Graphical Proof of the Riemann Hypothesis for the Dirichlet Beta Function

Abstract

(1) The problem of Zeros of the Dirichlet Beta function is reduced to the system of transcendental equations consisting of 4 equations with 2 real variables, by functional equation.

(2) On the critical line, certain 2 equations are identically 0, and the remaining 2 equations have simultaneous solutions.

(3) Except on the critical line, the two equations do not have simultaneous solutions in the critical strip. This can be illustrated by transition diagrams from above and below the contour line. And such transitions are more pronounced where the imaginary part of the variable is large.

(4) As a result of (3), the system of transcendental equations of (1) have no solution in the critical strip except on the critical line. Thus, the Riemann Hypothesis for the Dirichlet Beta Function holds true.

1 Introduction

Dirichle Beta Function

Dirichle Beta Function $\beta(z)$ is defined by the following Dirichlet series.

$$\beta(z) = \sum_{r=1}^{\infty} e^{-z\log(2r-1)} = \frac{1}{1^z} - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + - \cdots \qquad Re(z) > 1 \qquad (1.\beta)$$

This function is analytically continued to $Re(z) < 1$, and has trivial zeros $z = -(2n-1)$ (n = 1, 2, 3 , ···) and **non-trivial zeros** $z = 1/2 \pm b_n$ $(n=1,2,3,\cdots)$. So, it is the Riemann hypothesis for the Dirichlet Beta Function that there will be no non-trivial zeros other than these.

In addition, it is known that non-trivial zeros exist only in the **critical strip $0 < Re(z) < 1$** . Also, the center line **$Re(z) = 1/2$** is called the **critical line** .

2 Zeros of $\beta(z)$ and System of Equations

In this chapter, we consider the problem of zeros of the Dirichlet Beta function $\beta(z)$ from the point of view of the system of equations.

Lemma 2.1

When the set of real numbers is R and Dirichlet Beta functions is $\beta(z)$ $(z=x+iy\ ,\ x,y\in R)$, $\beta(z)=0$ in $0<x<1$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \beta(z) = \sum_{r=1}^{\infty}(-1)^{r-1}e^{-z\,log(2r-1)} = 0 & (2.1_+) \\ \beta(1-z) = \sum_{r=1}^{\infty}(-1)^{r-1}e^{-(1-z)\,log(2r-1)} = 0 & (2.1_-) \end{cases}$$

Proof

The following functional equation holds for the Dirichlet Beta function $\beta(z)$.

$$\beta(z) = \left(\frac{2}{\pi}\right)^{1-z} cos\frac{\pi z}{2}\Gamma(1-z)\beta(1-z) \qquad z\neq 1,2,3,\cdots$$

Here, gamma function and powers of $2/\pi$ have no zeros. Also, since the zero of $cos(\pi z/2)$ is $z = \pm 1\ ,\ \pm 3\ ,\ \pm 5\ ,\ \cdots$, $cos(\pi z/2)$ has no zero in the $0<Re(z)<1$.

Therefore, at the zeros of $\beta(z)$, the following expressions have to hold.

$$\beta(z) = \beta(1-z) = 0 \qquad 0<Re(z)<1$$

Representing $\beta(z)\ ,\beta(1-z)$ by the Dirichlet series respectively, we obtain the desired expressions.

Note1

Since there are 2 equations for 1 complex variable in the lemma, this system of equations is an overdetermined system. Such a system of equations generally has no solution. What forces this overdetermined system is the functional equation clearly.

Note2

(1) When $x=1/2$, the overdetermined property disappears. Because,

$$\begin{cases} \beta(1/2+iy) = \sum_{r=1}^{\infty}(-1)^{r-1}e^{-(1/2+iy)\,log(2r-1)} = 0 & (2.1_+) \\ \beta(1/2-iy) = \sum_{r=1}^{\infty}(-1)^{r-1}e^{-(1/2-iy)\,log(2r-1)} = 0 & (2.1_-) \end{cases}$$

i.e.

$$\begin{cases} \beta(1/2+iy) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{2r-1}}\left[cos\{y\,log(2r-1)\} - i\,sin\{y\,log(2r-1)\}\right] = 0 \\ \beta(1/2-iy) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{2r-1}}\left[cos\{y\,log(2r-1)\} + i\,sin\{y\,log(2r-1)\}\right] = 0 \end{cases}$$

At zero point $(1/2\ ,y)$,

$$-\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{2r-1}}sin\{y\,log(2r-1)\} = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{2r-1}}sin\{y\,log(2r-1)\} = 0$$

So, (2.1_+) and (2.1_-) become substantially the same equations.

(2) When $x \neq 1/2$, This system of equations is an overdetermined system.

Even though (2.1_+) and (2.1_+) are different equations, they must share one complex number. The Riemann hypothesis says that such a thing will not happen.

Replacing z with $1/2+z$, we obtain the following equivalent lemma.

Lemma 2.1'

When the set of real numbers is R and Dirichlet Beta function is $\beta(z)$ $(z = x+iy ,\ x,y \in R)$,
$\beta(1/2 \pm z) = 0$ in $-1/2 < x < 1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \beta\left(\dfrac{1}{2}+z\right) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} e^{-z\log(2r-1)} = 0 & (2.1'_+) \\ \beta\left(\dfrac{1}{2}-z\right) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} e^{z\log(2r-1)} = 0 & (2.1'_-) \end{cases}$$

Note

(1) The overdetermined property is the same as in Lemma 2.1.

(2) The known non-trivial zeros are moved parallel onto the **new critical line** $\boldsymbol{Re(z) = 0}$

(3) When $x = 0$, the overdetermined property disappears.

(4) When $x \neq 0$, if there are zeros, the set consists of the following four.

$$a \pm ib \ , \quad -a \pm ib \qquad (-1/2 < a < 1/2)$$

Hyperbolic Function Series

Lemma 2.1' is equivalent to the following

Lemma 2.2

When the set of real numbers is R and Dirichlet Beta function is $\beta(z)$ $(z = x+iy ,\ x,y \in R)$,
$\beta(1/2 \pm z) = 0$ in $-1/2 < x < 1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \beta_c(z) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \cosh\{z\log(2r-1)\} = 0 & (2.2c) \\ \beta_s(z) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{z\log(2r-1)\} = 0 & (2.2s) \end{cases}$$

Proof

From (2.1'+) , (2.1'-) ,

$$\begin{aligned} \frac{1}{2}\left\{\beta\left(\frac{1}{2}-z\right)+\beta\left(\frac{1}{2}+z\right)\right\} &= \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \frac{e^{z\log(2r-1)}+e^{-z\log(2r-1)}}{2} \\ &= \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \cosh\{z\log(2r-1)\} = 0 \end{aligned}$$

$$\frac{1}{2}\left\{\beta\left(\frac{1}{2}-z\right)-\beta\left(\frac{1}{2}+z\right)\right\} = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{2r-1}}\frac{e^{z\log(2r-1)}-e^{-z\log(2r-1)}}{2}$$

$$= \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{2r-1}}\sinh\{z\log(2r-1)\} = 0$$

Describing these as $\beta_c(z), \beta_s(z)$ respectively, we obtain the desired expressions.

Conversely, by adding or subtracting these, (2.1'+) , (2.1'-) are obtained.

Note

$\beta_c(z), \beta_s(z)$ are the sum and difference between Dirichlet series. Therefore, their convergence region is $-1/2 < x < 1/2$.

Hyperbolic Function Series (real part, imaginary part)

Theorem 2.3

When the set of real numbers is R and Dirichlet Beta function is $\beta(z)$ $(z=x+iy\ ,\ x,y\in R)$, $\beta(1/2\pm z)=0$ in $-1/2<x<1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} u_c(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{2r-1}}\cosh\{x\log(2r-1)\}\cos\{y\log(2r-1)\} = 0 \\ v_c(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{2r-1}}\sinh\{x\log(2r-1)\}\sin\{y\log(2r-1)\} = 0 \\ u_s(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{2r-1}}\sinh\{x\log(2r-1)\}\cos\{y\log(2r-1)\} = 0 \\ v_s(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{2r-1}}\cosh\{x\log(2r-1)\}\sin\{y\log(2r-1)\} = 0 \end{cases}$$

Proof

$$\cosh(x+iy) = \cosh x\cos y + i\sinh x\sin y$$
$$\sinh(x+iy) = \sinh x\cos y + i\cosh x\sin y$$

Replacing x with $x\log(2r-1)$ and y with $y\log(2r-1)$ respectively,

$$\cosh\{z\log(2r-1)\} = \cosh\{x\log(2r-1)\}\cos\{y\log(2r-1)\} + i\sinh\{x\log(2r-1)\}\sin\{y\log(2r-1)\}$$

$$\sinh\{z\log(2r-1)\} = \sinh\{x\log(2r-1)\}\cos\{y\log(2r-1)\} + i\cosh\{x\log(2r-1)\}\sin\{y\log(2r-1)\}$$

Substituting these for (2.2c) , (2.2s) respectively,

$$\beta_c(z) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{\sqrt{2r-1}}\cosh\{z\log(2r-1)\}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} cosh\{x\,log(2r-1)\} cos\{y\,log(2r-1)\}$$

$$+ i \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\} sin\{y\,log(2r-1)\}$$

$$\beta_s(z) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{z\,log(2r-1)\}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\} cos\{y\,log(2r-1)\}$$

$$+ i \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} cosh\{x\,log(2r-1)\} sin\{y\,log(2r-1)\}$$

Describing the real and imaginary parts as $u_c(x,y)$, $v_c(x,y)$, $u_s(x,y)$, $v_s(x,y)$ respectively, we obtain the desired expressions.

Overdetermined System

Since there are 4 equations for 2 real variable in Theorem 2.3 , this system of equations is an overdetermined system. Such a system of equations generally has no solution.

Zeros on the Critical Line

However, such a system of equations may exceptionally has solution. That is the case when $x = 0$. Note that $x = 0$ is the critical line of function $\beta(1/2+z)$. Substituting $x = 0$ for the equations in Theorem 2.3

$$\begin{cases} u_c(0,y) = 1 \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} cos\{y\,log(2r-1)\} = 0 \\ v_c(0,y) = 0 \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sin\{y\,log(2r-1)\} = 0 \\ u_s(0,y) = 0 \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} cos\{y\,log(2r-1)\} = 0 \\ v_s(0,y) = 1 \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sin\{y\,log(2r-1)\} = 0 \end{cases}$$

Since $v_c(0,y)$, $u_s(0,y)$ are equal to non-existent, the overdetermined property disappears. As the result,

$$0 = u_c(0,y) - i\,v_s(0,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} [cos\{y\,log(2r-1)\} - i\,sin\{y\,log(2r-1)\}]$$

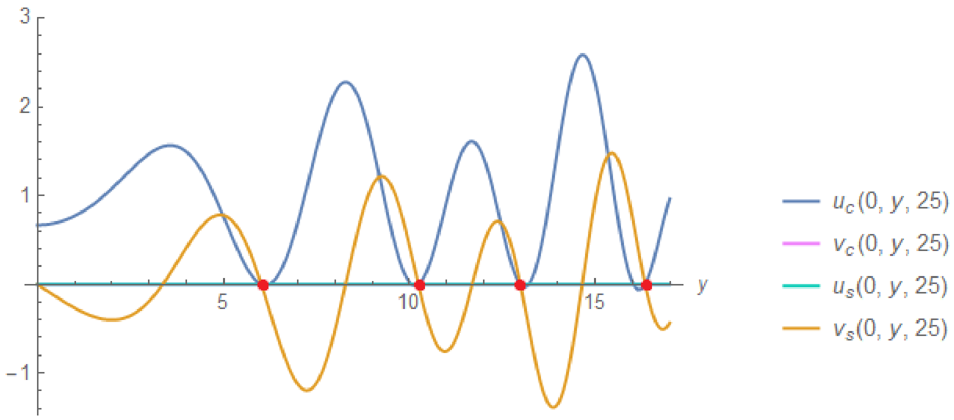
$$= \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} [cos\{y\,log(2r-1)\} + i\,sin\{y\,log(2r-1)\}]$$

i.e.

$$0 = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} e^{-y\,log(2r-1)} = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} e^{y\,log(2r-1)}$$

That is, they reduce to the case of $x = 0$ in Lemma 2.1'. These solutions are zeros on the critical line.

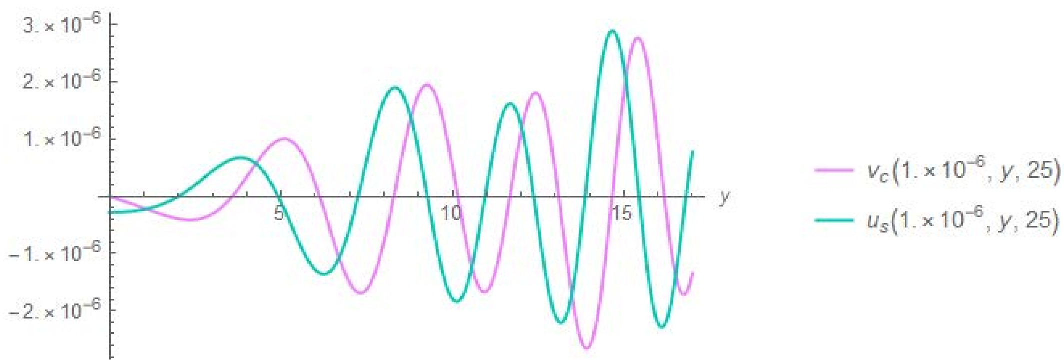
When $x = 0$, $u_c \sim v_s$ are drawn as follows. Blue is u_c and orange is v_s . The points (red) where these intersect on the y -axis are the zeros of $\beta(1/2 \pm z)$.



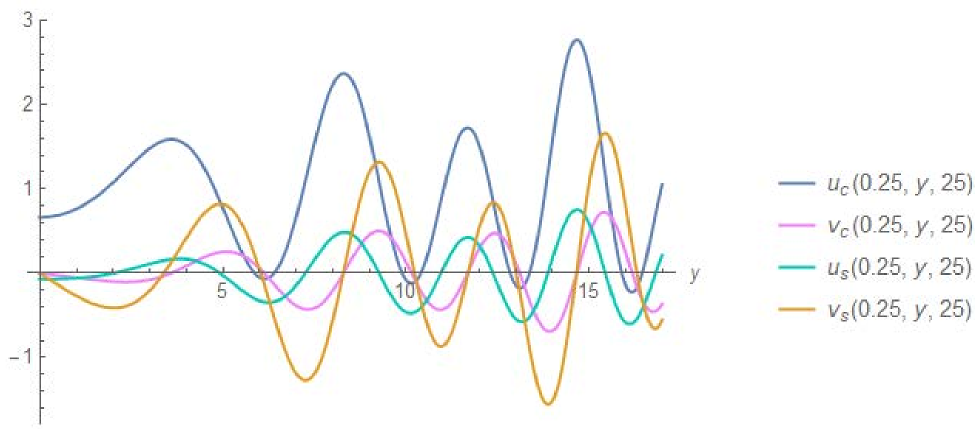
The points (red) where these intersect on the y -axis are the zeros of $\beta(\ 1/2 \pm z\ )$. Magenta is v_c and cyan is u_s . They overlap on the y -axis. Of course, these 2 straight lines also pass through the red points.

Zeros outside the Critical Line

If x deviates even slightly from 0 , v_c, u_s cease to be straight lines. For example, when $x = 0.000001$,



As the result, the property of overdetermination is restored. For example, when $x = 0.25$, $u_c \sim v_s$ are drawn as follows. It seems unlikely that the 4 curves would intersect at one point on the y -axis.



Propositions equivalent to the Riemann hypothesis

Theorem 2.3 is equivalent to that the following 6 pairs have a common solution. Each pair is one of the necessary conditions for $\beta(\ 1/2 + z\ )$ to have zeros.

$$\begin{cases} u_c = 0 \\ v_c = 0 \end{cases}, \quad \begin{cases} u_c = 0 \\ u_s = 0 \end{cases}, \quad \begin{cases} u_c = 0 \\ v_s = 0 \end{cases}, \quad \begin{cases} v_c = 0 \\ u_s = 0 \end{cases}, \quad \begin{cases} v_c = 0 \\ v_s = 0 \end{cases}, \quad \begin{cases} u_s = 0 \\ v_s = 0 \end{cases}$$

Therefore, to prove the Riemann hypothesis, it is sufficient to show that any one of these pairs does not have a solution such as $x \neq 0$.

The most interesting of these is $v_c = 0$ and $u_s = 0$ pair. This pair is unlikely to intersect at a point on the y-axis when $x \neq 0$, as seen in the 2D figure above. So, we can present the following proposition, which is equivalent to the Riemann hypothesis.

Proposition 2.4

When y is a real number, x is a real number s.t. $-1/2 < x < 1/2$, the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} v_c(x,y) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{x \log(2r-1)\} \sin\{y \log(2r-1)\} = 0 & (2.4c) \\ u_s(x,y) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{x \log(2r-1)\} \cos\{y \log(2r-1)\} = 0 & (2.4s) \end{cases}$$

If this proposition is proved, then by Theorem 2.3, $\beta(1/2+z)$ has no zeros such as $x \neq 0$.

Expression by β function

Since the convergence speed of the series in Proposition 2.4 is slow, accurate calculations and drawings are difficult at small y. In order to deal with this, in this paper we use the representation by the Dirichlet beta function $\beta(x,y)$. .Using this, $v_c(x,y)$, $u_s(x,y)$ are represented as follows.

$$v_c(x,y) = \frac{1}{2}\left[Im\left\{ \beta\left(\frac{1}{2} - x - iy \right) \right\} + Im\left\{ \beta\left(\frac{1}{2} + x + iy \right) \right\} \right] \qquad (2.4c')$$

$$u_s(x,y) = \frac{1}{2}\left[Re\left\{ \beta\left(\frac{1}{2} - x - iy \right) \right\} - Re\left\{ \beta\left(\frac{1}{2} + x + iy \right) \right\} \right] \qquad (2.4s')$$

However, Mathematica's $\beta(x,y)$ computation routines are unusable for large $|y|$. Therefore, where $|y|$ is large, we use (2.4c), (2.4s) as they are..

3 Amplitude of $v_c(x,y)$ with respect to y

Among the equations in Proposition 2.4 , $v_c(x,y)$ was as follows.

$$v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\} sin\{y\,log(2r-1)\} \tag{2.4c}$$

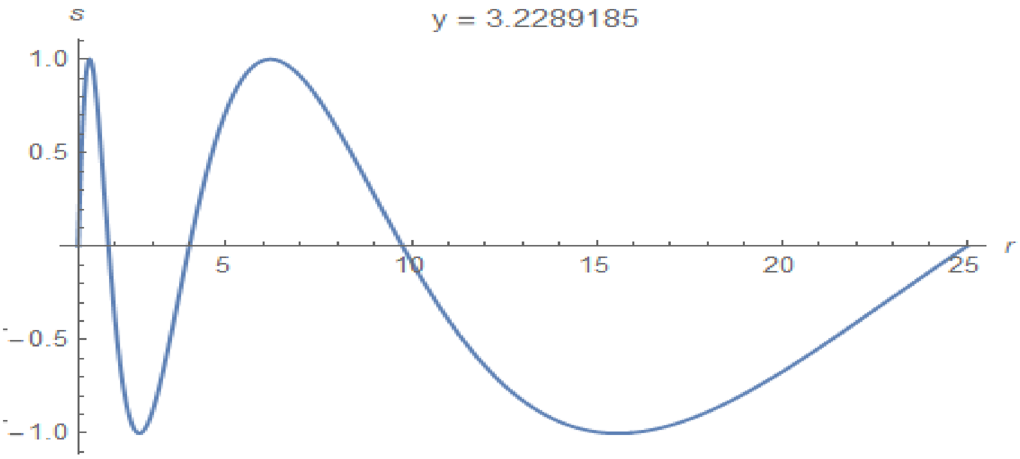
In this chapter, we consider the amplitude of this function with respect to y

3.1 sin(y log(2r-1))

Let r,y are positive numbers respectively, and consider the following function $s(r,y)$.

$$s(r,y) = sin\{y\,log(2r-1)\} \tag{3.1.1}$$

When $y = 3.2289185$, the 2D figures for $r = 1 \sim 25$ is drawn as follows.



