Graphical Proof of the Riemann Hypothesis

Abstract

(1) The problem of Zeros of the Riemann zeta function is reduced to the system of transcendental equations consisting of 4 equations with 2 real variables, by functional equation.

(2) On the critical line, certain 2 equations are identically 0, and the remaining 2 equations have simultaneous solutions.

(3) Except on the critical line, the two equations do not have simultaneous solutions in the critical strip. This can be illustrated by transition diagrams from above and below the contour line. And such transitions are more pronounced where the imaginary part of the variable is large.

(4) As a result of (3), the system of transcendental equations of (1) have no solution in the critical strip except on the critical line. Thus, the Riemann Hypothesis holds true.

1 Introduction

Riemann Zeta Function

Riemann Zeta Function $\zeta(z)$ is defined by the following Dirichlet series.

$$\zeta(z) = \sum_{r=1}^{\infty} e^{-z\log r} = \frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\cdots \qquad Re(z)>1 \qquad (1.\zeta)$$

This function is analytically continued to $Re(z)<1$, and has trivial zeros $z=-2n\ \ (n=1,2,3,\cdots)$ and **non-trivial zeros** $z=1/2\pm b_n\ \ (n=1,2,3,\cdots)$. So, it is the Riemann hypothesis that there will be no non-trivial zeros other than these. In addition, it is known that non-trivial zeros exist only in the **critical strip** $\mathbf{0<Re(z)<1}$. Also, the center line $\mathbf{Re(z)=1/2}$ is called the **critical line**.

Dirichlet Eta Function

Dirichlet Eta Function $\eta(z)$ is defined by the following Dirichlet series.

$$\eta(z) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-z\log r} = \frac{1}{1^z}-\frac{1}{2^z}+\frac{1}{3^z}-\frac{1}{4^z}+-\cdots \qquad Re(z)>0 \qquad (1.\eta)$$

This function is analytically continued to $Re(z)\le 0$, and has the following relation to $\zeta(z)$.

$$\zeta(z) = \frac{1}{1-2^{1-z}}\eta(z) \qquad z\neq 1$$

Therefore $\zeta(z)$ and $\eta(z)$ share trivial and non-trivial zeros. In addition, $\eta(z)$ has **$\eta(z)$-specific zeros** $\mathbf{z=1\pm 2n\pi/\log 2\ \ (n=1,2,3,\cdots)}$. These are the zeros of $1-2^{1-z}=0$.

Dirichlet Series to use

The right-hand sides of $(1.\zeta)$ and $(1.\eta)$ are called Dirichlet series. $(1.\zeta)$, which is the definition of $\zeta(z)$, is not suitable for analysis in the critical strip. This is because even if the Euler transformation or the like is applied, it will only be an asymptotic expansion. On the other hand, $(1.\eta)$, which is the definition of $\eta(z)$, can be used as it is in the critical strip. So, in this paper, we use $(1.\eta)$ to analyze the zeros of the Riemann zeta function $\zeta(z)$.

2 Zeros of $\eta(z)$ and System of Equations

In this chapter, we consider the problem of zeros of the Dirichlet Eta function $\eta(z)$ from the point of view of the system of equations.

Lemma 2.1

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy,\ x,y\in R)$, $\eta(z)=0$ in $0<x<1$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \eta(z) = \displaystyle\sum_{r=1}^{\infty}(-1)^{r-1}e^{-z\log r} = 0 & (2.1_{+}) \\ \eta(1-z) = \displaystyle\sum_{r=1}^{\infty}(-1)^{r-1}e^{-(1-z)\log r} = 0 & (2.1_{-}) \end{cases}$$

Proof

The following functional equation holds for the Dirichlet Eta function $\eta(z)$.

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left(1-2^{z}\right)\eta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\left(1-2^{1-z}\right)\eta(1-z) \qquad 0<Re(z)<1$$

Gamma function and powers of π have no zeros, and $1-2^{z}, 1-2^{1-z}$ have no zeros in $0<Re(z)<1$
Therefore, at the zero of $\eta(z)$,

$$\eta(z) = \eta(1-z) = 0 \qquad 0<Re(z)<1$$

Representing $\eta(z)$, $\eta(1-z)$ by Dirichlet series respectively, we obtain the desired expressions.

Note1

Since there are 2 equations for 1 complex variable in the lemma, this system of equations is an overdetermined system. Such a system of equations generally has no solution. What forces this overdetermined system is the functional equation clearly.

Note2

(1) When $x=1/2$, the overdetermined property disappears. Because,

$$\begin{cases} \eta(1/2+iy) = \displaystyle\sum_{r=1}^{\infty}(-1)^{r-1}e^{-(1/2+iy)\log r} = 0 & (2.1_{+}) \\ \eta(1/2-iy) = \displaystyle\sum_{r=1}^{\infty}(-1)^{r-1}e^{-(1/2-iy)\log r} = 0 & (2.1_{-}) \end{cases}$$

i.e.

$$\begin{cases} \eta(1/2+iy) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\{cos(y\log r) - i\,sin(y\log r)\} = 0 & (2.1_{+}) \\ \eta(1/2-iy) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\{cos(y\log r) + i\,sin(y\log r)\} = 0 & (2.1_{-}) \end{cases}$$

At zero point $(1/2, y)$,

$$-\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}sin(y\log r) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}sin(y\log r) = 0$$

So, (2.1_{+}) and (2.1_{-}) become substantially the same equation.

(2) When $x \neq 1/2$, This system of equations is an overdetermined system.

Even though (2.1_{+}) and (2.1_{+}) are different equations, they must share one complex number. The Riemann hypothesis says that such a thing will not happen.

Replacing z with $1/2+z$, we obtain the following equivalent lemma.

Lemma 2.1'

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy\ ,\ x,y\in R)$,
$\eta(1/2\pm z)=0$ in $-1/2<x<1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \eta\left(\dfrac{1}{2}+z\right) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}e^{-z\log r} = 0 & (2.1'_{+}) \\ \eta\left(\dfrac{1}{2}-z\right) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}e^{z\log r} = 0 & (2.1'_{-}) \end{cases}$$

Note

(1) The overdetermined property is the same as in Lemma 2.1.

(2) The known non-trivial zeros are moved parallel onto the **new critical line** $Re(z)=0$

(3) When $x=0$, the overdetermined property disappears.

(4) When $x\neq 0$, if there are zeros, the one set consists of the following four.

$$a\pm ib\ ,\quad -a\pm ib\qquad (\,-1/2<a<1/2\,)$$

Hyperbolic Function Series

Lemma 2.1' is equivalent to the following

Lemma 2.2

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy\ ,\ x,y\in R)$,
$\eta(1/2\pm z)=0$ in $-1/2<x<1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \eta_c(z) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\cosh(z\log r) = 0 & (2.2c) \\ \eta_s(z) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\sinh(z\log r) = 0 & (2.2s) \end{cases}$$

Proof

From (2.1'+) , (2.1'-) ,

$$\frac{1}{2}\left\{\eta\left(\frac{1}{2}-z\right)+\eta\left(\frac{1}{2}+z\right)\right\} = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\frac{e^{z\log r}+e^{-z\log r}}{2}$$

$$= \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\cosh(z\log r) = 0$$

$$\frac{1}{2}\left\{\eta\left(\frac{1}{2}-z\right)-\eta\left(\frac{1}{2}+z\right)\right\} = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\frac{e^{z\log r}-e^{-z\log r}}{2}$$

$$= \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\sinh\{z\log r\} = 0$$

Describing these as $\eta_c(z), \eta_s(z)$ respectively, we obtain the desired expressions.

Conversely, by adding or subtracting these, (2.1'+) , (2.1'-) are obtained.

Note

$\eta_c(z), \eta_s(z)$ are the sum and difference between Dirichlet series. Therefore, their convergence region is $-1/2 < x < 1/2$.

Hyperbolic Function Series (real part, imaginary part)

Theorem 2.3

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy\,,\ x,y\in R)$, $\eta(1/2\pm z)=0$ in $-1/2<x<1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} u_c(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\cosh(x\log r)\cos(y\log r) = 0 \\ v_c(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\sinh(x\log r)\sin(y\log r) = 0 \\ u_s(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\sinh(x\log r)\cos(y\log r) = 0 \\ v_s(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\cosh(x\log r)\sin(y\log r) = 0 \end{cases}$$

Proof

$$\cosh(x+iy) = \cosh x\cos y + i\sinh x\sin y$$

$$\sinh(x+iy) = \sinh x\cos y + i\cosh x\sin y$$

Replacing x with $x\log r$ and y with $y\log r$ respectively,

$$\cosh(z\log r) = \cosh(x\log r)\cos(y\log r) + i\sinh(x\log r)\sin(y\log r)$$

$$\sinh(z\log r) = \sinh(x\log r)\cos(y\log r) + i\cosh(x\log r)\sin(y\log r)$$

Substituting these for (2.2c) , (2.2s) respectively,

$$\eta_c(z) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\cosh(z\log r)$$

$$= \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{r}}\{\cosh(x\log r)\cos(y\log r) + i\sinh(x\log r)\sin(y\log r)\}$$

$$\begin{aligned}\eta_s(z) &= \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} sinh(z\log r) \\ &= \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \{sinh(x\log r)cos(y\log r) + i\,cosh(x\log r)sin(y\log r)\}\end{aligned}$$

Describing the real and imaginary parts as $u_c(x,y)$, $v_c(x,y)$, $u_s(x,y)$, $v_s(x,y)$ respectively, we obtain the desired expressions.

Overdetermined System

Since there are 4 equations for 2 real variable in Theorem 2.3 , this system of equations is an overdetermined system. Such a system of equations generally has no solution.

Zeros on the Critical Line

However, such a system of equations may exceptionally has solution. That is the case when $x=0$. Note that $x=0$ is the critical line of function $\eta(1/2+z)$. Substituting $x=0$ for the equations in Theorem 2.3

$$\begin{cases} u_c(0,y) = 1\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} cos(y\log r) = 0 \\ v_c(0,y) = 0\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} sin(y\log r) = 0 \\ u_s(0,y) = 0\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} cos(y\log r) = 0 \\ v_s(0,y) = 1\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} sin(y\log r) = 0 \end{cases}$$

Since $v_c(0,y)$, $u_s(0,y)$ are equal to non-existent, the overdetermined property disappears. As the result,

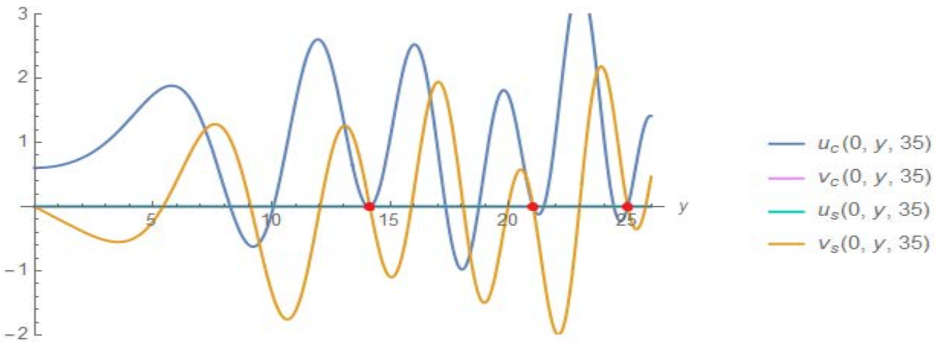
$$\begin{aligned}0 = u_c(0,y) - i\,v_s(0,y) &= \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \{cos(y\log r) - i\,sin(y\log r)\} \\ &= \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \{cos(y\log r) + i\,sin(y\log r)\}\end{aligned}$$

i.e.

$$0 = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} e^{-y\log r} = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} e^{y\log r}$$

That is, they reduce to the case of $x=0$ in Lemma 2.1'. These solutions are zeros on the critical line.

When $x=0$, $u_c \sim v_s$ are drawn as follows. Blue is u_c and orange is v_s .

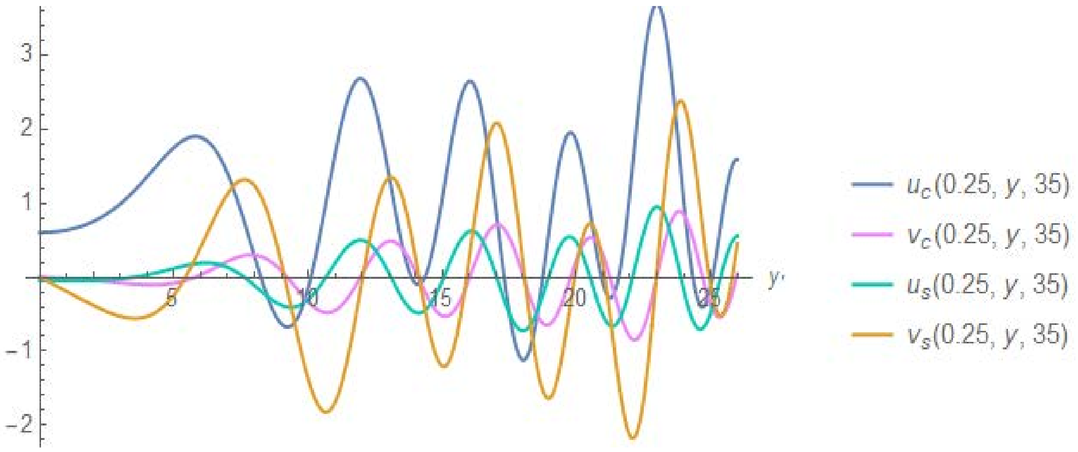


The points (red) where these intersect on the y -axis are the zeros of $\eta(1/2\pm z)$. Magenta is v_c and cyan is u_s . They overlap on the y -axis. Of course, these 2 straight lines also pass through the red points.

Zeros outside the Critical Line

If x deviates even slightly from 0 , v_c, u_s cease to be straight lines. For example, when $x=0.000001$,

As the result, the property of overdetermination is restored. For example, when $x=0.25$, $u_c \sim v_s$ are drawn as follows. It seems unlikely that the 4 curves would intersect at one point on the y -axis.



Propositions equivalent to the Riemann hypothesis

Theorem 2.3 is equivalent to that the following 6 pairs have a common solution. Each pair is one of the necessary conditions for $\eta(1/2+z)$ to have zeros.

$$\begin{cases} u_c = 0 \\ v_c = 0 \end{cases}, \quad \begin{cases} u_c = 0 \\ u_s = 0 \end{cases}, \quad \begin{cases} u_c = 0 \\ v_s = 0 \end{cases}, \quad \begin{cases} v_c = 0 \\ u_s = 0 \end{cases}, \quad \begin{cases} v_c = 0 \\ v_s = 0 \end{cases}, \quad \begin{cases} u_s = 0 \\ v_s = 0 \end{cases}$$

Therefore, to prove the Riemann hypothesis, it is sufficient to show that any one of these pairs does not have a solution such as $x \neq 0$.

The most interesting of these is $v_c = 0$ and $u_s = 0$ pair. This pair is unlikely to intersect at a point on the y -axis when $x \neq 0$, as seen in the 2D figure above. So, we can present the following proposition, which is equivalent to the Riemann hypothesis.

Proposition 2.4

When y is a real number, x is a real number s.t. $-1/2 < x < 1/2$, the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x\log r)\sin(y\log r) = 0 & (2.4c) \\ u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x\log r)\cos(y\log r) = 0 & (2.4s) \end{cases}$$

If this proposition is proved, then by Theorem 2.3 , $\eta(1/2+z)$ has no zeros such as $x \neq 0$.

Euler Transformation and Expression by η function

Since the convergence speed of the series in Theorem 2.3 is slow, accurate calculations and drawings are difficult at small y . In order to deal with this, in this paper we apply the Euler transformation to the series. This transformation accelerates the convergence of the series and even applies the summation method.

$$u_c(x,y,m) = \sum_{k=1}^{m} \sum_{r=1}^{k} \frac{1}{2^{k+1}} \binom{k}{r} \frac{(-1)^{r-1}}{\sqrt{r}} \cosh(x\log r)\cos(y\log r)$$

$$v_c(x,y,m) = \sum_{k=1}^{m} \sum_{r=1}^{k} \frac{1}{2^{k+1}} \binom{k}{r} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x\log r)\sin(y\log r)$$

$$u_s(x,y,m) = \sum_{k=1}^{m} \sum_{r=1}^{k} \frac{1}{2^{k+1}} \binom{k}{r} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x\log r)\cos(y\log r)$$

$$v_s(x,y,m) = \sum_{k=1}^{m} \sum_{r=1}^{k} \frac{1}{2^{k+1}} \binom{k}{r} \frac{(-1)^{r-1}}{\sqrt{r}} \cosh(x\log r)\sin(y\log r)$$

In addition, $v_c(x,y), u_s(x,y)$ are represented by the Dirichlet eta function $\eta(x,y)$, and the calculation routine of $\eta(x,y)$ in formula manipulation software ***Mathematica*** is used. These are represented as follows.

$$v_c(x,y) = \frac{1}{2}\left[Im\left\{ \eta\left(\frac{1}{2} - x - iy \right) \right\} + Im\left\{ \eta\left(\frac{1}{2} + x + iy \right) \right\} \right] \qquad (2.4c')$$

$$u_s(x,y) = \frac{1}{2}\left[Re\left\{ \eta\left(\frac{1}{2} - x - iy \right) \right\} - Re\left\{ \eta\left(\frac{1}{2} + x + iy \right) \right\} \right] \qquad (2.4s')$$

3 Amplitude of $v_c(x,y)$ with respect to y

Among the equations in Proposition 2.4 , $v_c(x,y)$ was as follows.

$$v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} sinh(x \log r)\ sin(y \log r) \tag{2.4c}$$

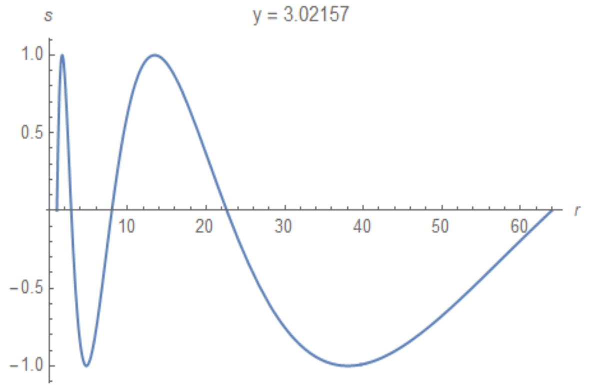
In this chapter, we consider the amplitude of this function with respect to y

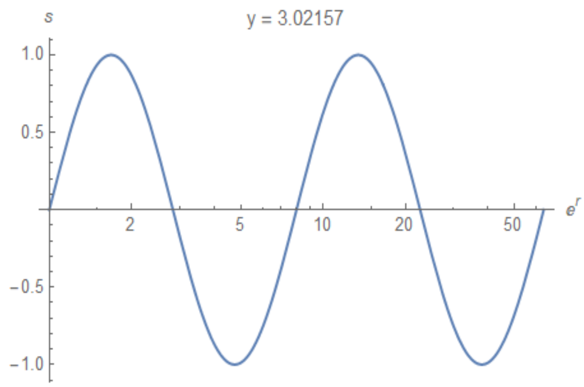
3.1 sin(y log r)

Let r,y are positive numbers respectively, and consider the following function $s(r,y)$.

$$s(r,y) = sin(y \log r) \tag{3.1.1}$$

When $y = 3.02157$, the 2D figures for $r=1\sim 64$ are drawn as follows. The left is normal scale and the right is semilogarithmic scale.





