}$$

$$\eta(p) = -\left(1-2^{1-p}\right) \cdot \frac{B_{1-p}}{1-p} \qquad p \neq 1 \tag{4.4}$$

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Alien's Mathematics