20 Higher Calculus of the product of many functions

20.1 Higher Derivative of the product of many functions

(1) Binomial Theorem and Leibniz Rule

According to the binomial theorem in **3.1** , the following expression holds for real numbers x_1, x_2 and natural number n .

$$(x_1+x_2)^n = \sum_{r=0}^{n} \binom{n}{r} x_1^{n-r} x_2^r$$

On the other hand, according to the Leibniz Rule, the following expression holds for functions f_1, f_2 of x and natural number n .

$$(f_1 f_2)^{(n)} = \sum_{r=0}^{n} \binom{n}{r} f_1^{(n-r)} f_2^{(r)}$$

(2) Multinomial Theorem and Higher Derivative of the product of many functions

According to the multiomial theorem in **3.3** , the following expression holds for real numbers $x_1, x_2, \cdots, x_\lambda$ and natural number n .

$$(x_1+x_2+\cdots+x_\lambda)^n = \sum_{r_1=0}^{n} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{n}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} x_1^{n-r_1} x_2^{r_1-r_2} \cdots x_\lambda^{r_{\lambda-1}}$$

Therefore, the following expression must hold for functions $f_1, f_2, \cdots, f_\lambda$ of x and natural number n .

$$(f_1 f_2 \cdots f_\lambda)^{(n)} = \sum_{r_1=0}^{n} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{n}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{(n-r_1)} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})}$$

20.1.1 Higher Derivative of the product of many functions

Theorem 20.1.1

When $f_k^{(r)}$ denotes the r th order derivative function of $f_k(x)$ $(k=1,2,\cdots,\lambda)$,

$$(f_1 f_2 \cdots f_\lambda)^{(n)} = \sum_{r_1=0}^{n} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{n}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{(n-r_1)} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})}$$

Proof

According to Theorem 18.1.1 (Leibniz) in **18.1** , the following expressions hold.

$$(f_1 f_2 f_3 f_4 \cdots f_\lambda)^{(n)} = \sum_{r_1=0}^{n} \binom{n}{r_1} f_1^{(n-r_1)} (f_2 f_3 f_4 \cdots f_\lambda)^{(r_1)} \tag{1}$$

$$(f_2 f_3 f_4 \cdots f_\lambda)^{(r_1)} = \sum_{r_2=0}^{r_1} \binom{r_1}{r_2} f_2^{(r_1-r_2)} (f_3 f_4 \cdots f_\lambda)^{(r_2)} \tag{2}$$

$$(f_3 f_4 \cdots f_\lambda)^{(r_2)} = \sum_{r_3=0}^{r2} \binom{r_2}{r_3} f_3^{(r_2-r_3)} (f_4 \cdots f_\lambda)^{(r_3)} \tag{3}$$

$\vdots$

$$\left(f_{\lambda-2}f_{\lambda-1}f_\lambda\right)^{(r_{\lambda-3})} = \sum_{r_{\lambda-2}=0}^{r_{\lambda-3}} \binom{r_{\lambda-3}}{r_{\lambda-2}} f_{\lambda-2}^{(r_{\lambda-3}-r_{\lambda-1})} \left(f_{\lambda-1}f_\lambda\right)^{(r_{\lambda-2})} \tag{λ-2}$$

$$\left(f_{\lambda-1}f_\lambda\right)^{(r_{\lambda-2})} = \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{r_{\lambda-2}}{r_{\lambda-1}} f_{\lambda-1}^{(r_{\lambda-2}-r_{\lambda-1})} f_\lambda^{(r_{\lambda-1})} \tag{λ-1}$$

Substituting (2), (3), ··· , (λ-2), (λ-1) for (1) one by one, we obtain the desired expression.

Example

$$\left(f_1f_2f_3\right)^{(n)} = \sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s} f_1^{(n-r)}f_2^{(r-s)}f_3^{(s)}$$

$$\left(f_1f_2f_3f_4\right)^{(n)} = \sum_{r=0}^{n}\sum_{s=0}^{r}\sum_{t=0}^{s}\binom{n}{r}\binom{r}{s}\binom{s}{t} f_1^{(n-r)}f_2^{(r-s)}f_3^{(s-t)}f_4^{(t)}$$

Example computation of the product of three functions

Since the combination of the product of many functions are numerous, we cannot calculate these one by one. Then we pick up some and calculate them.

Example1 Higher Derivative of $x^\alpha e^x \sin x$

Let $f_1 = x^\alpha$, $f_2 = e^x$, $f_3 = \sin x$. Then

$$\left(x^\alpha\right)^{(n-r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n+r)}x^{\alpha-n+r}$$

$$\left(e^x\right)^{(r-s)} = e^x$$

$$(\sin x)^{(s)} = \sin\left(x+\frac{s\pi}{2}\right)$$

Substituting these for Theorem 20.1.1 , we obtain

$$\left(x^\alpha e^x \sin x\right)^{(n)} = \sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n+r)}x^{\alpha-n+r}e^x\sin\left(x+\frac{s\pi}{2}\right) \tag{1.1}$$

Although this formula is troublesome in manual calculation, it is easy in mathematical software. When the 4th order derivative of $x^\alpha e^x \sin x$ is calculated using the mathematical software MuPad, it is as follows.

Differentiation by direct calculation

```
• fl := diff(x^a*E^x*sin(x) ,x,x,x,x):
• simplify(fl)
```

$$x^{a-4}\cdot e^x\cdot(11\cdot a^2\cdot\sin(x)-6\cdot a^3\cdot\sin(x)+a^4\cdot\sin(x)-4\cdot x^4\cdot\sin(x)-6\cdot a\cdot\sin(x)-12\cdot a\cdot x^2\cdot\cos(x)$$
$$-12\cdot a^2\cdot x\cdot\cos(x)+8\cdot a\cdot x^3\cdot\cos(x)+4\cdot a^3\cdot x\cdot\cos(x)-12\cdot a^2\cdot x\cdot\sin(x)-8\cdot a\cdot x^3\cdot\sin(x)$$
$$+4\cdot a^3\cdot x\cdot\sin(x)+12\cdot a^2\cdot x^2\cdot\cos(x)+8\cdot a\cdot x\cdot\cos(x)+8\cdot a\cdot x\cdot\sin(x))$$

Differentiation by the formula

```
• n:=4:
• delete r: fr:=0:
• for r from 0 to n do
     fs:=0:
     for s from 0 to r do
```

```
      fs := fs + binomial(n,r)*binomial(r,s)
                               *gamma(1+a)/gamma(1+a-n+r)*x^(a-n+r)
                               *E^x*sin(x+s*PI/2)
    end_for:
    fr:=fr+fs
  end_for:
• fr := expand(fr):
• simplify(fr)
```

$$x^{a-4}\cdot e^x\cdot(11\cdot a^2\cdot\sin(x)-6\cdot a^3\cdot\sin(x)+a^4\cdot\sin(x)-4\cdot x^4\cdot\sin(x)-6\cdot a\cdot\sin(x)-12\cdot a\cdot x^2\cdot\cos(x)$$
$$-12\cdot a^2\cdot x\cdot\cos(x)+8\cdot a\cdot x^3\cdot\cos(x)+4\cdot a^3\cdot x\cdot\cos(x)-12\cdot a^2\cdot x\cdot\sin(x)-8\cdot a\cdot x^3\cdot\sin(x)$$
$$+4\cdot a^3\cdot x\cdot\sin(x)+12\cdot a^2\cdot x^2\cdot\cos(x)+8\cdot a\cdot x\cdot\cos(x)+8\cdot a\cdot x\cdot\sin(x))$$

Verification of the equivalence

```
• testeq(fl,fr)
```

TRUE

Example2 Higher Derivative of $x^\alpha e^x \log x$

Let $f_1 = x^\alpha$, $f_2 = e^x$, $f_3 = \log x$. Then

$$\left(x^\alpha\right)^{(n-r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n+r)}x^{\alpha-n+r}, \qquad \left(e^x\right)^{(r-s)} = e^x$$

$$(\log x)^{(0)} = \log x \quad , \quad (\log x)^{(s)} = (-1)^{s-1}(s-1)!\,x^{-s}\ (s=1,2,3,\cdots)$$

Separating the terms containing $f_3^{(0)}$ from Theorem20.1.1 , we obtain

$$\left(f_1 f_2 f_3\right)^{(n)} = \sum_{r=0}^{n}\binom{n}{r} f_1^{(n-r)} f_2^{(r-s)} f_3^{(0)} + \sum_{r=1}^{n}\sum_{s=1}^{r}\binom{n}{r}\binom{r}{s} f_1^{(n-r)} f_2^{(r-s)} f_3^{(s)}$$

Substituting the abave expressions for this, we obtain

$$\left(x^\alpha e^x \log x\right)^{(n)} = \sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n+r)}x^{\alpha-n+r}e^x\log x$$
$$+\sum_{r=1}^{n}\sum_{s=1}^{r}\binom{n}{r}\binom{r}{s}(-1)^{s-1}\frac{\Gamma(1+\alpha)\Gamma(s)}{\Gamma(1+\alpha-n+r)}x^{\alpha-n+r-s}e^x \tag{1.2}$$

When $n=1$

$$\left(x^\alpha e^x \log x\right)^{(1)} = \sum_{r=0}^{1}\binom{1}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-1+r)}x^{\alpha-1+r}e^x\log x$$
$$+\sum_{r=1}^{1}\sum_{s=1}^{r}\binom{1}{r}\binom{r}{s}(-1)^{s-1}\frac{\Gamma(1+\alpha)\Gamma(s)}{\Gamma(1+\alpha-1+r)}x^{\alpha-1+r-s}e^x$$
$$=\left\{\binom{1}{0}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-1)}x^{\alpha-1}+\binom{1}{1}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)}x^\alpha\right\}e^x\log x$$
$$+\sum_{r=1}^{1}\sum_{s=1}^{1}\binom{1}{1}\binom{1}{1}(-1)^{1-1}\frac{\Gamma(1+\alpha)\Gamma(1)}{\Gamma(1+\alpha)}x^{\alpha-1}e^x$$
$$=\left(\alpha x^{\alpha-1}+x^\alpha\right)e^x\log x+x^{\alpha-1}e^x$$

