9 Higher Derivative

9.1 Higher Derivative and Higher Differentiation

9.1.1 Higher Derivative

Definition 9.1.1

When $f^{(n)}(x)$ denotes the derivative function of $f^{(n-1)}(x)$ for $n=1,2,3,\cdots$,
we call $f^{(n)}(x)$ ***Higher Order Derivative Function of*** $\boldsymbol{f(x)}$,
or for short , ***Higher Derivative of*** $\boldsymbol{f(x)}$.

9.1.2 Higher Differentiation

Definition 9.1.2

We call it ***Higher Differentiation*** to differentiate a function f with respect to an independent variable x repeatedly. And it is described as follows.

$$\frac{d^n}{dx^n}f(x) \qquad \left\{ = \frac{d}{dx}\left(\cdots \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}f(x)\right)\right)\cdots \right) \frac{d}{dx}:\ \text{n pieces}\right\}$$

9.1.3 Fundamental Theorem of Higher Differentiation

The following theorem holds from Theorem 4.1.3 in **4.1** .

Theorem 9.1.3

When $f^{(r)}$ $r=0,1,\cdots,n$ are continuous functions on a closed interval I and are the r th derivative functions of f , the following expression holds for $x \in I$.

$$\frac{d^n}{dx^n}f(x) = f^{(n)}(x) \tag{1.1}$$

Proof

Theorem 4.1.3 in **4.1** can be rewritten as follows.

$$f^{<n>}(x) = \int_{a_n}^{x}\cdots\int_{a_1}^{x} f(x)dx^n + \sum_{r=0}^{n-1} f^{<n-r>}(a_{n-r})\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

Differentiating both sides with respect to x n times, we obtain

$$\frac{d^n}{dx^n}f^{<n>}(x) = f^{<0>}(x) + \frac{d^n}{dx^n}\sum_{r=0}^{n-1} f^{<n-r>}(a_{n-r})\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

Here, since the constant-of-integration polynomial of the right side is degree $n-1$, if this is differentiated n times, it must become 0. Then we obtain

$$\frac{d^n}{dx^n}f^{<n>}(x) = f^{<0>}(x)$$

Shifting by $-n$ the index in the integration operator $<>$ and replacing $<>$ by differentiation operator $(\ )$, we obtain the desired expression.

Remark

Since differentiation is an inverse operation of integration, it cannot be unrelated to the constant-of-integration. In fact, the lineal and the collateral exist in Super Differentiation (non-integer times differentiation). (See later 12.1.2.) But this thorem guarantees that only the lineal exists in Higher Differentiation.

9.1.4 The basic formulas of Higher Differentiation

The following formulas hold like the 1st order differentiation.

$$\{c\,f(x)\}^{(n)} = c\,f^{(n)}(x) \qquad c \neq 0 \qquad \text{: constant multiple rule}$$

$$\{f(x)+g(x)\}^{(n)} = f^{(n)}(x) + g^{(n)}(x) \qquad \text{: sum rule}$$

$$\{f(x)g(x)\}^{(n)} = \sum_{r=0}^{n} {}_nC_r\, f^{(r)}(x)\, g^{(n-r)}(x) \qquad \text{: product rule (Leibniz rule)}$$

9.2 Higher Derivative of Elementary Functions

Formula 9.2.1 Higher Derivative of a power function

When $\Gamma(z)$ denots the zeta function, the following expressions hold.

(1) Basic form

$$\left(x^{\alpha}\right)^{(n)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)}x^{\alpha-n} \qquad (\alpha \geqq 0)$$

$$= (-1)^{-n}\frac{\Gamma(-\alpha+n)}{\Gamma(-\alpha)}x^{\alpha-n} \qquad (\alpha < 0)$$

(2) Linear form

$$\left\{(ax+b)^{\alpha}\right\}^{(n)} = \left(\frac{1}{a}\right)^{-n}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)}(ax+b)^{\alpha-n} \qquad (\alpha \geqq 0)$$

$$= \left(-\frac{1}{a}\right)^{-n}\frac{\Gamma(-\alpha+n)}{\Gamma(-\alpha)}(ax+b)^{\alpha-n} \qquad (\alpha < 0)$$

Proof

$$\left\{(ax+b)^{\alpha}\right\}^{(1)} = \left(\frac{1}{a}\right)^{-1}\alpha(ax+b)^{\alpha-1}$$

$$\left\{(ax+b)^{\alpha}\right\}^{(2)} = \left(\frac{1}{a}\right)^{-2}\alpha(\alpha-1)(ax+b)^{\alpha-2}$$

$$\vdots$$

$$\left\{(ax+b)^{\alpha}\right\}^{(n)} = \left(\frac{1}{a}\right)^{-n}\alpha(\alpha-1)\cdots\{\alpha-(n-1)\}x^{\alpha-n}$$

Substituting $\alpha(\alpha-1)\cdots\{\alpha-(n-1)\} = \dfrac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)}$ for this, we obtain (2).