Observing this shows that $s(r,y)$ is a variable periodic function with respect to r .

Amplitude (A)

The amplitude of this function is $A=1$.

Period (P)

This function is a periodic function. The first period starts at 0π and ends at 2π , the second period starts at 2π and ends at 4π , so

$$y\,log(2r_0-1)=0\pi\,,\,y\,log(2r_1-1)=2\pi\,,\,y\,log(2r_2-1)=4\pi\,,\,\cdots\,,\,y\,log(2r_n-1)=2n\pi\,,\cdots$$

From these,

$$2r_0-1 = e^{0\pi/y}\ ,\ 2r_1-1 = e^{2\pi/y}\ ,\ 2r_2-1 = e^{4\pi/y}\ ,\ \cdots\ ,\ 2r_n-1 = e^{2n\pi/y}\ ,\ \cdots$$

i.e.

$$r_0 = \frac{e^{0\pi/y}+1}{2}\ ,\ r_1 = \frac{e^{2\pi/y}+1}{2}\ ,\ r_2 = \frac{e^{4\pi/y}+1}{2}\ ,\ \cdots\ ,\ r_n = \frac{e^{2n\pi/y}+1}{2}\ ,\ \cdots$$

Therefore, the function $s(r,y)$ is separated into the following unit intervals.

$$\left[\frac{e^{0\pi/y}+1}{2}\,,\,\frac{e^{2\pi/y}+1}{2}\right)\,,\,\left[\frac{e^{2\pi/y}+1}{2}\,,\,\frac{e^{4\pi/y}+1}{2}\right)\,,\,\left[\frac{e^{4\pi/y}+1}{2}\,,\,\frac{e^{6\pi/y}+1}{2}\right)\,,\,\cdots$$

Each of these has one mountain and one valley. We will call these the 1 st period, the 2 nd period, $\cdots$. i.e.

$$P(n,y) = \left[\frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}}+1\right), \frac{1}{2}\left(e^{\frac{2n\pi}{y}}+1\right) \right)$$

In the figure above, the 1 st and the 2 nd periods of $s(r,y)$ are

$$P(1,3.2289185) = [1, 4) \quad , \quad P(2,3.2289185) = [4, 25)$$

Wavelength (λ)

The wavelength are the length of these periods. That is,

$$\frac{e^{0\pi/y}\left(e^{2\pi/y}-1\right)}{2}, \frac{e^{2\pi/y}\left(e^{2\pi/y}-1\right)}{2}, \cdots, \frac{e^{2\pi(n-1)/y}\left(e^{2\pi/y}-1\right)}{2}, \cdots$$

The wavelength is $e^{2\pi/y}$ times longer than the previous period in each period. So, **this function is a variable periodic function**. That is,

$$\lambda(n,y) = \frac{1}{2}e^{\frac{(2n-2)\pi}{y}}\left(e^{\frac{2\pi}{y}}-1\right)$$

In the figure above, the wavelengths of the 1 st and the 2 nd periods of $s(r,y)$ are

$$\lambda(1,3.2289185) = 3 \ , \quad \lambda(2,3.2289185) = 21$$

When $n=1$, y can be back calculated from λ. That is,

$$\lambda(1,y) = \frac{1}{2}\left(e^{2\pi/y}-1\right) \quad \Longrightarrow \quad 2\lambda+1 = e^{2\pi/y}$$

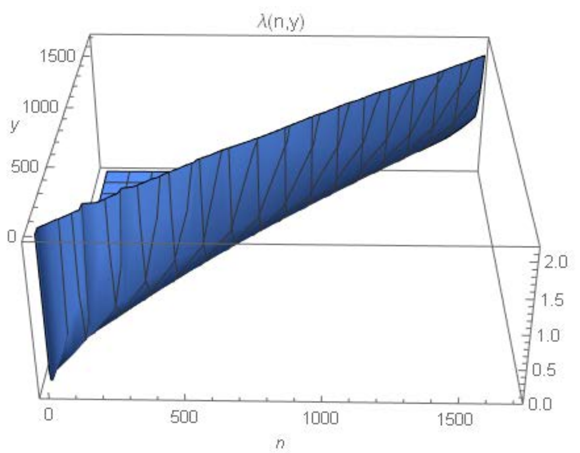
From this,

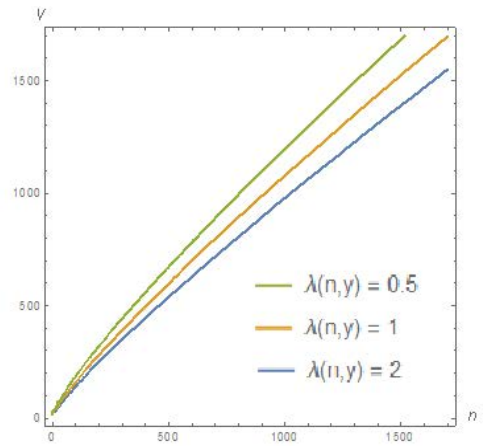
$$y = \frac{2\pi}{\log(2\lambda+1)}$$

When $\lambda(1,y)=3$, $\qquad y = \frac{2\pi}{\log 7} = 3.2289185$

When $\lambda(1,y)=1$, $\qquad y = \frac{2\pi}{\log 3} = 5.7192017$

A 3D view of $\lambda(n,y)$ at $n \neq 1$ is shown on the left. And the contour plots at $\lambda=0.5$, $\lambda=1.0$, $\lambda=2.0$ are shown on the right.





From these figures, we can see that the slope of the $\lambda(n,y)$ contour decreases as n increases. because,

$$\frac{\partial}{\partial n}\lambda(n,y) = \frac{\pi}{y}e^{(2n-2)\pi/y}\left(e^{2\pi/y}-1\right) > 0 \qquad for\ n,y > 0$$

Using this contour plot, we can find the n,y pair that gives the desired λ .

Zeros (Zs)

Since $s(r,y)$ is the sine function, the zeros are at the left edge and middle of the period. That is,

$$Zs(n,y) = \left\{ \frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}}+1\right) , \frac{1}{2}\left(e^{\frac{(2n-1)\pi}{y}}+1\right) \right\}$$

In the figure above, the zeros of the 1 st and the 2 nd periods of $s(r,y)$ are

$$Zs(1,3.2289185) = \{1, 1.8228757\} \quad , \quad Zs(2,3.2289185) = \{4, 9.7601297\}$$

Near zeros (Xs)

When the variable r of $s(r,y)$ is a discrete variable, we will call the integer r within ± 0.5 from the zero point **the neighborhood of the zero point.** That is,

$$Xs(n,y) = \left\{ Round\left(\frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}}+1\right)\right), Round\left(\frac{1}{2}\left(e^{\frac{(2n-1)\pi}{y}}+1\right)\right) \right\}$$

In the figure above,

$$Xs(1,3.2289185) = \{1,2\} \quad , \quad Xs(2,3.2289185) = \{4, 10\}$$

Mountain (Ms)

Since $s(r,y)$ is a sine function, the mountain is at $1/4$ of the period. That is,

$$Ms(n,y) = \frac{1}{2}\left(e^{\frac{(4n-3)\pi}{2y}} + 1\right)$$

In the figure above, the mountains of the 1 st and the 2 nd periods of $s(r,y)$ are

$$Ms(1,3.2289185) = 1.3132882 \quad , \quad Ms(2,3.2289185) = 6.1930180$$

Valley (Vs)

Since $s(r,y)$ is a sine function, the vallay is at $3/4$ of the period. That is,

$$Vs(n,y) = \frac{1}{2}\left(e^{\frac{(4n-1)\pi}{2y}} + 1\right)$$

In the figure above, the valleys of the 1 st and the 2 nd periods of $s(r,y)$ are

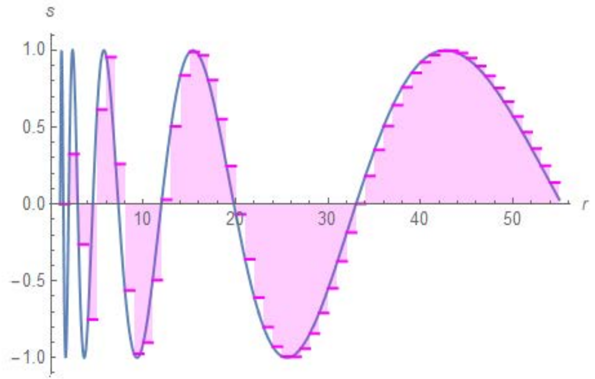
$$Vs(1,3.2289185) = 2.6517585 \quad , \quad Vs(2,3.2289185) = 15.5623100$$

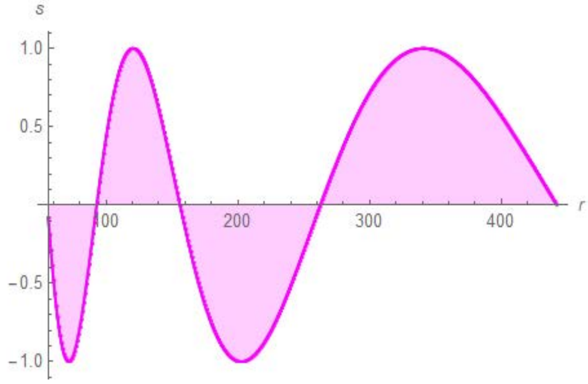
Dirichlet Rambda type Sine Series (when $y = 6.0209489\cdots$)

Let us consider the following Dirichlet Rambda type sine series.

$$v(y) = \sum_{r=1}^{\infty} sin\{y\, log(2r-1)\} \tag{3.1.2}$$

This is a series whose terms are $s(r,y)$ (3.1.1). For example, when $y=6.0209489\cdots$, $r=1,2,\cdots,55$ and $r=56,57,\cdots,442$ are drawn in succession, it is as follows.





The function value of (3.1.2) is the sum of the areas of magenta. In the left figure, this sum differs greatly from the integral value of (3.1.1). On the other hand, in the right figure this sum is close to the integral value of (3.1.1)

Divergence

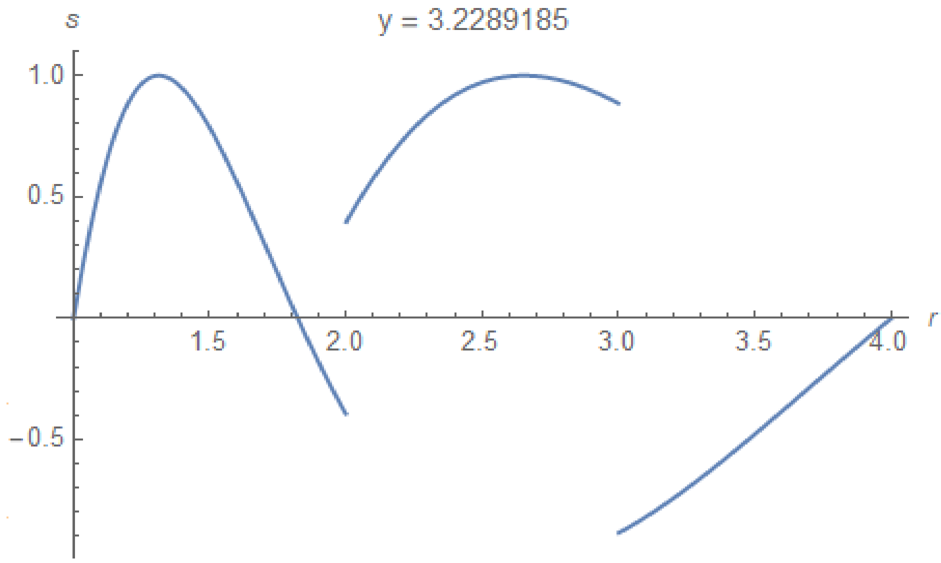
In the right figure, the area seems to be zero due to cancellation of plus and minus, but it is not. Because, the interval between waves expands and eventually becomes infinite. So, the series in (3.1.2) diverges.

3.2 ±sin(y log(2r-1))

Let r,y are positive numbers respectively, and consider the following function $s(r,y)$.

$$s(r,y) = (-1)^{\lfloor r-1 \rfloor} \sin\{y \log(2r-1)\} \quad (\lfloor\ \rfloor \text{ is floor function }) \tag{3.2.1}$$

When $y = 3.2289185$, the 2D figures for $r=1\sim 4$ is drawn as follows.



Unlike the previous section, $s(r,y)$ is a discontinuous function with respect to r.

Amplitude (A)

The amplitude of this function is

$$A(r) = \left|(-1)^{\lfloor r-1 \rfloor}\right| = 1$$

Period (P)

The period of this function is the same as in the previous section, That is,

$$P(n,y)=\left[\frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}}+1\right),\ \frac{1}{2}\left(e^{\frac{2n\pi}{y}}+1\right)\right)$$

In the figure above,

$$P(1,3.2289185)=[1,\ 4)$$

Wavelength (λ)

The wavelength of this function is the same as in the previous section, That is,

$$\lambda(n,y)=\frac{1}{2}e^{\frac{(2n-2)\pi}{y}}\left(e^{\frac{2\pi}{y}}-1\right)$$

In the figure above,

$$\lambda(1,3.2289185)=3$$

Zeros (Zs)

The zeros of this function are the same as in the previous section, That is,

$$Zs(n,y)=\left\{\frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}}+1\right),\ \frac{1}{2}\left(e^{\frac{(2n-1)\pi}{y}}+1\right)\right\}$$

In the figure above,

$$Zs(1,3.2289185)=\{1,\ 1.822875\}$$

Constriction (Xs)

Since this function $s(r,y)$ changes sign, the zero point looks like a constriction. So, we will call the integer r within ± 0.5 from the zero point **constriction**. That is,

$$Xs(n,y)=\left\{Round\left(\frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}}+1\right)\right),Round\left(\frac{1}{2}\left(e^{\frac{(2n-1)\pi}{y}}+1\right)\right)\right\}$$

In the figure above,

$$Xs(1,3.2289185)=\{1,2\}$$

Mountain or Valley (MVs)

Unlike the previous section, this function $s(r,y)$ changes sign. For this reason, mountains and valleys exist at most twice as many as in the previous section.

$$MVs(n,y)=\left\{\frac{1}{2}\left(e^{\frac{(4n-3)\pi}{2y}}+1\right),\ \frac{1}{2}\left(e^{\frac{(4n-1)\pi}{2y}}+1\right)\right\}$$

The mountain or valley is determined by the sign of $s(r,y)$ at $r=MVs(n,y)$.

In the figure above,

$$MVs(1,3.2289185)=\{1.31329,\ 2.65176\}$$

$$\{s(1.31329,3.2289185),\ s(2.65176,3.2289185)\}=(1,1)$$

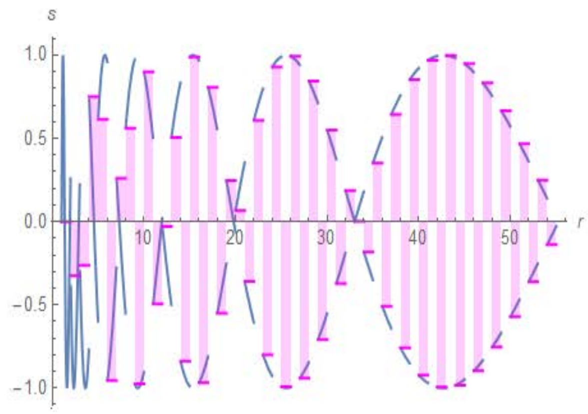
So, both the former and the latter are mountains.

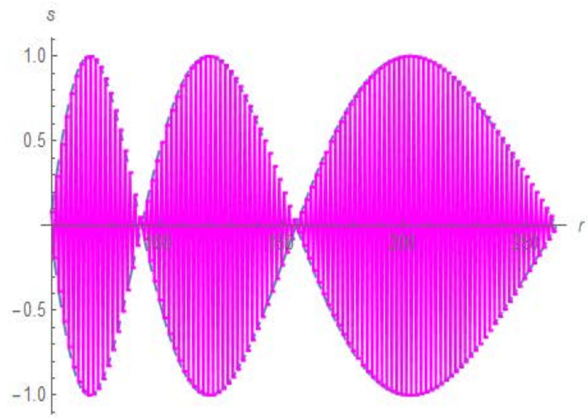
Dirichlet Beta type Sine Series (when $y = 6.0209489\cdots$)

We consider the following Dirichlet Beta type sine series

$$v(y) = \sum_{r=1}^{\infty} (-1)^{r-1} \sin\{y \log(2r-1)\} \tag{3.2.2}$$

This is a series whose terms are $s(r,y)$ (3.2.1). For example, when $y=6.0209489\cdots$, $r=1,2,\cdots,55$ and $r=56,57,\cdots,262$ are drawn in succession, it is as follows.





The function value of (3.2.2) is the sum of the areas of magenta. In the left figure, this sum differs greatly from the integral value of (3.2.1). On the other hand, in the right figure this sum is close to the integral value of (3.2.1)

Oscillations (Divergence)

This series oscillates (diverges). A good approximation can be obtained if this series is truncated at an appropriate constriction, but if it is truncated at mountains or valleys, an error of up to ± 0.5 will occur.

Summation Method

In such a case, the summation method says that an error of ± 0.5 should be regarded as 0 on average. One of the simplest summation methods is the Euler transformation. The Euler transformation accelerates the convergence of the series and also applies the summation method. If the Euler transform is applied to (3.2.2) ,

$$v(y,m) = \sum_{k=1}^{m} \sum_{r=1}^{k} \frac{1}{2^{k+1}} \binom{k}{r} (-1)^{r-1} \sin\{y \log(2r-1)\} \tag{3.2.2'}$$

If this formula is used, this series converges.