When $n=2$

$$\left(x^{\alpha}e^{x}\log x\right)^{(2)}=\sum_{r=0}^{2}\binom{2}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-2+r)}x^{\alpha-2+r}e^{x}\log x$$
$$+\sum_{r=1}^{2}\sum_{s=1}^{r}\binom{2}{r}\binom{r}{s}(-1)^{s-1}\frac{\Gamma(1+\alpha)\Gamma(s)}{\Gamma(1+\alpha-2+r)}x^{\alpha-2+r-s}e^{x}$$
$$=\left\{\binom{2}{0}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-2)}x^{\alpha-2}+\binom{2}{1}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-1)}x^{\alpha-1}+\binom{2}{2}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)}x^{\alpha}\right\}e^{x}\log x$$
$$+\binom{2}{1}\binom{1}{1}\frac{\Gamma(1+\alpha)\Gamma(1)}{\Gamma(1+\alpha-1)}x^{\alpha-2}e^{x}$$
$$+\binom{2}{2}\left\{\binom{2}{1}\frac{\Gamma(1+\alpha)\Gamma(1)}{\Gamma(1+\alpha)}x^{\alpha-1}e^{x}-\binom{2}{2}\frac{\Gamma(1+\alpha)\Gamma(2)}{\Gamma(1+\alpha)}x^{\alpha-2}e^{x}\right\}$$
$$=\left\{\alpha(\alpha-1)x^{\alpha-2}+2\alpha x^{\alpha-1}+x^{\alpha}\right\}e^{x}\log x$$
$$+\left(2\alpha x^{\alpha-2}+2x^{\alpha-1}-x^{\alpha-2}\right)e^{x}$$

20.1.2 Higher Derivative of the power of a function

Especially, when $f_1=f_2=\cdots=f_\lambda$ in Theorem 20.1.1 , the following theorem follows immediately.

Theorem 20.1.2

When $f^{(r)}$ denotes the r th order derivativefunction of $f(x)$ and λ is a natural number,

$$\left\{f^{\lambda}(x)\right\}^{(n)}=\sum_{r_1=0}^{n}\sum_{r_2=0}^{r_1}\cdots\sum_{r_{\lambda-1}=0}^{r_{\lambda-2}}\binom{n}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{\lambda-2}}{r_{\lambda-1}}f^{(n-r_1)}f^{(r_1-r_2)}\cdots f^{(r_{\lambda-1})}$$

Example1 Higher Derivative of $(e^x)^\lambda$

Since $(e^x)^{(r)}=e^x$, immediately from the theorem,

$$\left\{\left(e^{x}\right)^{\lambda}\right\}^{(n)}=\left(e^{x}\right)^{\lambda}\sum_{r_1=0}^{n}\sum_{r_2=0}^{r_1}\cdots\sum_{r_{\lambda-1}=0}^{r_{\lambda-2}}\binom{n}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{\lambda-2}}{r_{\lambda-1}}$$

Here, from (1.1") in **3.3** ,

$$\sum_{r_1=0}^{n}\sum_{r_2=0}^{r_3}\cdots\sum_{r_{\lambda-1}=0}^{r_{\lambda-2}}{}_{n}C_{r_1}\,{}_{r_1}C_{r_2}\cdots{}_{r_{\lambda-2}}C_{r_{\lambda-1}}=\lambda^{n}$$

Substituting this for the above, we obtain

$$\left\{\left(e^{x}\right)^{\lambda}\right\}^{(n)}=\lambda^{n}e^{\lambda x} \qquad \textbf{(2.1)}$$

Example2 Higher Derivative of $\log^3 x$

Let $f=\log x$. Then

$$f^{(0)}=\log x\ ,\qquad f^{(r)}=(-1)^{r-1}(r-1)!\,x^{-r}\qquad (r=1,2,3,\cdots)$$

Since $f^{(0)}$ is different from $f^{(r)}$ $r=1,2,\cdots$ in function type, we have to separate the terms containing $f^{(0)}$ from

$$\left(f^{3}\right)^{(n)}=\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}f^{(n-r)}f^{(r-s)}f^{(s)}$$

It is as follows.

$$\sum_{r=0}^{n}\binom{n}{r}f^{(n-r)}\sum_{s=0}^{r}\binom{r}{s}f^{(r-s)}f^{(s)}=\binom{n}{0}f^{(n)}\sum_{s=0}^{0}\binom{0}{s}f^{(0-s)}f^{(s)}$$

$$+\sum_{r=1}^{n-1}\binom{n}{r}f^{(n-r)}\sum_{s=0}^{r}\binom{r}{s}f^{(r-s)}f^{(s)}+\binom{n}{n}f^{(n-n)}\sum_{s=0}^{n}\binom{n}{s}f^{(n-s)}f^{(s)}$$

$$=f^{(n)}f^{(0)}f^{(0)}+\sum_{r=1}^{n-1}\binom{n}{r}f^{(n-r)}\sum_{s=0}^{r}\binom{r}{s}f^{(r-s)}f^{(s)}+f^{(0)}\sum_{s=0}^{n}\binom{n}{s}f^{(n-s)}f^{(s)} \tag{w}$$

Here,

$$\sum_{r=1}^{n-1}\binom{n}{r}f^{(n-r)}\sum_{s=0}^{r}\binom{r}{s}f^{(r-s)}f^{(s)}$$

$$=\binom{n}{1}f^{(n-1)}\sum_{s=0}^{1}\binom{1}{s}f^{(1-s)}f^{(s)}+\sum_{r=2}^{n-1}\binom{n}{r}f^{(n-r)}\sum_{s=0}^{r}\binom{r}{s}f^{(r-s)}f^{(s)}$$

$$=\binom{n}{1}f^{(n-1)}\left\{\binom{1}{0}f^{(1-0)}f^{(0)}+\binom{1}{1}f^{(1-1)}f^{(1)}\right\}$$

$$+\sum_{r=2}^{n-1}\binom{n}{r}f^{(n-r)}\left\{\binom{r}{0}f^{(r-0)}f^{(0)}+\sum_{s=1}^{r-1}\binom{r}{s}f^{(r-s)}f^{(s)}+\binom{r}{r}f^{(r-r)}f^{(r)}\right\}$$

$$=2f^{(0)}\binom{n}{1}f^{(n-1)}f^{(1)}+2f^{(0)}\sum_{r=2}^{n-1}\binom{n}{r}f^{(n-r)}f^{(r)}+\sum_{r=2}^{n-1}\sum_{s=1}^{r-1}\binom{n}{r}\binom{r}{s}f^{(n-r)}f^{(r-s)}f^{(s)}$$

$$=2f^{(0)}\sum_{r=1}^{n-1}\binom{n}{r}f^{(n-r)}f^{(r)}+\sum_{r=2}^{n-1}\sum_{s=1}^{r-1}\binom{n}{r}\binom{r}{s}f^{(n-r)}f^{(r-s)}f^{(s)}$$

$$f^{(0)}\sum_{s=0}^{n}\binom{n}{s}f^{(n-s)}f^{(s)}=f^{(0)}\left\{\binom{n}{0}f^{(n-0)}f^{(0)}+\sum_{s=1}^{n-1}\binom{n}{s}f^{(n-s)}f^{(s)}+\binom{n}{n}f^{(n-n)}f^{(n)}\right\}$$

$$=f^{(0)}\sum_{s=1}^{n-1}\binom{n}{s}f^{(n-s)}f^{(s)}+2f^{(0)}f^{(0)}f^{(n)}$$

Substitute these for (w) ,

$$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}f^{(n-r)}f^{(r-s)}f^{(s)}$$

$$=f^{(n)}f^{(0)}f^{(0)}+f^{(0)}\sum_{s=1}^{n-1}\binom{n}{s}f^{(n-s)}f^{(s)}+2f^{(0)}f^{(0)}f^{(n)}$$

$$+2f^{(0)}\sum_{r=1}^{n-1}\binom{n}{r}f^{(n-r)}f^{(r)}+\sum_{r=2}^{n-1}\sum_{s=1}^{r-1}\binom{n}{r}\binom{r}{s}f^{(n-r)}f^{(r-s)}f^{(s)}$$

i.e.

$$\left\{f^3(x)\right\}^{(n)}=3f^{(0)}f^{(0)}f^{(n)}+3f^{(0)}\sum_{r=1}^{n-1}\binom{n}{r}f^{(n-r)}f^{(r)}$$

$$+\sum_{r=2}^{n-1}\sum_{s=1}^{r-1}\binom{n}{r}\binom{r}{s}f^{(n-r)}f^{(r-s)}f^{(s)}$$

When $f(x)=\log x$

$$(\log x)^{(n-r)}=(-1)^{n-r-1}(n-r-1)!\,x^{-n+r} \qquad (r=0,1,2,\cdots,n-1)$$

$$(\log x)^{(r-s)}=(-1)^{r-s-1}(r-s-1)!\,x^{-r+s} \qquad (s=0,1,2,\cdots,r-1)$$

$$(\log x)^{(s)}=(-1)^{s-1}(s-1)!\,x^{-s} \qquad (s=1,2,3,\cdots)$$

Substituting these for the above,

$$\left\{\log^3 x\right\}^{(n)} = (-1)^{n-1}\frac{3(n-1)!}{x^n}\log^2 x + (-1)^n\frac{3\log x}{x^n}\sum_{r=1}^{n-1}\binom{n}{r}(n-r-1)!\,(r-1)!$$
$$+\frac{(-1)^{n-1}}{x^n}\sum_{r=2}^{n-1}\sum_{s=1}^{r-1}\binom{n}{r}\binom{r}{s}(n-r-1)!\,(r-s-1)!\,(s-1)!$$

Furthermore,

$$\binom{n}{r}(n-r-1)!\,(r-1)! = \frac{n!}{(n-r)!\,r!}(n-r-1)!\,(r-1)! = \frac{n!}{(n-r)r}$$
$$\binom{n}{r}\binom{r}{s}(n-r-1)!\,(r-s-1)!\,(s-1)! = \frac{n!}{(n-r)(r-s)s}$$

Using these, we obtain

$$\left\{\log^3 x\right\}^{(n)} = (-1)^{n-1}\frac{3(n-1)!}{x^n}\log^2 x + (-1)^n\frac{3\log x}{x^n}\sum_{r=1}^{n-1}\frac{n!}{(n-r)r}$$
$$+\frac{(-1)^{n-1}}{x^n}\sum_{r=2}^{n-1}\sum_{s=1}^{r-1}\frac{n!}{(n-r)(r-s)s} \tag{2.2}$$

When $n=1$

$$\left\{\log^3 x\right\}^{(1)} = (-1)^{1-1}\frac{3(1-1)!}{x^1}\log^2 x = \frac{3\log^2 x}{x^1}$$

When $n=2$

$$\left\{\log^3 x\right\}^{(2)} = (-1)^{2-1}\frac{3(2-1)!}{x^2}\log^2 x + (-1)^2\frac{3\cdot 2(2-2)!}{x^2}\log x$$
$$= -\frac{3\log^2 x}{x^2} + \frac{6\log x}{x^2}$$

When $n=3$

$$\left\{\log^3 x\right\}^{(3)} = (-1)^{3-1}\frac{3(3-1)!}{x^3}\log^2 x + (-1)^3\frac{3\log x}{x^3}\sum_{r=1}^{3-1}\frac{3!}{(3-r)r}$$
$$+\frac{(-1)^{3-1}}{x^3}\sum_{r=2}^{3-1}\sum_{s=1}^{r-1}\frac{3!}{(3-r)(r-s)s}$$
$$= \frac{6\log^2 x}{x^3} - \frac{18\log x}{x^3} + \frac{6}{x^3}$$

20.1.3 Higher Derivatives of $\cos^m x$, $\sin^m x$

Although this formula can also be derived from the Theorem20.1.2 , the proof is long and complicated. Therefore, we will use the following lemma.