Observing these shows that $s(r,y)$ is a variable periodic function with respect to r . However, the right figure looks like a fixed periodic function at first glance.

Amplitude (A)

The amplitude of this function is $A=1$.

Period (P)

This function is a periodic function. The first period starts at 0π and ends at 2π , the second period starts at 2π and ends at 4π , so

$$y \log r_0 = 0\pi\ ,\ y \log r_1 = 2\pi\ ,\ y \log r_2 = 4\pi\ ,\ \cdots\ ,\ y \log r_n = 2n\pi\ ,\ \cdots$$

From these,

$$r_0 = e^{0\pi/y}\quad ,\quad r_1 = e^{2\pi/y}\quad ,\quad r_2 = e^{4\pi/y}\quad ,\ \cdots\ ,\ r_n = e^{2n\pi/y}\quad ,\ \cdots$$

Therefore, the function $s(r,y)$ is separated into the following unit intervals.

$$\left[e^{0\pi/y}, e^{2\pi/y}\right)\ ,\ \left[e^{2\pi/y}, e^{4\pi/y}\right)\ ,\ \cdots\ ,\ \left[e^{(2n-2)\pi/y}, e^{2n\pi/y}\right)\ ,\ \cdots$$

Each of these has one mountain and one valley. We will call these the 1 st period, the 2 nd period, $\cdots$. i.e.

$$P(n,y) = \left[e^{(2n-2)\pi/y}, e^{2n\pi/y}\right)$$

In the figure above, the 1 st and the 2 nd periods of $s(r,y)$ are

$$P(1,3.02157) = [1, 8)\quad ,\quad P(2,3.02157) = [8, 64)$$

Wavelength (λ)

The wavelength are the length of these periods. That is,

$$e^{0\pi/y}\left(e^{2\pi/y}-1\right)\ ,\ e^{2\pi/y}\left(e^{2\pi/y}-1\right)\ ,\ \cdots\ ,\ e^{(2n-2)\pi/y}\left(e^{2\pi/y}-1\right)\ ,\ \cdots$$

The wavelength is $e^{2\pi/y}$ times longer than the previous period in each period. So, **this function is a variable periodic function**. That is,

$$\lambda(n,y)=e^{(2n-2)\pi/y}\left(e^{2\pi/y}-1\right)$$

In the figure above, the wavelengths of the 1 st and the 2 nd periods of $s(r,y)$ are

$$\lambda(1,3.02157)=7\quad,\quad\lambda(2,3.02157)=56$$

When $n=1$, y can be back calculated from λ.

$$y=\frac{2\pi}{\log(\lambda+1)}$$

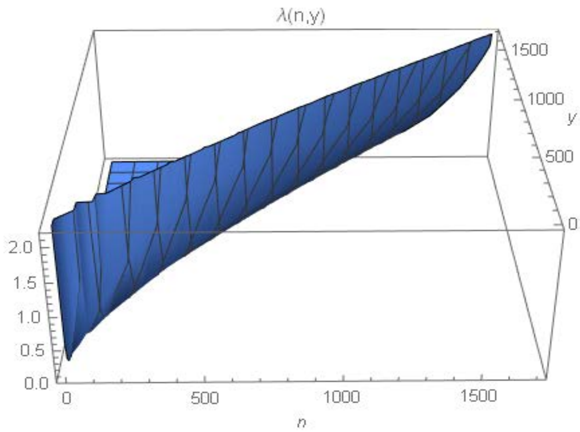
From this,

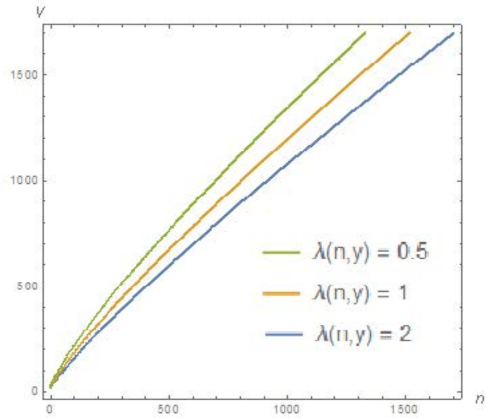
When $\lambda(1,y)=7$, $\quad y=\dfrac{2\pi}{\log 8}=3.02157$

When $\lambda(1,y)=1$, $\quad y=\dfrac{2\pi}{\log 2}=9.06472$

When $\lambda(1,y)=0.559743$, $\quad y=\dfrac{2\pi}{\log 1.559743}=14.1347$

A 3D view of $\lambda(n,y)$ at $n\neq 1$ is shown on the left. And the contour plots at $\lambda=0.5$, $\lambda=1.0$, $\lambda=2.0$ are shown on the right.





From these figures, we can see that the slope of the $\lambda(n,y)$ contour decreases as n increases. because,

$$\frac{\partial}{\partial n}\lambda(n,y)=\frac{2\pi}{y}e^{(2n-2)\pi/y}\left(e^{2\pi/y}-1\right)>0\qquad for\ n,y>0$$

Using this contour plot, we can find the n,y pair that gives the desired λ.

Mountain (Ms)

Since $s(r,y)$ is a sine function, the mountain is at $1/4$ of the period plotted on a semilogarithmic scale.

$$Ms(n,y)=e^{\frac{(4n-3)\pi}{2y}}$$

In the figure above, the mountains of the 1 st and the 2 nd periods of $s(r,y)$ are

$$Ms(1,3.02157) = 1.68179 \quad , \quad Ms(2,3.02157) = 13.4544$$

Valley (Vs)

Since $s(r,y)$ is a sine function, the vallay is at 3/4 of the period plotted on a semilogarithmic scale.

$$Vs(n,y) = e^{\frac{(4n-1)\pi}{2y}}$$

In the figure above, the valleys of the 1 st and the 2 nd periods of $s(r,y)$ are

$$Vs(1,3.02157)= 4.756843 \quad , \quad Vs(2,3.02157)= 38.0548$$

Zeros (Zs)

Since $s(r,y)$ is the sine function, the zeros are at the left edge and middle of the period plotted on a semi-logarithmic scale.

$$Zs(n,y) = \left\{ e^{\frac{(2n-2)\pi}{y}} , e^{\frac{(2n-1)\pi}{y}} \right\}$$

In the figure above, the zeros of the 1 st and the 2 nd periods of $s(r,y)$ are

$$Zs(1,3.02157) = \{1, 2.82843\} \quad , \quad Zs(2,3.02157) = \{8, 22.6275\}$$

Near zeros (Xs)

When the variable r of $s(r,y)$ is a discrete variable, we will call the integer r within ± 0.5 from the zero point **the neighborhood of the zero point**. That is,

$$Xs(n,y) = \left\{ Round\left(e^{\frac{(2n-2)\pi}{y}} \right) , Round\left(e^{\frac{(2n-1)\pi}{y}} \right) \right\}$$

In the figure above,

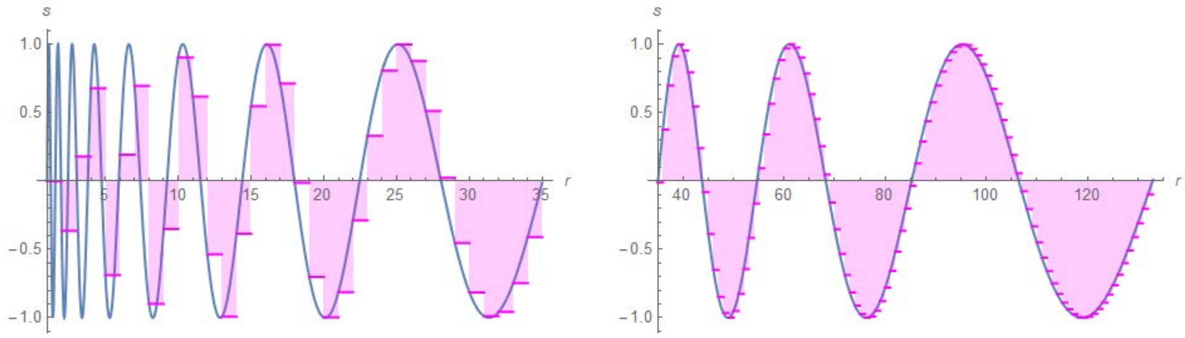
$$Xs(1,3.02157)= \{1,3\} \quad , \quad Xs(2,3.02157)= \{8, 23\}$$

Riemann Zeta type Sine Series (when $y = 14.1347\cdots$)

Let us consider the following Riemann Zeta type sine series.

$$v(y) = \sum_{r=1}^{\infty} \sin(y \log r) \tag{3.1.2}$$

This is a series whose terms are $s(r,y)$ (3.1.1). For example, when $y=14.1347\cdots$, $r=1,2,\cdots,34$ and $r=35,36,\cdots,132$ are drawn in succession, it is as follows.



The function value of (3.1.2) is the sum of the areas of magenta. In the left figure, this sum differs greatly from the integral value of (3.1.1). On the other hand, in the right figure this sum is close to the integral value of (3.1.1)

Divergence

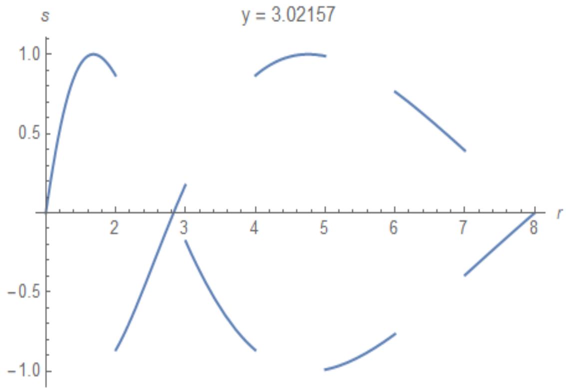
In the right figure, the area seems to be zero due to cancellation of plus and minus, but it is not. Because, the interval between waves expands and eventually becomes infinite. So, the series in (3.1.2) diverges.

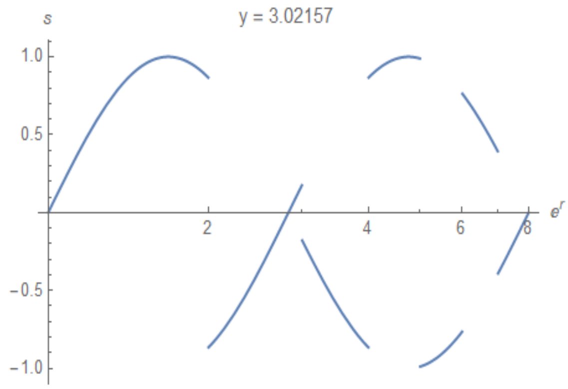
3.2 ±sin(y log r)

Let r,y are positive numbers respectively, and consider the following function $s(r,y)$.

$$s(r,y) = (-1)^{\lfloor r-1 \rfloor} \sin(y \log r) \qquad (\lfloor\ \rfloor \text{ is floor function }) \tag{3.2.1}$$

When $y = 3.02157$, the 2D figures for $r=1 \sim 8$ are drawn as follows. The left is normal scale and the right is semilogarithmic scale.





Unlike the previous section, $s(r,y)$ is a discontinuous function with respect to r .

Amplitude (A)

The amplitude of this function is

$$A(r) = \left|(-1)^{\lfloor r-1 \rfloor}\right| = 1$$

Period (P)

The period of this function is the same as in the previous section, That is,

$$P(n,y) = \left[e^{(2n-2)\pi/y} , e^{2n\pi/y}\right)$$

In the figure above,

$$P(1,3.02157) = [1 , 8)$$

Wavelength (λ)

The wavelength of this function is the same as in the previous section, That is,

$$\lambda(n,y) = e^{(2n-2)\pi/y}\left(e^{2\pi/y} - 1\right)$$

In the figure above,

$$\lambda(1,3.02157) = 7$$

Mountain or Valley (MVs)

Unlike the previous section, this function $s(r,y)$ changes sign. For this reason, mountains and valleys exist at most twice as many as in the previous section.

$$MVs(n,y) = \left\{ e^{\frac{(4n-3)\pi}{2y}} , e^{\frac{(4n-1)\pi}{2y}} \right\}$$

The mountain or valley is determined by the sign of $s(r,y)$ at $r = MVs(n,y)$.

In the figure above,

$$MVs(1,3.02157) = \{1.68179, 4.75684\}$$

$$\{s(1.68179, 3.02157), s(4.75684, 3.02157)\} = (1,1)$$

So, both the former and the latter are mountains.

Zeros (Zs)

The zeros of this function are the same as in the previous section, That is,

$$Zs(n,y) = \left\{ e^{\frac{(2n-2)\pi}{y}} , e^{\frac{(2n-1)\pi}{y}} \right\}$$

In the figure above,

$$Zs(1,3.02157) = \{1, 2.82843\}$$

Constriction (Xs)

Since this function $s(r,y)$ changes sign, the zero point looks like a constriction. So, we will call the integer r within ± 0.5 from the zero point **constriction**. That is,

$$Xs(n,y) = \left\{ Round\left(e^{\frac{(2n-2)\pi}{2y}} \right) , Round\left(e^{\frac{(2n-1)\pi}{2y}} \right) \right\}$$

In the figure above,

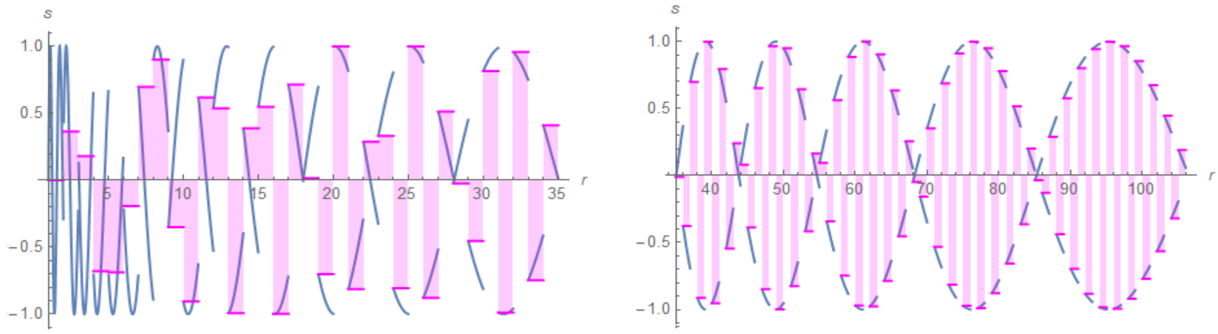
$$Xs(1,3.02157) = \{1,3\}$$

Dirichlet Eta type Sine Series (when $y = 14.1347\cdots$)

We consider the following Dirichlet Eta type sine series

$$v(y) = \sum_{r=1}^{\infty} (-1)^{r-1} \sin(y \log r) \tag{3.2.2}$$

This is a series whose terms are $s(r,y)$ (3.2.1). For example, when $y=14.1347\cdots$, $r=1,2,\cdots,34$ and $r=35,36,\cdots,105$ are drawn in succession, it is as follows.



The function value of (3.2.2) is the sum of the areas of magenta. In the left figure, this sum differs greatly from the integral value of (3.2.1). On the other hand, in the right figure this sum is close to the integral value of (3.2.1)

Convergence ?

The last two constrictions in the right figure belong to the 11 th period. The area between two constrictions seems to cancel out to zero. As a trial, when $y_1 = 14.1347\cdots$, the area between each constriction in the 11 th and the 25 th periods are calculated as follows.

The 11 th period $\quad Xs(11, y_1) = (85, 106)$, $\lambda(11, y_1) = 47.7$

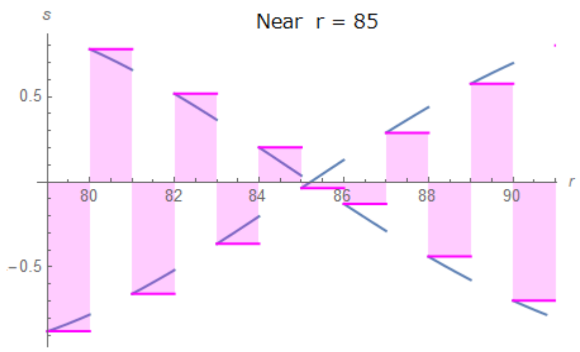
$$v_{11}(y_1) = \sum_{r=85}^{105} (-1)^{r-1} \sin(y_1 \log r) = 0.00208785$$

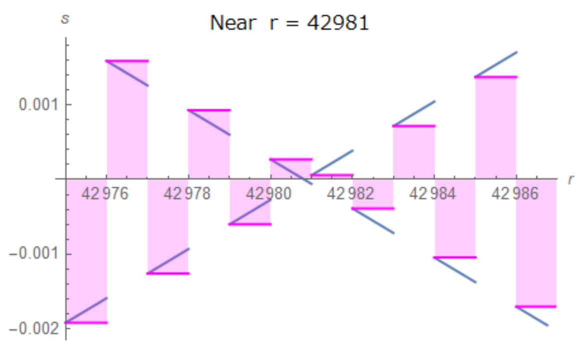
The 25 th period $\quad Xs(25, y_1) = (42981, 53679)$, $\lambda(25, y_1) = 24058.2$

$$v_{25}(y_1) = \sum_{r=42981}^{53678} (-1)^{r-1} \sin(y_1 \log r) = -0.0000684506$$

Certainly, the area between two constrictions approaches 0 as r increases.

To find out the cause of this, let us compare the enlarged images near $r=85$ and $r=42981$.