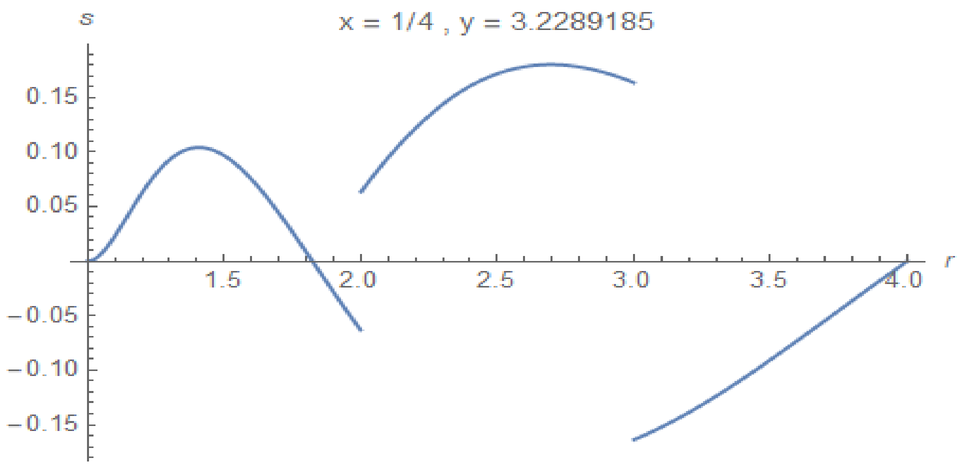
3.3 $v_c(x,y)$

Let r,x,y are positive numbers respectively, and consider the following function $s(r,x,y)$.

$$s(r,x,y) = \frac{(-1)^{\lfloor r-1 \rfloor}}{\sqrt{2r-1}} \sinh\{x \log(2r-1)\} \sin\{y \log(2r-1)\} \quad (\lfloor\ \rfloor \text{ is floor function })$$

(3.3.1)

When $x = 1/4$, $y = 3.2289185$, the 2D figures for $r=1 \sim 4$ is drawn as follows.



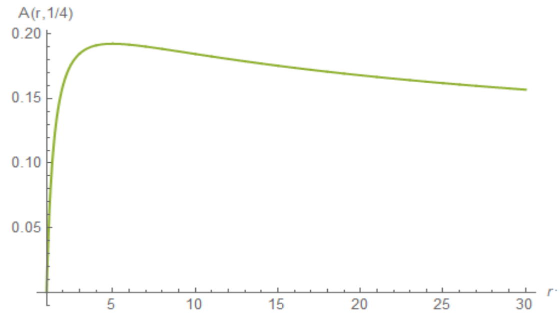
Amplitude (A)

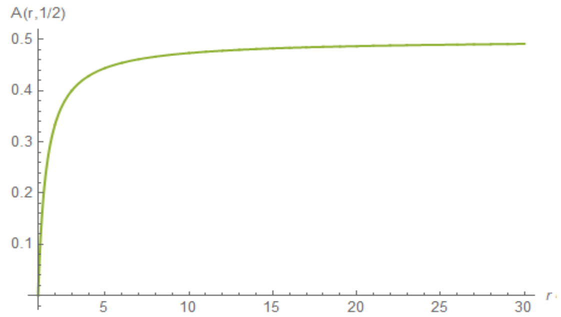
The amplitude of this function is

$$A(r,x) = \left| \frac{(-1)^{\lfloor r-1 \rfloor}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\} \right| = \frac{sinh\{x\,log(2r-1)\}}{\sqrt{2r-1}}$$

(1) When $0 < x < 1/2$, $\lim_{r\to\infty} sinh\{x\,log(2r-1)\}/\sqrt{2r-1} = 0$. This is shown on the left.

(2) When $x = 1/2$, $\lim_{r\to\infty} sinh\{x\,log(2r-1)\}/\sqrt{2r-1} = 1/2$. This is shown on the right.





Period (P)

The period of this function is the same as in the previous section, That is,

$$P(n,y) = \left[\frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}}+1\right), \frac{1}{2}\left(e^{\frac{2n\pi}{y}}+1\right) \right)$$

In the figure above,

$$P(1,3.2289185) = [1\ ,\ 4)$$

Wavelength (λ)

The wavelength of this function is the same as in the previous section, That is,

$$\lambda(n,y) = \frac{1}{2}e^{\frac{(2n-2)\pi}{y}}\left(e^{\frac{2\pi}{y}}-1\right)$$

In the figure above,

$$\lambda(1,3.2289185) = 3$$

Zeros (Zs)

The zeros of this function are the same as in the previous section, That is,

$$Zs(n,y) = \left\{ \frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}}+1\right), \frac{1}{2}\left(e^{\frac{(2n-1)\pi}{y}}+1\right) \right\}$$

In the figure above,

$$Zs(1,3.2289185) = \{1, 1.82288\}$$

Constriction (Xs)

The constrictions of this function are the same as in the previous section, That is,

$$Xs(n,y) = \left\{ Round\left(\frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}}+1\right)\right), Round\left(\frac{1}{2}\left(e^{\frac{(2n-1)\pi}{y}}+1\right)\right) \right\}$$

In the figure above,

$$Xs(1,3.2289185)= \{1,2\}$$

Mountain or Valley (MVs)

The mountains or valleys of this function are the same as in the previous section, That is,

$$MVs(n,y) = \left\{ \frac{1}{2}\left(e^{\frac{(4n-3)\pi}{2y}}+1\right), \frac{1}{2}\left(e^{\frac{(4n-1)\pi}{2y}}+1\right) \right\}$$

In the figure above,

$$MVs(1,3.2289185) = \{1.31329, 2.65176\}$$

$$\{s(1.31329, 1/4, 3.2289185), s(2.65176, 1/4, 3.2289185)\} = \{0.0955954, 0.179807\}$$

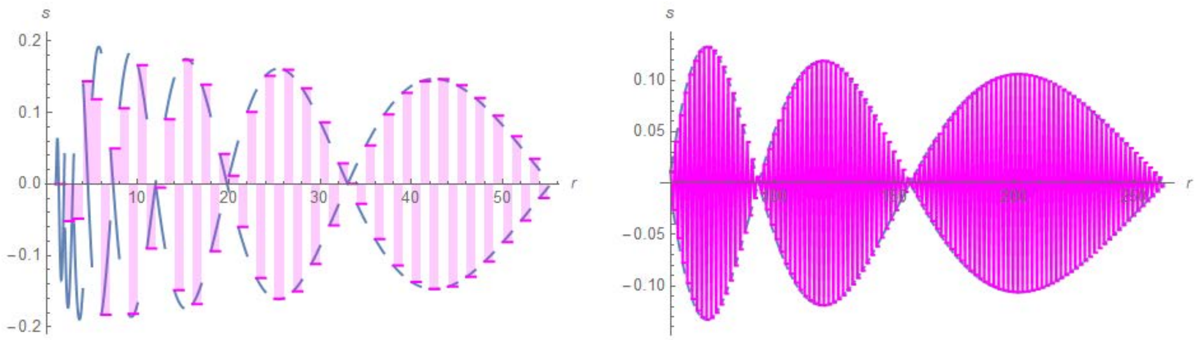
So, both the former and the latter are mountains.

Sine Series $v_c(x,y)$ (when $x=1/4$, $y=6.0209489\cdots$)

We consider the following sine series.

$$v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x \log(2r-1)\} sin\{y \log(2r-1)\} \qquad (2.4c)$$

This is a series whose terms are $s(r,x,y)$ (3.3.1) . For example, when $x=1/4$, $y=6.0209489\cdots$, $r=1,2,\cdots,55$ and $r=56,57,\cdots,262$ are drawn in succession, it is as follows.



The function value of (2.4c) is the sum of the areas of magenta. In the left figure, this sum differs greatly from the integral value of (3.3.1). On the other hand, in the right figure this sum is close to the integral value of (3.3.1)

Convergence

This series converges when $|x| < 1/2$ and diverges when $|x| \geq 1/2$.

(1) In either case, a good approximation is obtained if the series is truncated at a suitable constriction, and truncated at mountains or valleys yields errors of up to half the amplitude.

(2) When $|x| \geq 1/2$, this series converges if the summation method is applied.

3.4 Amplitude of $v_c(x,y)$ with respect to y

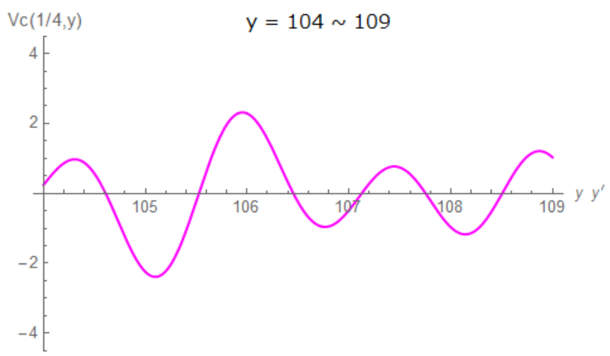
The sine function and series dealt with in **3.3** were as follows.

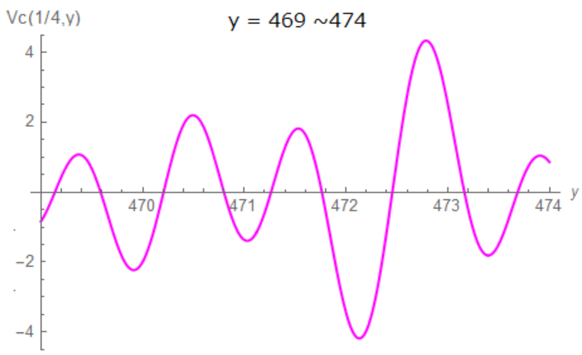
$$s(r,x,y) = \frac{(-1)^{\lfloor r-1 \rfloor}}{\sqrt{2r-1}} \sinh\{x \log(2r-1)\} \sin\{y \log(2r-1)\} \quad (\lfloor\ \rfloor \text{ is floor function}) \tag{3.3.1}$$

$$v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{x \log(2r-1)\} \sin\{y \log(2r-1)\} \tag{2.4c}$$

In this section, we explore the amplitude (mountain, valley) of (2.4c) with respect to y using (3.3.1).

Given x, $v_c(x,y)$ is a variable periodic function with respect to y. For example, when $x=1/4$, the 2D figures at $y=104 \sim 109$ and $y=469 \sim 474$ are drawn as follows.





Although the value of y is larger in the right figure than in the left figure, it cannot be said that the mountains in the right are higher than those in the left, or that the valleys in the right are deeper than those in the left.

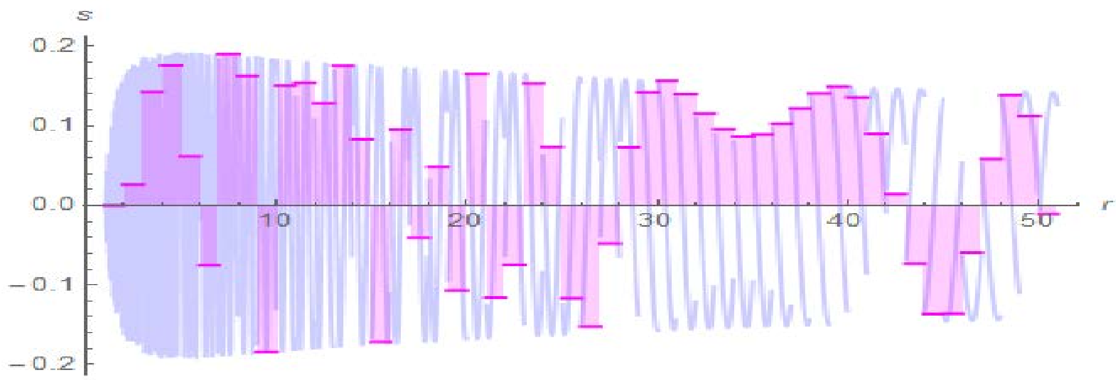
However, it can be said that the highest mountain in the right figure is higher than the highest one in the left figure, and the deepest valleys in the right figure is deeper than the deepest one in the left figure. That is, we can say that the amplitudes in the right figure are generally larger than those in the left figure. Below, this will be illustrated graphically.

3.4.1 Mountain of $v_c(1/4, y)$ (near y=106)

At $y=104 \sim 109$, the mountain near here is the highest. Accurate calculation of the mountain near here by (2.4c) is as follows.

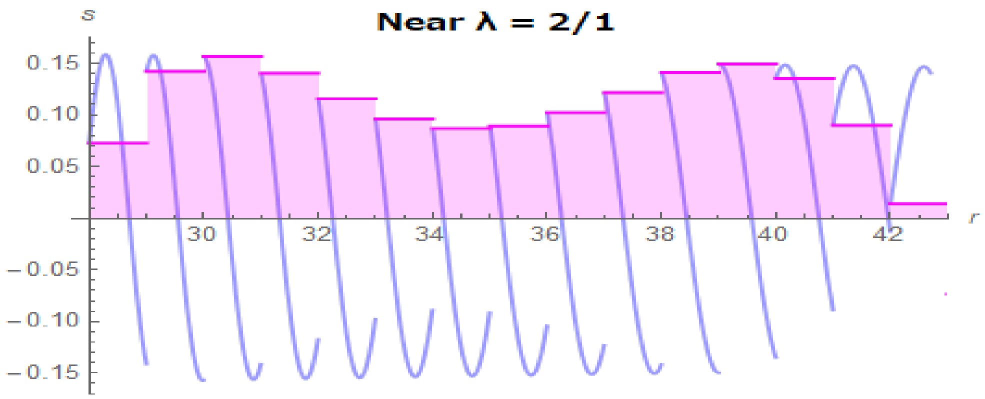
```
FindMaximum[vc[1 / 4, y, 200], {y, 106}]
{2.31542, {y → 105.954}}          yM := 105.954
```

Drawing $s(r, 1/4, y_M)$ by (3.3.1) is as follows. The horizontal axis is r. Cyan is drawn as a continuous variable and magenta as a discrete variable. The sum of the area of magenta becoms mountain 2.315 of (2.4c)



Looking at this figure, it can be seen that there are 4 intervals with consecutive positive terms, which contribute to the height of the mountain.

(1) An enlarged view near 2/1 wavelength is drawn as follows.



r = 28 ~ 42 are positive for 15 consecutive terms. Calculating the constriction that seems to be around here by trial and error,

```
{Xs[68, yM], Xs[75, yM]}
{{27, 28}, {41, 42}}
```

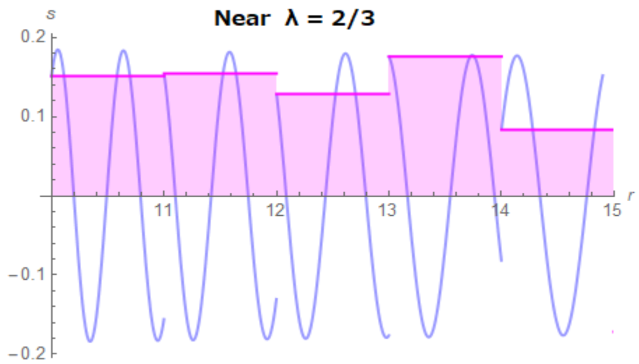
Then, r = 28 ~ 42 are included in the 68 ~ 75 th period. The wavelengths of these periods are

```
Table[λ[n, yM], {n, 68, 75}]
{1.62368, 1.72288, 1.82813, 1.93982, 2.05834, 2.18409, 2.31753, 2.45911}
```

That is, the wavelengths of r = 28 ~ 42 are 1.72 ~ 2.46

Here after, only the calculation results are described.

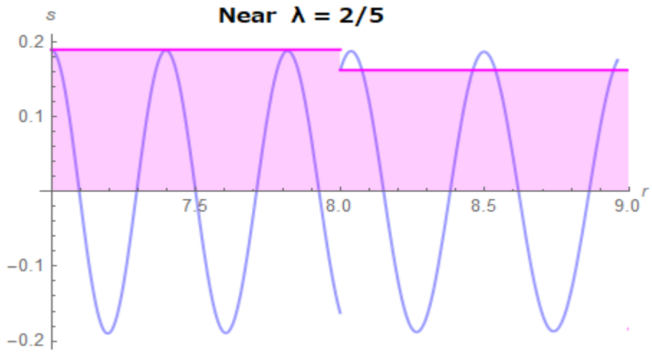
(2) An enlarged view near 2/3(=0.67) wavelength is drawn as follows.



r = 10 ~ 14 are positive for 5 consecutive terms.

These are included in the 51 ~ 56 th period and the wavelengths of the periods are 0.63 ~ 0.75.

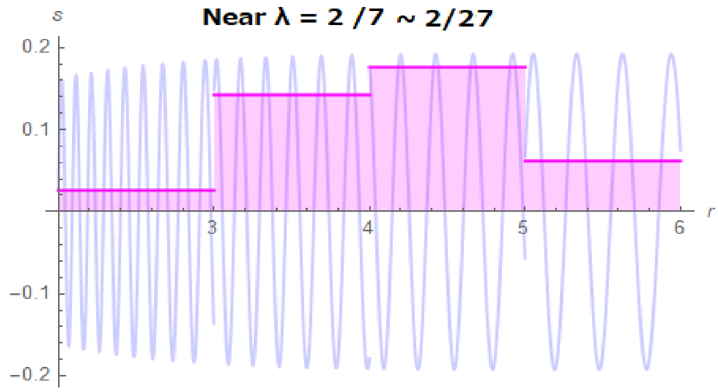
(3) An enlarged view near $2/5(=0.4)$ wavelength is drawn as follows.



$r = 7, 8$ are positive for 2 consecutive terms.

These are included in the $45, 46$ th period and the wavelengths of the periods are $0.42 \sim 0.44$.

(4) An enlarged view near $2/7 \sim 2/27$ $(0.074 \sim 0.286)$ wavelength is drawn as follows.



$r = 2 \sim 5$ are positive for 4 consecutive terms.

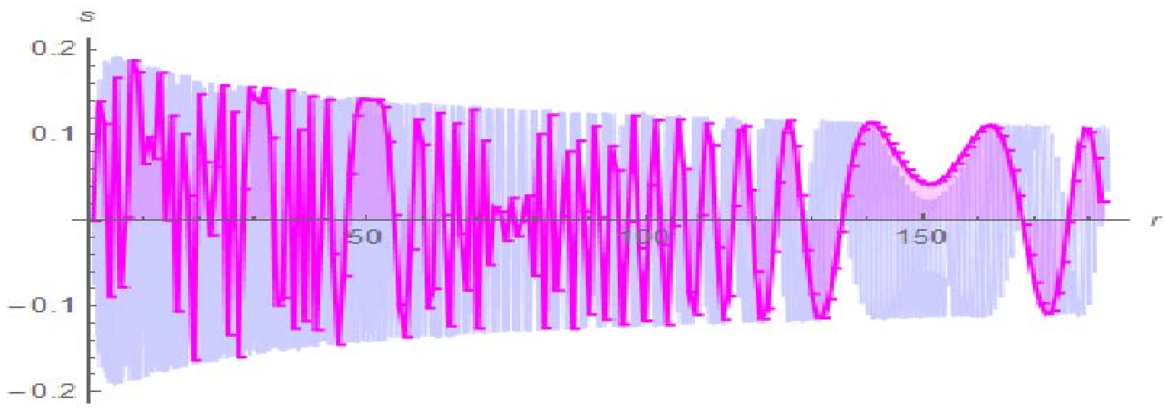
These are included in the $15 \sim 37$ th period and the wavelengths of the periods are $0.074 \sim 0.258$.

3.4.2 Mountain of $v_c(1/4, y)$ (near y=473)

At $y = 469 \sim 474$, the mountain near here is the highest. Accurate calculation of the mountain near here by (2.4c) is as follows.