Furthermore, substituting $a=1,\ b=0$ for this, we obtain (1).

However, when α is a negative integer, a denominator and a numerator become the infinite form of infinity, and as for this right side, the value can not be calculated.

Therfore, when $\alpha<0$ let $\beta=-\alpha$. . Then, since $(ax+b)^{\alpha} = (ax+b)^{-\beta}$,

$$\left\{(ax+b)^{-\beta}\right\}^{(1)} = \left(\frac{1}{a}\right)^{-1}-\beta(ax+b)^{-\beta-1}$$

$$\left\{(ax+b)^{-\beta}\right\}^{(2)} = \left(\frac{1}{a}\right)^{-2}+\beta(\beta+1)(ax+b)^{-\beta-2}$$

$$\vdots$$

$$\left\{(ax+b)^{-\beta}\right\}^{(n)} = \left(\frac{1}{a}\right)^{-n}(-1)^{n}\beta(\beta+1)\cdots(\beta+n-1)(ax+b)^{-\beta-n}$$

Here

$$\beta(\beta+1)\cdots(\beta+n-1) = \frac{\Gamma(\beta+n)}{\Gamma(\beta)}$$

we can substitute this for the above as follows.

$$\left\{ (ax+b)^{-\beta} \right\}^{(n)} = \left(\frac{1}{a} \right)^{-n} (-1)^n \frac{\Gamma(\beta+n)}{\Gamma(\beta)} (ax+b)^{-\beta-n}$$

Then replacing β with $-\alpha$ we botain (2), furthermore, substituting $a=1, b=0$ for this, we obtain (1).

Formula 9.2.2 : Higher Derivative of Exponential Functions

(1) Basic form

$$\left(e^{\pm x}\right)^{(n)} = (\pm 1)^{-n} e^{\pm x}$$

(2) Linear form

$$\left(e^{ax+b}\right)^{(n)} = \left(\frac{1}{a} \right)^{-n} e^{ax+b}$$

Proof

According to Formula 4.3.2 in **4.3** , it was as follows.

$$\int_{\mp\infty}^{x} \cdots \int_{\mp\infty}^{x} e^{\pm x} dx^n = (\pm 1)^n e^{\pm x}$$

$$\int_{\mp\infty}^{x} \cdots \int_{\mp\infty}^{x} e^{ax+b} dx^n = \left(\frac{1}{a} \right)^n e^{ax+b} \qquad \left(\begin{array}{l} a>0 : - \\ a<0 : + \end{array} \right)$$

Since the differentiation is a reverse-operation of integration, replacing the index n of the integration operator with $-n$, we obtain the desired expression.

Formula 9.2.3 : Higher Derivative of Logarithmic Functions

(1) Basic form

$$(\log x)^{(n)} = (-1)^{n-1} (n-1)!\ x^{-n}$$

(2) Linear form

$$\left\{ \log(ax+b) \right\}^{(n)} = (-1)^{n-1} (n-1)! \left(x + \frac{b}{a} \right)^{-n}$$

Proof

Differentiating log x with respect to x one by one, we obtain the desired expression. As for **(2)**, it is similar.

Formula 9.2.4 : Higher Derivatives of sin x, cos x

(1) Basic form

$$(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2} \right)$$

$$(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2} \right)$$

(2) Linear form

$$\left\{ \sin(ax+b) \right\}^{(n)} = \left(\frac{1}{a} \right)^{-n} \sin\left(ax+b + \frac{n\pi}{2} \right)$$

$$\left\{ \cos(ax+b) \right\}^{(n)} = \left(\frac{1}{a} \right)^{-n} \cos\left(ax+b + \frac{n\pi}{2} \right)$$

Proof

Replacing the index n of the integration operator with $-n$ in Formula 4.3.4 (**4.3**) , we obtain the desired expressions.

Formula 9.2.5 : Higher Derivatives of sinh x, cosh x

(1) Basic form

$$(\sinh x)^{(n)} = i^{-n}\sinh\left(x+\frac{n\pi i}{2}\right) = \frac{e^{x}-(-1)^{-n}e^{-x}}{2}$$

$$(\cosh x)^{(n)} = i^{-n}\cosh\left(x+\frac{n\pi i}{2}\right) = \frac{e^{x}+(-1)^{-n}e^{-x}}{2}$$

(2) Linear form

$$\{\sinh(ax+b)\}^{(n)} = \left(\frac{i}{a}\right)^{-n}\sinh\left(ax+b+\frac{n\pi i}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a}\right)^{-n}\left\{e^{ax+b}-(-1)^{-n}e^{-(ax+b)}\right\}$$

$$\{\cosh(ax+b)\}^{(n)} = \left(\frac{i}{a}\right)^{-n}\cosh\left(ax+b+\frac{n\pi i}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a}\right)^{-n}\left\{e^{ax+b}+(-1)^{-n}e^{-(ax+b)}\right\}$$

Proof

Replacing the index n of the integration operator with $-n$ in Formula 4.3.5 (**4.3**) , we obtain the desired expressions.