Lemma1

$$\cos^m x = \frac{1}{2^m}\sum_{r=0}^{m} {}_mC_r \cos\{(m-2r)x\} \qquad m=1,2,3,\cdots \tag{3.m}$$

Proof

The following formulas are known about the power of $\cos x$. (See "岩波 数学公式 II " p190.)

$$\cos^{2m}x = \frac{1}{2^{2m-1}}\sum_{r=0}^{m-1}{}_{2m}C_r\cos\{(2m-2r)x\} + \frac{{}_{2m}C_m}{2^{2m}}$$

$$\cos^{2m+1}x = \frac{1}{2^{2m}}\sum_{r=0}^{m}{}_{2m+1}C_r\cos\{(2m+1-2r)x\}$$

When the pwer is even,

$$\sum_{r=0}^{m-1}{}_{2m}C_r\cos\{(2m-2r)x\} = \sum_{r=m+1}^{2m}{}_{2m}C_r\cos\{(2m-2r)x\}$$

$${}_{2m}C_m = {}_{2m}C_m\cos\{(2m-2m)x\}$$

Using these,

$$\cos^{2m}x = \frac{1}{2^{2m-1}}\sum_{r=0}^{m-1}{}_{2m}C_r\cos\{(2m-2r)x\} + \frac{{}_{2m}C_m}{2^{2m}}$$

$$= \frac{1}{2^{2m}}\left[\sum_{r=0}^{m-1}{}_{2m}C_r\cos\{(2m-2r)x\} + \sum_{r=0}^{m-1}{}_{2m}C_r\cos\{(2m-2r)x\}\right] + \frac{{}_{2m}C_m}{2^{2m}}$$

$$= \frac{1}{2^{2m}}\left[\sum_{r=0}^{m-1}{}_{2m}C_r\cos\{(2m-2r)x\} + \sum_{r=m+1}^{2m}{}_{2m}C_r\cos\{(2m-2r)x\}\right]$$

$$+ \frac{1}{2^{2m}}{}_{2m}C_m\cos\{(2m-2m)x\}$$

$$= \frac{1}{2^{2m}}\sum_{r=0}^{2m}{}_{2m}C_r\cos\{(2m-2r)x\}$$

When the power is odd,

$$\sum_{r=0}^{m}{}_{2m+1}C_r\cos\{(2m+1-2r)x\} = \sum_{r=m+1}^{2m+1}{}_{2m+1}C_r\cos\{(2m+1-2r)x\}$$

Using this,

$$\cos^{2m+1}x = \frac{1}{2^{2m}}\sum_{r=0}^{m}{}_{2m+1}C_r\cos\{(2m+1-2r)x\}$$

$$= \frac{1}{2^{2m+1}}\left[\sum_{r=0}^{m}{}_{2m+1}C_r\cos\{(2m+1-2r)x\} + \sum_{r=0}^{m}{}_{2m+1}C_r\cos\{(2m+1-2r)x\}\right]$$

$$= \frac{1}{2^{2m+1}}\left[\sum_{r=0}^{m}{}_{2m+1}C_r\cos\{(2m+1-2r)x\} + \sum_{r=m+1}^{2m+1}{}_{2m+1}C_r\cos\{(2m+1-2r)x\}\right]$$

$$= \frac{1}{2^{2m+1}}\sum_{r=0}^{2m+1}{}_{2m+1}C_r\cos\{(2m+1-2r)x\}$$

Formula 20.1.3

When $\downarrow$ denotes the floor function, the following expressions hold for natural number m .

$$\left(\cos^{m}x\right)^{(n)} = \frac{1}{2^{m-1}}\sum_{r=0}^{m/2\downarrow}{}_{m}C_r\,(m-2r)^n\cos\left\{(m-2r)x+\frac{n\pi}{2}\right\} \qquad \text{(3.mc)}$$

$$\left(\sin^{m}x\right)^{(n)} = \frac{1}{2^{m-1}}\sum_{r=0}^{m/2\downarrow}{}_{m}C_r\,(m-2r)^n\cos\left\{(m-2r)\left(x-\frac{\pi}{2}\right)+\frac{n\pi}{2}\right\} \qquad \text{(3.ms)}$$

Proof

Differentiating the both sides of (3.m) in Lemma1 with respect to x n times,

$$\left(\cos^m x\right)^{(n)} = \frac{1}{2^m}\sum_{r=0}^{m}(m-2r)^n{}_mC_r\cos\left\{(m-2r)x+\frac{n\pi}{2}\right\}$$

When m is even,

$$\left(m-2\cdot\frac{m}{2}\right)^n{}_mC_{\frac{m}{2}}\cos\left\{\left(m-2\cdot\frac{m}{2}\right)x+\frac{n\pi}{2}\right\} = 0$$

$$\sum_{r=0}^{m/2-1}(m-2r)^n{}_mC_r\cos\left\{(m-2r)x+\frac{n\pi}{2}\right\} = \sum_{r=m/2+1}^{m}(m-2r)^n{}_mC_r\cos\left\{(m-2r)x+\frac{n\pi}{2}\right\}$$

Using these,

$$\begin{aligned}\left(\cos^m x\right)^{(n)} &= \frac{1}{2^m}\sum_{r=0}^{m}(m-2r)^n{}_mC_r\cos\left\{(m-2r)x+\frac{n\pi}{2}\right\}\\ &= \frac{2}{2^m}\sum_{r=0}^{m/2-1}(m-2r)^n{}_mC_r\cos\left\{(m-2r)x+\frac{n\pi}{2}\right\}\\ &\quad+\frac{2}{2^m}\left(m-2\cdot\frac{m}{2}\right)^n{}_mC_{\frac{m}{2}}\cos\left\{\left(m-2\cdot\frac{m}{2}\right)x+\frac{n\pi}{2}\right\}\\ &= \frac{1}{2^{m-1}}\sum_{r=0}^{m/2}(m-2r)^n{}_mC_r\cos\left\{(m-2r)x+\frac{n\pi}{2}\right\}\end{aligned}$$

When m is odd,

$$\sum_{r=0}^{m/2\downarrow}(m-2r)^n{}_mC_r\cos\left\{(m-2r)x+\frac{n\pi}{2}\right\} = \sum_{r=m/2\downarrow+1}^{m}(m-2r)^n{}_mC_r\cos\left\{(m-2r)x+\frac{n\pi}{2}\right\}$$

Using this,

$$\begin{aligned}\left(\cos^m x\right)^{(n)} &= \frac{1}{2^m}\sum_{r=0}^{m}(m-2r)^n{}_mC_r\cos\left\{(m-2r)x+\frac{n\pi}{2}\right\}\\ &= \frac{2}{2^m}\sum_{r=0}^{m/2\downarrow}(m-2r)^n{}_mC_r\cos\left\{(m-2r)x+\frac{n\pi}{2}\right\}\end{aligned}$$

Thus, we obtain (3.mc). (3.ms) follows by the replacing x with $x-\pi/2$.

Example The 7th order derivative of $\cos^6 x$

Differentiation by direct calculation

- `m:=6:`
- `f1 := diff(cos(x)^m ,x,x,x,x,x,x,x)`

$$64896\cdot\cos(x)^5\cdot\sin(x) - 174720\cdot\cos(x)^3\cdot\sin(x)^3 + 40320\cdot\cos(x)\cdot\sin(x)^5$$

Differentiation by the formula

- `n:=7:`
- `fr := 1/2^(m-1)*sum(binomial(m,r)*(m-2*r)^n*cos((m-2*r)*x+n*PI/2) , r=0..floor(m/2))`

$$60\cdot\cos\left(\frac{7\cdot\pi}{2}+2\cdot x\right) + 3072\cdot\cos\left(\frac{7\cdot\pi}{2}+4\cdot x\right) + 8748\cdot\cos\left(\frac{7\cdot\pi}{2}+6\cdot x\right)$$

Verification of the equivalence

- `testeq(f1,fr)`

TRUE

20.1.4 Higher Derivatives of $\cos^{\alpha} x$, $\sin^{\alpha} x$

Formula 20.1.4

The following expressions hold for a positive number α .

$$\left(\cos^{\alpha} x\right)^{(n)} = \frac{1}{2^{\alpha}} \sum_{r=0}^{\infty} \binom{\alpha}{r} (\alpha-2r)^{n} \cos\left\{ (\alpha-2r)x + \frac{n\pi}{2} \right\} \tag{4.c}$$

$$\left(\sin^{\alpha} x\right)^{(n)} = \frac{1}{2^{\alpha}} \sum_{r=0}^{\infty} \binom{\alpha}{r} (\alpha-2r)^{n} \cos\left\{ (\alpha-2r)\left(x - \frac{\pi}{2}\right) + \frac{n\pi}{2} \right\} \tag{4.s}$$

Proof

Differentiating the both sides of (3.m) in Lemma1 with respect to x ,

$$\left(\cos^{m} x\right)^{(n)} = \frac{1}{2^{m}} \sum_{r=0}^{m} (m-2r)^{n} {}_{m}C_{r} \cos\left\{ (m-2r)x + \frac{n\pi}{2} \right\}$$

Analytically continuing the power from the natural number m to the positive number α , we obtain (4.c) .
(4.s) follows by the replacing x with $x-\pi/2$.