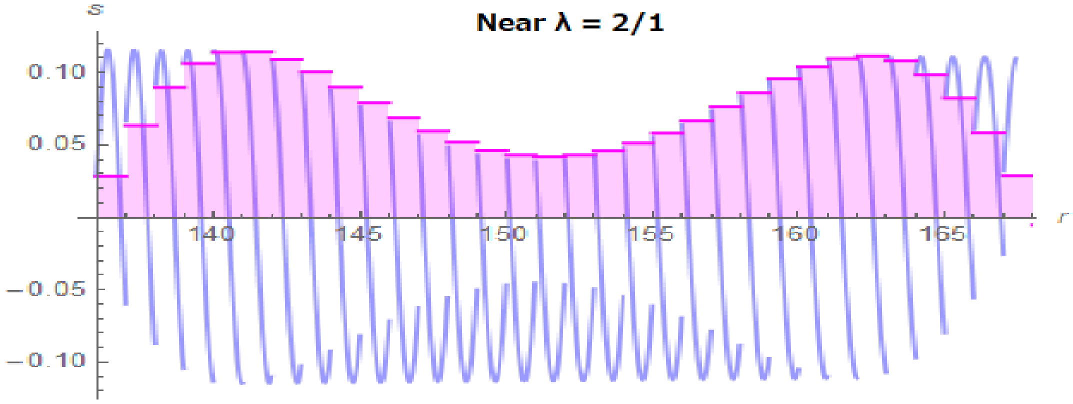
```
FindMaximum[vc[1 / 4, y, 600], {y, 473}]
{4.33386, {y → 472.787}}          yM := 472.787
```

Drawing $s(r, 1/4, y_M)$ by (3.3.1) is as follows. The horizontal axis is r . Cyan is drawn as a continuous variable and magenta as a discrete variable. The sum of the area of magenta becoms mountain 4.334 of (2.4c)



Looking at this figure, it can be seen that there are 5 intervals with consecutive positive terms, which contribute to the height of the mountain.

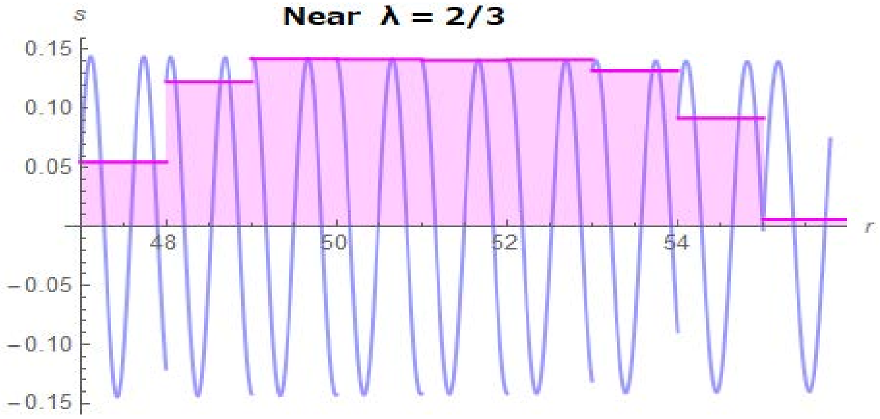
(1) An enlarged view near 2/1 wavelength is drawn as follows.



r = 136 ~ 167 are positive for 32 consecutive terms. This interval is 2.13 times one of **3.4.1 (1)**

These are included in the 422 ~ 438 th period and the wavelengths of the periods are 1.82 ~ 2.20 .

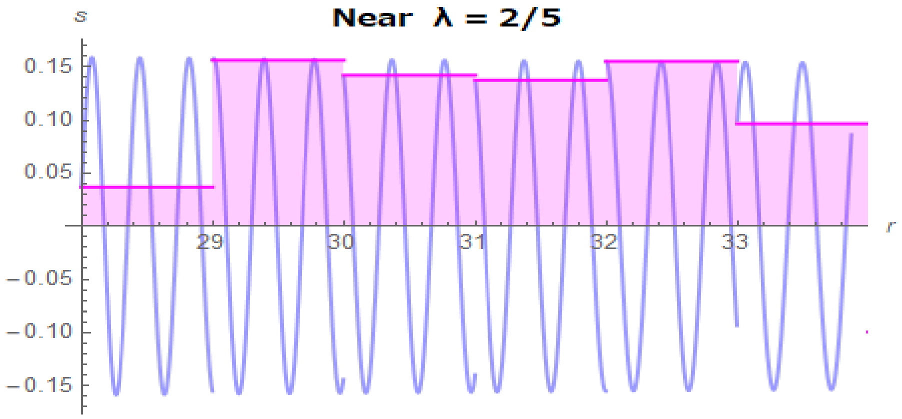
(2) An enlarged view near 2/3(=0.67) wavelength is drawn as follows.



r = 47 ~ 55 are positive for 9 consecutive terms. This interval is 1.8 times one of **3.4.1 (2)** .

These are included in the 342 ~ 353 th period and the wavelengths of the periods are 0.62 ~ 0.72 .

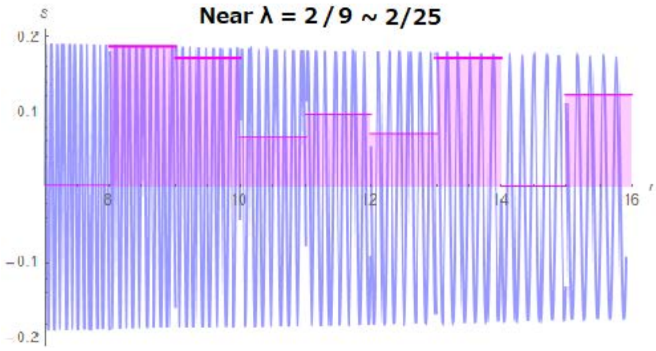
(3) An enlarged view near 2/5(=0.4) wavelength is drawn as follows.

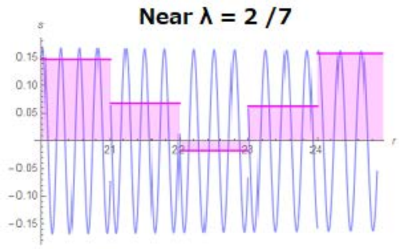


r = 28 ~ 33 are positive for 6 consecutive terms. This interval is 3 times one of **3.4.1 (3)** .

These are included in the 301 ~ 316 th period and the wavelengths of the periods are 0.37 ~ 0.43 .

(4) An enlarged view near $2/9 \sim 2/25$ $(0.08 \sim 0.22)$ and $2/7$ (0.29) wavelength are as follows

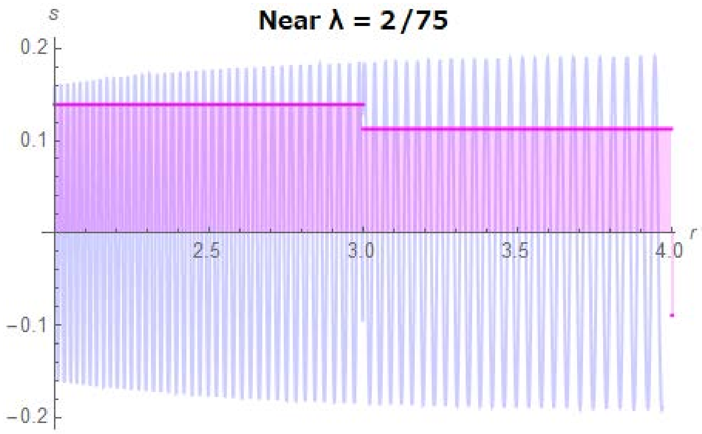




In the left figure, $r = 7 \sim 15$ is positive with 9 consecutive terms, and in the right figure, 4 terms out of $r = 20 \sim 24$ are positive. There are 13 positive terms in wavelengths $2/7 \sim 2/25$, which is 3.25 times one of **3.4.1 (4)** .

These are included in the $190 \sim 289$ th period and the wavelengths of the periods are $0.08 \sim 0.31$.

(5) An enlarged view near $2/75$ (0.027) wavelength is drawn as follows.



$r = 2, 3$ are positive for 2 consecutive terms. This interval is absent in **3.4.1** .

These are included in the 105 th period and the wavelength of the period is 0.027 .

3.4.3 Height of mountains near y=473 and y=106

The mountain near y=473 is higher than the one near y=106 . Because,

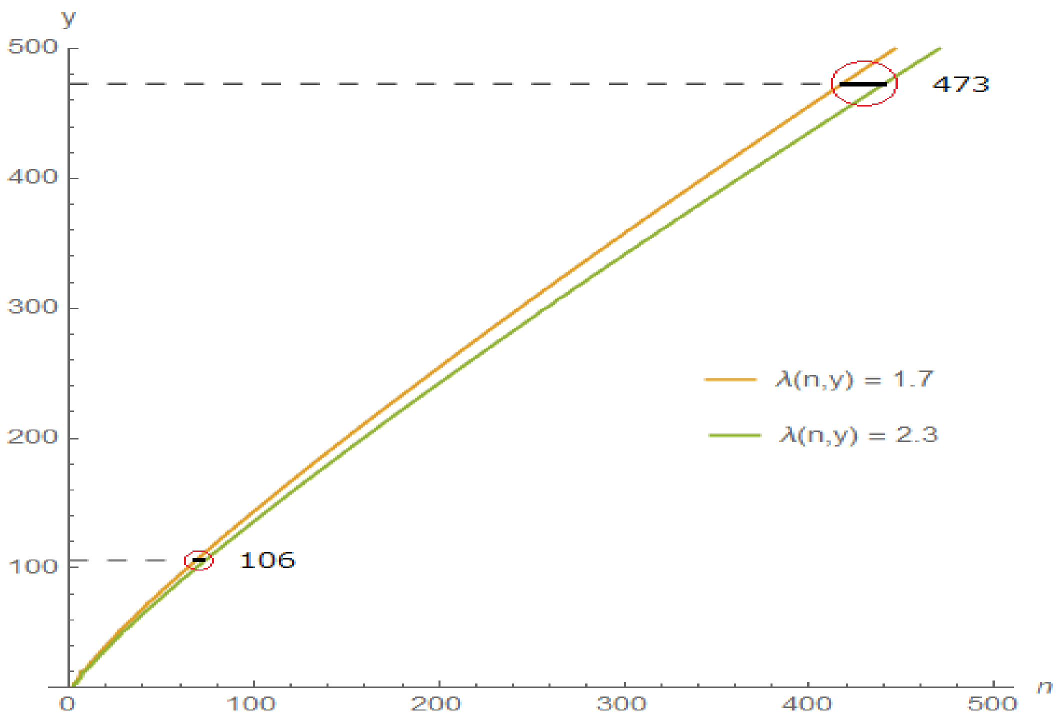
(1) The former is about $1.8 \sim 3.25$ times longer than the latter in the interval with consective positive terms near wavelength $2/1 \sim 2/25$.

(2) The intervals with consective positive terms near wavelength $2/75$ is added to the former.

The reason for (1) lies in the definition of wavelength. That is,

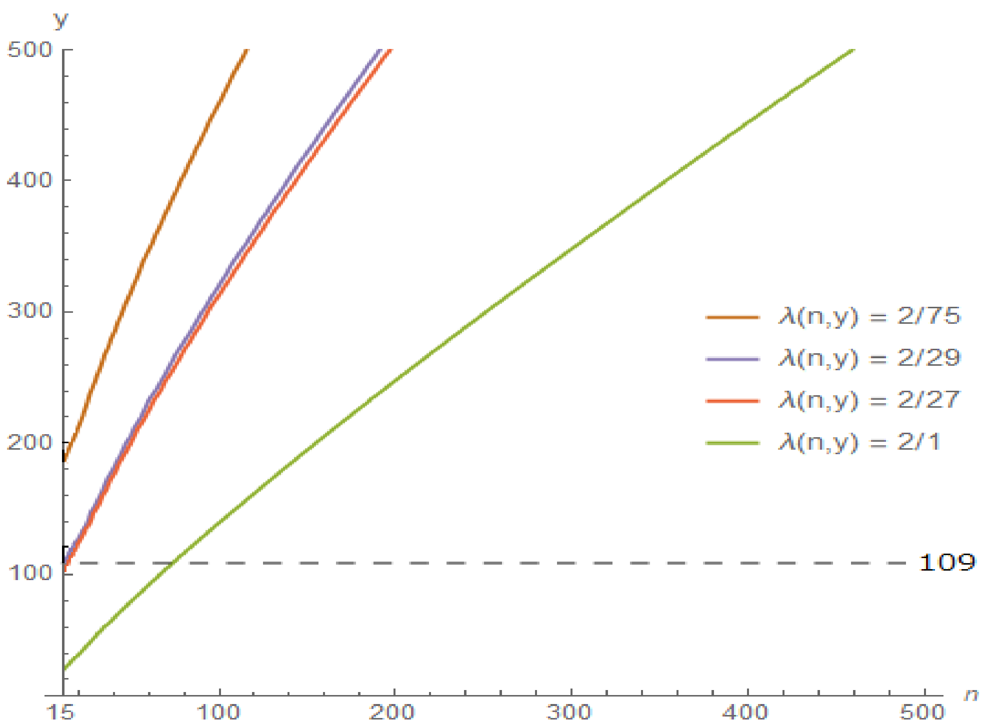
$$\lambda(n,y) = \frac{1}{2} e^{\frac{(2n-2)\pi}{y}} \left(e^{\frac{2\pi}{y}} - 1 \right)$$

When the near of $\lambda = 2$ is ± 0.3 , the contour plots of $\lambda(n,y) = 1.7$ and $\lambda(n,y) = 2.3$ are drawn as follows. The vertical axis is y and the horizontal axis is the period number n



The allowable range for the wavelength λ near $y=106$ is the lower left black horizontal line, and the one for the λ near $y=473$ is the upper right black horizontal line. Then, we can see that the allowable range near $y=473$ is wider than one near $y=106$. This is because the slope y/n of the contour decreases as the wavelength λ increases. This is the same for $\lambda=2/3$, $2/5$, ⋯ as well. Thus the number of consecutive positive terms near $y=473$ is greater than that near $y=106$. The above is the reason for (1).

The reason for (2) is the same. The contours for $\lambda=2/1$, $2/27$, $2/75$ are as follows.
Since the minimum value of the period number that gives these values near $y=106$ is 15 (**3.4.1 (4)**), the horizontal axis is drawn with $n \geq 15$.



Now, draw a horizontal line with a height of 109 with a dashed line. Since near $y=106$ is $y=104\sim 109$, it is below this chain line. And contours such as $\lambda \leq 2/29$ cannot exist here. The reason is that for a given period number n, the contours shift upwards as the wavelength λ decreases.
Thus, if y increases, k in $\lambda(15,y)=2/(2k-1)$ also increases. The above is the reason for (2) .

3.4.4 Depth of Valleys near $y=472$ and $y=105$

The valley near $y=472$ is deeper than the one near $y=105$. Because, **3.4.1** ~ **3.4.3** also hold for valley.

From **3.4.3** and **3.4.4** , we conclude that the amplitude at $y=469\sim 474$ is greater than that at $y=104\sim 109$. This can be described more generally as follows.

Law 3.4.5

Let x,y are real numbers and function $v_c(x,y)$ be as follows.

$$v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\} sin\{y\,log(2r-1)\} \tag{2.4c}$$

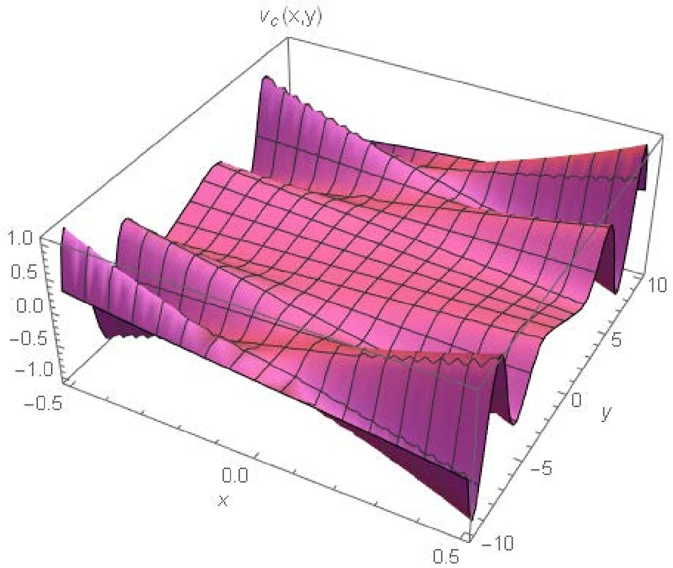
Then, given x, the amplitude of $v_c(x,y)$ is generally proportional to the absolute value of y .

Note

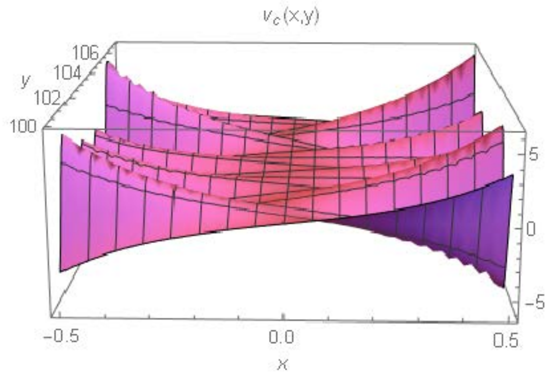
It is clear that this does not hold as a theorem. This is because there are quite a few exceptions. Nevertheless Law 3.4.5 holds. Because, it is due to the change in slope y/n of the contour line of wavelength λ and the shift of the contour line, as described above.
This law is similar to ***Bergmann's Law*** (Bears in high latitudes are generally larger than bears in low latitudes.).

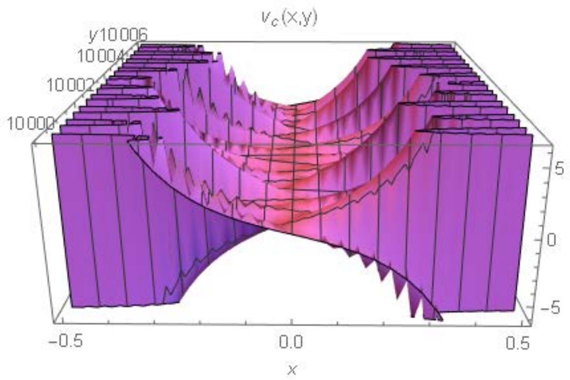
3.5 Shape and Properties of $v_c(x,y)$

From (2.4c) , we find that $v_c(x,y)$ is an odd function with respect to both x and y .
This shows that $v_c(x,y)$ is point symmetric with respect to both both x and y .



Next, when $-1/2 \leq x \leq 1/2$, the 3D view of $v_c(x,y)$ at $y = 100 \sim 107$ and $y = 10000 \sim 10007$ are drawn respectively as follows.