Formula 9.2.6 : Higher Derivatives of tan x, cot x

When ↑ denots ceiling function, the following expressions hold for a natural number n.

$$(\tan x)^{(n)} = \sum_{r=0}^{n/2\uparrow} {}_{n}T_{r}(\tan x)^{n+1-2r}$$

$$(\cot x)^{(n)} = (-1)^{n}\sum_{r=0}^{n/2\uparrow} {}_{n}T_{r}(\cot x)^{n+1-2r}$$

where ${}_{n}T_{r}$ are coefficients as follows.

$$\begin{matrix} {}_{0}T_{0} \\ {}_{1}T_{0}\quad {}_{1}T_{1} \\ {}_{2}T_{0}\quad {}_{2}T_{1} \\ {}_{3}T_{0}\quad {}_{3}T_{1}\quad {}_{3}T_{2} \\ {}_{4}T_{0}\quad {}_{4}T_{1}\quad {}_{4}T_{2} \\ {}_{5}T_{0}\quad {}_{5}T_{1}\quad {}_{5}T_{2}\quad {}_{5}T_{3} \\ {}_{6}T_{0}\quad {}_{6}T_{1}\quad {}_{6}T_{2}\quad {}_{6}T_{3} \\ \vdots \end{matrix} \quad = \quad \begin{matrix} 1 \\ 1\quad 1 \\ 2\quad 2 \\ 6\quad 8\quad 2 \\ 24\quad 40\quad 16 \\ 120\quad 240\quad 136\quad 16 \\ 720\quad 1680\quad 1232\quad 272 \\ \vdots \end{matrix}$$

And these coefficients are obtained by the following sequential computation

	Calculating formula
0!	
1	
1! 1	$1=0!\times 1$
2 0	
2! 2	$2=1!\times 2+1\times 0$
3 1	
3! 8 2	$8=2!\times 3+2\times 1$, $2=2\times 1$
4 2 0	
4! 40 16	$40=3!\times 4+8\times 2$, $16=8\times 2+2\times 0$
5 3 1	
5! 240 136 16	$240=4!\times 5+40\times 3$, $136=40\times 3+16\times 1$, $16=16\times 1$
6 4 2 0	
6! 1680 1232 272	$1680=5!\times 6+240\times 4$, $1232=240\times 4+136\times 2$, $272=136\times 2+16\times 0$
⋮	⋮

Note

When $T_1=1$, $T_3=2$, $T_5=16$, $T_7=272$, $T_9=7936$, ⋯ , T_{2n-1} , ⋯ are tangent numbers, there is the following relation between these and the above coefficients.

$$T_{2n-1} = {}_{2n-1}T_n \qquad n=1,2,3,\cdots$$

Formula 9.2.7 : Higher Derivatives of tanh x, coth x

When ↑ denots ceiling function, the following expressions hold for a natural number n.

$$(\tanh x)^{(n)} = (-1)^n \sum_{r=0}^{n/2\uparrow} (-1)^r \, {}_nT_r (\tanh x)^{n+1-2r}$$

$$(\coth x)^{(n)} = (-1)^n \sum_{r=0}^{n/2\uparrow} (-1)^r \, {}_nT_r (\coth x)^{n+1-2r}$$

where ${}_nT_r$ are same as ones in Formula 9.2.6 .

Formula 9.2.8 : Higher Derivatives of sec x, csc x

When ↓ denots floor function, the following expressions hold for a natural number n.

$$(\sec x)^{(n)} = \frac{1}{\cos x} \sum_{r=0}^{n/2\downarrow} {}_nE_r (\tan x)^{n-2r}$$

$$(\csc x)^{(n)} = \frac{(-1)^n}{\sin x} \sum_{r=0}^{n/2\downarrow} {}_nE_r (\cot x)^{n-2r}$$

where ${}_nE_r$ are coefficients as follows.

