3 Higher Calculus of Binomial Identity

The simplest one of Binomial Identity is given as follows.

$$(1+x)^n = \sum_{s=0}^{n} {}_nC_s x^s \tag{0}$$

Interestingly, to both sides, if higher differentiation is carried out, factorial is obtained, and if higher integration is carried out, the beta function is obtained.

3.1 Higher Differentiation of Binomial Identity

Formula 3.1.1 (zakii)

$$\sum_{s=0}^{n} (-1)^s {}_nC_s s^k = 0 \qquad k=0,1,\cdots,n-1 \tag{1.n-1}$$

$$\sum_{s=0}^{n} (-1)^s {}_nC_s s^n = (-1)^n n! \tag{1.n}$$

Proof

$$(1+x)^n = \sum_{s=0}^{n} {}_nC_s x^s \tag{0}$$

Substituting $x=-1$ for this,

$$0 = \sum_{s=0}^{n} (-1)^s {}_nC_s = \sum_{s=0}^{n} (-1)^s {}_nC_s s^0 \tag{1.0}$$

Differrentiating the both sides of (0) with respect to x 1 time,

$$n(1+x)^{n-1} = \sum_{s=0}^{n} {}_nC_s s x^{s-1}$$

Substituting $x=-1$ for this,

$$0 = \sum_{s=0}^{n} (-1)^{s-1} {}_nC_s s = -\sum_{s=0}^{n} (-1)^s {}_nC_s s^1 \tag{1.1}$$

Differrentiating the both sides of (0) with respect to x 2 times,

$$n(n-1)(1+x)^{n-2} = \sum_{s=0}^{n} {}_nC_s s(s-1) x^{s-2}$$

Substituting $x=-1$ for this,

$$0 = \sum_{s=0}^{n} (-1)^s {}_nC_s s(s-1) = \sum_{s=0}^{n} (-1)^s {}_nC_s s^2 - \sum_{s=0}^{n} (-1)^s {}_nC_s s^1$$

Since $\sum_{s=0}^{n} (-1)^s {}_nC_s s^k = 0 \quad (k=0,1)$ from (1.0) and (1.1) ,

$$0 = \sum_{s=0}^{n} (-1)^s {}_nC_s s^2 \tag{1.2}$$

$$\vdots$$

Differrentiating the both sides of (0) with respect to x $n-1$ times ,

$$n(n-1)(n-2)\cdots 2\cdot(1+x)^1 = \sum_{s=0}^{n} {}_nC_s s(s-1)(s-2)\cdots\{s-(n-2)\} x^{s-(n-1)}$$

Substituting $x=-1$ for this,

$$n(n-1)(n-2)\cdots 2\cdot 0^1 = (-1)^{n-1} \sum_{s=0}^{n} (-1)^s {}_nC_s s(s-1)(s-2)\cdots\{s-(n-2)\}$$

Since $0^1 = 0$, this expression can be rewritten as follows using integers $c_1, c_2, \cdots, c_{n-2}$.

$$0=\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}s^{n-1}+c_{n-2}\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}s^{n-2}+\cdots+c_{1}\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}s^{1}$$

Here, since $\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}s^{k}=0 \quad (k=0,1,n-2)$,

$$0=\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}s^{n-1} \tag{1.n-1}$$

Last, differrentiating the both sides of (0) with respect to x n times ,

$$n(n-1)\cdots1\cdot(1+x)^{n-n}=\sum_{s=0}^{n}{}_{n}C_{s}s(s-1)\cdots\{s-(n-1)\}\cdot x^{s-n}$$

Substituting $x=-1$ for this,

$$n(n-1)\cdots1\cdot0^{0}=\sum_{s=0}^{n}(-1)^{s-n}{}_{n}C_{s}s(s-1)\cdots\{s-(n-1)\}$$

Since $0^{0}=1$, this expression can be rewritten as follows using integers $c_1, c_2, \cdots, c_{n-1}$.

$$(-1)^{n}n!=\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}s^{n}+c_{n-1}\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}s^{n-1}+\cdots+c_{1}\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}s^{1}$$

Here, since $\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}s^{k}=0 \quad (k=0,1,n-1)$,

$$(-1)^{n}n!=\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}s^{n} \tag{1.n}$$

c.f.