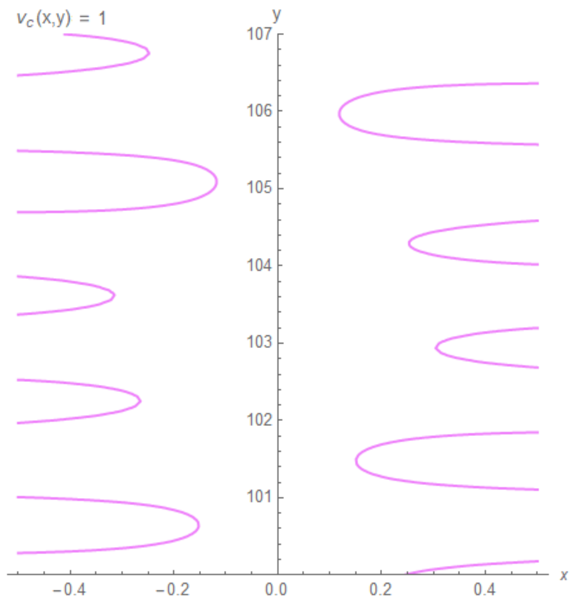


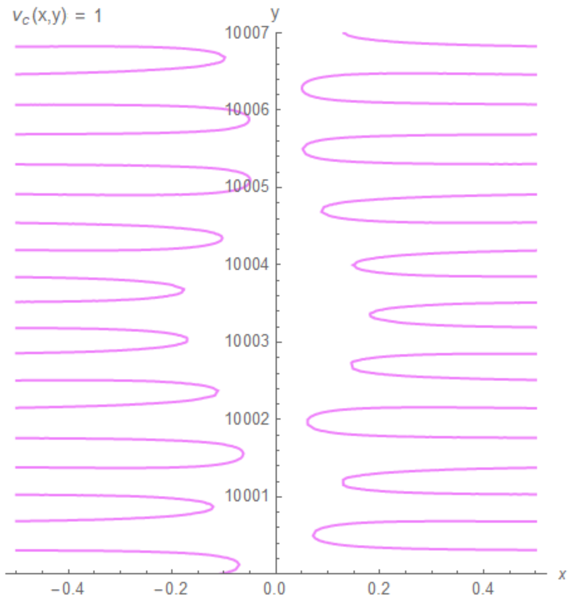
In both figures, the upper part looks like ∪ and the lower part looks like ∩. Then, we can see that both ∪ and ∩ generally have larger curvatures in the right figure than in the left figure. This is because mountains and valleys are generally steeper in the right figure than in the left figure according to Law 3.4.5 .

In addition, the right figure has more mountains and valleys than the left figure (about twice as many) . However, the reason for this is unknown.

3.6 Contour line of $v_c(x,y)$ with height 1

The height 1 contours of the two 3D views of $v_c(x,y)$ in the previous section are drawn as follows. The left figure is $y = 100 \sim 107$ and the right figure is $y = 10000 \sim 10007$.





In both figures, the contour line looks like ⊃ & ⊂. Then, we can see that both ⊃ & ⊂ are generally closer to the y-axis in the right figure than in the left figure. This is because mountains and valleys are generally steeper in the right figure than in the left figure according to Law 3.4.5 . Therefore, as $|y|$ increases, the tips ⊃ ⊂ of the contour approach the y -axis from both sides.

4 Amplitude of $u_s(x,y)$ with respect to y

Among the equations in Proposition 2.4 , $u_s(x,y)$ was as follows.

$$u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\} cos\{y\,log(2r-1)\} \tag{2.4s}$$

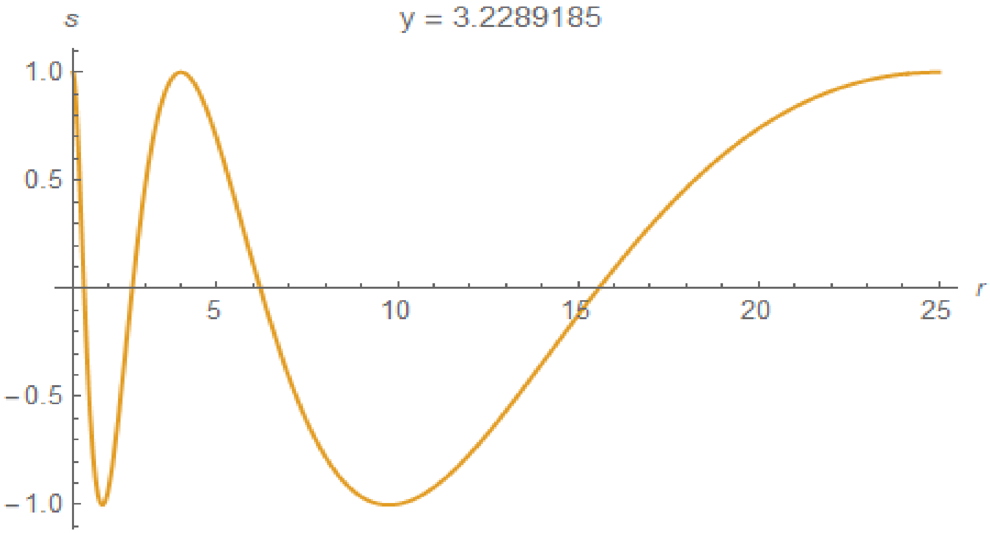
In this chapter, we consider the amplitude of this function with respect to y

4.1 cos (y log(2r-1))

Let r,y are positive numbers respectively, and consider the following function $c(r,y)$.

$$c(r,y) = cos\{y\,log(2r-1)\} \tag{4.1.1}$$

When $y = 3.2289185$, the 2D figure for $r=1 \sim 25$ is drawn as follows.



Observing this shows that $c(r,y)$ is a variable periodic function with respect to r .

Amplitude (A)

The amplitude of this function is $A=1$.

Period (P)

$$P(n,y) = \left[\frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}}+1\right), \frac{1}{2}\left(e^{\frac{2n\pi}{y}}+1\right)\right)$$

In the figure above, the 1 st and the 2 nd periods of $s(r,y)$ are

$$P(1,3.2289185) = [1\,,\,4) \quad , \quad P(2,3.2289185) = [4\,,\,25)$$

Wavelength (λ)

$$\lambda(n,y) = \frac{1}{2}e^{\frac{(2n-2)\pi}{y}}\left(e^{\frac{2\pi}{y}}-1\right)$$

In the figure above, the wavelengths of the 1 st and the 2 nd periods of $c(r,y)$ are

$$\lambda(1,3.2289185) = 3 \;, \quad \lambda(2,3.2289185) = 21$$

Zeros (Zc)

Since $c(r,y)$ is the cosine function, the zeros are at $1/4$ and $3/4$ of the period. That is,

$$Zc(n,y) = \left\{ \frac{1}{2}\left(e^{\frac{(4n-3)\pi}{2y}} + 1\right), \frac{1}{2}\left(e^{\frac{(4n-1)\pi}{2y}} + 1\right) \right\}$$

In the figure above, the zeros of the 1 st and the 2 nd periods of $c(r,y)$ are

$$Zc(1, 3.2289185) = \{1.3132882, 2.6517585\}$$
$$Zc(2, 3.2289185) = \{6.1930180, 15.5623100\}$$

Near zeros (Xc)

When the variable r of $c(r,y)$ is a discrete variable, we will call the integer r within ± 0.5 from the zero point **the neighborhood of the zero point.** That is,

$$Xc(n,y) = \left\{ Round\left(\frac{1}{2}\left(e^{\frac{(4n-3)\pi}{2y}} + 1\right)\right), Round\left(\frac{1}{2}\left(e^{\frac{(4n-1)\pi}{2y}} + 1\right)\right) \right\}$$

In the figure above,

$$Xc(1, 3.2289185) = \{1, 3\} \quad , \quad Xc(2, 3.2289185) = \{6, 16\}$$

Mountain (Mc)

Since $c(r,y)$ is a cosine function, there are half mountains at both ends of the period, but the tip is adopted.

$$Mc(n,y) = \frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}} + 1\right)$$

In the figure above, the mountains of the 1 st and the 2 nd periods of $c(r,y)$ are

$$Mc(1, 3.2289185) = 1 \quad , \quad Mc(2, 3.2289185) = 4$$

Valley (Vc)

Since $c(r,y)$ is a cosine function, there is a valley in the middle of the period. That is,

$$Vc(n,y) = \frac{1}{2}\left(e^{\frac{(2n-1)\pi}{y}} + 1\right)$$

In the figure above, the valleys of the 1 st and the 2 nd periods of $c(r,y)$ are

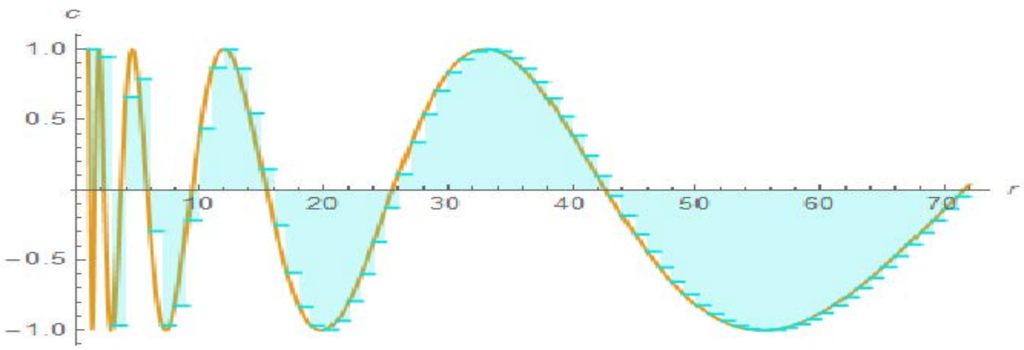
$$Vc(1, 3.2289185) = 1.8228757 \quad , \quad Vc(2, 3.2289185) = 9.7601297$$

Dirichlet Rambda type Cosine Series (when $y = 6.0209489\cdots$)

Let us consider the following Dirichlet Rambda type cosine series.

$$u(y) = \sum_{r=1}^{\infty} \cos\{y \log(2r-1)\} \tag{4.1.2}$$

This is a series whose terms are $c(r,y)$ (4.1.1) . For example, when $y=6.0209489\cdots$, $r=1,2,\cdots,72$ $c(r,y)$ is drawn as follows. The sum of the areas of cyan is the function value of (4.1.2) .
This series diverges, and even if the summation method is applied, it becomes only an asymptotic expansion.

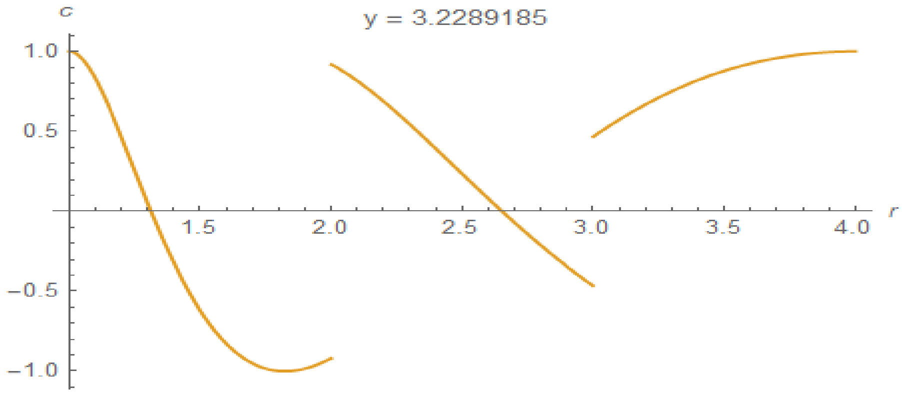


4.2 ±cos (y log(2r-1))

Let r, y are positive numbers respectively, and consider the following function $c(r,y)$.

$$c(r,y) = (-1)^{\lfloor r-1 \rfloor} \cos\{y \log(2r-1)\} \qquad (\lfloor\ \rfloor \text{ is floor function}) \tag{4.2.1}$$

When $y = 3.2289185$, the 2D figures for $r=1\sim 4$ is drawn as follows.



Unlike the previous section, $c(r,y)$ is a discontinuous function with respect to r .

Amplitude (A)

$$A(r) = \left|(-1)^{\lfloor r-1 \rfloor}\right| = 1$$

Period (P)

The period of this function is the same as in the previous section, That is,

$$P(n,y) = \left[\frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}}+1\right), \frac{1}{2}\left(e^{\frac{2n\pi}{y}}+1\right)\right)$$

In the figure above,

$$P(1, 3.2289185) = [1, 4)$$

Wavelength (λ)

The wavelength of this function is the same as in the previous section, That is,

$$\lambda(n,y) = \frac{1}{2} e^{\frac{(2n-2)\pi}{y}}\left(e^{\frac{2\pi}{y}}-1\right)$$

In the figure above,

$$\lambda(1, 3.2289185) = 3$$

Zeros (Zc)

The zeros of this function are the same as in the previous section, That is,

$$Zc(n,y) = \left\{ \frac{1}{2}\left(e^{\frac{(4n-3)\pi}{2y}} + 1\right), \frac{1}{2}\left(e^{\frac{(4n-1)\pi}{2y}} + 1\right) \right\}$$

In the figure above,

$$Zc(1,3.2289185) = \{1.31329, 2.65176\}$$

Constriction (Xc)

Since this function $c(r,y)$ changes sign, the zero point looks like a constriction. So, we will call the integer r within ± 0.5 from the zero point **constriction**. That is,

$$Xc(n,y) = \left\{ Round\left(\frac{1}{2}\left(e^{\frac{(4n-3)\pi}{2y}} + 1\right)\right), Round\left(\frac{1}{2}\left(e^{\frac{(4n-1)\pi}{2y}} + 1\right)\right) \right\}$$

In the figure above,

$$Xc(1,3.2289185) = \{1,3\}$$

Mountain or Valley (MVc)

Unlike the previous section, since this function $c(r,y)$ changes sign, mountains and valleys exist at most twice as many as in the previous section.

$$MVc(n,y) = \left\{ \frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}} + 1\right), \frac{1}{2}\left(e^{\frac{(2n-1)\pi}{y}} + 1\right) \right\}$$

The mountain or valley is determined by the sign of $c(r,y)$ at $r = MVc(n,y)$.
In the figure above,

$$MVc(1,3.2289185) = \{1, 1.82288\}$$

$$\{c(1,3.2289185), c(1.82288,3.2289185)\} = (1, -0.999999)$$

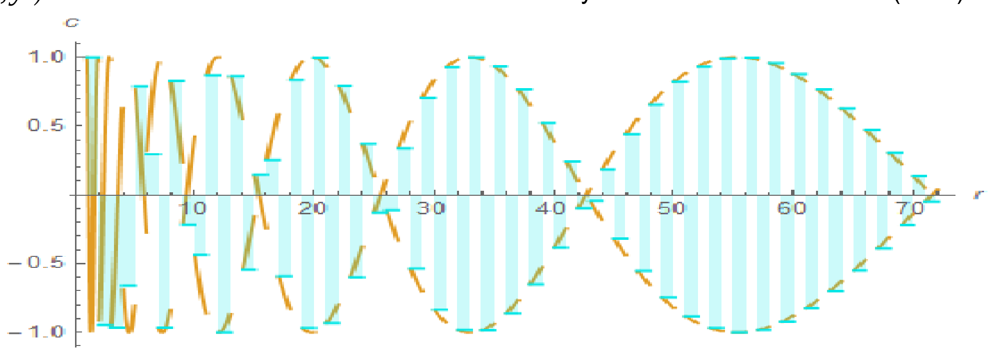
So, the former is mountain and the latter is valley.

Dirichlet Beta type Cosine Series (when $y = 6.0209489\cdots$)

Let us consider the following Dirichlet Eta type cosine function.

$$u(y) = \sum_{r=1}^{\infty} (-1)^{r-1} \cos\{y \log(2r-1)\} \tag{4.2.2}$$

This is a series whose terms are $c(r,y)$ (4.2.1). For example, when $y = 6.0209489\cdots$, $r = 1, 2, \cdots, 72$, $c(r,y)$ is drawn as follows. The sum of the areas of cyan is the function value of (4.2.2).



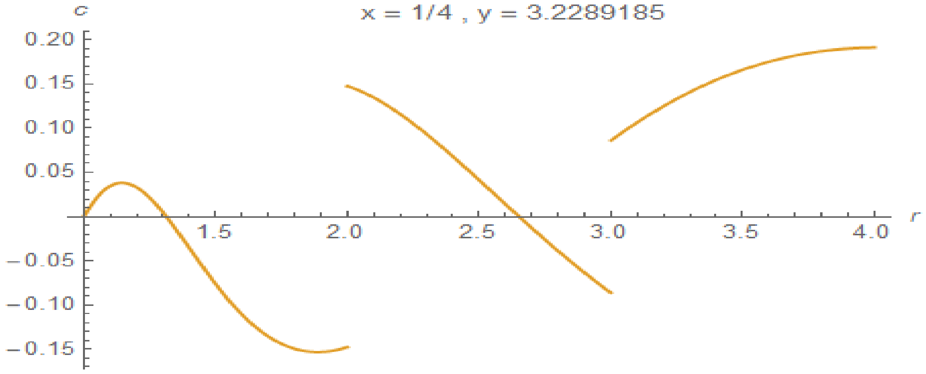
This series diverges, but converges if the summation method is applied..

4.3 $u_s(x,y)$

Let r,x,y are positive numbers respectively, and consider the following function $c(r,x,y)$.

$$c(r,x,y) = \frac{(-1)^{\lfloor r-1 \rfloor}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\} cos\{y\,log(2r-1)\} \quad (\lfloor\ \rfloor \text{ is floor function }) \tag{4.3.1}$$

When $x = 1/4$, $y = 3.02157$, the 2D figures for $r = 1 \sim 4$ is drawn as follows.



Amplitude (A)

The amplitude of this function is

$$A(r,x) = \left| \frac{(-1)^{\lfloor r-1 \rfloor}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\} \right| = \frac{sinh\{x\,log(2r-1)\}}{\sqrt{2r-1}}$$

(1) When $0 < x < 1/2$, $0 \le A(r,x) < 1/2 \quad for \quad r = 2,3,4,\cdots$.

(2) When $x = 1/2$, $\lim_{r \to \infty} sinh(x\,log(2r-1))/\sqrt{2r-1} = 1/2$.

Period (P)

The period of this function is the same as in the previous section, That is,

$$P(n,y) = \left[\frac{1}{2}\left(e^{\frac{(2n-2)\pi}{y}} + 1\right) , \frac{1}{2}\left(e^{\frac{2n\pi}{y}} + 1\right) \right)$$

In the figure above,

$$P(1, 3.2289185) = [1 , 4)$$

Wavelength (λ)

The wavelength of this function is the same as in the previous section, That is,

$$\lambda(n,y) = \frac{1}{2} e^{\frac{(2n-2)\pi}{y}} \left(e^{\frac{2\pi}{y}} - 1 \right)$$

In the figure above,

$$\lambda(1, 3.2289185) = 3$$

Zeros (Zc)

The zeros of this function are the same as in the previous section, That is,

$$Zc(n,y) = \left\{ \frac{1}{2}\left(e^{\frac{(4n-3)\pi}{2y}} + 1\right), \frac{1}{2}\left(e^{\frac{(4n-1)\pi}{2y}} + 1\right) \right\}$$

In the figure above,

$$Zc(1,3.2289185) = \{1.31329, 2.65176\}$$

Constriction (Xc)

The constrictions of this function are the same as in the previous section, That is,

$$Xc(n,y) = \left\{ Round\left(\frac{1}{2}\left(e^{\frac{(4n-3)\pi}{2y}} + 1\right)\right), Round\left(\frac{1}{2}\left(e^{\frac{(4n-1)\pi}{2y}} + 1\right)\right) \right\}$$

In the figure above,

$$Xc(1,3.2289185)= \{1,3\}$$

Mountain or Valley (MVc)

The position of mountains or valleys in this function is slightly different from the previous section. That is,

$$MVs(n,y) = \left\{ \frac{1}{2}\left(e^{\frac{(4n-3)\pi}{2y}} +1\right), \frac{1}{2}\left(e^{\frac{(4n-1)\pi}{2y}} +1\right) \right\}$$

In the figure above,

$$MVc(1,3.2289185) = \{1, 2.82288\}$$

$$\left\{ c\left(1, \frac{1}{4}, 3.2289185\right), c\left(1.82288, \frac{1}{4}, 3.2289185\right) \right\} = \{0, -0.15102\}$$

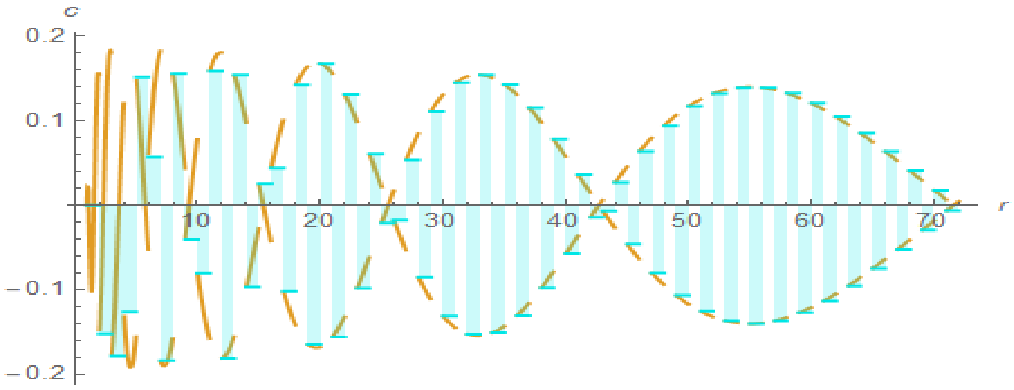
Let 0 be not a valley, and only negative number be valley. This is an exception for the first period only.