$$\begin{array}{ccccccc} & & & {}_0E_0 & & & \\ & & & {}_1E_0 & & & \\ & & {}_2E_0 & & {}_2E_1 & & \\ & & {}_3E_0 & & {}_3E_1 & & \\ & {}_4E_0 & & {}_4E_1 & & {}_4E_2 & \\ & {}_5E_0 & & {}_5E_1 & & {}_5E_2 & \\ {}_6E_0 & & {}_6E_1 & & {}_6E_2 & & {}_6E_3 \\ {}_7E_0 & & {}_7E_1 & & {}_7E_2 & & {}_7E_3 \\ & & & \vdots & & & \end{array} = \begin{array}{cccc} & & & 1 \\ & & & 1 \\ & & 2 & 1 \\ & & 6 & 5 \\ & 24 & 28 & 5 \\ & 120 & 180 & 61 \\ 720 & 1320 & 662 & 61 \\ 5040 & 10920 & 7266 & 1385 \\ & & \vdots & \end{array}$$

And these coefficients are obtained by the following sequential computation

$$\begin{array}{lll}
\mathbf{1!} & & \\
1 & & \\
\mathbf{2!}\quad \mathbf{1} & & 1=1!\times 1 \\
2\quad 1 & & \\
\mathbf{3!}\quad \mathbf{5} & & 5=2!\times 2+1\times 1 \\
3\quad 2\quad 1 & & \\
\mathbf{4!}\quad \mathbf{28}\quad \mathbf{5} & & 28=3!\times 3+5\times 2,\ 5=5\times 1 \\
4\quad 3\quad 2\quad 1 & & \\
\mathbf{5!}\quad \mathbf{180}\quad \mathbf{61} & & 180=4!\times 4+28\times 3,\ 61=28\times 2+5\times 1 \\
5\quad 4\quad 3\quad 2\quad 1 & & \\
\mathbf{6!}\quad \mathbf{1320}\quad \mathbf{662}\quad \mathbf{61} & & 1320=5!\times 5+180\times 4,\ 662=180\times 3+61\times 2,\ 61=61\times 1 \\
6\quad 5\quad 4\quad 3\quad 2\quad 1 & & \\
\mathbf{7!}\quad \mathbf{10920}\quad \mathbf{7266}\quad \mathbf{1385} & & 10920=6!\times 6+1320\times 5,\ 7266=1320\times 4+662\times 3, \\
\vdots & \vdots & 1385=662\times 2+61\times 1
\end{array}$$

Note

When $E_0=1$, $E_2=-1$, $E_4=5$, $E_6=-61$, $E_8=1385$, $\cdots$, E_{2n}, $\cdots$ are Euler numbers, there is the following relation between these and the above coefficients.

$$E_{2n} = (-1)^n {}_{2n}E_n \qquad n=0,1,2,\cdots$$

Formula 9.2.9 : Higher Derivatives of sech x, csch x

When $\downarrow$ denots floor function, the following expressions hold for a natural number n.

$$(\operatorname{sech} x)^{(n)} = \frac{(-1)^n}{\cosh x}\sum_{r=0}^{n/2\downarrow}(-1)^r {}_nE_r(\tanh x)^{n-2r}$$

$$(\operatorname{csch} x)^{(n)} = \frac{(-1)^n}{\sinh x}\sum_{r=0}^{n/2\downarrow}(-1)^r {}_nE_r(\coth x)^{n-2r}$$

where ${}_nE_r$ are same as ones in Formula 9.2.8 .

9.3 Higher Derivative of Inverse Trigonometric Functions

Formula 9.3.1 : Higher Derivatives of arctan x , arccot x

When ↑ denots ceiling function, the following expressions hold for a natural number n.

$$\left(\tan^{-1}x\right)^{(n)} = (-1)^n \frac{(n-1)!}{\left(x^2+1\right)^n} \sum_{r=1}^{n/2\uparrow} (-1)^r {}_nC_{n+1-2r}\, x^{n+1-2r}$$

$$\left(\cot^{-1}x\right)^{(n)} = (-1)^{n-1} \frac{(n-1)!}{\left(x^2+1\right)^n} \sum_{r=1}^{n/2\uparrow} (-1)^r {}_nC_{n+1-2r}\, x^{n+1-2r}$$

Proof

Differentiating arctan x with respect to x one by one, it is as follows.