Example The 3rd order derivative of $\sin^{5.3} x$

The differential quotient on arbitrary point x= 0.9 was calculated by the direct calculation and the formula. Both are corresponding very well.

```
a = 5.3; n = 3;
fl[x_] = ∂x ∂x ∂x Sin[x]^a;
fr[x_] := 1/2^a Sum[Binomial[a, r] (a - 2 r)^n Cos[(a - 2 r) (x - π/2) + n π/2], {r, 0, ∞}]

N[fl[0.9]]          N[fr[0.9]]
-5.72269            -5.72269 + 3.60713 × 10^-16 i
```

20.2 Higher Integral of the product of many functions

(1) Generalized binomial theorem and Higher integral of the product of 2 functions

According to generalized binomial theorem in **3.2** , the following expression holds for real numbers x_1, x_2 such that $|x_1| > |x_2|$ and natural number n .

$$(x_1+x_2)^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} x_1^{-n-r} x_2^{r}$$

On the other hand, according to Formula 16.1.2 in **16.1** , the following expression holds for functions f_1, f_2 of x and natural number n .

$$\int_a^x \cdots \int_a^x f_1 f_2 \, dx^n = \sum_{r=0}^{m-1} \binom{-n}{r} f_1^{<n+r>} f_2^{(r)} + R_m^n$$

$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \int_a^x \cdots \int_a^x f_1^{<m+k>} f_2^{(m+k)} dx^n$$

Reversing the sign of the index of the differentiation operator (n) in the Leibniz Rule ,

$$(f_1 f_2)^{(-n)} = \sum_{r=0}^{\infty} \binom{-n}{r} f_1^{(-n-r)} f_2^{(r)}$$

Next, Replacing $(-n)$ with the intagration operator $<n>$, and dividing the series into a polynomial and a remainder, we obtain the above formula.

(2) Generalized multinomial theorem and Higher Integral of the product of many functions

According to the generalized multinomial theorem in **3.4** , the following expression holds for real numbers $x_1, x_2, \cdots, x_\lambda$ such that $|x_1| > |x_2 + x_3 + \cdots + x_\lambda|$ and natural number n .

$$(x_1+x_2+\cdots+x_\lambda)^{-n} = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{-n}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} x_1^{-n-r_1} x_2^{r_1-r_2} \cdots x_\lambda^{r_{\lambda-1}}$$

Therefore, the following expression must hold for functions $f_1, f_2, \cdots, f_\lambda$ of x and natural number n .

$$\int_a^x \cdots \int_a^x (f_1 f_2 \cdots f_\lambda) dx^n = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{-n}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{(n-r_1)} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})} + R_m^n$$

20.2.1 Higher Integral of the product of many functions

Theorem 20.2.1

Let $f_k^{(r)}$ be the r th order derivative function of $f_k(x)$ $(k=1,2,\cdots,\lambda)$, $f_k^{<r>}$ be the arbitrary r th order primitive function of $f_k(x)$, m, n are natural numbers and $B(n,m)$ be the beta function. If there is a number a such that $f_1^{<r>}(a)=0$ $(r=1,2,\cdots,m+n-1)$ or $f_k^{(s)}(a)=0$ $(s=0,1,\cdots,m+n-2)$ for at least one $k>1$, then the following expression holds.

$$\int_a^x \cdots \int_a^x f_1 f_2 \cdots f_\lambda dx^n = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{-n}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{\langle n+r_1 \rangle} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})} + R_m^n$$

$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \frac{{}_{n-1}C_{k_1}}{m+k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}}$$
$$\times \int_a^x \cdots \int_a^x f_1^{\langle m+k_1 \rangle} f_2^{(m+k_1-k_2)} f_3^{(k_2-k_3)} \cdots f_\lambda^{(k_{\lambda-1})} dx^n$$

Proof

Let us integrate $f_1 f_2 f_3 \cdots f_\lambda$ with respect to x from a to x n times. Then according to Theorem 16.1.2 in **16.1** , we obtain

$$\int_a^x \cdots \int_a^x f_1 f_2 f_3 \cdots f_\lambda dx^n = \sum_{r_1=0}^{m-1} \binom{-n}{r_1} f_1^{\langle n+r_1 \rangle} (f_2 f_3 \cdots f_\lambda)^{(r_1)} + R_m^n$$
$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_{k_1}}{m+k_1} \int_a^x \cdots \int_a^x f_1^{\langle m+k_1 \rangle} (f_2 f_3 \cdots f_\lambda)^{(m+k_1)} dx^n$$

According to Theorem20.1.1 ,

$$(f_2 f_3 \cdots f_\lambda)^{(r_1)} = \sum_{r_2=0}^{r_1} \sum_{r_3=0}^{r_2} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{r_1}{r_2} \binom{r_2}{r_3} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_2^{(r_1-r_2)} f_3^{(r_2-r_3)} \cdots f_\lambda^{(r_{\lambda-1})}$$
$$(f_2 f_3 \cdots f_\lambda)^{(m+k_1)} = \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}} f_2^{(k_1-k_2)} f_3^{(k_2-k_3)} \cdots f_\lambda^{(k_{\lambda-1})}$$

Substituting these for the above, we obtain the disired expression.

Example

$$\int_a^x \cdots \int_a^x f_1 f_2 f_3 \, dx^n = \sum_{r=0}^{m-1} \sum_{s=0}^{r} \binom{-n}{r} \binom{r}{s} f_1^{<n+r>} f_2^{(r-s)} f_3^{(s)} + R_m^n$$
$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{r=0}^{n-1} \sum_{s=0}^{m+r} \frac{{}_{n-1}C_r}{m+r} \binom{m+r}{s} \binom{r}{s} \int_a^x \cdots \int_a^x f_1^{<m+r>} f_2^{(m+r-s)} f_3^{(s)} dx^n$$
$$\int_a^x \cdots \int_a^x f_1 f_2 f_3 f_4 dx^n = \sum_{r=0}^{m-1} \sum_{s=0}^{r} \sum_{t=0}^{s} \binom{-n}{r} \binom{r}{s} \binom{s}{t} f_1^{<n+r>} f_2^{(r-s)} f_3^{(s-t)} f_4^{(t)} + R_m^n$$
$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{r=0}^{n-1} \sum_{s=0}^{m+r} \sum_{t=0}^{s} \frac{{}_{n-1}C_r}{m+r} \binom{m+r}{s} \binom{r}{s} \binom{s}{t}$$
$$\times \int_a^x \cdots \int_a^x f_1^{<m+r>} f_2^{(m+r-s)} f_3^{(s-t)} f_4^{(t)} dx^n$$

Example computation of the product of three functions

Since the combination of the product of many functions are numerous, we cannot calculate these one by one. Then we pick up some and calculate them.

Example1 Higher Integral of $x^\alpha e^x \sin x$

The common zero of the higher order integral of $x^\alpha e^x \sin x$ is $x=-\infty$.

Then let $f_1 = x^\alpha$, $f_2 = e^x$, $f_3 = \sin x$,

$$(x^{\alpha})^{<n+r>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)}x^{\alpha+n+r} \quad , \quad (x^{\alpha})^{<m+r>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)}x^{\alpha+m+r}$$

$$(e^x)^{(r-s)} = (e^x)^{(m+r-s)} = e^x$$

$$(\sin x)^{(s)} = \sin\left(x+\frac{s\pi}{2}\right)$$

Substituting these for Theorem 20.2.1 , we obtain

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} x^{\alpha}e^x\sin x\,dx^n$$

$$= \sum_{r=0}^{m-1}\sum_{s=0}^{r}\binom{-n}{r}\binom{r}{s}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)}x^{\alpha+n+r}e^x\sin\left(x+\frac{s\pi}{2}\right) + R_m^n \qquad \textbf{(1.1)}$$

$$R_m^n = \frac{(-1)^m}{B(n,m)}\sum_{r=0}^{n-1}\sum_{s=0}^{m+r}\frac{1}{m+r}\binom{n-1}{r}\binom{m+r}{s}$$
$$\times\int_{-\infty}^{x}\cdots\int_{-\infty}^{x}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)}x^{\alpha+m+r}e^x\sin\left(x+\frac{s\pi}{2}\right)dx^n \qquad \textbf{(1.1r)}$$

And when $n=2$

$$\int_{-\infty}^{x}\int_{-\infty}^{x} x^{\alpha}e^x\sin x\,dx^2 = \sum_{r=0}^{m-1}\sum_{s=0}^{r}\binom{-2}{r}\binom{r}{s}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3+r)}x^{\alpha+2+r}e^x\sin\left(x+\frac{s\pi}{2}\right) + R_m^2$$

$$R_m^2 = \frac{(-1)^m}{B(2,m)}\sum_{r=0}^{1}\sum_{s=0}^{m+r}\frac{1}{m+r}\binom{1}{r}\binom{m+r}{s}$$
$$\times\int_{-\infty}^{x}\int_{-\infty}^{x}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)}x^{\alpha+m+r}e^x\sin\left(x+\frac{s\pi}{2}\right)dx^2$$

This remainder term is an automorphism. And the calculation is difficult when m is large. So let $m=1$, then the above expression is simplified as follows.

$$\int_{-\infty}^{x}\int_{-\infty}^{x} x^{\alpha}e^x\sin x\,dx^2 = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3)}x^{\alpha+2}e^x\sin x + R_1^2$$

$$R_1^2 = \frac{(-1)^1}{B(2,1)}\sum_{s=0}^{1}\frac{1}{1}\binom{1}{0}\binom{1}{s}\int_{-\infty}^{x}\int_{-\infty}^{x}\frac{\Gamma(1+\alpha)}{\Gamma(\alpha+2)}x^{\alpha+1}e^x\sin\left(x+\frac{s\pi}{2}\right)dx^2$$
$$+\frac{(-1)^1}{B(2,1)}\sum_{s=0}^{2}\frac{1}{2}\binom{1}{1}\binom{2}{s}\int_{-\infty}^{x}\int_{-\infty}^{x}\frac{\Gamma(1+\alpha)}{\Gamma(\alpha+3)}x^{\alpha+2}e^x\sin\left(x+\frac{s\pi}{2}\right)dx^2$$
$$= -\frac{2\Gamma(\alpha+1)}{\Gamma(\alpha+2)}\left(\int_{-\infty}^{x}\int_{-\infty}^{x}x^{\alpha+1}e^x\sin x\,dx^2+\int_{-\infty}^{x}\int_{-\infty}^{x}x^{\alpha+1}e^x\cos x\,dx^2\right)$$
$$-\frac{2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}\int_{-\infty}^{x}\int_{-\infty}^{x}x^{\alpha+2}e^x\cos x\,dx^2$$

When $\alpha=3$, if the values of the both sides on arbitrary point $x=4$ are calculated, it is as follows. Although the remainder R_1^2 is large because of $m=1$, both sides are corresponding exactly.