Then, at a glance, it can be seen that the scale of the vertical axis is an order of magnitude. Why? The reason is simple. Because, the wavelength becomes longer as r moves away from the origin. Since the amplitude is 1 , the longer the wavelength, the slower the slope of the variable-length sine curve. However, even so, this series $v(y)$ is a divergent series. i.e. it just oscillates and never converges. So, if this series is truncated at mountain or valley, there will be a maximum error of ± 0.5 . That is, this series (3.2.2) oscillates within ± 0.5

Summation Method

In such a case, the summation method says that an error of ± 0.5 should be regarded as 0 on average. One of the simplest summation methods is the Euler transformation. The Euler transformation accelerates the convergence of the series and also applies the summation method. If the Euler transform is applied to (3.2.2) ,

$$v(y, m) = \sum_{k=1}^{m} \sum_{r=1}^{k} \frac{1}{2^{k+1}} \binom{k}{r} (-1)^{r-1} \sin(y \log r) \tag{3.2.2'}$$

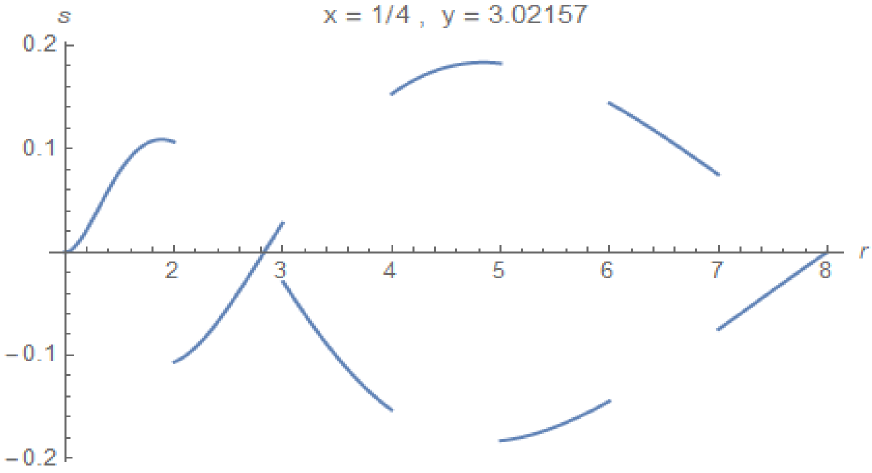
If this formula is used, this series converges.

3.3 $v_c(x,y)$

Let r, x, y are positive numbers respectively, and consider the following function $s(r, x, y)$.

$$s(r, x, y) = (-1)^{\lfloor r-1 \rfloor} \frac{\sinh(x \log r)}{\sqrt{r}} \sin(y \log r) \qquad (\lfloor\ \rfloor \text{ is floor function }) \tag{3.3.1}$$

When $x = 1/4, y = 3.02157$, the 2D figures for $r = 1 \sim 8$ is drawn as follows.



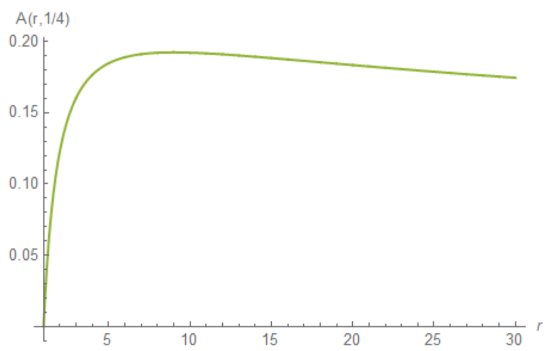
Amplitude (A)

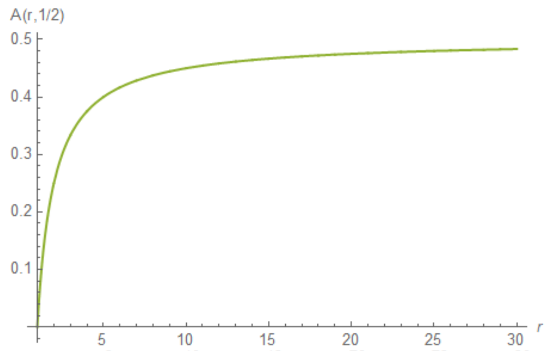
The amplitude of this function is

$$A(r,x) = \left| (-1)^{\lfloor r-1 \rfloor} \frac{\sinh(x \log r)}{\sqrt{r}} \right| = \frac{\sinh(x \log r)}{\sqrt{r}}$$

(1) When $0 < x < 1/2$, $\lim_{r \to \infty} \sinh(x \log r)/\sqrt{r} = 0$. This is shown on the left.

(2) When $x = 1/2$, $\lim_{r \to \infty} \sinh(x \log r)/\sqrt{r} = 1/2$. This is shown on the right.





Period (P)

The period of this function is the same as in the previous section, That is,

$$P(n,y) = \left[e^{(2n-2)\pi/y}, \ e^{2n\pi/y} \right)$$

In the figure above,

$$P(1, 3.02157) = [1, 8)$$

Wavelength (λ)

The wavelength of this function is the same as in the previous section, That is,

$$\lambda(n,y) = e^{(2n-2)\pi/y} \left(e^{2\pi/y} - 1 \right)$$

In the figure above,

$$\lambda(1, 3.02157) = 7$$

Mountain or Valley (MVs)

The mountains or valleys of this function are the same as in the previous section, That is,

$$MVs(n,y) = \left\{ e^{\frac{(4n-3)\pi}{2y}}, e^{\frac{(4n-1)\pi}{2y}} \right\}$$

In the figure above,

$$MVs(1,3.02157) = \{1.68179, 4.75684\}$$

$$\{s(1.68179, 1/4, 3.02157), s(4.75684, 1/4, 3.02157)\} = \{0.100499, 0.183332\}$$

So, both the former and the latter are mountains.

Zeros (Zs)

The zeros of this function are the same as in the previous section, That is,

$$Zs(n,y) = \left\{ e^{\frac{(2n-2)\pi}{y}}, e^{\frac{(2n-1)\pi}{y}} \right\}$$

In the figure above,

$$Zs(1,3.02157) = \{1, 2.82843\}$$

Constriction (Xs)

The constrictions of this function are the same as in the previous section, That is,

$$Xs(n,y) = \left\{ Round\left(e^{\frac{(2n-2)\pi}{2y}} \right), Round\left(e^{\frac{(2n-1)\pi}{2y}} \right) \right\}$$

In the figure above,

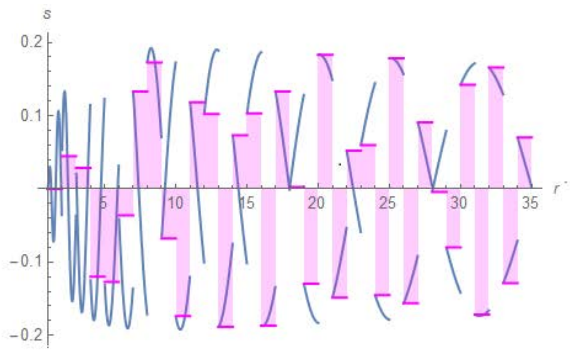
$$Xs(1,3.02157) = \{1,3\}$$

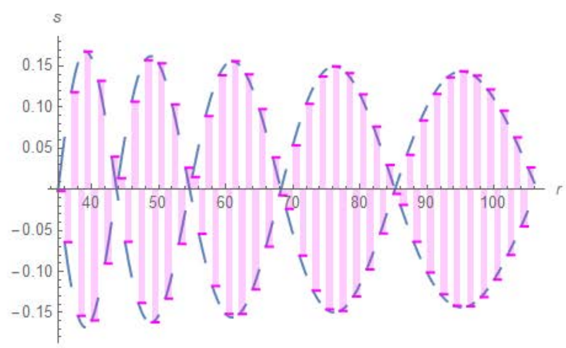
Sine Series $v_c(x,y)$ (when $x = 1/4$, $y = 14.1347\cdots$)

We consider the following sine series.

$$v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} sinh(x \log r)\ sin(y \log r) \tag{2.4c}$$

This is a series whose terms are $s(r,x,y)$ (3.3.1) . For example, when $x=1/4$, $y=14.1347\cdots$, $r=1,2,\cdots,34$ and $r=35,36,\cdots,105$ are drawn in succession, it is as follows.





The function value of (2.4c) is the sum of the areas of magenta. In the left figure, this sum differs greatly from the integral value of (3.3.1). On the other hand, in the right figure this sum is close to the integral value of (3.3.1)

Convergence

The last two constrictions in the right figure belong to the 11 th period. The area between two constrictions seems to cancel out to zero. As a trial, when $x=1/4$, $y_1=14.1347\cdots$, the area between each constriction in the 11 th and the 25 th periods are calculated as follows.

The 11 th period $Xs(11, y_1) = (85, 106)$, $\lambda(11, y_1) = 47.7$

$$v_{11}\left(\frac{1}{4}, y_1\right) = \sum_{r=85}^{105} (-1)^{r-1} \frac{\sinh(y_1 \log r)}{\sqrt{r}} \sin(y_1 \log r) = -0.0000708924$$

The 25 th period $Xs(25, y_1) = (42981, 53679)$, $\lambda(25, y_1) = 24058.2$

$$v_{25}\left(\frac{1}{4}, y_1\right) = \sum_{r=42981}^{53678} (-1)^{r-1} \frac{\sinh(y_1 \log r)}{\sqrt{r}} \sin(y_1 \log r) = -0.00000233757$$

Comparing the two, the area between the two constrictions converges to 0 as r increases. The cause is clear. Even when the amplitude is 1 , the area between constrictions decreases as r increases. In addition, the amplitude approaches 0 . Due to these synergistic effects, the area between the constrictions has to approach 0 even more. However, since the amplitude does not approach 0 at $x \geq 1/2$, the summation method is needed for convergence.

3.4 Amplitude of $v_c(x,y)$ with respect to y

The sine function and series dealt with in **3.3** were as follows.

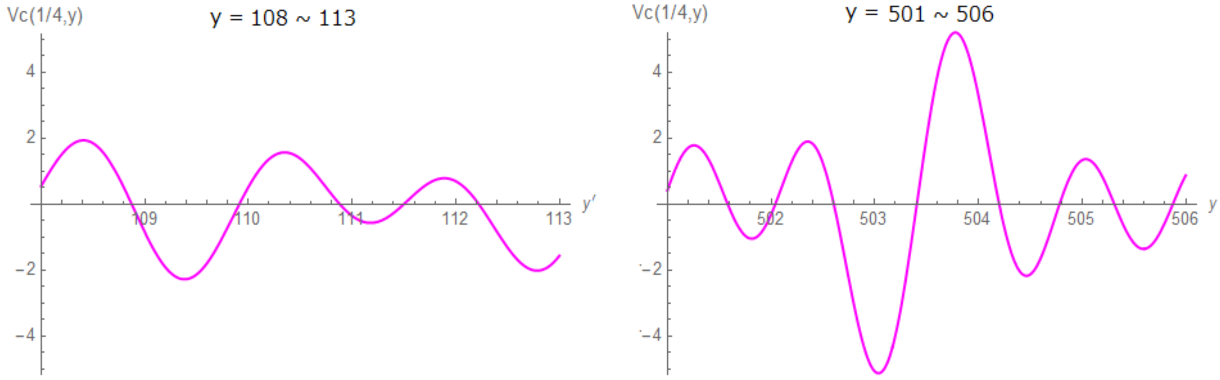
$$s(r,x,y) = (-1)^{\lfloor r-1 \rfloor} \frac{\sinh(x \log r)}{\sqrt{r}} \sin(y \log r) \qquad (\lfloor\ \rfloor \text{ is floor function }) \qquad (3.3.1)$$

$$v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x \log r) \sin(y \log r) \qquad (2.4c)$$

$$= \frac{1}{2}\left[Im\left\{ \eta\left(\frac{1}{2} - x - iy\right)\right\} + Im\left\{ \eta\left(\frac{1}{2} + x + iy\right)\right\}\right] \qquad (2.4c')$$

In this section, we explore the amplitude (mountain, valley) of (2.4c) with respect to y using (3.3.1) . To ensure accuracy, calculation and drawing are performed by (2.4c') .

Given x , $v_c(x,y)$ is a variable periodic function with respect to y . For example, when $x=1/4$, the 2D figures at $y = 108 \sim 113$ and $y = 501 \sim 506$ are drawn as follows.



Although the value of y is larger in the right figure than in the left figure, it cannot be said that the mountains in the right are higher than those in the left, or that the valleys in the right are deeper than those in the left.

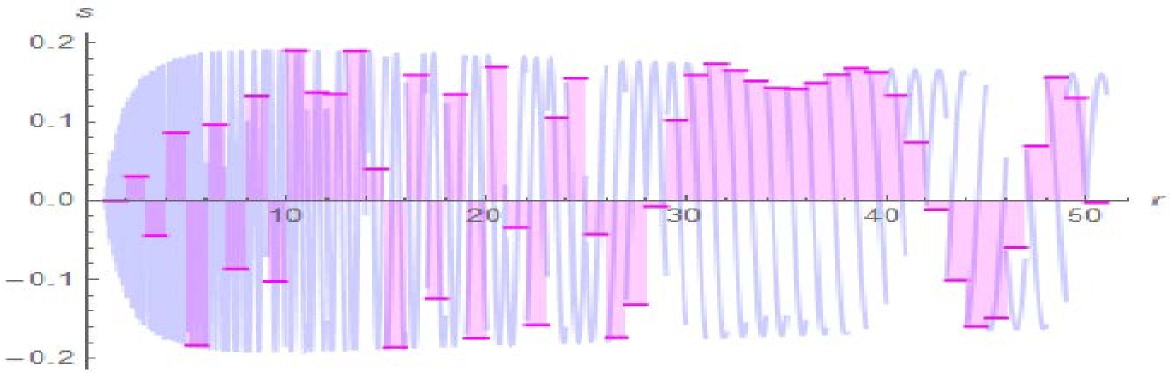
However, it can be said that the highest mountain in the right figure is higher than the highest one in the left figure, and the deepest valley in the right figure is deeper than the deepest one in the left figure. That is, we can say that the amplitude in the right figure is generally larger than one in the left figure. Below, this will be illustrated graphically.

3.4.1 Mountain of $v_c(1/4, y)$ (near y=108.4)

At $y = 108 \sim 113$, the mountain near here is the highest. Accurate calculation of the mountain near here by (2.4c') is as follows.

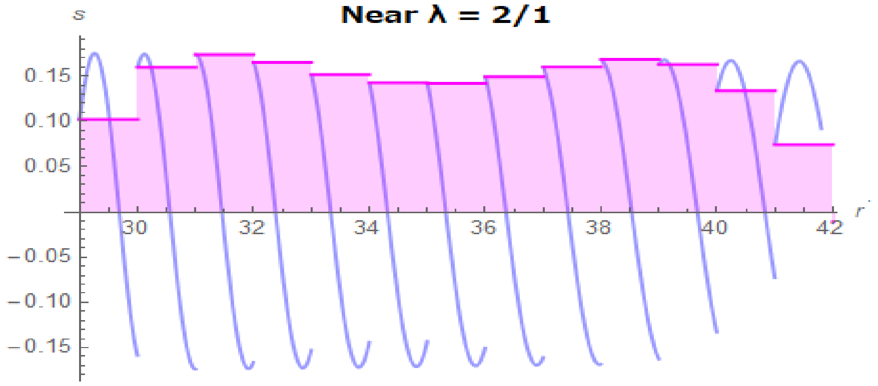
```
FindMaximum[vc[1 / 4, y], {y, 108.5}]
{1.92904, {y → 108.409}}          yM := 108.409
```

Drawing $s(r, 1/4, y_M)$ by (3.3.1) is as follows. The horizontal axis is r. Cyan is drawn as a continuous variable and magenta as a discrete variable. The sum of the area of magenta becoms mountain 1.929 of (2.4c)



Looking at this figure, it can be seen that there are 2 intervals with consecutive positive terms, which contribute to the height of the mountain.

(1) An enlarged view near 2/1 wavelength is drawn as follows.



$r = 29 \sim 41$ are positive for 13 consecutive terms. Calculating the constriction that seems to be around here by trial and error,

```
{Xs[59, yM], Xs[65, yM]}
{{29, 30}  , {41, 42}}
```

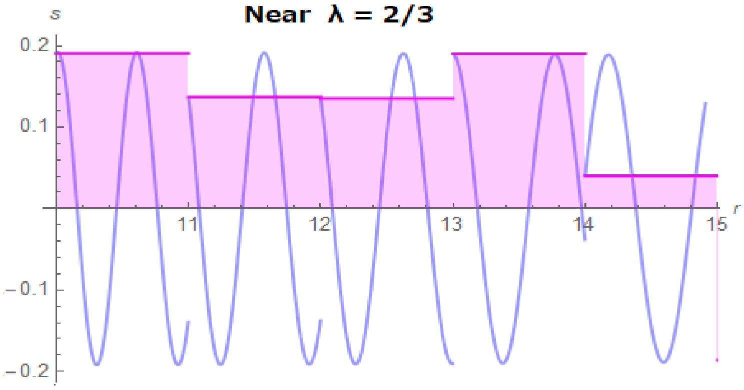
Then, $r = 29 \sim 41$ are included in the $59 \sim 65$ th period. The wavelengths of these periods are

```
Table[λ[n, yM], {n, 59, 65}]
{1.72054, 1.82321, 1.932, 2.04728, 2.16944, 2.29889, 2.43607}
```

That is, the wavelengths of $r = 29 \sim 41$ are $1.72 \sim 2.44$

Here after, only the calculation results are described.

(2) An enlarged view near $2/3(=0.67)$ wavelength is drawn as follows.



$r = 10 \sim 14$ are positive for 5 consecutive terms.

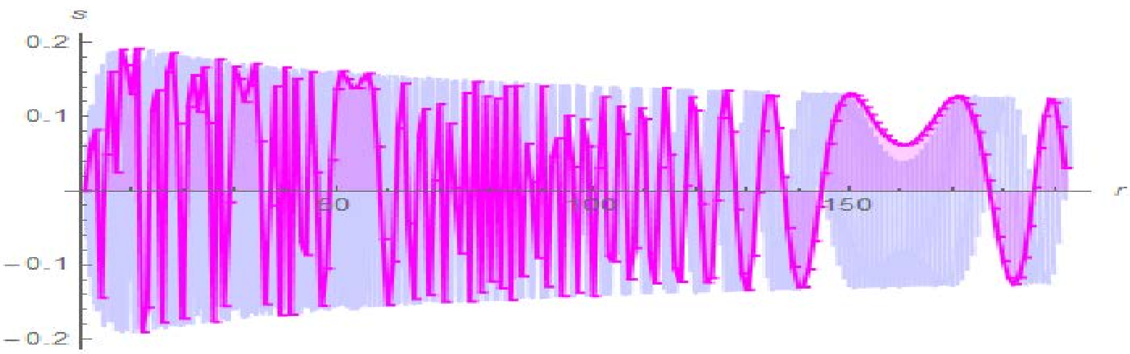
These are included in the $42 \sim 46$ th period and the wavelengths of the periods are $0.64 \sim 0.81$.

3.4.2 Mountain of $v_c(1/4, y)$ (near y= 503.8)

At $y = 501 \sim 506$, the mountain near here is the highest. Accurate calculation of the mountain near here by (2.4c') is as follows.