$$\left(\tan^{-1}x\right)^{(2)} = -\frac{1!\cdot 2x}{\left(x^2+1\right)^2} = -\frac{1!\,{}_2C_1\, x}{\left(x^2+1\right)^2}$$

$$\left(\tan^{-1}x\right)^{(3)} = \frac{2!\left(3x^2-1\right)}{\left(x^2+1\right)^3} = \frac{2!\left({}_3C_2\, x^2 - {}_3C_0\right)}{\left(x^2+1\right)^3}$$

$$\left(\tan^{-1}x\right)^{(4)} = -\frac{3!\left(4x^3-4x\right)}{\left(x^2+1\right)^4} = -\frac{3!\left({}_4C_3\, x^3 - {}_4C_1\, x\right)}{\left(x^2+1\right)^4}$$

$$\left(\tan^{-1}x\right)^{(5)} = \frac{4!\left(5x^4-10x^2+1\right)}{\left(x^2+1\right)^5} = \frac{4!\left({}_6C_4\, x^4 - {}_5C_2\, x^2 + {}_5C_0\right)}{\left(x^2+1\right)^5}$$

$$\left(\tan^{-1}x\right)^{(6)} = -\frac{5!\left(6x^5-20x^3+6x\right)}{\left(x^2+1\right)^6} = -\frac{5!\left({}_6C_5\, x^5 - {}_6C_3\, x^3 + {}_6C_1\, x\right)}{\left(x^2+1\right)^6}$$

$$\vdots$$

Hereafter by induction, we obtain the desired expression.

c.f.

According to 「岩波数学公式Ｉ」 p39,40, the following expressions hold for a natural number n.

$$\left(\tan^{-1}x\right)^{(n)} = (n-1)!\cos^n\!\left(\tan^{-1}x\right)\sin\left(n\left(\tan^{-1}x+\frac{\pi}{2}\right)\right)$$

$$\left(\cot^{-1}x\right)^{(n)} = (-1)^n (n-1)!\ \sin^n\!\left(\cot^{-1}x\right)\sin\left(n\cot^{-1}x\right)$$

Therefore, combining with Formula 9.3.1, these formulas lead to the following equations.

$$\sum_{k=1}^{n/2\uparrow} (-1)^k {}_nC_{n+1-2k}\, x^{n+1-2k} = -\left(x^2+1\right)^n \sin^n\!\left(\cot^{-1}x\right)\sin\left(n\cot^{-1}x\right)$$

$$\sum_{k=1}^{n/2\uparrow} (-1)^k {}_nC_{n+1-2k} = -2^{\frac{n}{2}}\sin\frac{n\pi}{4}$$

Example

$$-{}_5C_4\, x^4 + {}_5C_2\, x^2 - {}_5C_0\, x^0 = -\left(x^2+1\right)^5 \sin^5\!\left(\cot^{-1}x\right)\sin\left(5\cot^{-1}x\right)$$

$$-{}_5C_4 + {}_5C_2 - {}_5C_0 = -2^{\frac{5}{2}}\sin\frac{5\pi}{4} = 4$$

Formula 9.3.2 : Higher Derivatives of arcsin x, arccos x

When $\downarrow$ denots floor function, the following expressions hold for a natural number n.

$$\left(\sin^{-1}x\right)^{(n)} = \sum_{r=0}^{n/2\downarrow}\binom{n-1}{n-1-2r}\frac{(2r-1)!!\,(2n-3-2r)!!\,x^{n-1-2r}}{\left(1-x^2\right)^{n-r-\frac{1}{2}}}$$

$$\left(\cos^{-1}x\right)^{(n)} = -\sum_{r=0}^{n/2\downarrow}\binom{n-1}{n-1-2r}\frac{(2r-1)!!\,(2n-3-2r)!!\,x^{n-1-2r}}{\left(1-x^2\right)^{n-r-\frac{1}{2}}}$$

Proof

Differentiating arcsin x with respect to x one by one, it is as follows.

$$\left(\sin^{-1}x\right)^{(1)} = \frac{1}{\left(1-x^2\right)^{\frac{1}{2}}}$$

$$\left(\sin^{-1}x\right)^{(2)} = \frac{x}{\left(1-x^2\right)^{\frac{3}{2}}}$$

$$\left(\sin^{-1}x\right)^{(3)} = \frac{3x^2}{\left(1-x^2\right)^{\frac{5}{2}}} + \frac{1}{\left(1-x^2\right)^{\frac{3}{2}}}$$

$$\left(\sin^{-1}x\right)^{(4)} = \frac{15x^3}{\left(1-x^2\right)^{\frac{7}{2}}} + \frac{9x}{\left(1-x^2\right)^{\frac{5}{2}}}$$

$$\left(\sin^{-1}x\right)^{(5)} = \frac{105x^4}{\left(x^2+1\right)^{\frac{9}{2}}} + \frac{90x^2}{\left(x^2+1\right)^{\frac{7}{2}}} + \frac{9}{\left(x^2+1\right)^{\frac{5}{2}}}$$

$$\left(\sin^{-1}x\right)^{(6)} = \frac{945x^5}{\left(x^2+1\right)^{\frac{11}{2}}} + \frac{1050x^3}{\left(x^2+1\right)^{\frac{9}{2}}} + \frac{225x}{\left(x^2+1\right)^{\frac{7}{2}}}$$

$$\vdots$$

These coefficients are expressed as follows using binomial coefficients and double factorials.