The 2nd order integral of x^3 * e^x * sin x

Left: direct integration

```
• a:=3:
```

```
fl := int(int(x^a*E^x*sin(x) ,x),x):
float(subs(fl, x=4))
```

20.68491162

Series

```
fr := gamma(a+1)/gamma(a+3)*x^(a+2)*E^x*sin(x):
float(subs(fr, x=4))
```

−2115.584829

Remainder

```
S10 := int(int(x^(a+1)*E^x*sin(x) ,x),x):
S11 := int(int(x^(a+1)*E^x*cos(x) ,x),x):
S21 := int(int(x^(a+2)*E^x*cos(x) ,x),x):
R21 := -2*gamma(a+1)/gamma(a+2)*(S10+S11)
       -2*gamma(a+1)/gamma(a+3)*S21:
float(subs(R21,x=4))
```

2136.26974

Right: Series+Remainder

```
float(subs(fr+R21, x=4))
```

20.68491162

When $m \to \infty$

On the contrary, if whether to be $R_\infty^2 = 0$ at the time of $m \to \infty$ is asked, unfortunately, it is not so. Surprisingly, when $\alpha=3$, it becomes $R_\infty^2 = -3$ regardless of the value of x. Then

$$\int_{-\infty}^{x}\int_{-\infty}^{x} x^3 e^x \sin x\, dx^2 = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{-2}{r}\binom{r}{s}\frac{\Gamma(3+1)}{\Gamma(3+3+r)}x^{3+2+r}e^x\sin\left(x+\frac{s\pi}{2}\right) - 3$$

As the result of much calculation, it turned out that the following formula holds.

Formula 20.2.1'

When $\alpha = 0,1,2,\cdots$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} x^\alpha e^x \sin x\, dx^n = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{-n}{r}\binom{r}{s}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)}x^{\alpha+n+r}e^x\sin\left(x+\frac{s\pi}{2}\right)$$
$$+\frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\, c_{\alpha+r}\, x^{n-1-r} \qquad \text{(1.1')}$$

Where

$$c_{4k+0} = -(-1)^k 2^{k-1} k!\cdot(2k-1)!!\cdot(4k-1)!!$$
$$c_{4k+1} = (-1)^k 2^{k-1} k!\cdot(2k-1)!!\cdot(4k+1)!!$$
$$c_{4k+2} = -(-1)^k 2^{k-1} k!\cdot(2k+1)!!\cdot(4k+1)!!$$
$$c_{4k+3} = 0$$

According to this formula, R_∞^2 at the time of $\alpha=3$ is obtained as follows.

$$R_\infty^2 = \frac{1}{(2-1)!}\sum_{r=0}^{2-1}(-1)^r{}_{2-1}C_r\, c_{3+r}\, x^{2-1-r} = {}_1C_0\, c_{3+0}\, x^1 - {}_1C_1\, c_{3+1}\, x^0$$

Here, from the proviso,

$$c_{3+0} = c_{4\cdot 0+3} = 0$$

$$c_{3+1} = c_{4\cdot 1+0} = -(-1)^1 2^{1-1} 1!\cdot(2\cdot 1-1)!!\cdot(4\cdot 1-1)!! = 3$$

Substituting these for the above, we obtain $R_\infty^2 = -3$.

By reference, when R_∞^n is shown for $n=1,2,3$, $\alpha=1,2,\cdots,10$, it is as followins.

α	$R_\infty^1(=c_\alpha)$	R_∞^2	R_∞^3
1	$1/2$	$x/2+1/2$	$x^2/4+x/2$
2	$-1/2$	$-x/2$	$-x^2/4+3/2$
3	0	-3	$-3x-15/2$
4	3	$3x+15$	$3x^2/2+15x+45/2$
5	-15	$-15x-45$	$-15x^2/2-45x$
6	45	$45x$	$45x^2/2-630$
7	0	1260	$1260x+5670$
8	-1260	$-1260x-11340$	$-630x^2-11340x-28350$
9	11340	$11340x+56700$	$5670x^2+56700x$
10	-56700	$-56700x$	$-28350x^2+1871100$

Example2 Higher Integral of $x^\alpha e^x \log x$

A common zero of the higher order integral of $x^\alpha e^x \log x$ is $x=-\infty$.

Then let $f_1 = x^\alpha$, $f_2 = e^x$, $f_3 = \log x$

$$(x^\alpha)^{<n+r>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)} x^{\alpha+n+r} \quad , \quad (e^x)^{(r-s)} = e^x$$

$$(x^\alpha)^{<m+r>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r} \quad , \quad (e^x)^{(m+r-s)} = e^x$$

$$(\log x)^{(0)} = \log x \quad (s=0) \quad , \quad (\log x)^{(s)} = (-1)^{s-1}(s-1)!\,x^{-s} \quad (s\neq 0)$$

First, we need to separate the terms containing $f_3^{(0)}$ from Theorem 20.2.1 . Then

$$\int_a^x\cdots\int_a^x f_1 f_2 f_3 dx^n = f_3^{(0)} \sum_{r=0}^{m-1} \binom{-n}{r} f_1^{<n+r>} f_2^{(r)} + \sum_{r=1}^{m-1}\sum_{s=1}^{r} \binom{-n}{r}\binom{r}{s} f_1^{<n+r>} f_2^{(r-s)} f_3^{(s)} + R_m^n$$

$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{r=0}^{n-1} \frac{{}_{n-1}C_r}{m+r} \binom{m+r}{0} \int_a^x\cdots\int_a^x f_1^{<m+r>} f_2^{(m+r)} f_3^{(0)} dx^n$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{r=0}^{n-1}\sum_{s=1}^{m+r} \frac{{}_{n-1}C_r}{m+r} \binom{m+r}{s} \int_a^x\cdots\int_a^x f_1^{<m+r>} f_2^{(m+r-s)} f_3^{(s)} dx^n$$

Substituting the above expressins for this,

$$\int_{-\infty}^x\cdots\int_{-\infty}^x x^\alpha e^x \log x\, dx^n = e^x \log x \sum_{r=0}^{m-1} \binom{-n}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)} x^{\alpha+n+r}$$

$$+ \sum_{r=1}^{m-1}\sum_{s=1}^{r} (-1)^{s-1} \binom{-n}{r}\binom{r}{s} \frac{\Gamma(1+\alpha)\Gamma(s)}{\Gamma(1+\alpha+n+r)} x^{\alpha+n+r-s} e^x + R_m^n \qquad \textbf{(1.2)}$$

$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{r=0}^{n-1} \frac{{}_{n-1}C_r}{m+r} \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r} e^x \log x \, dx^n$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{r=0}^{n-1} \sum_{s=1}^{m+r} (-1)^{s-1} \frac{{}_{n-1}C_r}{m+r} \binom{m+r}{s} \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} \frac{\Gamma(1+\alpha)\Gamma(s)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r-s} e^x dx^n \tag{1.2r}$$

When $\alpha=3/2$, $n=3$, $m=80$, the values of the both sides on arbitrary point x=25 are as follows. The right side of (1.2) seems to be asymptotic expansion.

```
a = 3 / 2; n = 3; m = 80;
f1[x_] := 1/Gamma[n] ∫_{-∞}^{x} (x - t)^(n-1) t^a e^t Log[t] dt
fr[x_] := e^x Log[x] Sum_{r=0}^{m-1} Binomial[-n, r] Gamma[1 + a]/Gamma[1 + a + n + r] x^(a+n+r) -
   e^x Sum_{r=1}^{m-1} Sum_{s=1}^{r} (-1)^s Binomial[-n, r] Binomial[r, s] Gamma[1 + a] Gamma[s]/Gamma[1 + a + n + r] x^(a+n+r-s)
N[f1[25]]
2.30674 × 10^13 - 391.836 i
N[fr[25]]
2.30674 × 10^13
```

20.2.2 Higher Integral of the power of a function

Especially, when $f_1 = f_2 = \cdots = f_\lambda$ in Theorem 20.2.1 , the following theorem follows immediately.