Replacing s with $n-s$ in this formula, we obtain as follows.

$$\sum_{s=0}^{n}(-1)^{n-s}{}_{n}C_{s}(n-s)^{n-1}=0$$

$$\sum_{s=0}^{n}(-1)^{n-s}{}_{n}C_{s}(n-s)^{n}=(-1)^{n}n!$$

These are corresponding to Formula 2.1.4 in " 02 Multiple Series & Exponential Function "

3.2 Higher Integration of Binomial Identity

Formula 3.2.1

$$\sum_{s=0}^{n}\frac{(-1)^s}{s+1}{}_nC_s = \frac{0!}{n+1} \tag{1.1}$$

$$\sum_{s=0}^{n}\frac{(-1)^s}{s+2}{}_nC_s = \frac{1!}{(n+1)(n+2)} \tag{1.2}$$

$$\sum_{s=0}^{n}\frac{(-1)^s}{s+3}{}_nC_s = \frac{2!}{(n+1)(n+2)(n+3)} \tag{1.3}$$

$$\vdots$$

$$\sum_{s=0}^{n}\frac{(-1)^s}{s+m}{}_nC_s = \frac{(m-1)!}{(n+1)(n+2)\cdots(n+m)} \qquad \{\ = B(1+n,m)\ \} \tag{1.m}$$

Proof

$$(1+x)^n = \sum_{s=0}^{n}{}_nC_s x^s \tag{0}$$

Integratiating the both sides of (0) with respect to x from 0 to x 1 time,

Left: $$\int_0^x (1+x)^n dx = \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^x = \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1}\frac{x^0}{0!}$$

Right: $$\int_0^x \sum_{s=0}^{n}{}_nC_s x^s dx = \sum_{s=0}^{n}{}_nC_s\left[\frac{x^{s+1}}{s+1}\right]_0^x = \sum_{s=0}^{n}{}_nC_s\frac{x^{s+1}}{s+1}$$

Substituting $x=-1$ for this,

$$\frac{1}{n+1} = \sum_{s=0}^{n}\frac{(-1)^s}{s+1}{}_nC_s \tag{1.1}$$

Integratiating the both sides of (0) with respect to x from 0 to x 2 times,

Left: $$\int_0^x\int_0^x (1+x)^n dx^2 = \left[\frac{(1+x)^{n+2}}{(n+1)(n+2)} - \frac{1}{n+1}\frac{x^1}{1!}\right]_0^x$$

$$= \frac{(1+x)^{n+2}}{(n+1)(n+2)} - \frac{1}{n+1}\frac{x^1}{1!} - \frac{1}{(n+1)(n+2)}\frac{x^0}{0!}$$

Right: $$\int_0^x\int_0^x \sum_{s=0}^{n}{}_nC_s\frac{x^{s+1}}{s+1}dx^2 = \sum_{s=0}^{n}{}_nC_s\frac{x^{s+2}}{(s+1)(s+2)}$$

Substituting $x=-1$ for this,

$$\frac{1}{n+1} - \frac{1}{(n+1)(n+2)} = \sum_{s=0}^{n}\frac{(-1)^s}{(s+1)(s+2)}{}_nC_s$$

Here, the following partial fraction decomposition holds.

$$\frac{1}{(s+1)(s+2)} = \frac{1}{1!}\left(\frac{{}_1C_0}{s+1} - \frac{{}_1C_1}{s+2}\right)$$

Using this,

$$\frac{1}{n+1}-\frac{1}{(n+1)(n+2)}=\frac{{}_1C_0}{1!}\sum_{s=0}^{n}\frac{(-1)^s}{s+1}{}_nC_s-\frac{{}_1C_1}{1!}\sum_{s=0}^{n}\frac{(-1)^s}{s+2}{}_nC_s$$

Substituting (1.1) for this,

$$\frac{1!}{(n+1)(n+2)}=\sum_{s=0}^{n}\frac{(-1)^s}{s+2}{}_nC_s \tag{1.2}$$

Integratiating the both sides of (0) with respect to x from 0 to x 3 times,

Left:

$$\int_0^x\int_0^x\int_0^x(1+x)^n dx^3=\frac{(1+x)^{n+3}}{(n+1)(n+2)(n+3)}$$

$$-\frac{1}{n+1}\frac{x^2}{2!}-\frac{1}{(n+1)(n+2)}\frac{x^1}{1!}-\frac{1}{(n+1)(n+2)(n+3)}\frac{x^0}{0!}$$