1	${}_0C_0(-1)!!\,(-1)!!$		
2	${}_1C_1(-1)!!\,1!!$		
3	${}_2C_2(-1)!!\,3!!$	${}_2C_0 1!!\,1!!$	
4	${}_3C_3(-1)!!\,5!!$	${}_3C_1 1!!\,3!!$	
5	${}_4C_4(-1)!!\,7!!$	${}_4C_2 1!!\,5!!$	${}_4C_0 3!!\,3!!$
6	${}_5C_5(-1)!!\,9!!$	${}_5C_3 1!!\,7!!$	${}_5C_1 3!!\,5!!$
	$\vdots$		

Substituting these for the above,

$$\left(\sin^{-1}x\right)^{(1)} = \frac{{}_0C_0\,(-1)!!\,(-1)!!\,x^0}{\left(x^2+1\right)^{1-\frac{1}{2}}}$$

$$\left(\sin^{-1}x\right)^{(2)} = \frac{{}_1C_1\,(-1)!!\,1!!\,x^1}{\left(x^2+1\right)^{2-\frac{1}{2}}}$$

$$\left(\sin^{-1}x\right)^{(3)} = \frac{{}_2C_2\,(-1)!!\,3!!\,x^2}{\left(x^2+1\right)^{3-\frac{1}{2}}} + \frac{{}_2C_0\,1!!\,1!!\,x^0}{\left(x^2+1\right)^{2-\frac{1}{2}}}$$

$$\left(\sin^{-1}x\right)^{(4)} = \frac{{}_3C_3\,(-1)!!\,5!!\,x^3}{\left(x^2+1\right)^{4-\frac{1}{2}}} + \frac{{}_3C_1\,1!!\,3!!\,x^1}{\left(x^2+1\right)^{3-\frac{1}{2}}}$$

$$\left(\sin^{-1}x\right)^{(5)} = \frac{{}_4C_4\,(-1)!!\,7!!\,x^4}{\left(x^2+1\right)^{5-\frac{1}{2}}} + \frac{{}_4C_2\,1!!\,5!!\,x^2}{\left(x^2+1\right)^{4-\frac{1}{2}}} + \frac{{}_4C_0\,3!!\,3!!\,x^0}{\left(x^2+1\right)^{3-\frac{1}{2}}}$$

$$\left(\sin^{-1}x\right)^{(6)} = \frac{{}_5C_5\,(-1)!!\,9!!\,x^5}{\left(x^2+1\right)^{6-\frac{1}{2}}} + \frac{{}_5C_3\,1!!\,7!!\,x^3}{\left(x^2+1\right)^{5-\frac{1}{2}}} + \frac{{}_5C_1\,3!!\,5!!\,x^1}{\left(x^2+1\right)^{4-\frac{1}{2}}}$$

$$\vdots$$

$$\left(\sin^{-1}x\right)^{(n)} = \sum_{r=0}^{n/2\downarrow} \binom{n-1}{n-1-2r} \frac{(2r-1)!!\,(2n-3-2r)!!\,x^{n-1-2r}}{\left(1-x^2\right)^{n-r-\frac{1}{2}}}$$

And reversing this sighn , we obtain $\left(\cos^{-1}x\right)^{(n)}$.

Example: the 9th order derivative of $\sin^{-1}x$

Repeated differentiation

- `g := diff(arcsin(x),x,x,x,x,x,x,x,x,x)`

$$\frac{396900 \cdot x^2}{\left(1-x^2\right)^{\frac{11}{2}}} + \frac{2182950 \cdot x^4}{\left(1-x^2\right)^{\frac{13}{2}}} + \frac{3783780 \cdot x^6}{\left(1-x^2\right)^{\frac{15}{2}}} + \frac{2027025 \cdot x^8}{\left(1-x^2\right)^{\frac{17}{2}}} + \frac{11025}{\left(1-x^2\right)^{\frac{9}{2}}}$$

Formula

- ```
  f := n-> sum(binomial(n-1,n-1-2*r)
                    *(2*r-1)!!*(2*n-3-2*r)!!*x^(n-1-2*r)
                    /(1-x^2)^(n-r-1/2), r=0..floor(n/2))
  ```

$$n \to \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\binom{n-1}{n-1-2\cdot r} \cdot 2 \cdot r - 1!! \cdot 2 \cdot n - 3 - 2 \cdot r!! \cdot x^{n-1-2\cdot r}}{\left(1-x^2\right)^{n-r-\frac{1}{2}}}$$