Theorem 20.2.2

Let $f^{(r)}$ be the r th order derivative function of $f(x)$, $f_k^{<r>}$ be the arbitrary r th order primitive function of $f(x)$, m,n are natural numbers and $B(n,m)$ be the beta function. At this time , if there is a number a such that

$$f^{<r>}(a)=0 \ \ (r=1,2,\cdots,m+n-1) \quad or \quad f^{(s)}(a)=0 \ \ (s=0,1,\cdots,m+n-2)$$

then the following expression holds for $\lambda=2,3,4,\cdots$.

$$\int_a^x \cdots \int_a^x f^\lambda dx^n = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{-n}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f^{\langle n+r_1 \rangle} f^{(r_1-r_2)} \cdots f^{(r_{\lambda-1})} + R_m^n$$

$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \frac{{}_{n-1}C_{k_1}}{m+k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}}$$

$$\times \int_a^x \cdots \int_a^x f^{\langle m+k_1 \rangle} f^{(m+k_1-k_2)} f^{(k_2-k_3)} \cdots f^{(k_{\lambda-1})} dx^n$$

Example Higher Integral of $\log^3 x$

Let $f = \log x$. Then

$$(\log x)^{<n+r>} = \frac{\log x - \psi(1+n+r) - \gamma}{\Gamma(1+n+r)} x^{n+r}$$

$$(\log x)^{(r-s)} = \log x \quad (r=s) \ , \qquad (\log x)^{(r-s)} = (-1)^{r-s-1}(r-s-1)!\, x^{-r+s} \quad (r \neq s)$$

$$(\log x)^{(0)} = \log x \quad (s=0) \ , \qquad (\log x)^{(s)} = (-1)^{s-1}(s-1)!\, x^{-s} \quad (r \neq 0)$$

Since $f^{(0)}$ is different from $f^{(r)}$ $r=1,2,\cdots$ in function type, we have to separate the terms containing $f^{(0)}$

from

$$\int_a^x \cdots \int_a^x f^3 dx^n = \sum_{r=0}^{m-1} \sum_{s=0}^{r} \binom{-n}{r} \binom{r}{s} f^{<n+r>} f^{(r-s)} f^{(s)} + R_m^n$$

$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{r=0}^{n-1} \sum_{s=0}^{m+r} \frac{1}{m+r} \binom{n-1}{r} \binom{m+r}{s} \int_a^x \cdots \int_a^x f^{<m+r>} f^{(m+r-s)} f^{(s)} dx^n$$

It is as follows.

$$\sum_{r=0}^{m-1} \binom{-n}{r} f^{<n+r>} \sum_{s=0}^{r} \binom{r}{s} f^{(r-s)} f^{(s)}$$

$$= \binom{-n}{0} f^{<n+0>} \binom{0}{0} f^{(0)} f^{(0)} + \binom{-n}{1} f^{<n+1>} \left\{ \binom{1}{0} f^{(1)} f^{(0)} + \binom{1}{1} f^{(0)} f^{(1)} \right\}$$

$$+ \sum_{r=2}^{m-1} \binom{-n}{r} f^{<n+r>} \left\{ \binom{r}{0} f^{(r)} f^{(0)} + \sum_{s=1}^{r-1} \binom{r}{s} f^{(r-s)} f^{(s)} + \binom{r}{r} f^{(0)} f^{(r)} \right\}$$

$$= f^{<n>} f^{(0)} f^{(0)} + 2 \sum_{r=1}^{m-1} \binom{-n}{r} f^{<n+r>} f^{(r)} f^{(0)} + \sum_{r=2}^{m-1} \binom{-n}{r} f^{<n+r>} \sum_{s=1}^{r-1} \binom{r}{s} f^{(r-s)} f^{(s)}$$

$$\sum_{r=0}^{n-1} \binom{n-1}{r} f^{<m+r>} \sum_{s=0}^{m+r} \binom{m+r}{s} f^{(m+r-s)} f^{(s)}$$

$$= \sum_{r=0}^{n-1} \binom{n-1}{r} f^{<m+r>} \left\{ f^{(m+r)} f^{(0)} + \sum_{s=1}^{m+r-1} \binom{m+r}{s} f^{(m+r-s)} f^{(s)} + f^{(0)} f^{(m+r)} \right\}$$

$$= 2 \sum_{r=0}^{n-1} \binom{n-1}{r} f^{<m+r>} f^{(m+r)} f^{(0)} + \sum_{r=0}^{n-1} \binom{n-1}{r} f^{<m+r>} \sum_{s=1}^{m+r-1} \binom{m+r}{s} f^{(m+r-s)} f^{(s)}$$

Substituting thes for the above ,

$$\int_a^x \cdots \int_a^x f^3 dx^n = f^{<n>} f^{(0)} f^{(0)} + 2 \sum_{r=1}^{m-1} \binom{-n}{r} f^{<n+r>} f^{(r)} f^{(0)}$$

$$+ \sum_{r=2}^{m-1} \sum_{s=1}^{r-1} \binom{-n}{r} \binom{r}{s} f^{<n+r>} f^{(r-s)} f^{(s)}$$

$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{r=0}^{n-1} \frac{2}{m+r} \binom{n-1}{r} \int_a^x \cdots \int_a^x f^{<m+r>} f^{(m+r)} f^{(0)} dx^n$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{r=0}^{n-1} \sum_{s=1}^{m+r-1} \frac{1}{m+r} \binom{n-1}{r} \binom{m+r}{s} \int_a^x \cdots \int_a^x f^{<m+r>} f^{(m+r-s)} f^{(s)} dx^n$$

Substituting the above expressions and the following expressions for this

$$(\log x)^{<m+r>} = \frac{\log x - \psi(1+m+r) - \gamma}{\Gamma(1+m+r)} x^{m+r}$$

$$(\log x)^{(m+r)} = (-1)^{m+r-1} (m+r-1)!\, x^{-m-r} \qquad (r = 0, \cdots, n-1)$$

$$(\log x)^{(m+r-s)} = (-1)^{m+r-s-1} (m+r-s-1)!\, x^{-m-r+s} \qquad (s = 1, \cdots, m+r-1)$$

we obtain

$$\int_0^x \cdots \int_0^x \log^3 x\, dx^n = \frac{\log x - \psi(1+n) - \gamma}{\Gamma(1+n)} x^n (\log x)^2$$

$$- 2 x^n \log x \sum_{r=1}^{m-1} (-1)^r \binom{-n}{r} \frac{\log x - \psi(1+n+r) - \gamma}{\Gamma(1+n+r)} \Gamma(r)$$

$$+ x^n \sum_{r=2}^{m-1} \sum_{s=1}^{r-1} (-1)^r \binom{-n}{r} \binom{r}{s} \frac{\log x - \psi(1+n+r) - \gamma}{\Gamma(1+n+r)} \Gamma(r-s) \Gamma(s)$$

$$+R_m^n \tag{2.1}$$

$$R_m^n = -\frac{2}{B(n,m)}\sum_{r=0}^{n-1}\frac{(-1)^r}{(m+r)^2}\binom{n-1}{r}\int_0^x\cdots\int_0^x\{\log x-\psi(1+m+r)-\gamma\}\log x\,dx^n$$

$$+\frac{1}{B(n,m)}\sum_{r=0}^{n-1}\sum_{s=1}^{m+r-1}\frac{(-1)^r}{m+r}\binom{n-1}{r}\binom{m+r}{s}$$

$$\times\int_0^x\cdots\int_0^x\frac{\log x-\psi(1+m+r)-\gamma}{\Gamma(1+m+r)}\Gamma(m+r-s)\Gamma(s)\,dx^n \tag{2.1r}$$

Although this formula is complicated, fortunately $\lim_{m\to\infty} R_m^n = 0$ holds. However, the proof is difficult and the convergence speed is very slow.

When $n=2$, $m=275$, the values of the both sides of (2.1) on arbitrary point $x=0.3$ are as follows. The both sides are corresponding up to only 1 digit below the decimal point. If larger m is given, the accuracy can be raised. However, it takes a lot of time.

```
n = 2;  m = 275;
fl[x_] := 1/Gamma[n] ∫_0^x (x - t)^(n-1) Log[t]^3 dt
fr[x_] := (Log[x] - PolyGamma[1 + n] - EulerGamma)/Gamma[1 + n] x^n Log[x]^2 -
   2 x^n Log[x] Sum_{r=1}^{m-1} (-1)^r Binomial[-n, r] (Log[x] - PolyGamma[1 + n + r] - EulerGamma)/Gamma[1 + n + r] Gamma[r] +
   x^n Sum_{r=2}^{m-1} Sum_{s=1}^{r-1} (-1)^r Binomial[-n, r] Binomial[r, s] (Log[x] - PolyGamma[1 + n + r] - EulerGamma)/Gamma[1 + n + r]
N[fl[0.3]]              N[fr[0.3]]
-1.44719                -1.40044
```

20.2.3 Higher Integrls of $\cos^m x$, $\sin^m x$

Formula 20.2.3c

(1) Oddth power

$$\int_{\frac{(n-1)\pi}{2}}^x\cdots\int_{\frac{1\pi}{2}}^x\int_{\frac{0\pi}{2}}^x\cos^{2m+1}x\,dx^n = \frac{1}{2^{2m}}\sum_{r=0}^{m}\frac{{}_{2m+1}C_r}{(2m-2r+1)^n}\cos\left\{(2m-2r+1)x-\frac{n\pi}{2}\right\}$$

(3.co)

(2) Eventh power

$$\int_{a_n}^x\cdots\int_{a_2}^x\int_{a_1}^x\cos^{2m}x\,dx^n = \frac{1}{2^{2m-1}}\sum_{r=0}^{m-1}\frac{{}_{2m}C_r}{(2m-2r)^n}\cos\left\{(2m-2r)x-\frac{n\pi}{2}\right\}$$

$$+\frac{{}_{2m}C_m}{2^{2m}}\int_{a_n}^x\cdots\int_{a_2}^x\int_{a_1}^x dx^n \tag{3.ce}$$

Where, $a_1, a_2, \cdots, a_n$ are the solutions of the following transcendental equation.

$$\sum_{r=0}^{m-1}\frac{{}_{2m}C_r}{(2m-2r)^k}\cos\left\{(2m-2r)x-\frac{k\pi}{2}\right\} = 0 \qquad k=1,2,\cdots,n$$

Especially when $m=1$,

$$\int_{\frac{(n-1)\pi}{4}}^{x}\cdots\int_{\frac{1\pi}{4}}^{x}\int_{\frac{0\pi}{4}}^{x}\cos^2 x\,dx^n = \frac{1}{2^{n+1}}\cos\left(2x-\frac{n\pi}{2}\right) + \frac{1}{2}\int_{\frac{(n-1)\pi}{4}}^{x}\cdots\int_{\frac{1\pi}{4}}^{x}\int_{\frac{0\pi}{4}}^{x}dx^n \tag{3.c2}$$

Proof

The following expression holds from Lemma1 .

$$\cos^{2m+1}x = \frac{1}{2^{2m}}\sum_{r=0}^{m}{}_{2m+1}C_r\cos\{(2m-2r+1)x\}$$

Integrating the both sides of this with respect to x n times,

$$\left(\cos^{2m+1}x\right)^{<n>} = \frac{1}{2^{2m}}\sum_{r=0}^{m}\frac{{}_{2m+1}C_r}{(2m-2r+1)^n}\cos\left\{(2m-2r+1)x-\frac{n\pi}{2}\right\}$$

Here, let us consider the solutions of the following transcendental equations.

$$\sum_{r=0}^{m}\frac{{}_{2m+1}C_r}{(2m-2r+1)^k}\cos\left\{(2m-2r+1)x-\frac{k\pi}{2}\right\} = 0 \qquad k=1,2,\cdots,n$$

Then

$$x = a_k = \frac{(k-1)\pi}{2} \quad k=1,2,\cdots,n$$

are the solutions of the transcendental equations. If it is why,

$$\cos\left\{(2m-2r+1)\cdot\frac{(k-1)\pi}{2}-\frac{k\pi}{2}\right\} = \cos\left\{(m-r)(k-1)\pi-\frac{\pi}{2}\right\}$$

$$= (-1)^{(m-r)(k-1)}\cos\frac{\pi}{2} = 0$$

These solutions are not dependent on m . Thus we obtain (3.co) .