Right:

$$\int_0^x\int_0^x\int_0^x\sum_{s=0}^{n}{}_nC_s\frac{x^{s+2}}{(s+1)(s+2)}dx^3=\sum_{s=0}^{n}{}_nC_s\frac{x^{s+3}}{(s+1)(s+2)(s+3)}$$

Substituting $x=-1$ for this,

$$-\frac{1}{n+1}\frac{1}{2!}+\frac{1}{(n+1)(n+2)}\frac{1}{1!}-\frac{1}{(n+1)(n+2)(n+3)}\frac{1}{0!}$$

$$=-\sum_{s=0}^{n}\frac{(-1)^s}{(s+1)(s+2)(s+3)}{}_nC_s$$

Here, the following partial fraction decomposition holds.

$$\frac{1}{(s+1)(s+2)(s+3)}=\frac{1}{2!}\left(\frac{{}_2C_0}{s+1}-\frac{{}_2C_1}{s+2}+\frac{{}_2C_2}{s+3}\right)$$

Using this,

$$-\frac{1}{n+1}\frac{1}{2!}+\frac{1}{(n+1)(n+2)}\frac{1}{1!}-\frac{1}{(n+1)(n+2)(n+3)}\frac{1}{0!}$$

$$=-\frac{{}_2C_0}{2!}\sum_{s=0}^{n}\frac{(-1)^s}{s+1}{}_nC_s+\frac{{}_2C_1}{2!}\sum_{s=0}^{n}\frac{(-1)^s}{s+2}{}_nC_s-\frac{{}_2C_2}{2!}\sum_{s=0}^{n}\frac{(-1)^s}{s+3}{}_nC_s$$

Substituting (1.1) and (1.2) for this,

$$\frac{2!}{(n+1)(n+2)(n+3)}=\sum_{s=0}^{n}\frac{(-1)^s}{s+3}{}_nC_s \tag{1.3}$$

Hereafter, in a similar way, we obtain

$$\frac{(m-1)!}{(n+1)(n+2)\cdots(n+m)}=\sum_{s=0}^{n}\frac{(-1)^s}{s+m}{}_nC_s \tag{1.m}$$

And,

$$\frac{(m-1)!}{(n+1)(n+2)\cdots(n+m)}=\frac{n!\,(m-1)!}{(n+m)!}=\frac{\Gamma(1+n)\Gamma(m)}{\Gamma(1+n+m)}=B(1+n,m)$$

Example: $n=5$

When this is calculated with the expression software, it is as follows.

Sum of Fractions with alternating sign

```
fl := n-> sum((-1)^s/(s+m)*binomial(n,s), s=0..n)
```

$$n\to\sum_{s=0}^{n}\frac{(-1)^s}{s+m}\cdot\binom{n}{s}$$

- `fl(5)`

$$\frac{10}{m+2}-\frac{5}{m+1}-\frac{10}{m+3}+\frac{5}{m+4}-\frac{1}{m+5}+\frac{1}{m}$$

Factorial x Product of Fractions

- `fr := n-> (m-1)!*product(1/(n+s), s=1..m)`

$$n \rightarrow (m-1)!\cdot\left(\prod_{s=1}^{m}\frac{1}{n+s}\right)$$

- `fr(5)`

$$\frac{120\cdot(m-1)!}{(m+5)!}$$

Verification of equivalence

- `testeq(fl(5),fr(5))`

TRUE

c.f.

The following expressions follow from (1.m) .

$$\sum_{r=0}^{n-1}\frac{(-1)^r}{m+r}\,{}_{n-1}C_r = B(m,n)$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r}{p+r}\binom{q-1}{r} = B(p,q)$$

These are corresponding to Formula 7.3.5 and Formula 7.3.6 in " 07 Super Integral ".

By-products (Partial Fraction Decomposition)

The following expression holds for $s \neq -1,-2,-3,\cdots$.

$$\prod_{t=1}^{n}\frac{1}{s+t} = \frac{1}{(n-1)!}\sum_{t=1}^{n}\frac{(-1)^{t-1}}{s+t}\binom{n-1}{t-1}$$

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K. Kono

Alien's Mathematics