- `f(9)`

$$\frac{396900 \cdot x^2}{\left(1-x^2\right)^{\frac{11}{2}}} + \frac{2182950 \cdot x^4}{\left(1-x^2\right)^{\frac{13}{2}}} + \frac{3783780 \cdot x^6}{\left(1-x^2\right)^{\frac{15}{2}}} + \frac{2027025 \cdot x^8}{\left(1-x^2\right)^{\frac{17}{2}}} + \frac{11025}{\left(1-x^2\right)^{\frac{9}{2}}}$$
  ```

Formula 9.3.2 can also be expressed as follows, if partial fraction decomposition is applied to the right side.

Formula 9.3.2'

$$\left(\sin^{-1}x\right)^{(n)} = \frac{1}{2^{n-1}\sqrt{1-x^2}}\sum_{r=0}^{n-1}(-1)^r\binom{n-1}{r}\frac{(2r-1)!!\,(2n-2r-3)!!}{(1+x)^r(1-x)^{n-1-r}}$$

$$\left(\cos^{-1}x\right)^{(n)} = -\frac{1}{2^{n-1}\sqrt{1-x^2}}\sum_{r=0}^{n-1}(-1)^r\binom{n-1}{r}\frac{(2r-1)!!\,(2n-2r-3)!!}{(1+x)^r(1-x)^{n-1-r}}$$

For example,

$$\left(\sin^{-1}x\right)^{(3)} = \frac{3x^2}{\left(1-x^2\right)^{\frac{5}{2}}} + \frac{1}{\left(1-x^2\right)^{\frac{3}{2}}} = \frac{1}{\sqrt{1-x^2}}\frac{2x^2+1}{(1+x)^2(1-x)^2}$$

Let us decompose this to the partial fractions. then,

$$\frac{2x^2+1}{(1+x)^2(1-x)^2} = \frac{B_0}{(1-x)^2} + \frac{B_1}{(1+x)^1(1-x)^1} + \frac{B_2}{(1+x)^2}$$

Using Heaviside cover-up method,

$$\frac{2x^2+1}{(1+x)^2} = B_0 + \frac{B_1(1-x)^1}{(1+x)^1} + \frac{B_2(1-x)^2}{(1+x)^2} \qquad (1)$$

$$\frac{2x^2+1}{(1+x)^1(1-x)^1} = \frac{B_0(1+x)^1}{(1-x)^1} + B_1 + \frac{B_2(1-x)^1}{(1+x)^1} \qquad (2)$$

$$\frac{2x^2+1}{(1-x)^2} = \frac{B_0(1+x)^2}{(1-x)^2} + \frac{B_1(1+x)^1}{(1-x)^1} + B_2 \qquad (3)$$

Giving $x=1,\ 0\ ,\ -1$ to (1),(2),(3) respectively,

$$\frac{3}{2^2} = B_0\ ,\quad 1 = B_0 + B_1 + B_2\ ,\quad \frac{3}{2^2} = B_2$$

From these $B_1 = -\frac{2}{2^2}$. Then,

$$\left(\sin^{-1}x\right)^{(3)} = \frac{1}{2^2\sqrt{1-x^2}}\left\{\frac{3}{(1-x)^2} - \frac{2}{(1+x)^1(1-x)^1} + \frac{3}{(1+x)^2}\right\}$$

$$= \frac{1}{2^2\sqrt{1-x^2}}\left\{\frac{(-1)!!\,3!!}{(1+x)^0(1-x)^2} - \frac{2\cdot 1!!\,1!!}{(1+x)^1(1-x)^1} + \frac{3!!\,(-1)!!}{(1+x)^2(1-x)^0}\right\}$$

$$= \frac{1}{2^2\sqrt{1-x^2}}\sum_{r=0}^{3-1}(-1)^r\binom{2}{r}\frac{(2r-1)!!\,(2\cdot 3-2r-3)!!}{(1+x)^r(1-x)^{3-1-r}}$$

Formula 9.3.3 : Higher Derivatives of arcsec x, arccsc x

The following expressions hold for a natural number n.

$$\left(\sec^{-1}x\right)^{(n)} = (-1)^{n-1}\sum_{r=1}^{n}\frac{{}_nA_r}{x^{n-1+2r}\left(1-x^{-2}\right)^{r-\frac{1}{2}}}$$

$$\left(\csc^{-1}x\right)^{(n)} = (-1)^n \sum_{r=1}^{n} \frac{{}_nA_r}{x^{n-1+2r}\left(1-x^{-2}\right)^{r-\frac{1}{2}}}$$

where ${}_nA_r$ are coefficients as follows.