Next, the following expression holds from Lemma1 .

$$\cos^{2m}x = \frac{1}{2^{2m-1}}\sum_{r=0}^{m-1}{}_{2m}C_r\cos\{(2m-2r)x\} + \frac{{}_{2m}C_m}{2^{2m}}$$

Integrating the both sides of this with respect to x n times,

$$\left(\cos^{2m}x\right)^{<n>} = \frac{1}{2^{2m-1}}\sum_{r=0}^{m-1}\frac{{}_{2m}C_r}{(2m-2r)^n}\cos\left\{(2m-2r)x-\frac{n\pi}{2}\right\} + \frac{{}_{2m}C_m}{2^{2m}}\sum_{k=0}^{n-1}c_{k+1}x^k$$

Here, let a_k $k=1,2,\cdots,n$ are the solutions of the following transcendental equations.

$$\sum_{r=0}^{m-1}\frac{{}_{2m}C_r}{(2m-2r)^k}\cos\left\{(2m-2r)x-\frac{k\pi}{2}\right\} = 0 \qquad k=1,2,\cdots,n$$

Then, (3.ce) follows immediately.

Especially when $m=1$, these transcendental equations are as follows.

$$\frac{{}_2C_0}{2^k}\cos\left(2x-\frac{k\pi}{2}\right) = 0 \qquad k=1,2,\cdots,n$$

Then

$$x = a_k = \frac{(k-1)\pi}{4} \quad k=1,2,\cdots,n$$

are solutions of the transcendental equations. If it is why,

$$\frac{{}_2C_0}{2^k}\cos\left\{2\cdot\frac{(k-1)\pi}{4}-\frac{k\pi}{2}\right\}=\frac{1}{2^k}\cos\frac{\pi}{2}=0 \qquad k=1,2,\cdots,n$$

These solutions are unrelated to m . Thus, (3.c2) holds.

Note1

According to Formula20.1.3 , the n th order derivative of $\cos^{2m+1}x$ is as follows.

$$\left(\cos^{2m+1}x\right)^{(n)}=\frac{1}{2^{2m}}\sum_{r=0}^{m}{}_{2m+1}C_r\,(2m+1-2r)^n\cos\left\{(2m+1-2r)x+\frac{n\pi}{2}\right\}$$

Since an integration is the inverse operation of a differentiation, replacing n with $-n$,

$$\left(\cos^{2m+1}x\right)^{(-n)}=\frac{1}{2^{2m}}\sum_{r=0}^{m}\frac{{}_{2m+1}C_r}{(2m-2r+1)^n}\cos\left\{(2m-2r+1)x-\frac{n\pi}{2}\right\}$$

Replacing the differentiation operator with the integration operator, we can obtain (3.co) .

Note2

The general notation of $\int_{a_n}^{x}\cdots\int_{a_1}^{x}dx^n$ is difficult. By reference, if these are shown for $n=1,2,3$, it is as follws. The 4th order or more are unmanageable.

$$\int_{a_1}^{x}dx^1=\frac{(x-a_1)}{1!}\ ,\qquad \int_{a_2}^{x}\int_{a_1}^{x}dx^2=\frac{(x-a_2)(x-2a_1+a_2)}{2!}$$

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x}dx^3=\frac{(x-a_3)\left(x^2+a_3x-3a_1x+6a_1a_2-3a_1a_3-3a_2^2+a_3^2\right)}{3!}$$

These resemble a part of constant-of-integration in **4.1.3** . However, these are completely another. That is, these are undoubted parts of the lineal higher primitive function that originates in $\cos^2x=(\cos 2x-1)/2$ $etc.$.

Example1 The 3rd order integral of $\cos^5x$

```
m = 2; n = 3;
```

$$\text{f1}[x\_] := \int_{\frac{2\pi}{2}}^{x}\left(\int_{\frac{1\pi}{2}}^{v}\left(\int_{\frac{0\pi}{2}}^{u}\text{Cos}[t]^{2m+1}\,dt\right)du\right)dv$$

$$\text{fr}[x\_] := \frac{1}{2^{2m}}\sum_{r=0}^{m}\frac{\text{Binomial}[2m+1,r]}{(2m-2r+1)^n}\text{Cos}\left[(2m-2r+1)x-\frac{n\pi}{2}\right]$$

f1[x] $\quad -\frac{5\,\text{Sin}[x]}{8}-\frac{5}{432}\text{Sin}[3x]-\frac{\text{Sin}[5x]}{2000}$

Expand[fr[x]] $\quad -\frac{5\,\text{Sin}[x]}{8}-\frac{5}{432}\text{Sin}[3x]-\frac{\text{Sin}[5x]}{2000}$

Example2 The 3rd order integral of $\cos^2x$

$$\text{f1}[x\_] := \int_{\frac{2\pi}{4}}^{x}\left(\int_{\frac{1\pi}{4}}^{v}\left(\int_{\frac{0\pi}{4}}^{u}\text{Cos}[t]^{2}\,dt\right)du\right)dv$$

```
fr[x_] := 1/2^(3+1) Cos[2 x - 3π/2] + 1/2 ∫_{2π/4}^{x} (∫_{1π/4}^{v} (∫_{0π/4}^{u} dt) du) dv
```

```
Expand[fl[x]]
```

$$-\frac{\pi^3}{384}-\frac{\pi^2 x}{64}+\frac{x^3}{12}-\frac{1}{16}\mathrm{Sin}[2x]$$

```
Expand[fr[x]]
```

$$-\frac{\pi^3}{384}-\frac{\pi^2 x}{64}+\frac{x^3}{12}-\frac{1}{16}\mathrm{Sin}[2x]$$

Formula 20.2.3s

(1) Oddth power

$$\int_{\frac{n\pi}{2}}^{x}\cdots\int_{\frac{2\pi}{2}}^{x}\int_{\frac{1\pi}{2}}^{x}\sin^{2m+1}x\,dx^n=\frac{1}{2^{2m}}\sum_{r=0}^{m}\frac{(-1)^{m-r}{}_{2m+1}C_r}{(2m-2r+1)^n}\sin\left\{(2m-2r+1)x-\frac{n\pi}{2}\right\} \tag{3.so}$$

(2) Eventh power

$$\int_{a_n}^{x}\cdots\int_{a_2}^{x}\int_{a_1}^{x}\sin^{2m}x\,dx^n=\frac{1}{2^{2m-1}}\sum_{r=0}^{m-1}\frac{(-1)^{m-r}{}_{2m}C_r}{(2m-2r)^n}\cos\left\{(2m-2r)x-\frac{n\pi}{2}\right\}+\frac{{}_{2m}C_m}{2^{2m}}\int_{a_n}^{x}\cdots\int_{a_2}^{x}\int_{a_1}^{x}dx^n \tag{3.se}$$

Where, $a_1, a_2, \cdots, a_n$ are the solutions of the following transcendental equation.

$$\sum_{r=0}^{m-1}\frac{(-1)^{m-r}{}_{2m}C_r}{(2m-2r)^k}\cos\left\{(2m-2r)x-\frac{k\pi}{2}\right\}=0 \qquad k=1,2,\cdots,n$$

Especially when $m=1$,

$$\int_{\frac{(n+1)\pi}{4}}^{x}\cdots\int_{\frac{3\pi}{4}}^{x}\int_{\frac{2\pi}{4}}^{x}\sin^2x\,dx^n=-\frac{1}{2^{n+1}}\cos\left(2x-\frac{n\pi}{2}\right)+\frac{1}{2}\int_{\frac{(n+1)\pi}{4}}^{x}\cdots\int_{\frac{3\pi}{4}}^{x}\int_{\frac{2\pi}{4}}^{x}dx^n \tag{3.s2}$$

Proof

Replacing x with $x-\pi/2$ in (3.co) in Formula 20.2.3c ,

$$\int_{\frac{n\pi}{2}}^{x}\cdots\int_{\frac{2\pi}{2}}^{x}\int_{\frac{1\pi}{2}}^{x}\sin^{2m+1}x\,dx^n=\frac{1}{2^{2m}}\sum_{r=0}^{m}\frac{{}_{2m+1}C_r}{(2m-2r+1)^n}\cos\left\{(2m-2r+1)\left(x-\frac{\pi}{2}\right)-\frac{n\pi}{2}\right\}$$

Here

$$\begin{aligned}\cos\left\{(2m-2r+1)\left(x-\frac{\pi}{2}\right)-\frac{n\pi}{2}\right\}&=\cos\left\{(2m-2r+1)x-\frac{(2m-2r+1)\pi}{2}-\frac{n\pi}{2}\right\}\\&=\sin\left\{(2m-2r+1)x-\frac{n\pi}{2}-(m-r)\pi\right\}\\&=(-1)^{m-r}\sin\left\{(2m-2r+1)x-\frac{n\pi}{2}\right\}\end{aligned}$$

Substituting this for the above, we obtain (3.so) . (3.se) and (3.s2) are obtained from Formul20.2.3c in a similar way.