Coine Series $u_s(x,y)$ (when $x = 1/4, y = 6.0209489\cdots$)

We consider the following sine series.

$$u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x\log(2r-1)\} cos\{y\log(2r-1)\} \qquad (2.4s)$$

This is a series whose terms are $c(r,x,y)$ (4.3.1) . For example, when $x=1/4, y=6.0209489\cdots$ and $r=1,2,\cdots,72$, $c(r,y)$ is drawn as follows.



The sum of the areas of cyan is the function value of (2.4s) . When $0 < x < 1/2$, this series converges . When $x \geq 1/2$, this converges if the summation method is applied.

4.4 Amplitude of $u_s(x,y)$ with respect to y

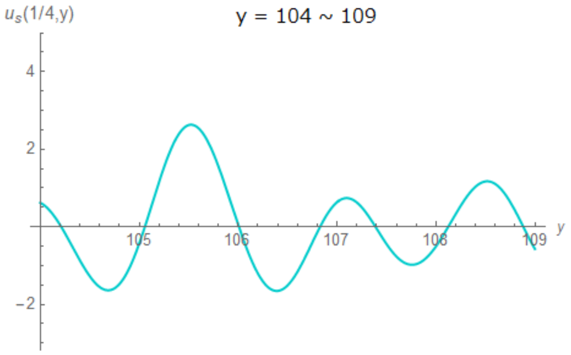
The cosine function and series dealt with in **4.3** were as follows.

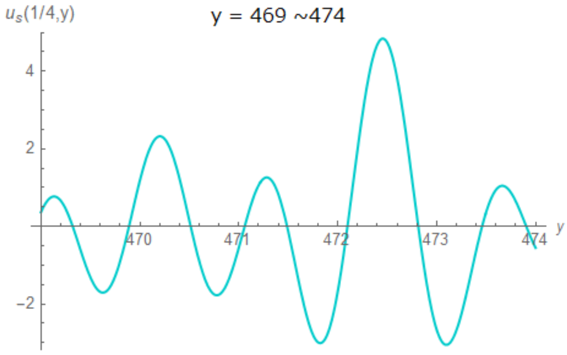
$$c(r,x,y) = \frac{(-1)^{\lfloor r-1 \rfloor}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\}cos\{y\,log(2r-1)\} \quad (\lfloor\ \rfloor \text{ is floor function }) \tag{4.3.1}$$

$$u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\}\,cos\{y\,log(2r-1)\} \tag{2.4s}$$

In this section, we explore the amplitude (mountain, valley) of (2.4s) with respect to y using (4.3.1) .

Given x , $u_s(x,y)$ is a variable periodic function with respect to y . For example, when $x=1/4$, the 2D figures at $y=104 \sim 109$ and $y=469 \sim 474$ are drawn as follows.





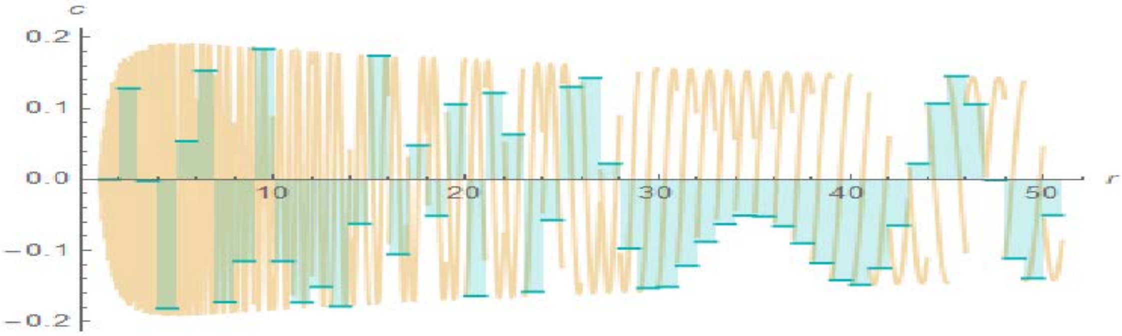
Although the value of y is larger in the right figure than in the left figure, it cannot be said that the valleys in the right are deeper than those in the left, or that the mountains in the right are heigher than those in the left. However, it can be said that the deepest valley in the right figure is deeper than the deepest one in the left figure, and the highest mountain in the right figure is higher than the highest one in the left figure. Below, this will be illustrated graphically.

4.4.1 Valley of $u_s(1/4,y)$ (near y= 106)

At $y=y=104 \sim 109$, the valley near here is the deepest. Accurate calculation of the valley near here by (2.4s) is as follows.

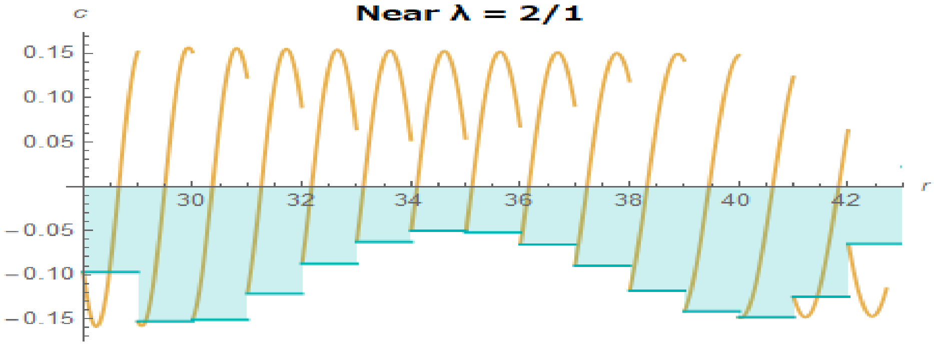
```
FindMinimum[us[1 / 4, y, 1000], {y, 106.6}]
{-1.65359, {y → 106.391}}          yv := 106.391
```

Drawing $c(r,1/4,y_v)$ by (4.3.1) is as follows. The horizontal axis is r . Orange is drawn as a continuous variable and cyan as a discrete variable. The sum of the area of cyan becoms valley -1.654 of (2.4s) .



Looking at this figure, it can be seen that there are 3 intervals with consecutive negative terms, which contribute to the depth of the valley.

(1) An enlarged view near 2/1 wavelength is drawn as follows.



r = 28 ~ 42 are negative for 15 consecutive terms. Calculating the constriction that seems to be around here by trial and error,

```
{Xc[68, yv], Xc[75, yv]}
{{27, 28}, {41, 42}}
```

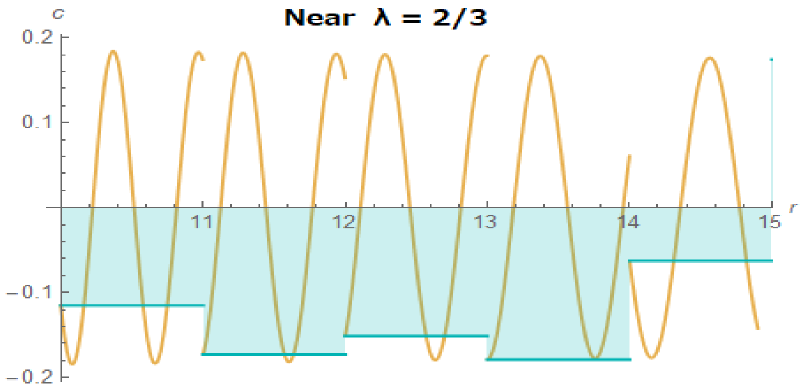
Then, r = 28 ~ 42 are included in the 68～75 th period. The wavelengths of these periods are

```
Table[λ[n, yv], {n, 68, 75}]
{1.59064, 1.68741, 1.79006, 1.89896, 2.01449, 2.13704, 2.26705, 2.40497}
```

That is, the wavelengths of r = 28 ~ 42 are 1.69 ~ 2.40

Here after, only the calculation results are described.

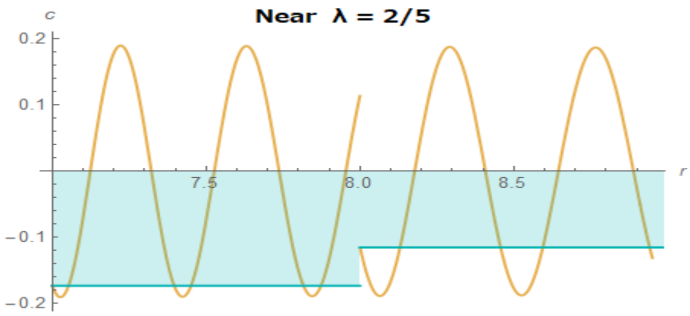
(2) An enlarged view near 2/3(=0.67) wavelength is drawn as follows.



r = 10 ~ 14 are negative for 5 consecutive terms.

These are included in the 51～56 th period and the wavelengths of the periods are 0.58 ~ 0.74 .

(3) An enlarged view near 2/5(=0.4) wavelength is drawn as follows.



r = 7, 8 are negative for 2 consecutive terms.

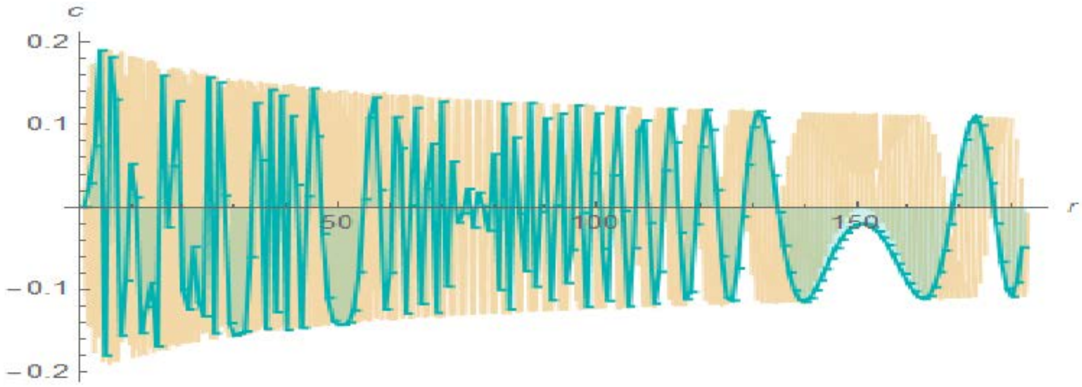
These are included in the 45 th period and the wavelength of the periods are 0.41.

4.4.2 Valley of $u_s(1/4, y)$ (near y=473)

At $y = 469 \sim 474$, the valley near here is the deepest. Accurate calculation of the valley near here by (2.4s) is as follows.

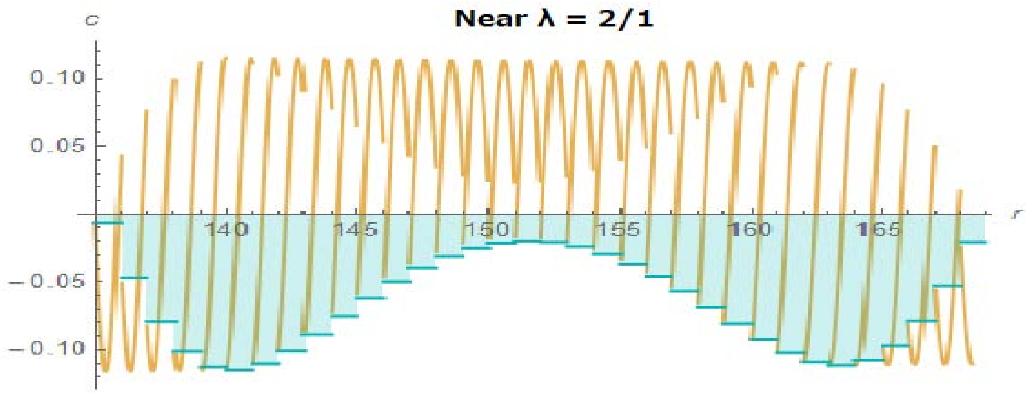
```
FindMinimum[us[1 / 4, y, 1000], {y, 473}]
{-3.05334, {y → 473.098}}          yv := 473.098
```

Drawing $c(r, 1/4, y_v)$ by (4.3.1) is as follows. The horizontal axis is r . Orange is drawn as a continuous variable and cyan as a discrete variable. The sum of the area of cyan becomes valley **-3.053** of (2.4s) .



Looking at this figure, it can be seen that there are 5 intervals with consecutive negative terms, which contribute to the depth of the valley.

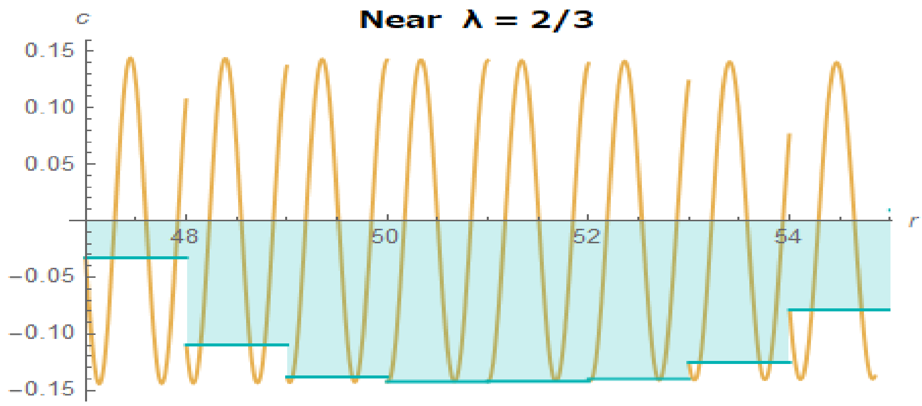
(1) An enlarged view near 2/1 wavelength is drawn as follows.



r = 135 ~ 168 are negative for 34 consecutive terms. This interval is 2.3 times one of **4.4.1 (1)** .

These are included in the 422 ~ 438 th period and the wavelengths of the periods are 1.79 ~ 2.22 .

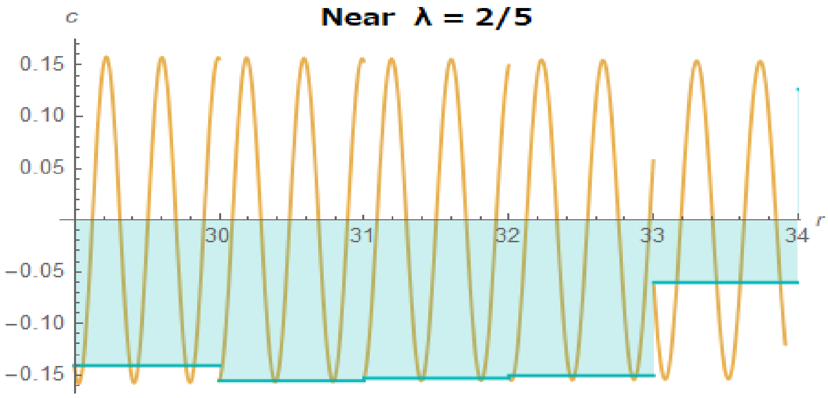
(2) An enlarged view near 2/3(=0.67) wavelength is drawn as follows.



r = 47 ~ 54 are negative for 8 consecutive terms. This interval is 1.6 times one of **4.4.1 (2)** .

These are included in the 342~352 th period and the wavelengths of the periods are 0.61~0.71 .

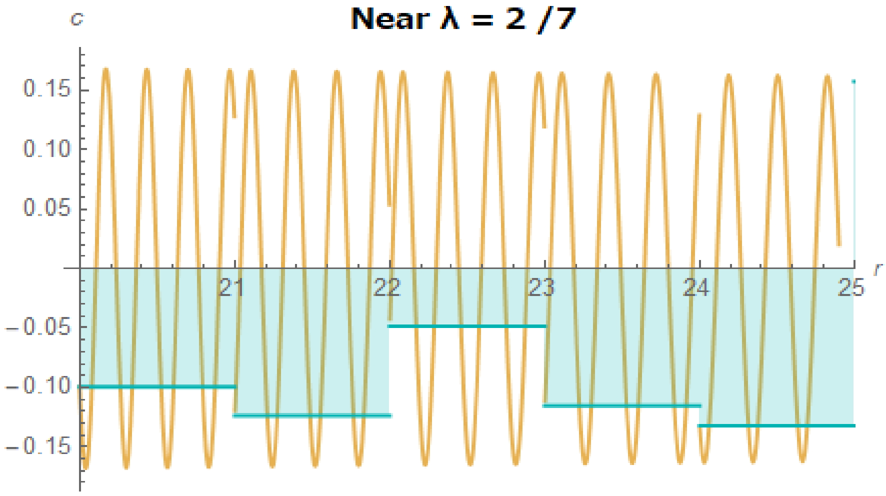
(3) An enlarged view near $2/5(=0.40)$ wavelength is drawn as follows.



$r = 29 \sim 33$ are negative for 5 consecutive terms. This interval is 2.5 times one of **4.4.1 (3)** .

These are included in the $306 \sim 314$ th period and the wavelengths of the periods are $0.38 \sim 0.42$.

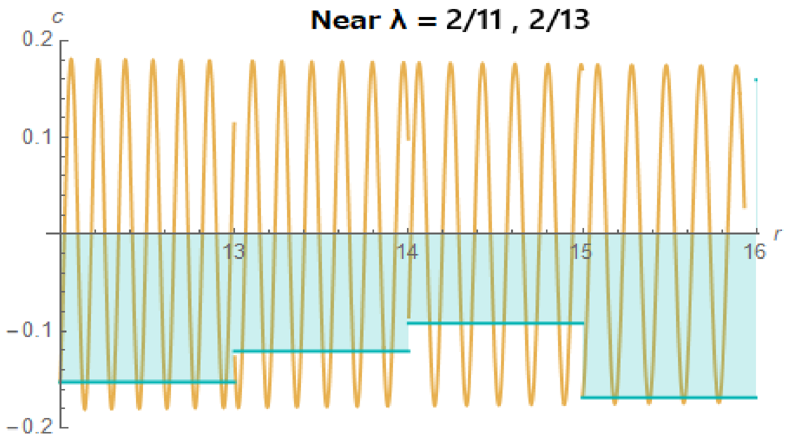
(4) An enlarged view near $2/7(=0.294)$ wavelength is drawn as follows.