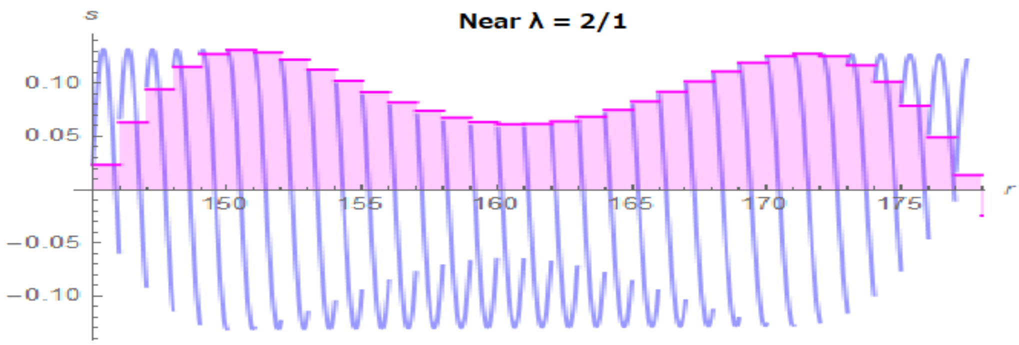
```
FindMaximum[vc[1 / 4, y], {y, 503.8}]
{5.19826, {y → 503.778}}        yM := 503.778
```

Drawing $s(r, 1/4, y_M)$ by (3.3.1) is as follows. The horizontal axis is r. Cyan is drawn as a continuous variable and magenta as a discrete variable. The sum of the area of magenta becoms mountain 5.198 of (2.4c)



Looking at this figure, it can be seen that there are 5 intervals with consecutive positive terms, which contribute to the height of the mountain.

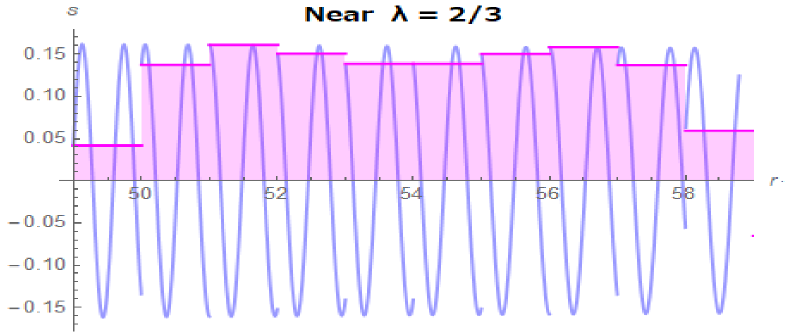
(1) An enlarged view near $2/1$ wavelength is drawn as follows.



$r = 145 \sim 177$ are positive for 33 consecutive terms. This interval is 2.54 times one of **3.4.1 (1)**

These are included in the $400 \sim 416$ th period and the wavelengths of the periods are $1.82 \sim 2.22$.

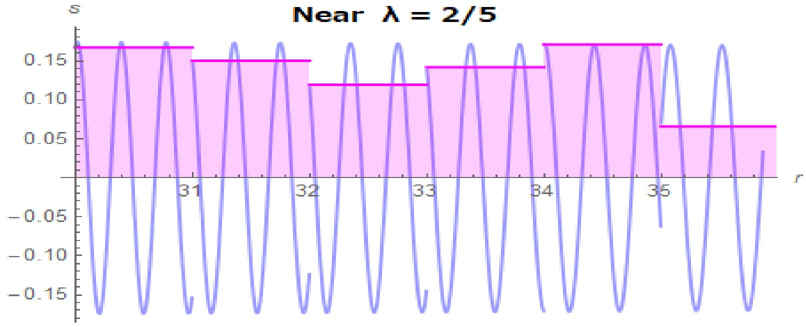
(2) An enlarged view near $2/3(=0.67)$ wavelength is drawn as follows.



$r = 49 \sim 58$ are positive for 10 consecutive terms. This interval is 2 times one of **3.4.1 (2)** .

These are included in the $312 \sim 326$ th period and the wavelengths of the periods are $0.61 \sim 0.71$.

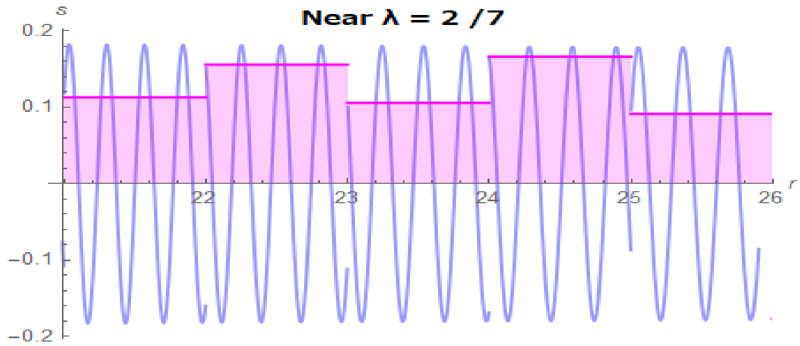
(3) An enlarged view near $2/5(=0.4)$ wavelength is drawn as follows.



$r = 30 \sim 35$ are positive for 6 consecutive terms. This interval is absent in **3.4.1** .

These are included in the $272 \sim 285$ th period and the wavelengths of the periods are $0.37 \sim 0.43$.

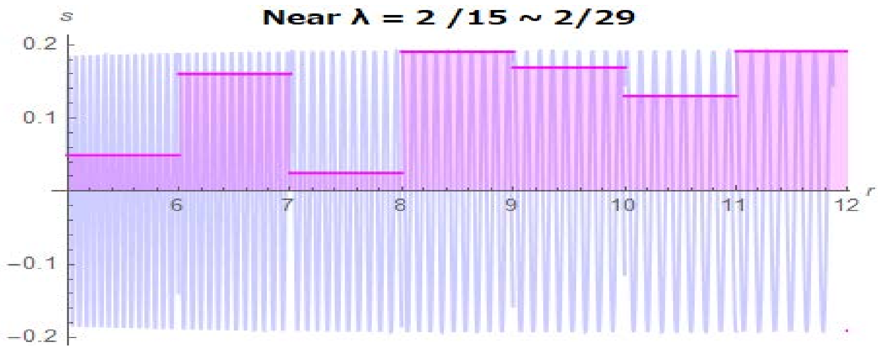
(4) An enlarged view near $2/7(=0.29)$ wavelength is drawn as follows.



$r = 21 \sim 25$ are positive for 5 consecutive terms. This interval is absent in **3.4.1** .

These are included in the $246 \sim 257$ th period and the wavelengths of the periods are $0.27 \sim 0.31$.

(5) An enlarged view near $2/15 \sim 2/29$ $(0.07 \sim 0.13)$ wavelength is drawn as follows.



$r = 5 \sim 11$ are positive for 7 consecutive terms. This interval is absent in **3.4.1** .

These are included in the $137 \sim 190$ th period and the wavelengths of the periods are $0.07 \sim 0.13$.

3.4.3 Height of mountains near y=503.8 and y=108.4

The mountain near y=503.8 is higher than the one near y=108.4 . because,

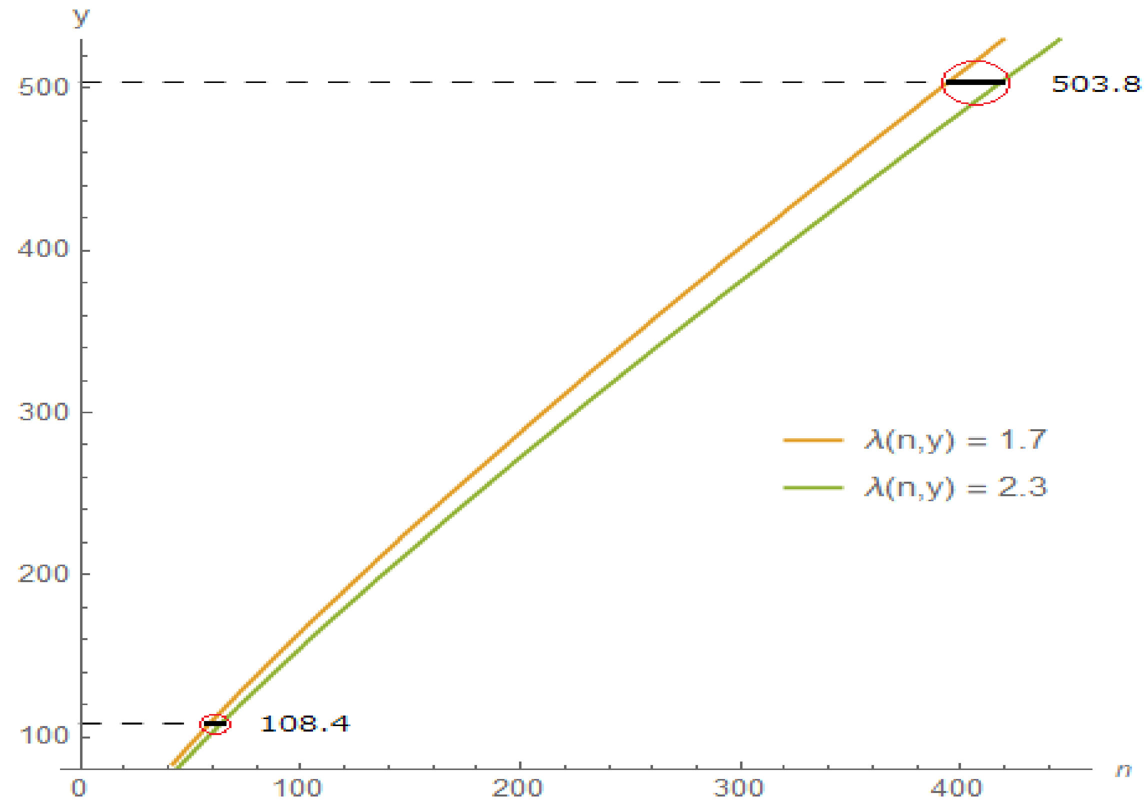
(1) The former is about $2 \sim 2.5$ times longer than the latter in the interval with consective positive terms near wavelength $2/1$, $2/3$.

(2) The intervals with consective positive terms near wavelength $2/5 \sim 2/29$ are added to the former.

The reason for (1) lies in the definition of wavelength. That is,

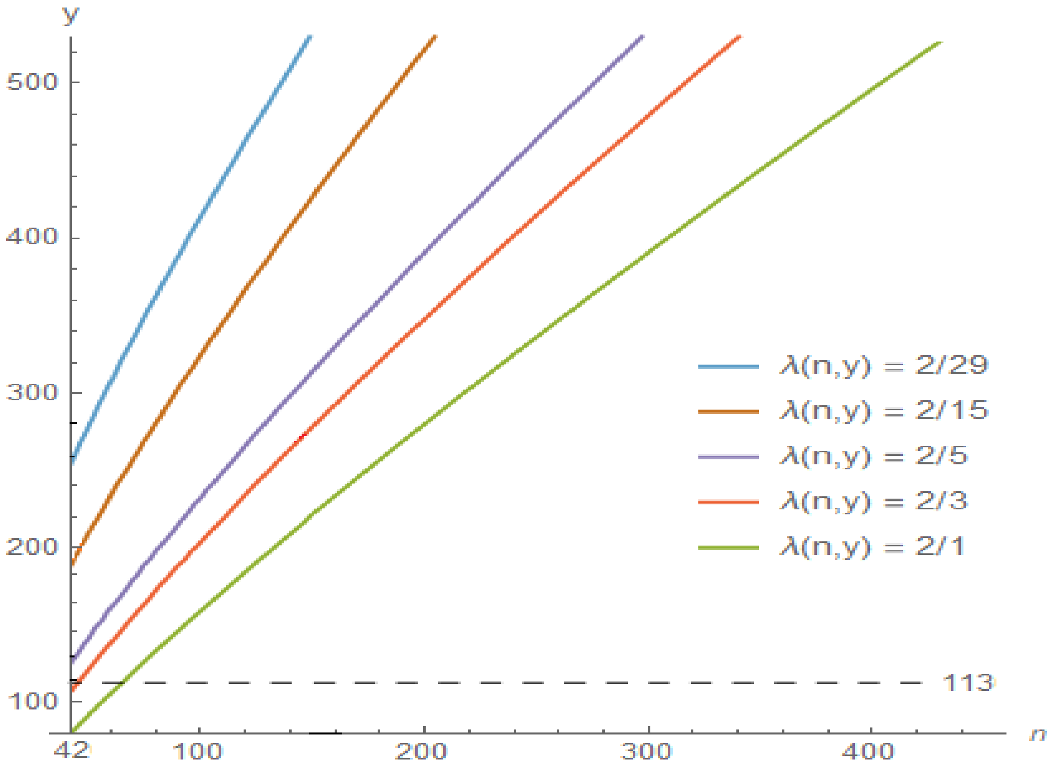
$$\lambda(n,y) = e^{(2n-2)\pi/y}\left(e^{2\pi/y} - 1\right)$$

When the near of $\lambda = 2$ is ± 0.3 , the contour plots of $\lambda(n,y) = 1.7$ and $\lambda(n,y) = 2.3$ are drawn as follows. The vertical axis is y and the horizontal axis is the period number n



The allowable range for the wavelength λ near y=108.4 is the lower left black horizontal line, and the one for the λ near y=503.8 is the upper right black horizontal line. Then, we can see that the allowable range near y=503.8 is wider than one near y=108.4 . This is because the slope y/n of the contour decreases as the wavelength λ increases. This is the same for $\lambda = 2/3$, $2/5$, $\cdots$ as well. Thus the number of consecutive positive terms near y=503.8 is greater than that near y=108.4 . The above is the reason for (1) .

The reason for (2) is the same. The contours for $\lambda = 2/1$, $2/3$, $2/5$, $2/15$, $2/29$ are as follows. Since the minimum value of the period number that gives these values near $y = 108.4$ is 42 (**3.4.1 (2)**), the horizontal axis is drawn with $n \geq 42$.



Now, draw a horizontal line with a height of 113 with a dashed line. Since near $y = 108.4$ is $y = 108 \sim 113$, it is below this chain line, and $\lambda = 2/5$, $2/15$, $2/29$ cannot exist here. The reason is that for a given period number n , the contours shift upwards as the wavelength λ decreases. Thus, if y increases, k in $\lambda(42, y) = 2/(2k-1)$ also increases. The above is the reason for (2) .

3.4.4 Depth of Valleys near $y = 503$ and $y = 109.4$

The valley near $y = 503$ is deeper than the one near $y = 109.4$. Because, **3.4.1** ~ **3.4.3** also hold for valley.

From **3.4.3** and **3.4.4** , we conclude that the amplitude at $y = 501 \sim 506$ is greater than that at $y = 108 \sim 113$. This can be described more generally as follows.

Law 3.4.5

Let x, y are real numbers and function $v_c(x, y)$ be as follows.

$$v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x \log r) \sin(y \log r) \tag{2.4c}$$

Then, given x , the amplitude of $v_c(x, y)$ is generally proportional to the absolute value of y .

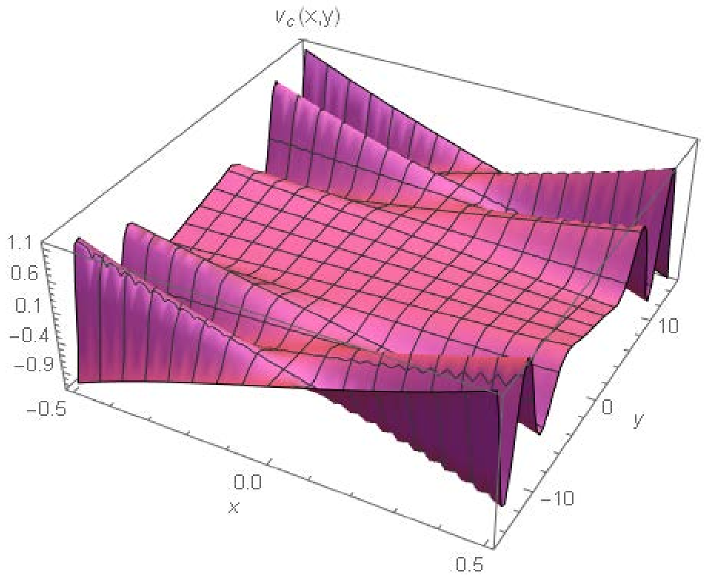
Note

It is clear that this does not hold as a theorem. This is because there are quite a few exceptions. Nevertheless Law 3.4.5 holds. Because, it is due to the change in slope y/n of the contour line of wavelength λ and the shift of the contour line, as described above.

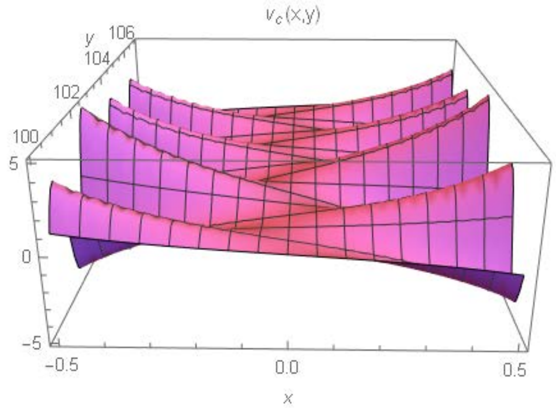
This law is similar to ***Bergmann's Law*** (Bears in high latitudes are generally larger than bears in low latitudes.).

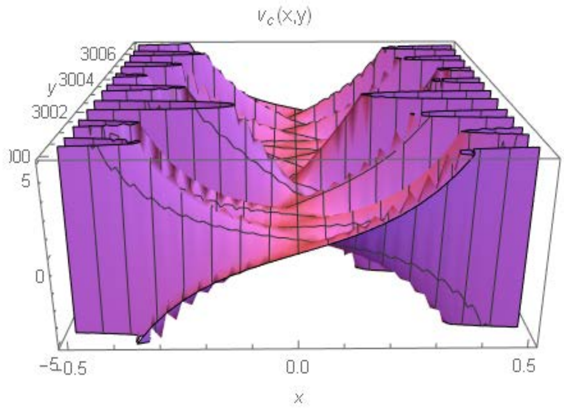
3.5 Shape and Properties of $v_c(x,y)$

From (2.4c) , we find that $v_c(x,y)$ is an odd function with respect to both x and y .
This shows that $v_c(x,y)$ is point symmetric with respect to both both x and y .



Next, when $-1/2 \leq x \leq 1/2$, the 3D view of $v_c(x,y)$ at $y = 100 \sim 107$ and $y = 3000 \sim 3007$ are drawn respectively as follows.