Example1 The 3rd order integral of $\sin^7 x$

```
m = 3; n = 3;
```

$$fl[x\_] := \int_{\frac{3\pi}{2}}^{x}\left(\int_{\frac{2\pi}{2}}^{v}\left(\int_{\frac{1\pi}{2}}^{u}\mathrm{Sin}[t]^{2m+1}\,dt\right)du\right)dv$$

$$fr[x\_] := \frac{1}{2^{2m}}\sum_{r=0}^{m}\frac{(-1)^{m-r}\,\mathrm{Binomial}[2m+1,r]}{(2m-2r+1)^{n}}\,\mathrm{Sin}\left[(2m-2r+1)x-\frac{n\pi}{2}\right]$$

fl[x]

$$\frac{35\,\mathrm{Cos}[x]}{64}-\frac{7}{576}\mathrm{Cos}[3x]+\frac{7\,\mathrm{Cos}[5x]}{8000}-\frac{\mathrm{Cos}[7x]}{21952}$$

Expand[fr[x]]

$$\frac{35\,\mathrm{Cos}[x]}{64}-\frac{7}{576}\mathrm{Cos}[3x]+\frac{7\,\mathrm{Cos}[5x]}{8000}-\frac{\mathrm{Cos}[7x]}{21952}$$

Example2 The 4th order integral of $sin^2 x$

$$fl[x\_] := \int_{\frac{5\pi}{4}}^{x}\left(\int_{\frac{4\pi}{4}}^{t4}\left(\int_{\frac{3\pi}{4}}^{t3}\left(\int_{\frac{2\pi}{4}}^{t2}\mathrm{Sin}[t1]^{2}\,dt1\right)dt2\right)dt3\right)dt4$$

$$fr[x\_] := -\frac{1}{2^{4+1}}\mathrm{Cos}\left[2x-\frac{4\pi}{2}\right]+\frac{1}{2}\int_{\frac{5\pi}{4}}^{x}\left(\int_{\frac{4\pi}{4}}^{t4}\left(\int_{\frac{3\pi}{4}}^{t3}\left(\int_{\frac{2\pi}{4}}^{t2}dt1\right)dt2\right)dt3\right)dt4$$

Expand[fl[x]]

$$\frac{5\pi^4}{12288}-\frac{\pi^3 x}{192}+\frac{3\pi^2 x^2}{128}-\frac{\pi x^3}{24}+\frac{x^4}{48}-\frac{1}{32}\mathrm{Cos}[2x]$$

Expand[fr[x]]

$$\frac{5\pi^4}{12288}-\frac{\pi^3 x}{192}+\frac{3\pi^2 x^2}{128}-\frac{\pi x^3}{24}+\frac{x^4}{48}-\frac{1}{32}\mathrm{Cos}[2x]$$

20.2.4 Higher Integrals of $cos^{\alpha}x$, $sin^{\alpha}x$

Lemma1 is a polynomial including binomial coefficients. Such a formula can be easily extended to a real number region.

Lemma2

$$cos^{\alpha}x=\frac{1}{2^{\alpha}}\sum_{r=0}^{\infty}\binom{\alpha}{r}cos\{(\alpha-2r)x\} \qquad \alpha>0\ ,\ \ |x|\le\frac{\pi}{2} \tag{4.α}$$

Formula 20.2.4

$$\int_0^x cos^{\alpha}x\,dx=\frac{1}{2^{\alpha}}\sum_{r=0}^{\infty}\binom{\alpha}{r}\frac{1}{\alpha-2r}sin\{(\alpha-2r)x\} \tag{4.c}$$

$$\int_{\frac{\pi}{2}}^x sin^{\alpha}x\,dx=\frac{1}{2^{\alpha}}\sum_{r=0}^{\infty}\binom{\alpha}{r}\frac{1}{\alpha-2r}sin\left\{(\alpha-2r)\left(x-\frac{\pi}{2}\right)\right\} \tag{4.s}$$

Especially when $\alpha=2m$,

$$\int_0^x cos^{2m}x\,dx=\frac{1}{2^{2m-1}}\sum_{r=0}^{m-1}\frac{{}_{2m}C_r}{2m-2r}sin\{(2m-2r)x\}+\frac{{}_{2m}C_m}{2^{2m}}x \tag{4.ce}$$

$$\int_{\frac{\pi}{2}}^{x} \sin^{2m} x\, dx = \frac{1}{2^{2m-1}} \sum_{r=0}^{m-1} \frac{(-1)^{m-r}\, {}_{2m}C_r}{2m-2r} \sin\{(2m-2r)x\} + \frac{{}_{2m}C_m}{2^{2m}} \left(x - \frac{\pi}{2} \right) \tag{4.se}$$

Proof

Integrating both sides of $(4.\alpha)$ in Lemma2 with respect to x from 0 to x, we obtain

$$\int_0^x \cos^{\alpha} x\, dx = \frac{1}{2^{\alpha}} \sum_{r=0}^{\infty} \binom{\alpha}{r} \frac{1}{\alpha - 2r} \sin\{(\alpha - 2r)x\} \tag{4.c}$$

Especially when $\alpha = 2m$,

$$\sum_{r=0}^{m-1} \frac{{}_{2m}C_r}{2m-2r} \sin\{(2m-2r)x\} = \sum_{r=m+1}^{2m} \frac{{}_{2m}C_r}{2m-2r} \sin\{(2m-2r)x\}$$

Using this,

$$\begin{aligned}\int_0^x \cos^{2m} x\, dx &= \frac{1}{2^{2m}} \sum_{r=0}^{2m} \frac{{}_{2m}C_r}{2m-2r} \sin\{(2m-2r)x\} \\ &= \frac{1}{2^{2m}} \left[\sum_{r=0}^{m-1} \frac{{}_{2m}C_r}{2m-2r} \sin\{(2m-2r)x\} + \sum_{r=m+1}^{2m} \frac{{}_{2m}C_r}{2m-2r} \sin\{(2m-2r)x\} \right] \\ &\qquad + \frac{{}_{2m}C_m}{2^{2m}} \lim_{r \to m} \frac{\sin\{(2m-2r)x\}}{2m-2r} \\ &= \frac{2}{2^{2m}} \sum_{r=0}^{m-1} \frac{{}_{2m}C_r}{2m-2r} \sin\{(2m-2r)x\} + \frac{{}_{2m}C_m}{2^{2m}} x \end{aligned} \tag{4.ce}$$

(4.s), (4.se) are obtained by the replacing x with $x - \pi/2$.

Note

Even if the number of α is non-even number, the following formula does not hold generally.

$$\int_{a_n}^{x} \cdots \int_{a_2}^{x} \int_{a_1}^{x} \cos^{\alpha} x\, dx^n = \frac{1}{2^{\alpha}} \sum_{r=0}^{\infty} \binom{\alpha}{r} \frac{1}{(\alpha - 2r)^n} \cos\left\{ (\alpha - 2r)x - \frac{n\pi}{2} \right\}$$

It is because that the right side is always a real valued function although the left side is generally a complex valued function for the real number α . Although the left side can be a real valued function for the real number α , when all $a_1, a_2, \cdots, a_n$ is below $\pi/2$, it is difficult to find out such a case. This is the same also about $\sin^{\alpha} x$.

Example1 The 1st order integral of $\sqrt[3]{\sin x}$

Integration by the direct calculation

```
• a:=1/3:
• f1 := x-> int(sin(t)^a,t=PI/2..x)
```

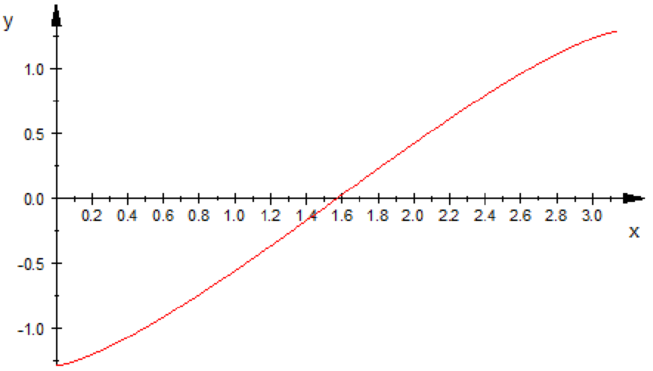
$$x \to \int_{\frac{\pi}{2}}^{x} \sin(t)^a\, dt$$

Integration by the formula

```
• m:=80: MAXDEPTH:=1000:
• fr := x-> 1/2^a*sum(binomial(a,r)/(a-2*r)
                  *sin((a-2*r)*(x-PI/2)), r=0..m)
```

$$x \to \frac{1}{2^a} \cdot \left(\sum_{r=0}^{m} \frac{\binom{a}{r}}{a-2 \cdot r} \cdot \sin\left((a-2 \cdot r) \cdot \left(x - \frac{\pi}{2} \right) \right) \right)$$

Bulu e: Direct, Red: Form ula



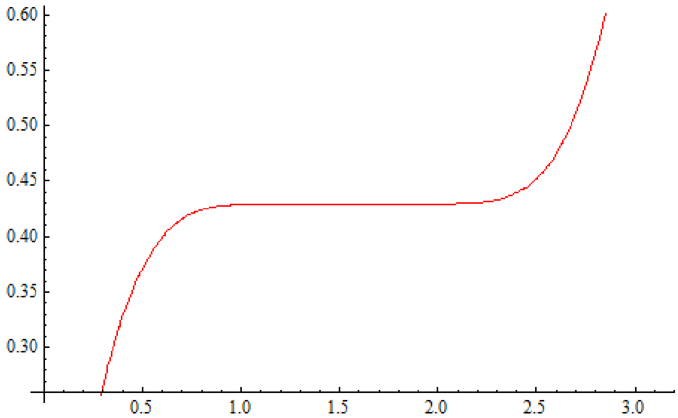
Since both sides have overlapped, the left side (blue) can not be seen.

Example2 The 1st order integral of $cos^8 x$

```
m = 4;
fl[x_] := ∫_0^x Cos[t]^(2 m) ⅆt
fr[x_] := 1/2^(2 m-1) Σ_(r=0)^(m-1) Binomial[2 m, r]/(2 m - 2 r) Sin[(2 m - 2 r) x] + Binomial[2 m, m]/2^(2 m) x
```

$$fl[x\_] := \int_0^x Cos[t]^{2m} \, dt$$

$$fr[x\_] := \frac{1}{2^{2m-1}} \sum_{r=0}^{m-1} \frac{Binomial[2m, r]}{2m-2r} Sin[(2m-2r)\, x] + \frac{Binomial[2m, m]}{2^{2m}} x$$



Since both sides have overlapped, the left side (blue) can not be seen.

2010.12.21

K. Kono

Alien's Mathematics