$r = 20 \sim 24$ are negative for 5 consecutive terms. This interval is absent in **4.4.1** .

These are included in the $278 \sim 289$ th period and the wavelengths of the periods are $0.27 \sim 0.31$.

(5) An enlarged view near $2/11$, $2/13(=0.15 \sim 0.18)$ wavelength is drawn as follows.



$r = 12 \sim 15$ are negative for 4 consecutive terms. This interval is absent in **4.4.1** .

These are included in the $240 \sim 252$ th period and the wavelengths of the periods are $0.16 \sim 0.18$.

4.4.3 Depth of Valleys near $y=473$ and $y=106$

The valley near $y=473$ is deeper than the one near $y=106$. Because,

(1) The former is $1.6 \sim 2.5$ times longer than the latter in the interval with consective negative terms near wavelength $2/1$, $2/3$, $2/5$.

(2) The interval with consective negative terms near wavelength $2/7, 2/11, 2/13$ are added to the former.

These causes are as seen in **3.4.3** .

4.4.4 Height of mountains near $y=472.5$ and $y=105.5$

The mountain near $y=472.5$ is higher than the one near $y=105.5$. because, **4.4.1** ~ **4.4.3** also hold for mountain.

From 4.4.3 and 4.4.4 , we conclude that the amplitude at $y=469 \sim 474$ is greater than that at $y=104 \sim 109$. This can be described more generally as follows.

Law 4.4.5

Let x, y are real numbers and function $u_s(x,y)$ be as follows.

$$u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x\, log(2r-1)\} cos\{y\, log(2r-1)\} \qquad (2.4s)$$

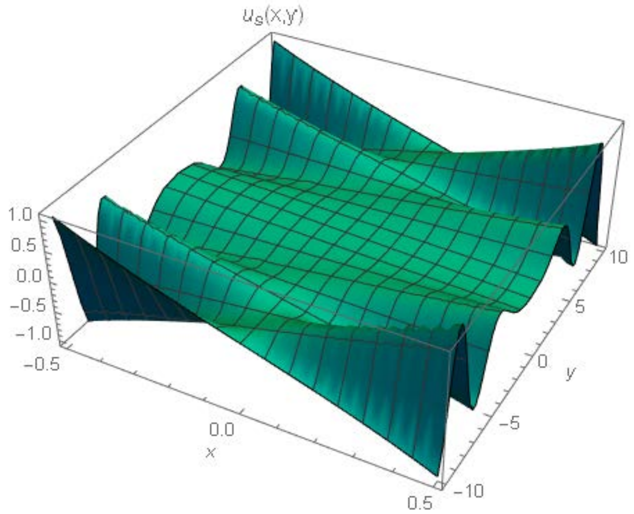
Then, given x , the amplitude of $u_s(x,y)$ is generally proportional to the absolute value of y .

Note

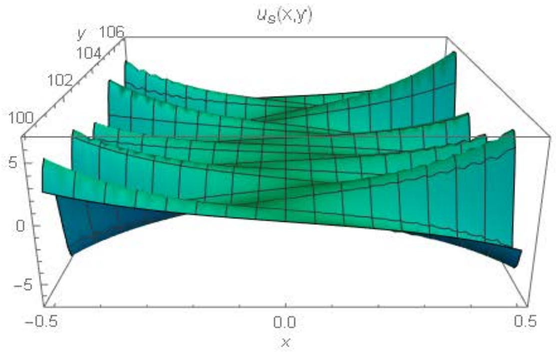
This does not hold as a theorem, but holds as a law. Because, it depends on the change of the slope y/n of the contour line at wavelength λ and the shift of the contour line. This law is similar to ***Bergmann's Law*** (Bears in high latitudes are generally larger than bears in low latitudes) .

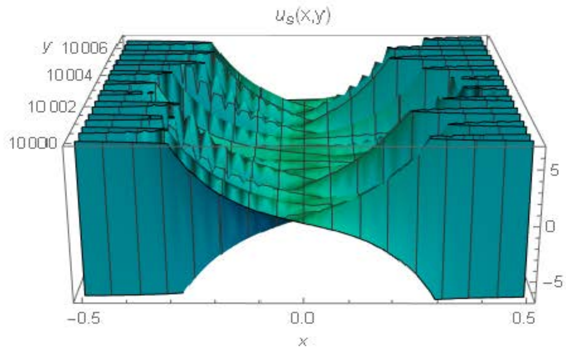
4.5 Shape and Properties of $u_s(x,y)$

From (2.4s) , $u_s(x,y)$ is an odd function with respect to x and an even function with respect to y . This shows that $u_s(x,y)$ is point symmetric with respect to x and line symmetric with respect to y .



Next, when $-1/2 \leq x \leq 1/2$, the 3D view of $u_s(x,y)$ at $y=100 \sim 107$ and $y=10000 \sim 10007$ are drawn respectively as follows.



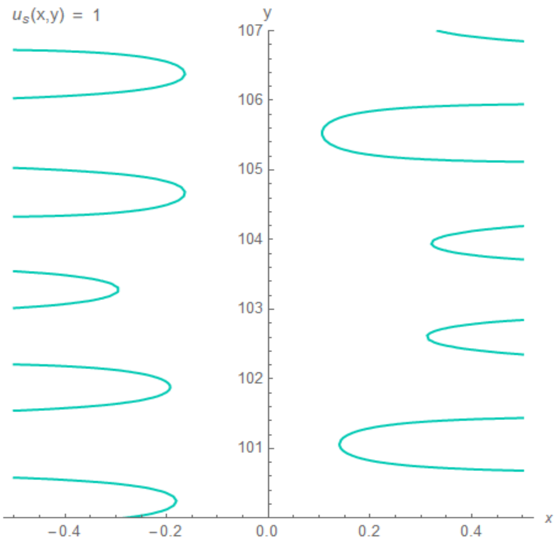


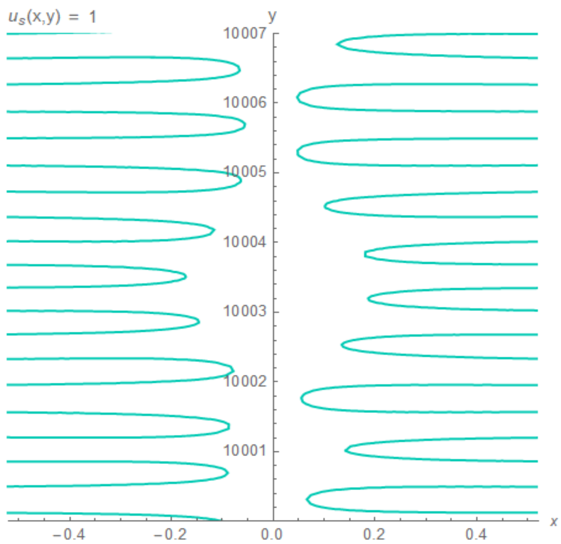
In both figures, the upper part looks like ∪ and the lower part looks like ∩. Then, we can see that both ∪ and ∩ generally have larger curvatures in the right figure than in the left figure. This is because mountains and valleys are generally steeper in the right figure than in the left figure according to Law 4.4.5 .

In addition, the right figure has more mountains and valleys than the left figure (about twice as many), but the reason for this is unknown.

4.6 Contour line of $v_c(x,y)$ with height 1

The height 1 contours of the two 3D views of $u_s(x,y)$ in the previous section are drawn as follows.





The left figure is $y=100 \sim 107$ and the right figure is $y=10000 \sim 10007$.

In both figures, the contour line looks like ⊃ & ⊂. Then, we can see that both ⊃ & ⊂ are generally closer to the y-axis in the right figure than in the left figure. This is because mountains and valleys are generally steeper in the right figure than in the left figure according to Law 4.4.5 . Therefore, as $|y|$ increases, the tips ⊃ ⊂ of the contour approach the y -axis from both sides.

5 Contour Lines of $v_c(x,y)$, $u_s(x,y)$ and the Transitions

5.1 Contour Lines of $v_c(x,y)$, $u_s(x,y)$

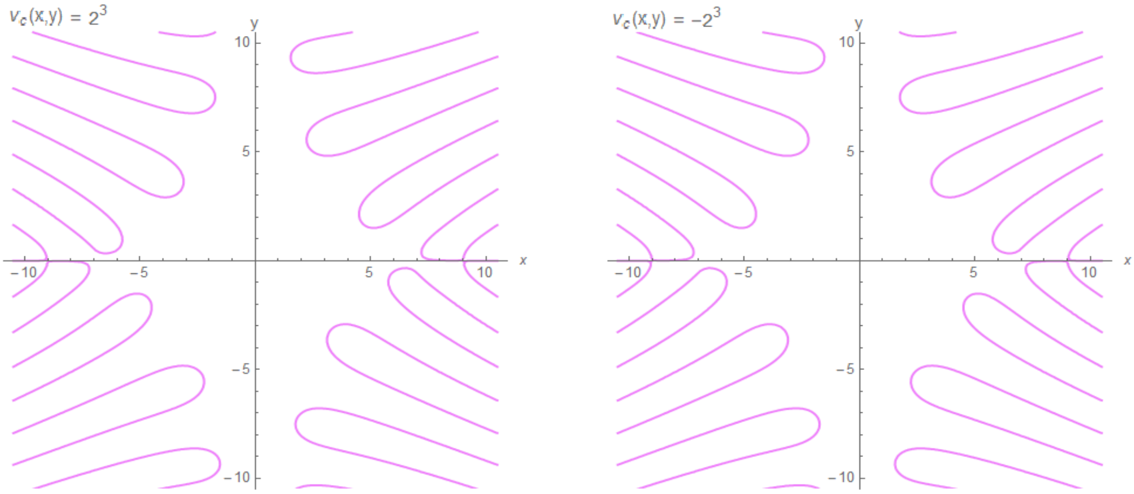
The functions $v_c(x,y)$, $u_s(x,y)$ of Propositions 2.4 were as follows, respectively.

$$v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\} sin\{y\,log(2r-1)\} \tag{2.4c}$$

$$u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\} cos\{y\,log(2r-1)\} \tag{2.4s}$$

5.1.1 Contour line of $v_c(x,y)$ with height ± 8

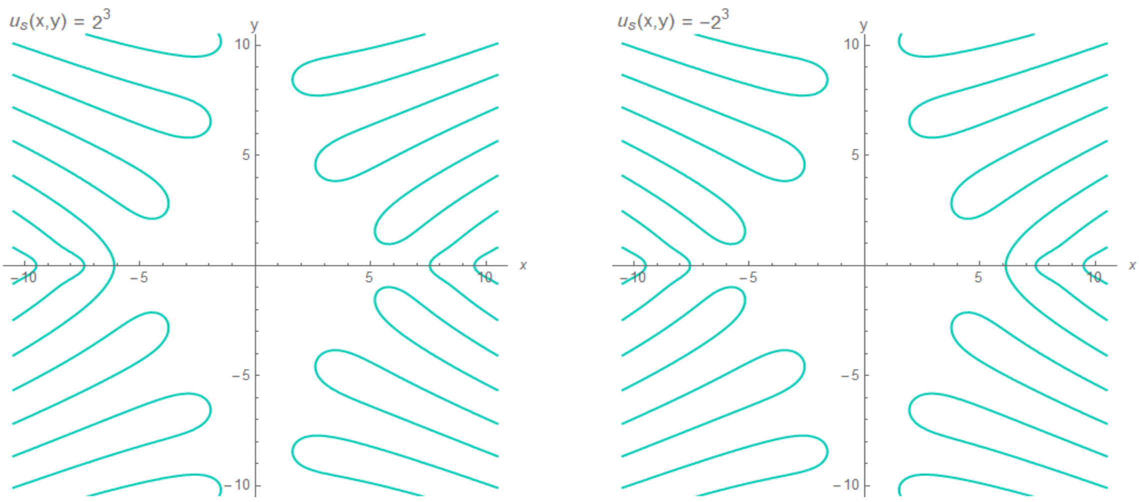
Contour plots of $v_c(x,y)$ at height ± 8 are as follows. The left figure is $+8$ and the right figure is -8 .



Since $v_c(x,y)$ is an odd function with respect to both x and y , the left and right figures have a mirror image relationship with respect to both the y -axis and the x -axis.

5.1.2 Contour line of $u_s(x,y)$ with height ± 8

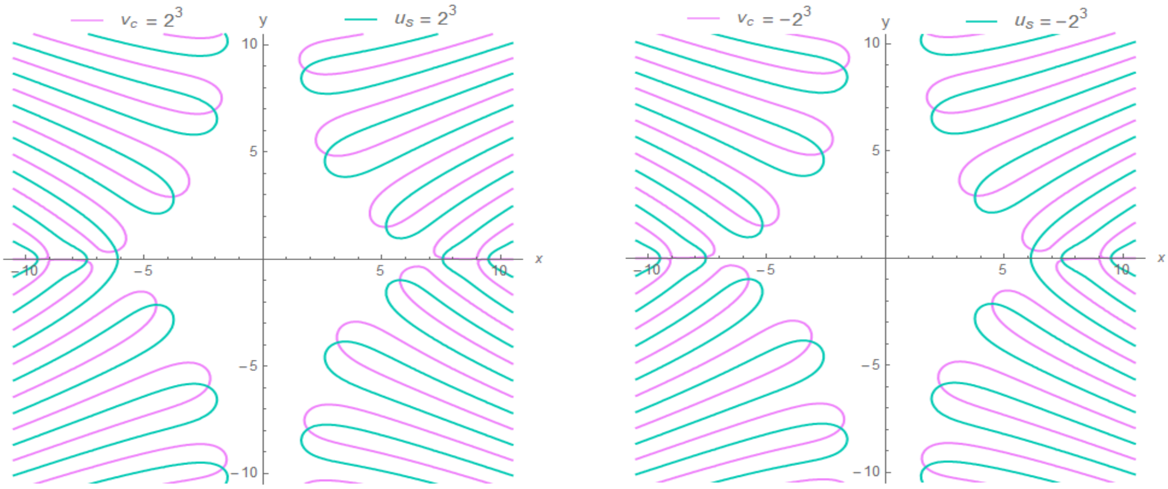
Contour plots of $u_s(x,y)$ at height ± 8 are as follows. The left figure is $+8$ and the right figure is -8 .



Since $u_s(x,y)$ is an odd function with respect to x the left and right figures have a mirror image relationship with respect to the y -axis

5.1.3 Contour lines of $v_c(x,y)$, $u_s(x,y)$ with height ± 8

When **5.1.1** and **5.1.2** are overlapped, it becomes as follows.



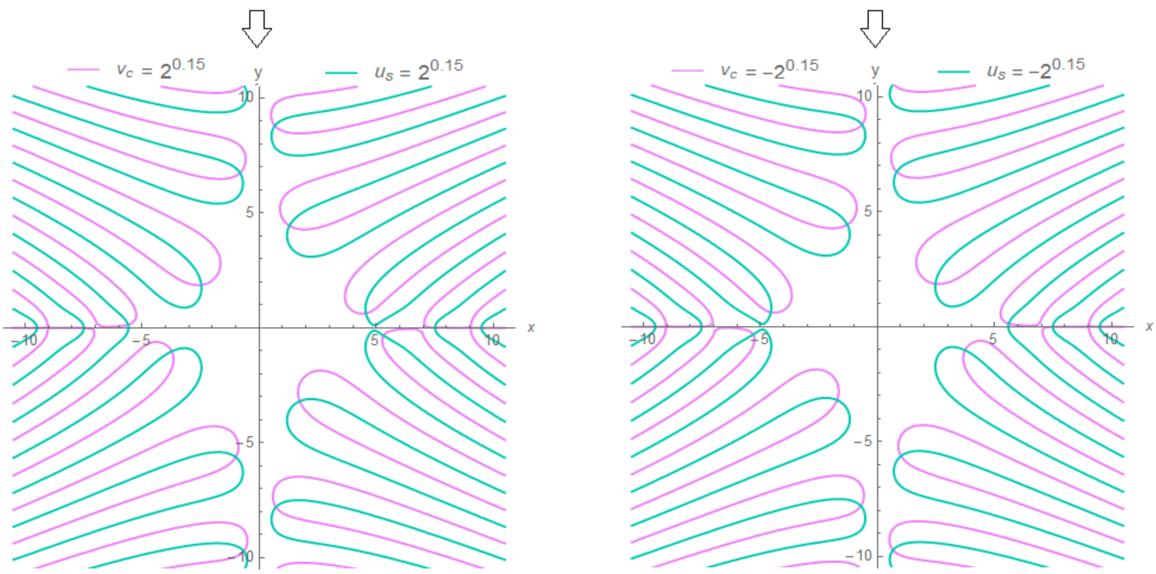
Since $v_c(x,y), u_s(x,y)$ are odd functions with respect to x the left and right figures have a mirror image relationship with respect to the y -axis. Both figures never overlap by translation or rotation in the plane.