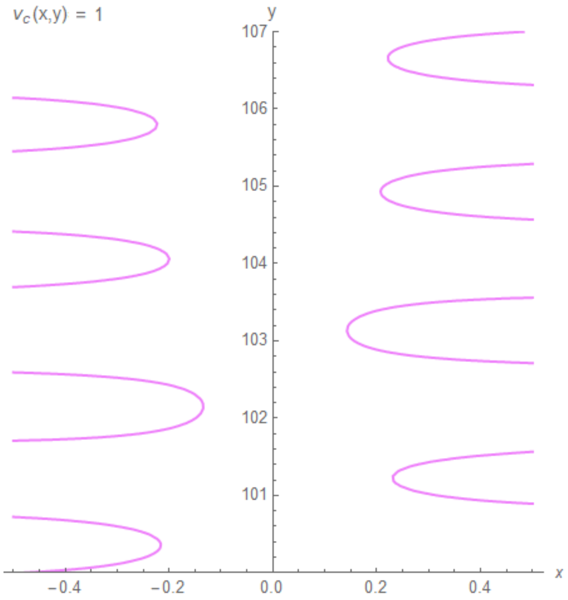


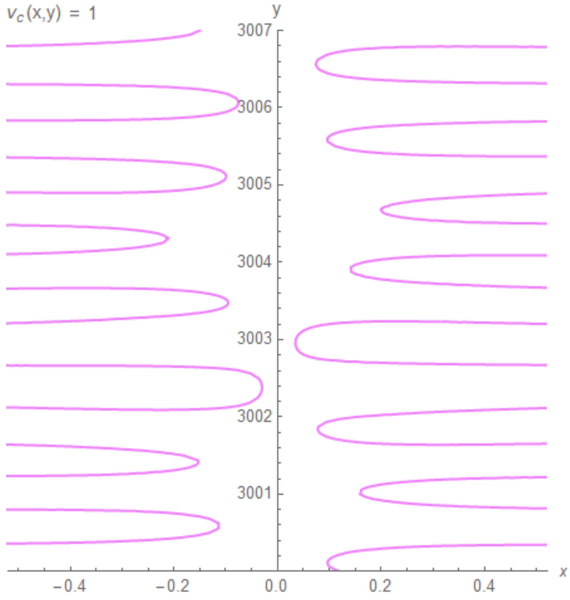
In both figures, the upper part looks like $\cup$ and the lower part looks like $\cap$. Then, we can see that both $\cup$ and $\cap$ generally have larger curvatures in the right figure than in the left figure. This is because mountains and valleys are steeper in the right figure than in the left figure according to Law 3.4.5 . In addition, the right figure has more mountains and valleys than the left figure (about twice as many), but the reason for this is unknown.

3.6 Contour line of $v_c(x,y)$ with height 1

The height 1 contours of the two 3D views of $v_c(x,y)$ in the previous section are drawn as follows.

The left figure is $y = 100 \sim 107$ and the right figure is $y = 3000 \sim 3007$.





In both figures, the contour line looks like $\supset$ & $\subset$. Then, we can see that both $\supset$ & $\subset$ are generally closer to the y-axis in the right figure than in the left figure. This is because mountains and valleys are generally steeper in the right figure than in the left figure according to Law 3.4.5 . Therefore, as $|y|$ increases, the tips $\supset \subset$ of the contour approach the y -axis from both sides.

4 Amplitude of $u_s(x,y)$ with respect to y

Among the equations in Proposition 2.4 , $u_s(x,y)$ was as follows.

$$u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x \log r) \cos(y \log r) \tag{2.4s}$$

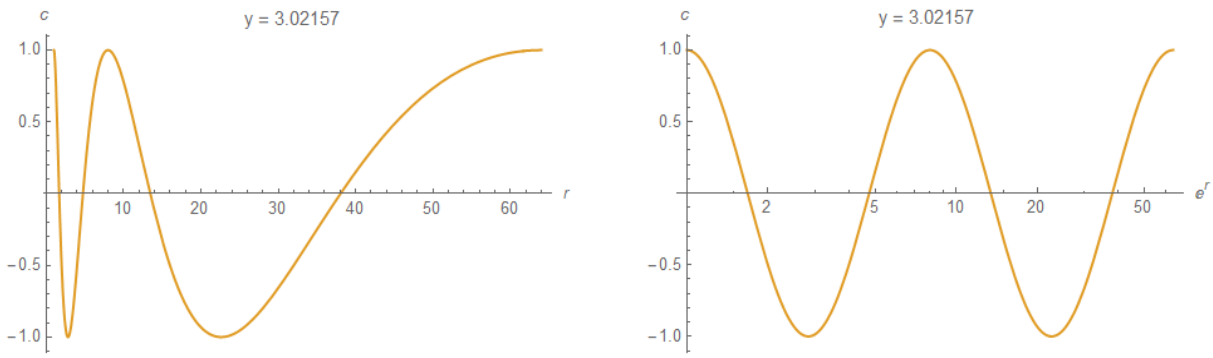
In this chapter, we consider the amplitude of this function with respect to y

4.1 cos (y log r)

Let r, y are positive numbers respectively, and consider the following function $c(r,y)$.

$$c(r,y) = \cos(y \log r) \tag{4.1.1}$$

When $y = 3.02157$, the 2D figures for $r = 1 \sim 64$ are drawn as follows. The left is normal scale and the right is semilogarithmic scale.



Observing these shows that $c(r,y)$ is a variable periodic function with respect to r . However, the right figure looks like a fixed periodic function at first glance.

Amplitude (A)

The amplitude of this function is $A = 1$.

Period (P)

$$P(n,y) = \left[e^{(2n-2)\pi/y} ,\ e^{2n\pi/y} \right)$$

In the figure above, the 1 st and the 2 nd periods of $s(r,y)$ are

$$P(1, 3.02157) = [1\ ,\ 8) \quad , \quad P(2, 3.02157) = [8\ ,\ 64)$$

Wavelength (λ)

$$\lambda(n,y) = e^{(2n-2)\pi/y} \left(e^{2\pi/y} - 1 \right)$$

In the figure above, the wavelengths of the 1 st and the 2 nd periods of $c(r,y)$ are

$$\lambda(1, 3.02157) = 7 \quad , \quad \lambda(2, 3.02157) = 56$$

Mountain (Mc)

Since $c(r,y)$ is a cosine function, there are half mountains at both ends of the period, but the tip is adopted.

$$Mc(n,y) = e^{(2n-2)\pi/y}$$

In the figure above, the mountains of the 1 st and the 2 nd periods of $c(r,y)$ are

$$Mc(1,3.02157) = 1 \quad , \quad Mc(2,3.02157) = 8$$

Valley (Vc)

Since $c(r,y)$ is a cosine function, there is a valley in the middle of the period plotted on a semi-logarithmic scale

$$Vc(n,y) = e^{(2n-1)\pi/y}$$

In the figure above, the valleys of the 1 st and the 2 nd periods of $c(r,y)$ are

$$Vc(1,3.02157)= 2.82843 \quad , \quad Vc(2,3.02157)= 22.6275$$

Zeros (Zc)

Since $c(r,y)$ is the cosine function, the zeros are at $1/4$ and $3/4$ of the period plotted on a semi-logarithmic scale.

$$Zc(n,y) = \left\{ e^{\frac{(4n-3)\pi}{2y}} , e^{\frac{(4n-1)\pi}{2y}} \right\}$$

In the figure above, the zeros of the 1 st and the 2 nd periods of $c(r,y)$ are

$$Zc(1,3.02157) = \{1.68179 , 4.75684\}$$

$$Zc(2,3.02157) = \{13.4544 , 38.0548\}$$

Near zeros (Xc)

When the variable r of $c(r,y)$ is a discrete variable, we will call the integer r within ± 0.5 from the zero point **the neighborhood of the zero point.** That is,

$$Xc(n,y) = \left\{ Round\left(e^{\frac{(4n-3)\pi}{2y}} \right) , Round\left(e^{\frac{(4n-1)\pi}{2y}} \right) \right\}$$

In the figure above,

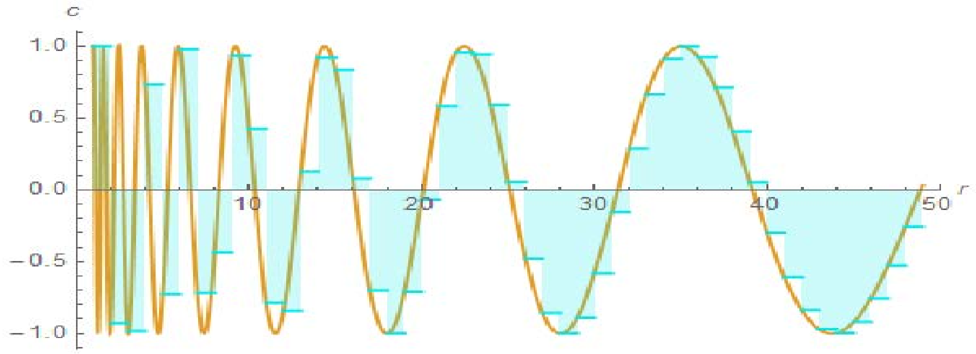
$$Xc(1,3.02157)= \{2,5\} \quad , \quad Xc(2,3.02157)= \{13 , 38\}$$

Riemann Zeta type Cosine Series (when $y = 14.1347\cdots$)

Let us consider the following Riemann Zeta type cosine series.

$$u(y) = \sum_{r=1}^{\infty} cos(y \log r) \tag{4.1.2}$$

This is a series whose terms are $c(r,y)$ (4.1.1). For example, when $y=14.1347\cdots$, $r=1,2,\cdots,49$ $c(r,y)$ is drawn as follows.



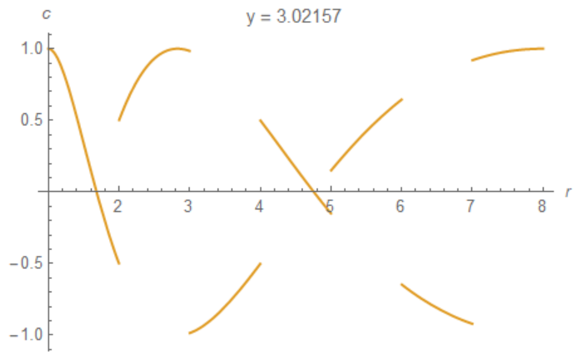
The sum of the areas of cyan is the function value of (4.1.2) .
This series diverges, and the summation method only leads to an asymptotic expansion.

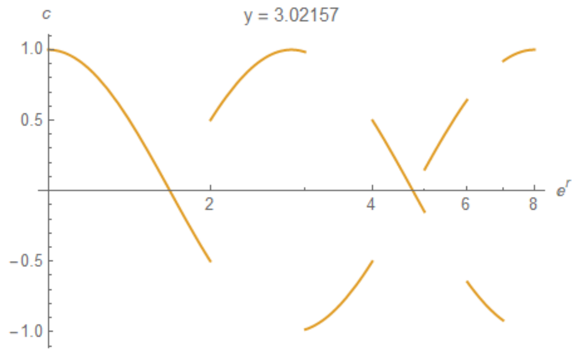
4.2 ±cos (y log r)

Let r,y are positive numbers respectively, and consider the following function $c(r,y)$.

$$c(r,y) = (-1)^{\lfloor r-1 \rfloor} \cos(y \log r) \qquad (\lfloor\ \rfloor \text{ is floor function}) \tag{4.2.1}$$

When $y = 3.02157$, the 2D figures for $r=1 \sim 8$ are drawn as follows. The left is normal scale and the right is semilogarithmic scale.





Unlike the previous section, $c(r,y)$ is a discontinuous function with respect to r .

Amplitude (A)

The amplitude of this function is

$$A(r) = \left|(-1)^{\lfloor r-1 \rfloor}\right| = 1$$

Period (P)

The period of this function is the same as in the previous section, That is,

$$P(n,y) = \left[e^{(2n-2)\pi/y} ,\ e^{2n\pi/y}\right)$$

In the figure above,

$$P(1,3.02157) = [1 ,\ 8)$$

Wavelength (λ)

The wavelength of this function is the same as in the previous section, That is,

$$\lambda(n,y) = e^{(2n-2)\pi/y}\left(e^{2\pi/y} - 1\right)$$

In the figure above,

$$\lambda(1,3.02157) = 7$$

Mountain or Valley (MVc)

Unlike the previous section, this function $c(r,y)$ changes sign. For this reason, mountains and valleys exist at most twice as many as in the previous section.

$$MVc(n,y) = \left\{ e^{\frac{(2n-2)\pi}{y}} ,\ e^{\frac{(2n-1)\pi}{y}} \right\}$$

The mountain or valley is determined by the sign of $c(r,y)$ at $r = MVc(n,y)$.
In the figure above,

$$MVc(1,3.02157) = \{1, 2.8284\}$$
$$\{c(1,3.02157), c(2.8284,3.02157)\} = (1,1)$$

So, both the former and the latter are mountains.

Zeros (Zc)

The zeros of this function are the same as in the previous section, That is,

$$Zc(n,y) = \left\{ e^{\frac{(4n-3)\pi}{2y}}, e^{\frac{(4n-1)\pi}{2y}} \right\}$$

In the figure above,

$$Zc(1,3.02157) = \{1.68179, 4.75684\}$$

Constriction (Xc)

Since this function $c(r,y)$ changes sign, the zero point looks like a constriction. So, we will call the integer r within ± 0.5 from the zero point **constriction**. That is,

$$Xc(n,y) = \left\{ Round\left(e^{\frac{(4n-3)\pi}{2y}} \right), Round\left(e^{\frac{(4n-1)\pi}{2y}} \right) \right\}$$

In the figure above,

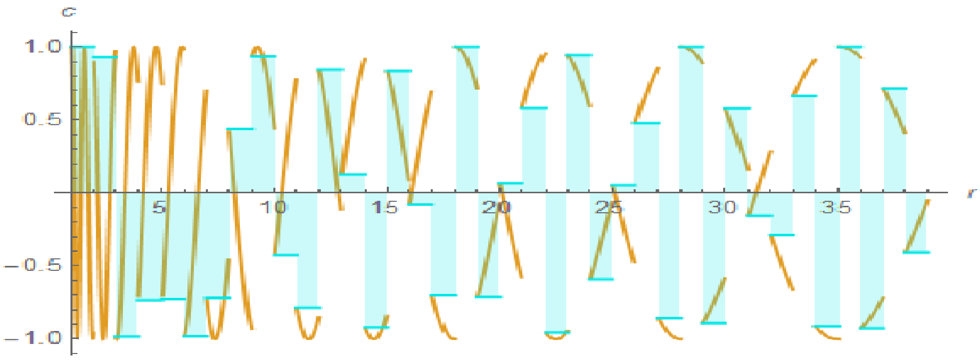
$$Xc(1,3.02157) = \{2,5\}$$

Dirichlet Eta type Cosine Series (when $y = 14.1347\cdots$)

Let us consider the following Dirichlet Eta type cosine function.

$$u(y) = \sum_{r=1}^{\infty} (-1)^{r-1} \cos(y \log r) \tag{4.2.2}$$

This is a series whose terms are $c(r,y)$ (4.2.1). For example, when $y = 14.1347\cdots$, $r = 1, 2, \cdots, 38$ $c(r,y)$ is drawn as follows.



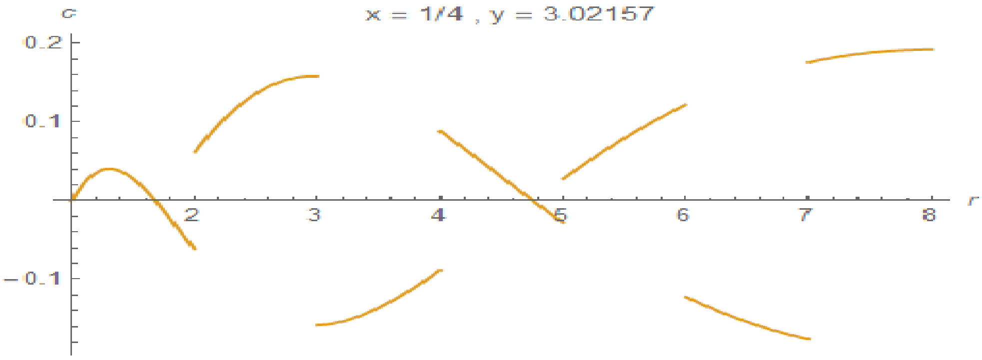
The sum of the areas of cyan is the function value of (4.2.2).
This series diverges, but converges if the summation method is applied. .

4.3 $u_s(x,y)$

Let r,x,y are positive numbers respectively, and consider the following function $c(r,x,y)$.

$$c(r,x,y) = (-1)^{\lfloor r-1 \rfloor} \frac{\sinh(x \log r)}{\sqrt{r}} \cos(y \log r) \qquad (\lfloor\ \rfloor \text{ is floor function}) \qquad (4.3.1)$$

When $x = 1/4, y = 3.02157$, the 2D figures for $r=1 \sim 8$ is drawn as follows.



Amplitude (A)

The amplitude of this function is

$$A(r,x) = \left| (-1)^{\lfloor r-1 \rfloor} \frac{\sinh(x \log r)}{\sqrt{r}} \right| = \frac{\sinh(x \log r)}{\sqrt{r}}$$

(1) When $0 < x < 1/2$, $0 \le A(r,x) < 1/2 \quad for \ r=2,3,4,\cdots$.

(2) When $x = 1/2$, $\lim_{r \to \infty} \sinh(x \log r)/\sqrt{r} = 1/2$.

Period (P)

The period of this function is the same as in the previous section, That is,

$$P(n,y) = \left[e^{(2n-2)\pi/y},\ e^{2n\pi/y} \right)$$

In the figure above,

$$P(1, 3.02157) = [1\ ,\ 8)$$

Wavelength (λ)

The wavelength of this function is the same as in the previous section, That is,

$$\lambda(n,y) = e^{(2n-2)\pi/y}\left(e^{2\pi/y} - 1\right)$$

In the figure above,

$$\lambda(1, 3.02157) = 7$$

Mountain or Valley (MVc)

The position of mountains or valleys in this function is slightly different from the previous section. That is,

$$MVc(n,y) = \left\{ e^{\frac{(2n-2)\pi}{y}},\ e^{\frac{(2n-1)\pi}{y}} \right\}$$

In the figure above,

$$MVc(1, 3.02157) = \{1,\ 2.8284\}$$

$$\left\{ c\left(1, \frac{1}{4}, 3.02157\right), c\left(2.8284, \frac{1}{4}, 3.02157\right) \right\} = \{0, 0.156302\}$$

Let 0 be not a mountain, and only positive number be mountain. This is an exception for the first period only.

Zeros (Zc)

The zeros of this function are the same as in the previous section, That is,

$$Zc(n,y) = \left\{ e^{\frac{(4n-3)\pi}{2y}}, e^{\frac{(4n-1)\pi}{2y}} \right\}$$

In the figure above,

$$Zc(1,3.02157) = \{1.68179, 4.75684\}$$

Constriction (Xc)

The constrictions of this function are the same as in the previous section, That is,

$$Xc(n,y) = \left\{ Round\left(e^{\frac{(4n-3)\pi}{2y}} \right), Round\left(e^{\frac{(4n-1)\pi}{2y}} \right) \right\}$$

In the figure above,

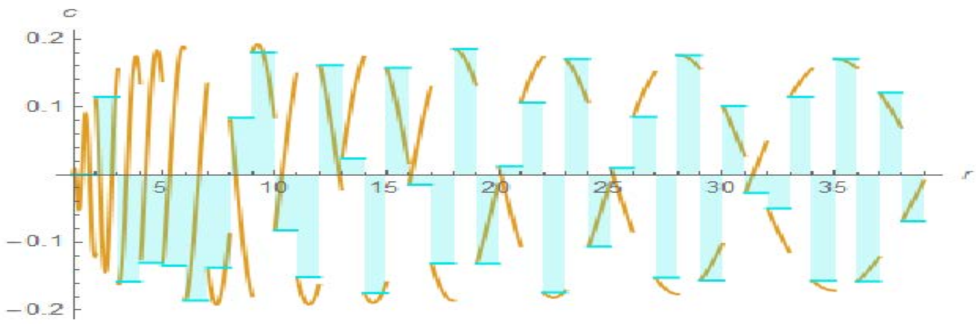
$$Xc(1,3.02157) = \{2,5\}$$

Coine Series $u_s(x,y)$ (when $x = 1/4$, $y = 14.1347\cdots$)

We consider the following sine series.

$$u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} sinh(x \log r) \cos(y \log r) \tag{2.4s}$$

This is a series whose terms are $c(r,x,y)$ (4.3.1) . For example, when $x=1/4$, $y=14.1347\cdots$ and $r=1,2,\cdots,38$, $c(r,y)$ is drawn as follows.



The sum of the areas of cyan is the function value of (2.4s) . When $0 < x < 1/2$, this series converges . When $x \geq 1/2$, this converges by applying the summation method.

4.4 Amplitude of $u_s(x,y)$ with respect to y

The cosine function and series dealt with in **4.3** were as follows.

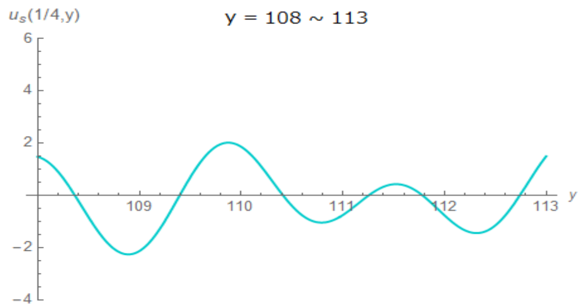
$$c(r,x,y) = (-1)^{\lfloor r-1 \rfloor} \frac{sinh(x \log r)}{\sqrt{r}} \cos(y \log r) \qquad (\lfloor\ \rfloor \text{ is floor function}) \tag{4.3.1}$$

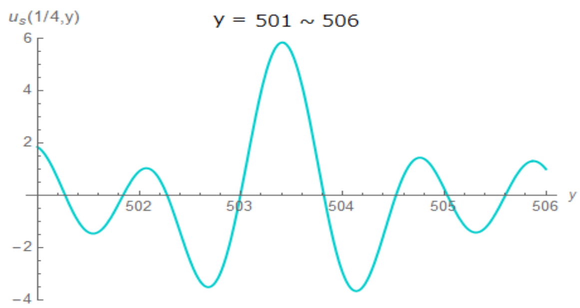
$$u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} sinh(x \log r) \cos(y \log r) \tag{2.4s}$$

$$= \frac{1}{2}\left[Re\left\{ \eta\left(\frac{1}{2} - x - i\,y \right) \right\} - Re\left\{ \eta\left(\frac{1}{2} + x + i\,y \right) \right\} \right] \tag{2.4s'}$$

In this section, we explore the amplitude (mountain, valley) of (2.4s) with respect to y using (4.3.1) .
To ensure accuracy, calculation and drawing are performed by (2.4s') .

Given x , $u_s(x,y)$ is a variable periodic function with respect to y . For example, when $x=1/4$, the 2D figures at $y=108 \sim 113$ and $y=501 \sim 506$ are drawn as follows.





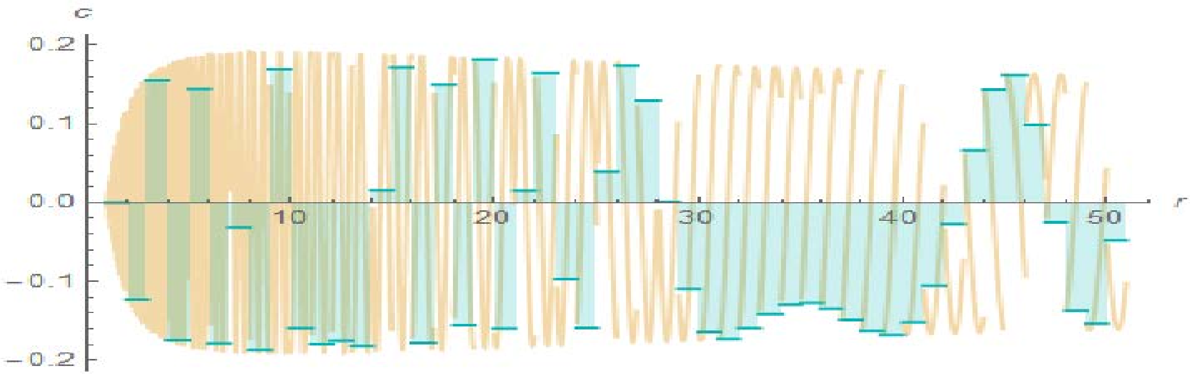
Although the value of y is larger in the right figure than in the left figure, it cannot be said that the valleys in the right are deeper than those in the left, or that the mountains in the right are heigher than those in the left. However, it can be said that the deepest valley in the right figure is deeper than the deepest one in the left figure, and the highest mountain in the right figure is higher than the highest one in the left figure. Below, this will be illustrated graphically.

4.4.1 Valley of $u_s(1/4, y)$ (near y= 108.9)

At $y=108 \sim 113$, the valley near here is the deepest. Accurate calculation of the valley near here by (2.4s') is as follows.

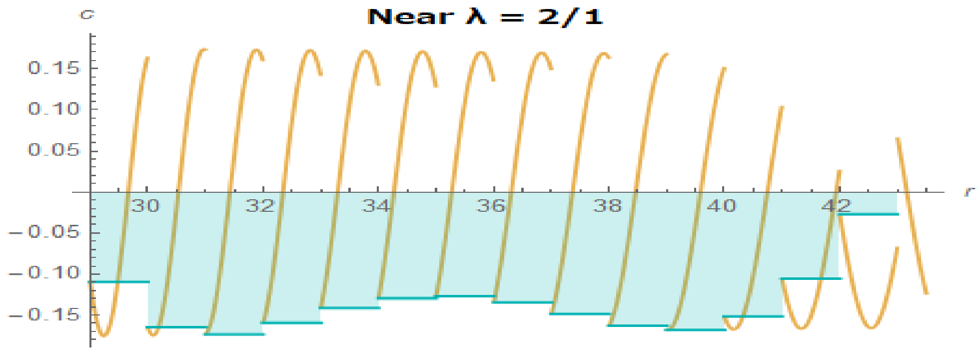
```
FindMinimum[us[1 / 4, y], {y, 108.9}]
{-2.25541, {y → 108.892}}          yv := 108.892
```

Drawing $c(r,1/4,y_v)$ by (4.3.1) is as follows. The horizontal axis is r . Orange is drawn as a continuous variable and cyan as a discrete variable. The sum of the area of cyan becoms valley -2.255 of (2.4s) .



Looking at this figure, it can be seen that there are 3 intervals with consecutive negative terms, which contribute to the depth of the valley.

(1) An enlarged view near $2/1$ wavelength is drawn as follows.



$r = 29 \sim 42$ are negative for 14 consecutive terms. Calculating the constriction that seems to be around here by trial and error,

```
{Xc[59, yv], Xc[65, yv]}
{{29, 30}  , {41, 42}}
```

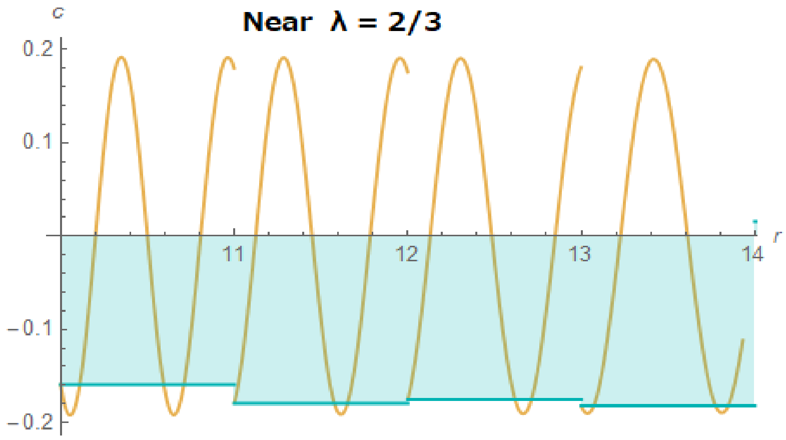
Then, $r = 29 \sim 42$ are included in the $59 \sim 65$ th period. The wavelengths of these periods are

```
Table[λ[n, yv], {n, 59, 65}]
{1.68739, 1.78762, 1.8938, 2.00629, 2.12546, 2.25171, 2.38545}
```

That is, the wavelengths of $r = 29 \sim 42$ are $1.69 \sim 2.39$

Here after, only the calculation results are described.

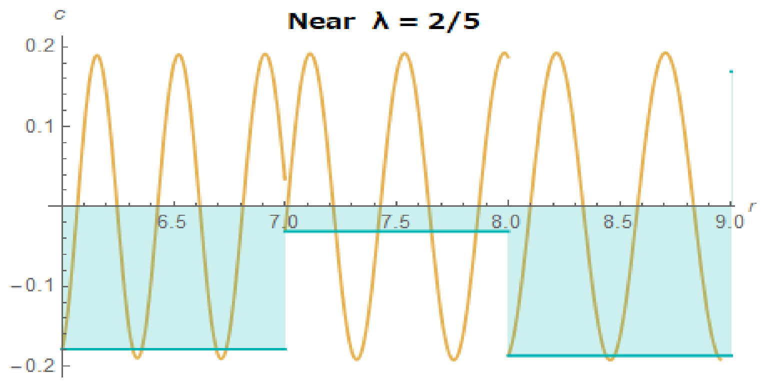
(2) An enlarged view near $2/3(=0.67)$ wavelength is drawn as follows.



$r = 10 \sim 13$ are negative for 4 consecutive terms.

These are included in the $41 \sim 45$ th period and the wavelengths of the periods are $0.63 \sim 0.71$.

(3) An enlarged view near $2/5(=0.4)$ wavelength is drawn as follows.



$r = 6 \sim 8$ are negative for 3 consecutive terms.

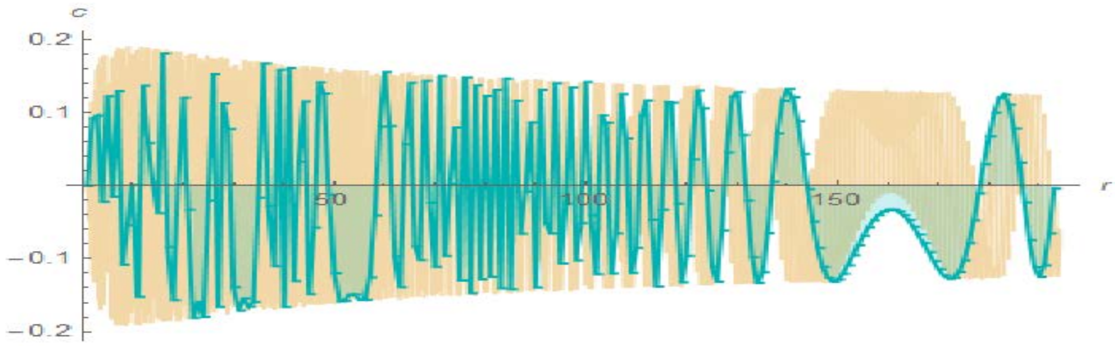
These are included in the $33 \sim 36$ th period and the wavelengths of the periods are $0.38 \sim 0.42$.

4.4.2 Valley of $u_s(1/4, y)$ (near y= 504.1)

At $y = 501 \sim 506$, the valley near here is the deepest. Accurate calculation of the valley near here by (2.4s') is as follows.

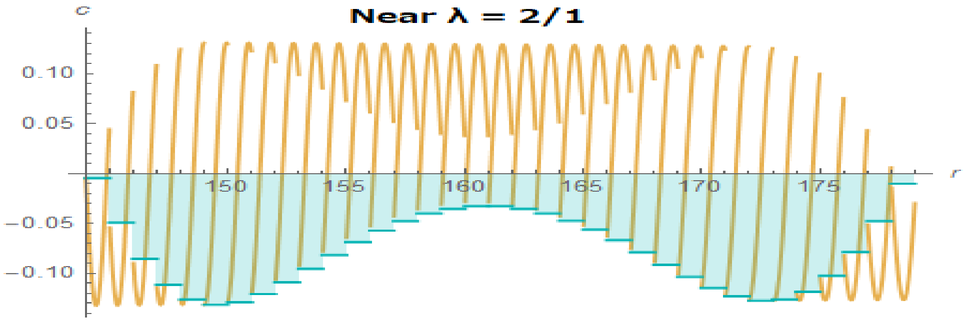
```
FindMinimum[us[1 / 4, y], {y, 504.1}]
{-3.65549, {y → 504.135}}        yv := 504.135
```

Drawing $c(r, 1/4, y_v)$ by (4.3.1) is as follows. The horizontal axis is r . Orange is drawn as a continuous variable and cyan as a discrete variable. The sum of the area of cyan becoms valley -3.655 of (2.4s) .



Looking at this figure, it can be seen that there are 4 intervals with consecutive negative terms, which contribute to the depth of the valley.

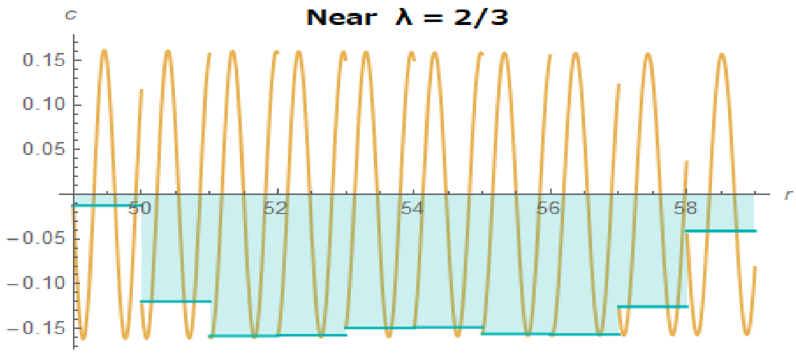
(1) An enlarged view near $2/1$ wavelength is drawn as follows.



$r = 144 \sim 178$ are negative for 35 consecutive terms. This interval is 2.5 times one of **4.4.1 (1)** .

These are included in the $399 \sim 416$ th period and the wavelengths of the periods are $1.79 \sim 2.21$.

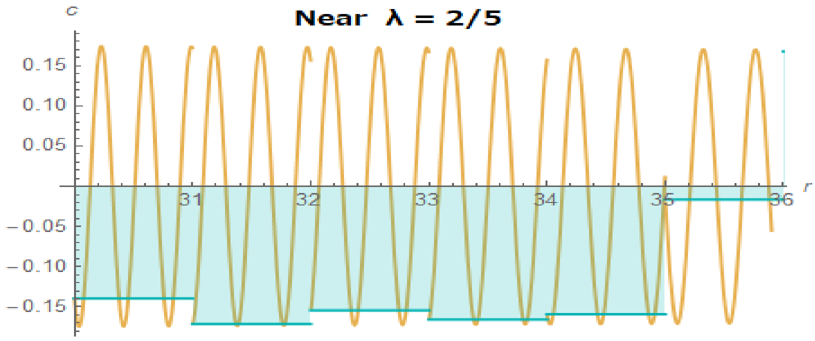
(2) An enlarged view near $2/3(=0.67)$ wavelength is drawn as follows.



$r = 49 \sim 58$ are negative for 10 consecutive terms. This interval is 2.5 times one of **4.4.1 (2)** .

These are included in the $312 \sim 326$ th period and the wavelengths of the periods are $0.60 \sim 0.71$.

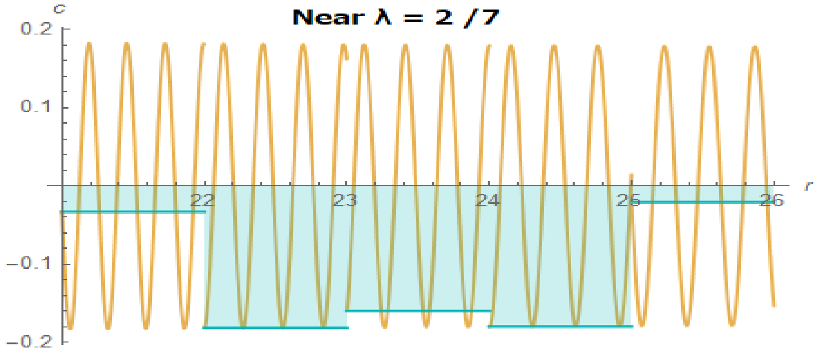
(3) An enlarged view near $2/5(=0.40)$ wavelength is drawn as follows.



$r = 30 \sim 35$ are negative for 6 consecutive terms. This interval is 2 times one of **4.4.1 (3)** .

These are included in the $272 \sim 285$ th period and the wavelengths of the periods are $0.37 \sim 0.43$.

(4) An enlarged view near $2/7\,(=0.294)$ wavelength is drawn as follows.



$r = 21 \sim 25$ are negative for 5 consecutive terms. This interval is absent in **4.4.1** .

These are included in the $246 \sim 257$ th period and the wavelengths of the periods are $0.27 \sim 0.30$.

4.4.3 Depth of Valleys near $y=504.1$ and $y=108.9$

The valley near $y=504.1$ is deeper than the one near $y=108.9$. Because,

(1) The former is $2 \sim 2.5$ times longer than the latter in the interval with consective negative terms near wavelength $2/1$, $2/3$, $2/5$.

(2) The interval with consective negative terms near wavelength $2/7$ is added to the former.

These causes are as seen in **3.4.3** .

4.4.4 Height of mountains near $y=503.4$ and $y=109.9$

The mountain near $y=503.4$ is higher than the one near $y=109.9$. because, **4.4.1** ~ **4.4.3** also hold for mountain.

From 4.4.3 and 4.4.4 , we conclude that the amplitude at $y=501 \sim 506$ is greater than that at $y=108 \sim 113$. This can be described more generally as follows.

Law 4.4.5

Let x, y are real numbers and function $u_s(x,y)$ be as follows.

$$u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x \log r) \cos(y \log r) \tag{2.4s}$$

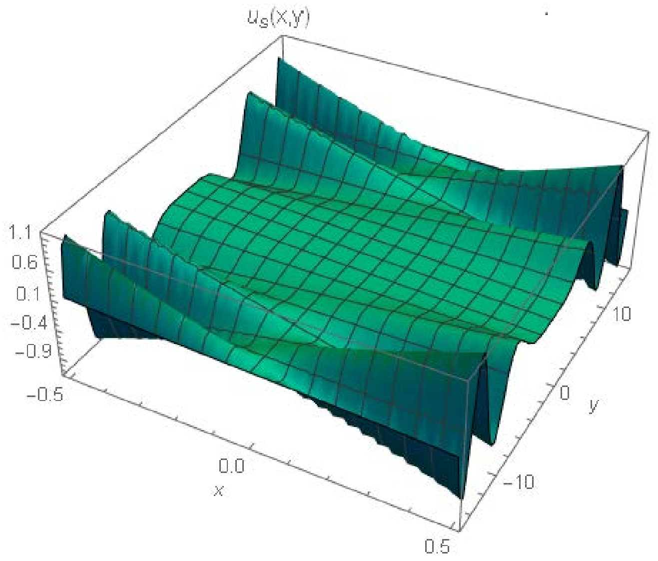
Then, given x , the amplitude of $u_s(x,y)$ is generally proportional to the absolute value of y .