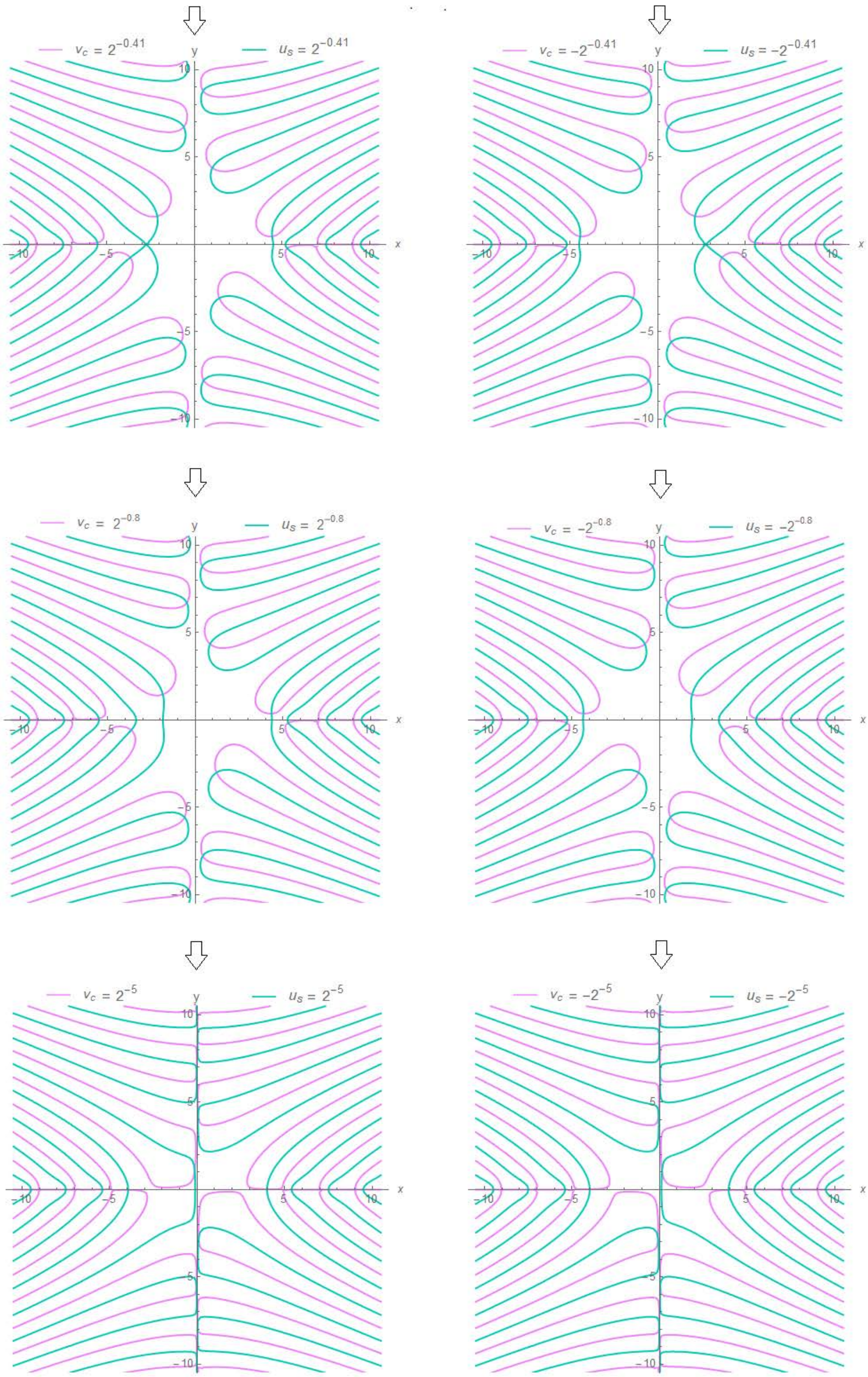
5.2 Transitions of contour lines of $v_c(x,y)$, $u_s(x,y)$

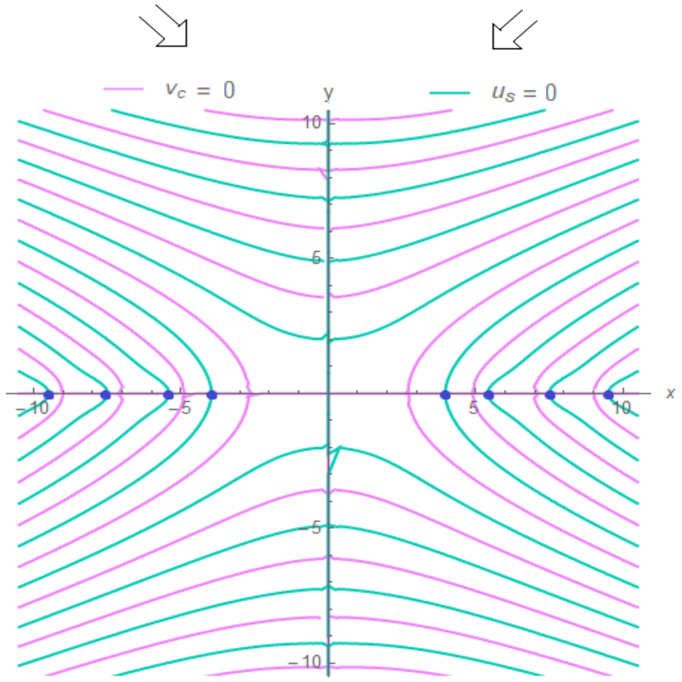
Nevertheless, at height ± 0 , the left and right figures have to overlap with no translation or rotation. For the purpose, the contour lines in both figures have to be deformed as the height approaches ± 0 from above and below. And, at height ± 0 , both figures must be symmetrical about both the y -axis and the x -axis.

This forces contour lines that were alternate at height $\neq 0$ to be opposite at height ± 0 . This also applies to the x -axis. Thus, At height ± 0 , the right and left edges of $\supset$ $\subset$ must be absorbed into the y -axis, and the lower and upper edges of $\cup$ $\cap$ must be absorbed into the x -axis.

In fact, when the height is changed to $\pm 2^{0.15}$, $\pm 2^{-0.41}$, $\pm 2^{-0.8}$, $\pm 2^{-5}$, $\pm 2^{-\infty}$, the above figures are deformed as follows.




$v_c = 2^{-0.41}$
$u_s = 2^{-0.41}$
$v_c = -2^{-0.41}$
$u_s = -2^{-0.41}$
$v_c = 2^{-0.8}$
$u_s = 2^{-0.8}$
$v_c = -2^{-0.8}$
$u_s = -2^{-0.8}$
$v_c = 2^{-5}$
$u_s = 2^{-5}$
$v_c = -2^{-5}$
$u_s = -2^{-5}$
x
y
-10
-5
5
10



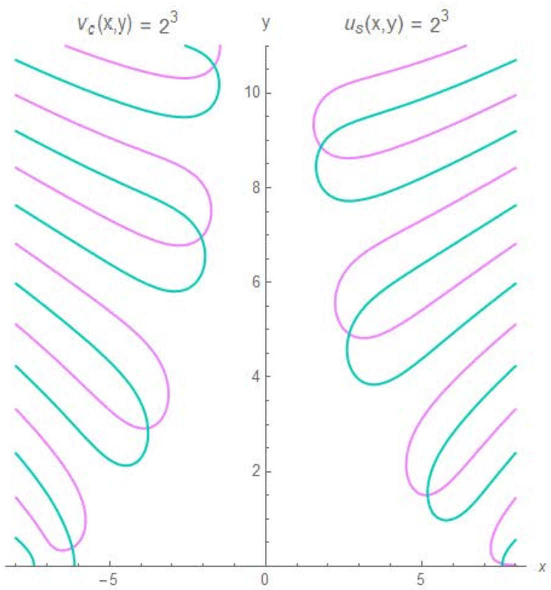
For the animation of the above, click here. AnimB5218.gif

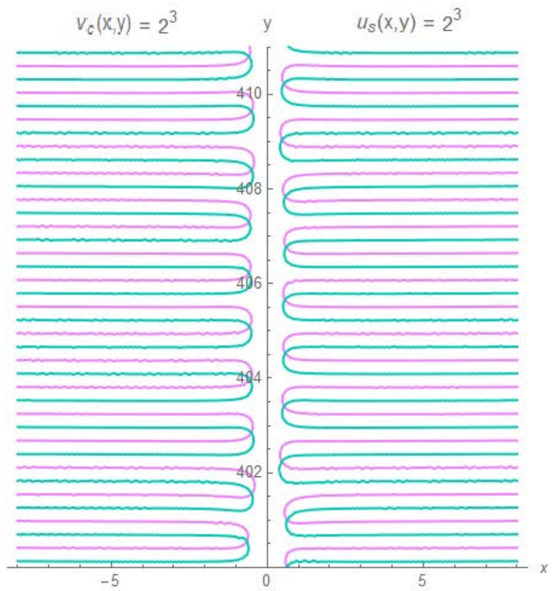
Consistent with theory, the contour parts asymmetric with respect to the y and x -axis were absorbed in both axes. As the result,

(1) Trivial solutions $(\pm 3.9709, 0), (\pm 5.41062, 0), (\pm 7.50465, 0), \cdots$ (blue points) of $v_c = u_s = 0$ arose countless on the x -axis. However, they do not satisfy $u_c = 0$

(2) All solutions (intersections of v_c & u_s) except (1) moved on the y -axis.

The figures above are for $|y| \le 10.5$, but what about when $|y|$ is large ? As an example, drawing contour lines of height 8 of v_c & u_s for $y = 0 \sim 11$ and $y = 400 \sim 411$ are as follows. The left figure is $y = 0 \sim 11$ and the right figure is $y = 400 \sim 411$. It is observed that both $\supset$ and $\subset$ are generally closer to the y -axis in the right figure than in the left figure. As stated in the previous two chapters, this is due to Law 3.4.5 and Law 4.4.5 .





Both figures show that the above phenomenon (2) becomes more pronounced where $|y|$ is large. That is,
The above (2) occurs in the whole domain $|y| > 0$.

So, the system of equations $v_c(x,y) = u_s(x,y) = 0$ has no solution in the critical strip $-1/2 < x < 1/2$ except on the critical line $x = 0$.

Note

$x = 0$ is equivalent to the absence of v_c and u_s .

6 Proof of the Riemann Hypothesis for the Dirichlet Beta Function

In this chapter, we prove the Riemann hypothesis for the Dirichlet Beta Function by summarizing the above.

Proposition 6.1 (Riemann Hypothesis)

Let $\beta(z)$ be the function defined by the following Dirichlet series.

$$\beta(z) = \sum_{r=1}^{\infty} e^{-z\log r} = \frac{1}{1^z} - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + - \cdots \qquad Re(z) > 1 \qquad (1.\beta)$$

This function has no non-trivial zeros except on the critical line $Re(z) = 1/2$.

Proof

First, by the functional equation, the solution for $\beta(z) = 0$ is consistent with the solution of the following system of equations. (Lemma 2.1)

$$\begin{cases} \beta(z) = \displaystyle\sum_{r=1}^{\infty} (-1)^{r-1} e^{-z\log(2r-1)} = 0 \\ \beta(1-z) = \displaystyle\sum_{r=1}^{\infty} (-1)^{r-1} e^{-(1-z)\log(2r-1)} = 0 \end{cases} \qquad 0 < Re(z) < 1$$

Second, by translation, the solution for $\beta(1/2+z) = 0$ is consistent with the solution of the following system of equations. (Lemma 2.1 ')

$$\begin{cases} \beta\left(\dfrac{1}{2}+z\right) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} e^{-z\log(2r-1)} = 0 \\ \beta\left(\dfrac{1}{2}-z\right) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} e^{z\log(2r-1)} = 0 \end{cases} \qquad -\frac{1}{2} < Re(z) < \frac{1}{2}$$

Third, by addition and subtraction, the solution for $\beta(1/2+z) = 0$ is consistent with the solution of the following system of equations. (Lemma 2.2)

$$\begin{cases} \beta_c(z) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \cosh\{z\log(2r-1)\} = 0 \\ \beta_s(z) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{z\log(2r-1)\} = 0 \end{cases} \qquad -\frac{1}{2} < Re(z) < \frac{1}{2}$$

Last, expressing these by real and imaginary parts, we obtain the following theorem.

Theorem 2.3 (reprint)

When the set of real numbers is R and Dirichlet Beta functions is $\beta(z)$ $(z = x+iy,\ x,y \in R)$, $\beta(1/2 \pm z) = 0$ in $-1/2 < x < 1/2$ if and only if the following system of equations has a solution on the domain..

$$\begin{cases} u_c(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \cosh\{x\log(2r-1)\}\cos\{y\log(2r-1)\} = 0 \\ v_c(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{x\log(2r-1)\}\sin\{y\log(2r-1)\} = 0 \\ u_s(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{x\log(2r-1)\}\cos\{y\log(2r-1)\} = 0 \\ v_s(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \cosh\{x\log(2r-1)\}\sin\{y\log(2r-1)\} = 0 \end{cases}$$

According to this theorem, if a system of equations consisting of any two of these equations does not have a solution in the critical strip except on the critical line, the Riemann hypothesis holds. Therefore, the following proposition equivalent to the Riemann hypothesis can be presented.

Proposition 2.4 (reprint)

When y is a real number, x is a real number s.t. $-1/2 < x < 1/2$, the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} v_c(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\} sin\{y\,log(2r-1)\} = 0 & (2.4c) \\ u_s(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x\,log(2r-1)\} cos\{y\,log(2r-1)\} = 0 & (2.4s) \end{cases}$$

From **3.1** to **5.2**, as evidenced mainly by figures, this system of equations has no solution in the critical strip $(-1/2 < x < 1/2)$ except for the critical line $(x = 0)$.

Thus since Proposition 2.4 has been graphically proved, according to Theorem 2.3 , the Riemann hypothesis for the Dirichlet Beta Function holds. Q.E.D.

Appendix

The 2 functions $v_c(x,y), u_s(x,y)$ that are central to this paper are expressed by the following formulas using the Dirichlet Beta function $\beta(x,y)$.

$$v_c(x,y) = \frac{1}{2}\left[Im\left\{ \beta\left(\frac{1}{2} - x - iy \right) \right\} + Im\left\{ \beta\left(\frac{1}{2} + x + iy \right) \right\} \right] \qquad (2.4c')$$

$$u_s(x,y) = \frac{1}{2}\left[Re\left\{ \beta\left(\frac{1}{2} - x - iy \right) \right\} - Re\left\{ \beta\left(\frac{1}{2} + x + iy \right) \right\} \right] \qquad (2.4s')$$

1 For $\eta(x,y)$

The discussion in this paper is valid even if the functions in these formulas are replaced by the Dirichlet Eta function $\eta(x,y)$. That is,

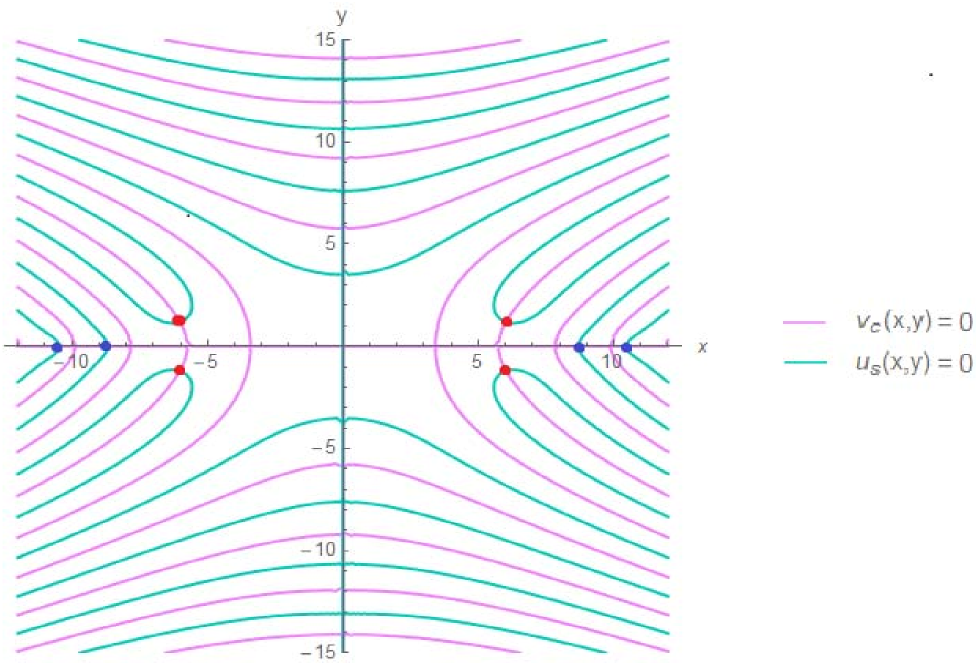
$$v_c(x,y) = \frac{1}{2}\left[Im\left\{ \eta\left(\frac{1}{2} - x - iy \right) \right\} + Im\left\{ \eta\left(\frac{1}{2} + x + iy \right) \right\} \right] \qquad (\eta.c')$$

$$u_s(x,y) = \frac{1}{2}\left[Re\left\{ \eta\left(\frac{1}{2} - x - iy \right) \right\} - Re\left\{ \eta\left(\frac{1}{2} + x + iy \right) \right\} \right] \qquad (\eta.c')$$

Where,

$$\eta(z) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-z\log(2r+1)} = \frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + - \cdots \qquad Re(z) > 0$$

Using these, the contour lines of $v_c(x,y), u_s(x,y)$ at height 0 were drawn as follows.



Blue points are the trivial solutions. These exist innumerably on the x -axis as $(\pm 8.69593, 0)$, $(\pm 10.4734, 0)$, $\cdots$.

Red points are non-trivial solutions of $v_c = u_s = 0$. They are 4 in 1 sets of $(\pm 6.01956, \pm 1.19483)$. These exist near the boundary between hyperbola and parabola, that is, around the origin. So, There are no non-trivial solutions other than these 4. Since these are outside the critical strip, the Riemann Hypothesis must hold.

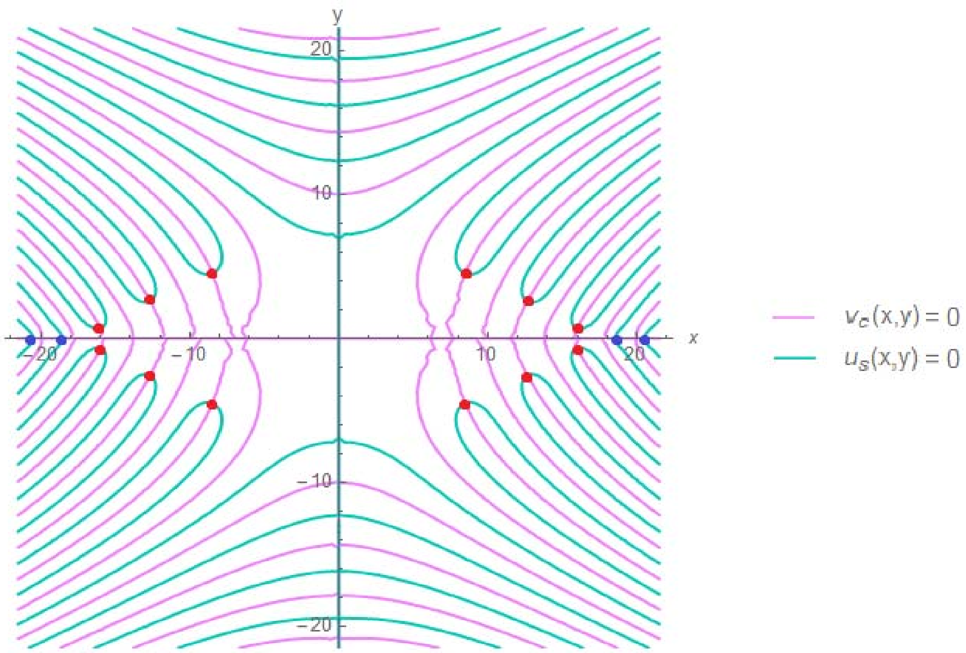
2 For $\zeta(x,y)$

The discussion in this paper is valid even if the functions in these formulas are replaced by the Riemann zeta function $\zeta(x,y)$. That is,

$$v_c(x,y) = \frac{1}{2}\left[Im\left\{ \zeta\left(\frac{1}{2} - x - i\,y \right) \right\} + Im\left\{ \zeta\left(\frac{1}{2} + x + i\,y \right) \right\} \right] \qquad (\zeta.c')$$

$$u_s(x,y) = \frac{1}{2}\left[Re\left\{ \zeta\left(\frac{1}{2} - x - i\,y \right) \right\} - Re\left\{ \zeta\left(\frac{1}{2} + x + i\,y \right) \right\} \right] \qquad (\zeta.s')$$

Using these, the contour lines of $v_c(x,y)$, $u_s(x,y)$ at height 0 were drawn as follows.



Blue points are the trivial solutions. These exist innumerably on the x -axis as $(\pm 18.5678, 0)$, $(\pm 20.4924, 0)$, … .

Red points are non-trivial solutions of $v_c = u_s = 0$. They are 12 in 3 sets of $(\pm 8.49059, \pm 4.51058)$, $(\pm 12.6627, \pm 2.58053)$, $(\pm 15.9781, \pm 0.679408)$. These exist near the boundary between hyperbola and parabola, that is, around the origin. So, there are no non-trivial solutions other than these 12.

Since these are outside the critical strip, the Riemann Hypothesis must hold.

2023.08.30

Kano Kono

Hiroshima, Japan

Alien's Mathematics