Note

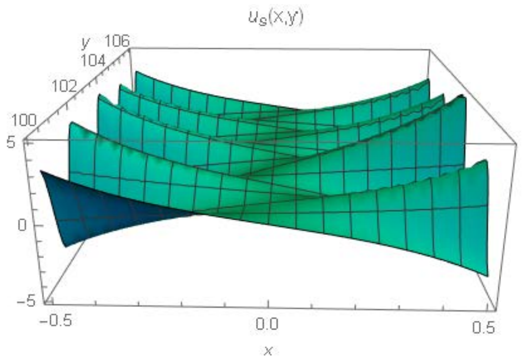
This does not hold as a theorem, but holds as a law. Because, it depends on the change of the slope y/n of the contour line at wavelength λ and the shift of the contour line. This law is similar to ***Bergmann's Law*** (Bears in high latitudes are generally larger than bears in low latitudes.).

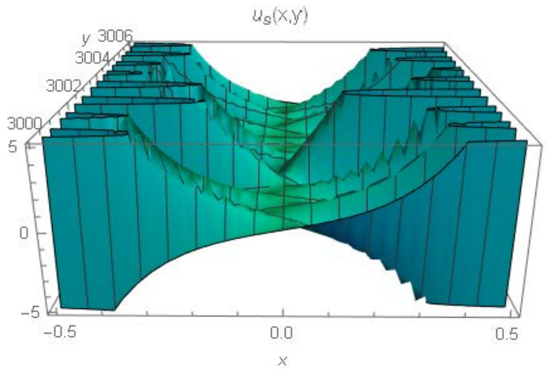
4.5 Shape and Properties of $u_s(x,y)$

From (2.4s) , $u_s(x,y)$ is an odd function with respect to x and an even function with respect to y . This shows that $u_s(x,y)$ is point symmetric with respect to x and line symmetric with respect to y .



Next, when $-1/2 \le x \le 1/2$, the 3D view of $u_s(x,y)$ at $y = 100 \sim 107$ and $y = 3000 \sim 3007$ are drawn respectively as follows.

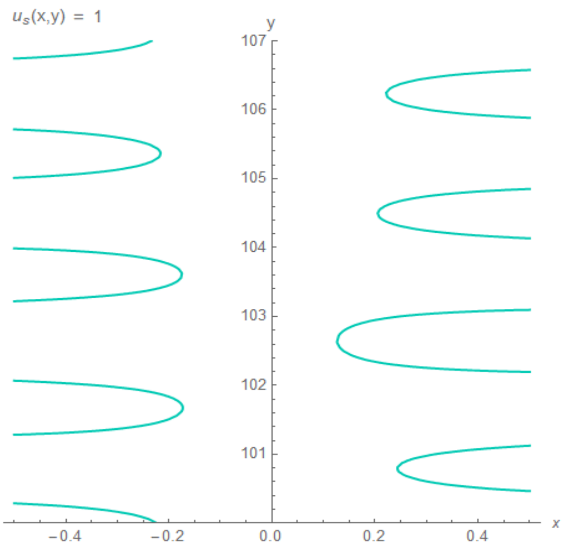


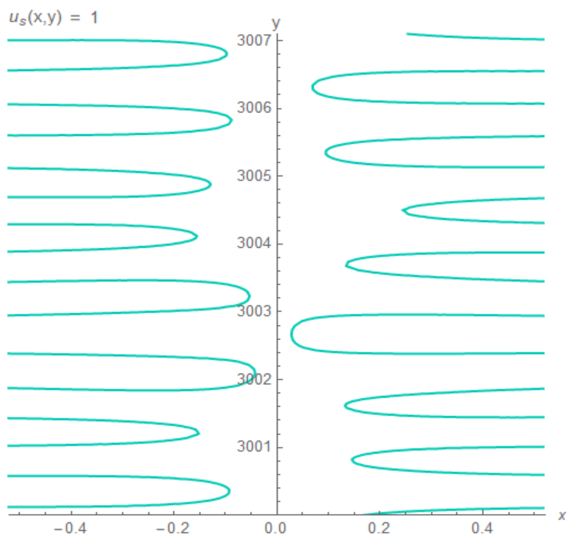


In both figures, the upper part looks like $\cup$ and the lower part looks like $\cap$. Then, we can see that both $\cup$ and $\cap$ generally have larger curvatures in the right figure than in the left figure. This is because mountains and valleys are steeper in the right figure than in the left figure according to Law 4.4.5 . In addition, the right figure has more mountains and valleys than the left figure (about twice as many), but the reason for this is unknown.

4.6 Contour line of $v_c(x,y)$ with height 1

The height 1 contours of the two 3D views of $u_s(x,y)$ in the previous section are drawn as follows.





The left figure is $y = 100 \sim 107$ and the right figure is $y = 3000 \sim 3007$.

In both figures, the contour line looks like $\supset$ & $\subset$. Then, we can see that both $\supset$ & $\subset$ are generally closer to the y-axis in the right figure than in the left figure. This is because mountains and valleys are generally steeper in the right figure than in the left figure according to Law 4.4.5 . Therefore, as $|y|$ increases, the tips $\supset$ $\subset$ of the contour approach the y -axis from both sides.

5 Contour Lines of $v_c(x,y)$, $u_s(x,y)$ and the Transitions

5.1 Contour Lines of $v_c(x,y)$, $u_s(x,y)$

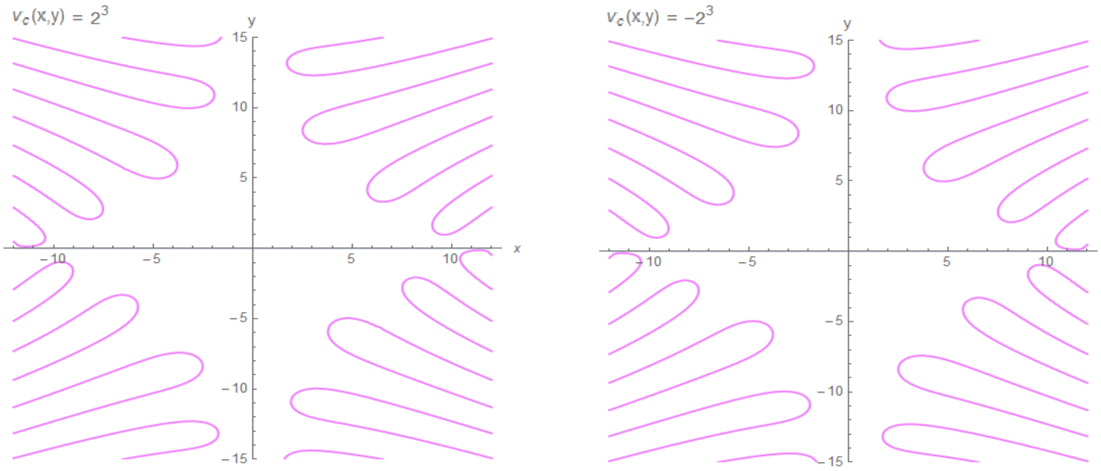
The functions $v_c(x,y)$, $u_s(x,y)$ of Propositions 2.4 were as follows, respectively.

$$v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x \log r) \sin(y \log r) \tag{2.4c}$$

$$u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x \log r) \cos(y \log r) \tag{2.4s}$$

5.1.1 Contour line of $v_c(x,y)$ with height ± 8

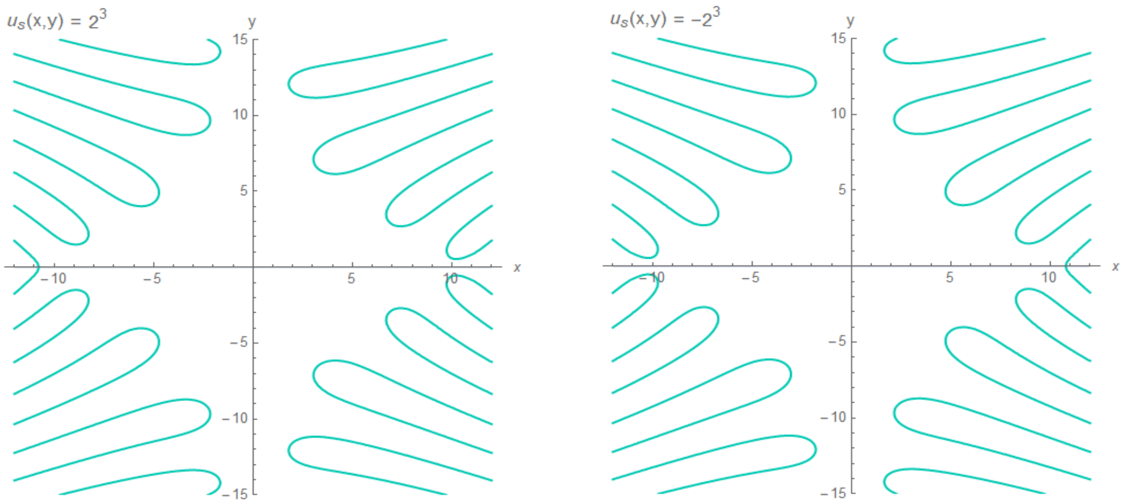
Contour plots of $v_c(x,y)$ at height ± 8 are as follows. The left figure is $+8$ and the right figure is -8 .



Since $v_c(x,y)$ is an odd function with respect to both x and y , the left and right figures have a mirror image relationship with respect to both the y -axis and the x -axis.

5.1.2 Contour line of $u_s(x,y)$ with height ± 8

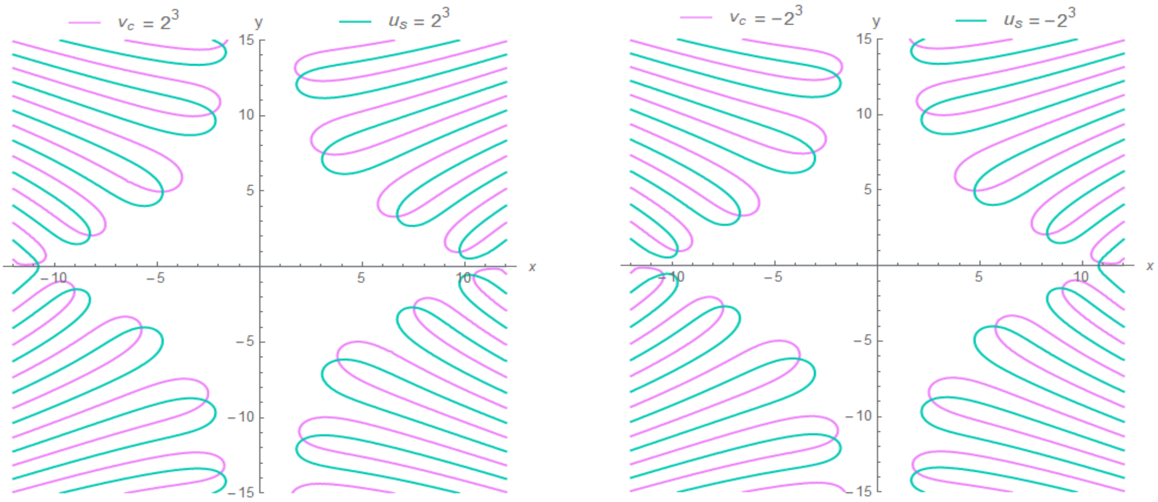
Contour plots of $u_s(x,y)$ at height ± 8 are as follows. The left figure is $+8$ and the right figure is -8 .



Since $u_s(x,y)$ is an odd function with respect to x the left and right figures have a mirror image relationship with respect to the y -axis

5.1.3 Contour lines of $v_c(x,y)$, $u_s(x,y)$ with height ± 8

When **5.1.1** and **5.1.2** are overlapped, it becomes as follows.



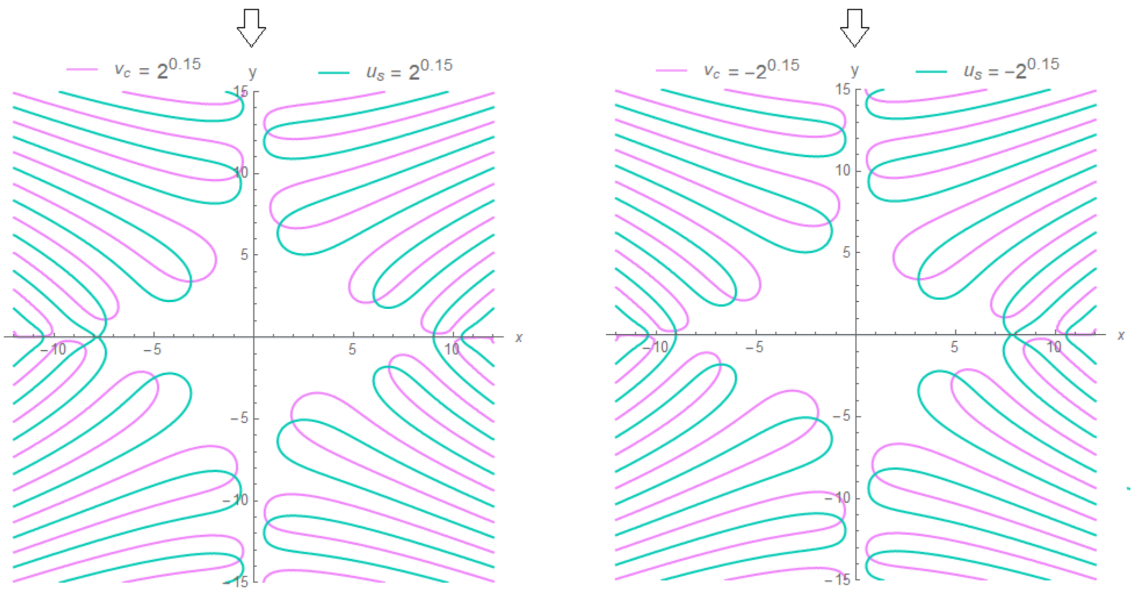
Since $v_c(x,y)$,$u_s(x,y)$ are odd functions with respect to x the left and right figures have a mirror image relationship with respect to the y -axis. Both figures never overlap by translation or rotation in the plane.

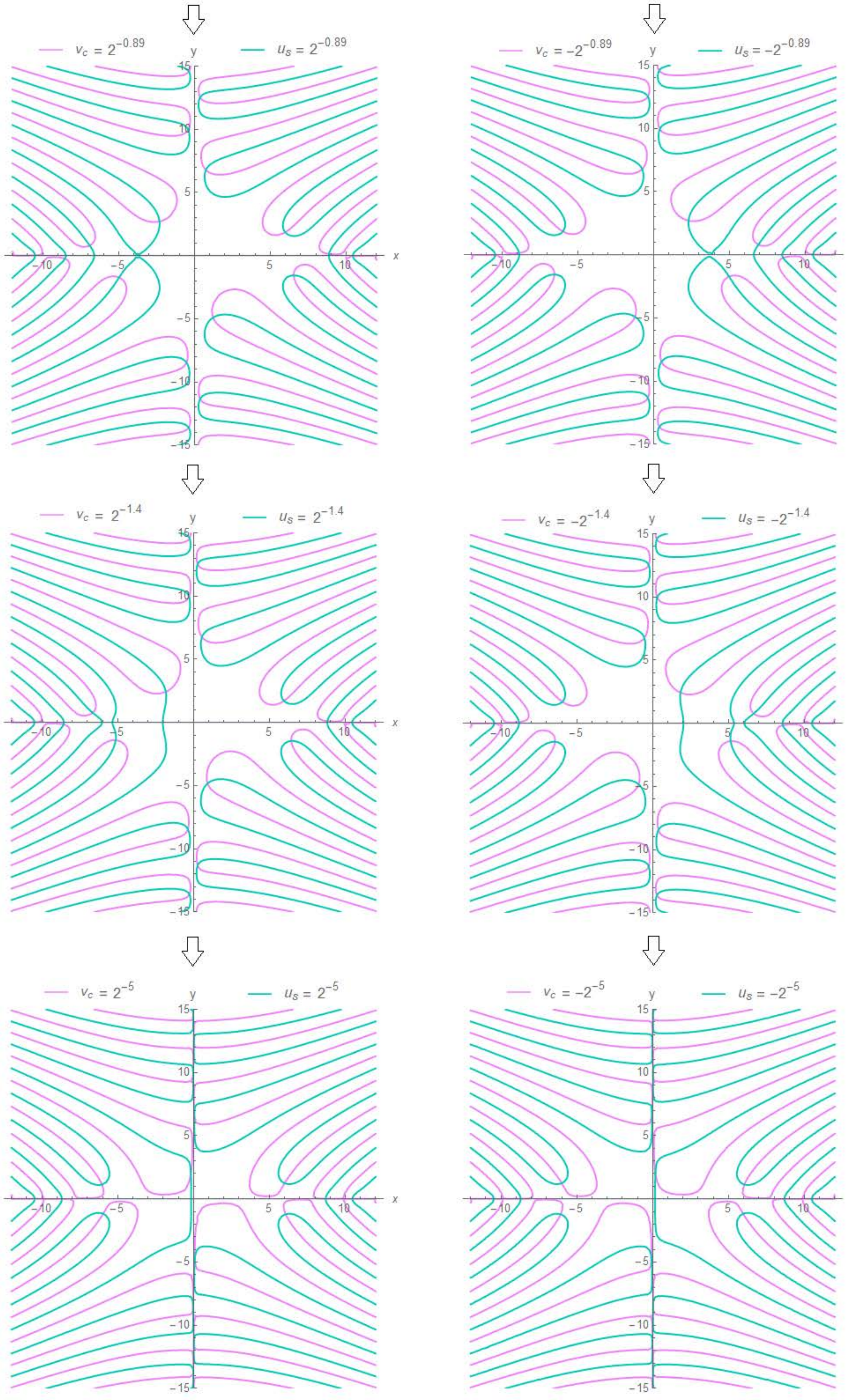
5.2 Transitions of contour lines of $v_c(x,y)$, $u_s(x,y)$

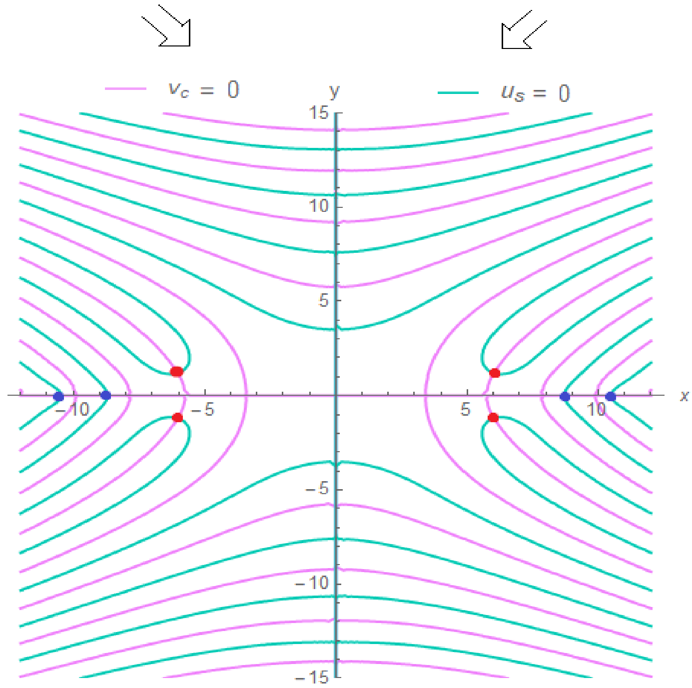
Nevertheless, at height ± 0 , the left and right figures have to overlap with no translation or rotation. For the purpose, the contour lines in both figures have to be deformed as the height approaches ± 0 from above and below. And, at height ± 0 , both figures must be symmetrical about both the y -axis and the x -axis.

This forces contour lines that were alternate at height $\neq 0$ to be opposite at height ± 0 . This also applies to the x -axis. Thus, At height ± 0 , the right and left edges of $\supset$ $\subset$ must be absorbed into the y -axis, and the lower and upper edges of $\cup$ $\cap$ must be absorbed into the x -axis.

In fact, when the height is changed to $\pm 2^{0.15}$, $\pm 2^{0.15}$, $\pm 2^{-0.89}$, $\pm 2^{-1.4}$, $\pm 2^{-5}$, the above figures are deformed as follows.




$v_c = 2^{-0.89}$
$u_s = 2^{-0.89}$
$v_c = -2^{-0.89}$
$u_s = -2^{-0.89}$
$v_c = 2^{-1.4}$
$u_s = 2^{-1.4}$
$v_c = -2^{-1.4}$
$u_s = -2^{-1.4}$
$v_c = 2^{-5}$
$u_s = 2^{-5}$
$v_c = -2^{-5}$
$u_s = -2^{-5}$



For the animation from $v_c = u_s = \pm 1$ to $v_c = u_s = \pm 0$, click here. AnimZ5219.gif

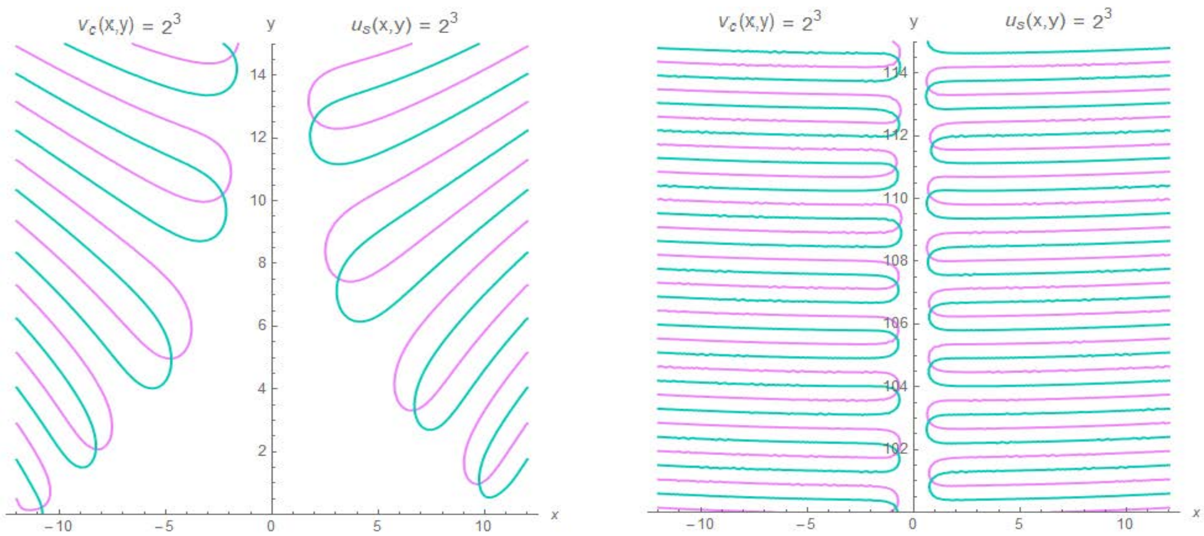
Consistent with the theory, the contour parts asymmetric with respect to the y and x -axis were absorbed in both axes. As the result,

(1) Trivial solutions $(\pm 8.69593 , 0)$, $(\pm 10.4734 , 0)$, ⋯ (blue point) of $v_c = u_s = 0$ arose countless on the x -axis. However, they do not satisfy $u_c = 0$

(2) Non-trivial solutions $(\pm 6.01956 , \pm 1.19483)$ (red point) of $v_c = u_s = 0$ remained. However, they are not in the critical strip, and do not satisfy $v_s = u_c = 0$.

(3) All solutions (intersections of v_c & u_s) except (1) and (2) moved on the y -axis.

The figures above are for $|y| \le 15$, but what about when $|y|$ is large ? As an example, drawing contour lines of height 8 of v_c & u_s for $y = 0 \sim 15$ and $y = 100 \sim 115$ is as follows. The left figure is $0 \sim 15$ and the right figure is $100 \sim 115$



It is observed that both $\supset$ and $\subset$ are generally closer to the y -axis in the right figure than in the left figure. As stated in the previous two chapters, this is due to Law 3.4.5 and Law 4.4.5 ..

Both figures show that the above phenomenon (3) becomes more pronounced where $|y|$ is large. That is, The above (3) occurs in the whole domain $|y| > 1.19483\cdots$.

So, the system of equations $v_c(x,y) = u_s(x,y) = 0$ has no solution in the critical strip $-1/2 < x < 1/2$ except on the critical line $x = 0$.

Note

$x = 0$ is equivalent to the absence of v_c and u_s .

6 Proof of the Riemann Hypothesis

In this chapter, we will prove the Riemann hypothesis by organizing and summarizing the above.

Proposition 6.1 (Riemann Hypothesis)

Let $\zeta(z)$ be the function defined by the following Dirichlet series.

$$\zeta(z) = \sum_{r=1}^{\infty} e^{-z\log r} = \frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\cdots \qquad Re(z)>1 \qquad (1.\zeta)$$

This function has no non-trivial zeros except on the critical line $Re(z)=1/2$.

Proof

Dirichlet Eta Function $\eta(z)$ is defined by the following Dirichlet series.

$$\eta(z) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-z\log r} = \frac{1}{1^z}-\frac{1}{2^z}+\frac{1}{3^z}-\frac{1}{4^z}+-\cdots \qquad Re(z)>0 \qquad (1.\eta)$$

This function is analytically continued to $Re(z) \le 0$, and has the following relation to $\zeta(z)$.

$$\zeta(z) = \frac{1}{1-2^{1-z}}\eta(z) \qquad z \ne 1$$

Therefore, the non-trivial zeros of $\zeta(z)$ and $\eta(z)$ coincide in the critical strip $0<Re(z)<1$.

First, by functional equation, the solution for $\eta(z)=0$ is consistent with the solution of the following system of equations. (Lemma 2.1)

$$\begin{cases} \eta(z) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-z\log r} = 0 \\ \eta(1-z) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-(1-z)\log r} = 0 \end{cases} \qquad 0<Re(z)<1$$

Second, by translation, the solution for $\eta(1/2+z)=0$ is consistent with the solution of the following system of equations. (Lemma 2.1 ')

$$\begin{cases} \eta\left(\frac{1}{2}+z\right) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} e^{-z\log r} = 0 \\ \eta\left(\frac{1}{2}-z\right) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} e^{z\log r} = 0 \end{cases} \qquad -\frac{1}{2}<Re(z)<\frac{1}{2}$$

Third, by addition and subtraction, the solution for $\eta(1/2+z)=0$ is consistent with the solution of the following system of equations. (Lemma 2.2)

$$\begin{cases} \eta_c(z) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \cosh(z\log r) = 0 \\ \eta_s(z) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(z\log r) = 0 \end{cases} \qquad -\frac{1}{2}<Re(z)<\frac{1}{2}$$

Last, expressing these by real and imaginary parts, we obtain the following theorem.

Theorem 2.3 (reprint)

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy,\ x,y \in R)$, $\eta(1/2 \pm z)=0$ in $-1/2<x<1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} u_c(x,y) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{r}} \cosh(x\log r)\cos(y\log r) = 0 \\ v_c(x,y) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{r}} \sinh(x\log r)\sin(y\log r) = 0 \\ u_s(x,y) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{r}} \sinh(x\log r)\cos(y\log r) = 0 \\ v_s(x,y) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{r}} \cosh(x\log r)\sin(y\log r) = 0 \end{cases}$$

According to this theorem, if a system of equations consisting of any two of these equations does not have a solution in the critical strip except on the critical line, the Riemann hypothesis holds. Therefore, the following proposition equivalent to the Riemann hypothesis can be presented.

Proposition 2.4 (reprint)

When y is a real number, x is a real number s.t. $-1/2 < x < 1/2$, the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} v_c(x,y) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{r}} \sinh(x\log r)\sin(y\log r) = 0 & (2.4c) \\ u_s(x,y) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{r}} \sinh(x\log r)\cos(y\log r) = 0 & (2.4s) \end{cases}$$

From **3.1** to **5.2**, as evidenced mainly by figures, this system of equations has only four non-trivial solutions $(\pm 6.01956, \pm 1.19483)$. In other words, this system of equations has no solution in the critical strip $(-1/2 < x < 1/2)$ except for the critical line $(x=0)$.

Thus since Proposition 2.4 has been graphically proved, according to Theorem 2.3 , the Riemann hypothesis holds. Q.E.D.

Appendix

The 2 functions $v_c(x,y), u_s(x,y)$ that are central to this paper are expressed by the following formulas using the Dirichlet Eta function $\eta(x,y)$.

$$v_c(x,y) = \frac{1}{2}\left[Im\left\{ \eta\left(\frac{1}{2}-x-iy \right) \right\} + Im\left\{ \eta\left(\frac{1}{2}+x+iy \right) \right\} \right] \tag{2.4c'}$$

$$u_s(x,y) = \frac{1}{2}\left[Re\left\{ \eta\left(\frac{1}{2}-x-iy \right) \right\} - Re\left\{ \eta\left(\frac{1}{2}+x+iy \right) \right\} \right] \tag{2.4s'}$$

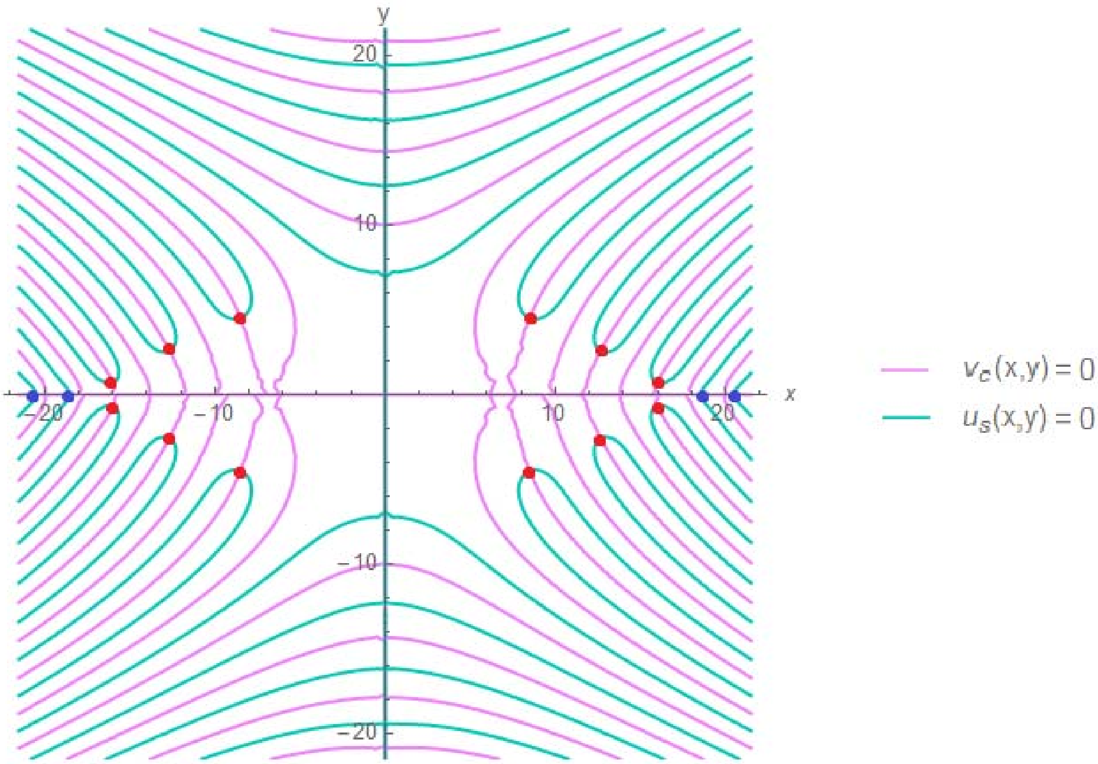
1 For $\zeta(x,y)$

The discussion in this paper is valid even if the functions in these formulas are replaced by the Riemann zeta function $\zeta(x,y)$. That is,

$$v_c(x,y) = \frac{1}{2}\left[Im\left\{ \zeta\left(\frac{1}{2}-x-iy \right) \right\} + Im\left\{ \zeta\left(\frac{1}{2}+x+iy \right) \right\} \right] \tag{ζ.c'}$$

$$u_s(x,y) = \frac{1}{2}\left[Re\left\{ \zeta\left(\frac{1}{2}-x-iy \right) \right\} - Re\left\{ \zeta\left(\frac{1}{2}+x+iy \right) \right\} \right] \tag{ζ.s'}$$

Using these, the contour lines of $v_c(x,y), u_s(x,y)$ at height 0 were drawn as follows.



Blue points are the trivial solutions. These exist innumerably on the x -axis as $(\pm 18.5678, 0)$, $(\pm 20.4924, 0)$, $\cdots$.

Red points are non-trivial solutions of $v_c = u_s = 0$. They are 12 in 3 sets of $(\pm 8.49059, \pm 4.51058)$, $(\pm 12.6627, \pm 2.58053)$, $(\pm 15.9781, \pm 0.679408)$. These exist near the boundary between hyperbola and parabola, that is, around the origin. So, There are no non-trivial solutions other than these 12.

Since these are outside the critical strip, the Riemann Hypothesis must hold.

2 For $\beta(x,y)$

The discussion in this paper is valid even if the functions in these formulas are replaced by the Dirichlet beta function $\beta(x,y)$. That is,

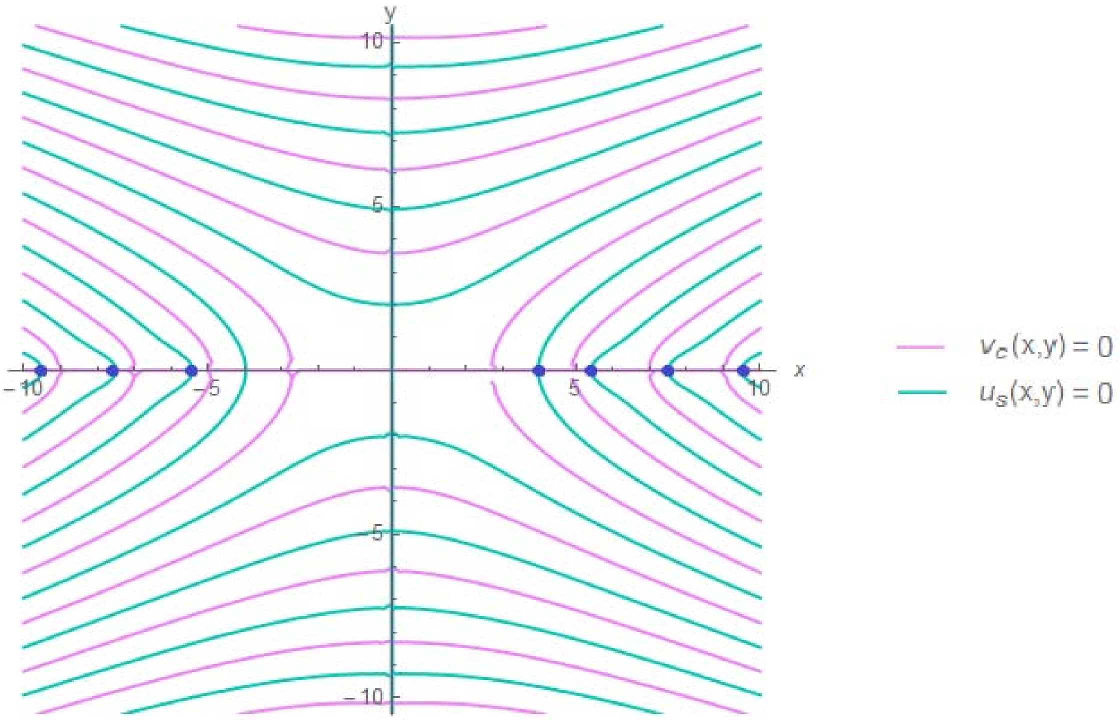
$$v_c(x,y) = \frac{1}{2}\left[Im\left\{\beta\left(\frac{1}{2}-x-iy\right)\right\} + Im\left\{\beta\left(\frac{1}{2}+x+iy\right)\right\}\right] \qquad (\beta.c')$$

$$u_s(x,y) = \frac{1}{2}\left[Re\left\{\beta\left(\frac{1}{2}-x-iy\right)\right\} - Re\left\{\beta\left(\frac{1}{2}+x+iy\right)\right\}\right] \qquad (\beta.c')$$

Where,

$$\beta(z) = \sum_{r=1}^{\infty}(-1)^{r-1}e^{-z\log(2r+1)} = \frac{1}{1^z}-\frac{1}{3^z}+\frac{1}{5^z}-\frac{1}{7^z}+- \cdots \qquad Re(z)>0$$

Using these, the contour lines of $v_c(x,y)$, $u_s(x,y)$ at height 0 were drawn as follows.



Blue points are the trivial solutions. These exist innumerably on the x-axis as $(\pm 3.970898, 0)$, $(\pm 5.410623, 0)$, $\cdots$.

There is no non-trivial solution for $v_c = u_s = 0$ near the boundary between hyperbola and parabola. They do not even exist around the y-axis. Therefore, the Riemann hypothesis also holds for the Dirichlet beta function.

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