$$\begin{array}{c} {}_1A_1 \\ {}_2A_1 \quad {}_2A_2 \\ {}_3A_1 \quad {}_3A_2 \quad {}_3A_3 \\ {}_4A_1 \quad {}_4A_2 \quad {}_4A_3 \quad {}_4A_4 \\ {}_5A_1 \quad {}_5A_2 \quad {}_5A_3 \quad {}_5A_4 \quad {}_5A_5 \\ {}_6A_1 \quad {}_6A_2 \quad {}_6A_3 \quad {}_6A_4 \quad {}_6A_5 \quad {}_6A_6 \\ \vdots \end{array} = \begin{array}{c} 1 \\ 2 \quad 1 \\ 6 \quad 7 \quad 3 \\ 24 \quad 48 \quad 45 \quad 15 \\ 120 \quad 360 \quad 549 \quad 390 \quad 105 \\ 720 \quad 3000 \quad 6570 \quad 7425 \quad 4200 \quad 945 \\ \vdots \end{array}$$

And these coefficients are obtained by the following sequential computation

$$\begin{array}{c} 2! \ \ 1!! \\ 1,5 \\ 3! \ \ 7 \ \ 3!! \\ 1,6 \quad 3,8 \\ 4! \ \ 48 \ \ 45 \ \ 5!! \\ 1,7 \quad 3,9 \quad 5,11 \\ 5! \ \ 360 \ \ 549 \ \ 390 \ \ 7!! \\ 1,8 \quad 3,10 \quad 5,12 \quad 7,14 \\ 6! \ \ 3000 \ \ 6570 \ \ 7425 \ \ 4200 \ \ 9!! \\ \vdots \end{array}$$

$7=2!\cdot 1+1!!\cdot 5$

$48=3!\cdot 1+7\cdot 6$, $45=7\cdot 3+3!!\cdot 8$

$360=4!\cdot 1+48\cdot 7$, $549=48\cdot 3+45\cdot 9$, $390=45\cdot 5+5!!\cdot 11$

$3000=5!\cdot 1+360\cdot 8$, $6570=360\cdot 3+549\cdot 10$, $7425=549\cdot 5+390\cdot 12$, $4200=390\cdot 7+7!!\cdot 14$

$\vdots$

Proof

Differentiating arcsec x with respect to x one by one, it is as follows.

$$\left(\sec^{-1}x\right)^{(1)} = \frac{1}{x^2\left(1-x^{-2}\right)^{\frac{1}{2}}}$$

$$\left(\sec^{-1}x\right)^{(2)} = -\frac{2}{x^3\left(1-x^{-2}\right)^{\frac{1}{2}}} - \frac{1}{x^5\left(1-x^{-2}\right)^{\frac{3}{2}}}$$

$$\left(\sec^{-1}x\right)^{(3)} = \frac{6}{x^4\left(1-x^{-2}\right)^{\frac{1}{2}}} + \frac{7}{x^6\left(1-x^{-2}\right)^{\frac{3}{2}}} + \frac{3}{x^8\left(1-x^{-2}\right)^{\frac{5}{2}}}$$

$$\left(\sec^{-1}x\right)^{(4)} = -\frac{24}{x^5\left(1-x^{-2}\right)^{\frac{1}{2}}} - \frac{48}{x^7\left(1-x^{-2}\right)^{\frac{3}{2}}} - \frac{45}{x^9\left(1-x^{-2}\right)^{\frac{5}{2}}} - \frac{15}{x^{11}\left(1-x^{-2}\right)^{\frac{7}{2}}}$$

$\vdots$

These coefficients are calculable with the algorithm in the formula. Then, by induction, we obtain the desired expressions.

Note

These coefficients and the direct calculation method is not known. However, these have the following character.

$$
\begin{array}{rcl}
1 & = & 0! \\
2 - 1 & = & 1! \\
6 - 7 + 3 & = & 2! \\
24 - 48 + 45 - 15 & = & 3! \\
120 - 360 + 549 - 390 + 105 & = & 4! \\
720 - 3000 + 6570 - 7425 + 4200 - 945 & = & 5! \\
\vdots & & \vdots
\end{array}
$$

9.4 Higher Derivative of Inverse Hyperbolic Functions

Formula 9.4.1 : Higher Derivatives of arctanh x , arccoth x

When ↑ denots ceiling function, the following expressions hold for a natural number n.

$$\left(\tanh^{-1}x\right)^{(n)} = \left(\coth^{-1}x\right)^{(n)} = (-1)^n \frac{(n-1)!}{\left(x^2+1\right)^n} \sum_{r=1}^{n/2\uparrow} {}_nC_{n+1-2r}\, x^{n+1-2r}$$

Proof

If we differentiate arctanh x with respect to x one by one, these coefficients become the same as the ones of $\left(\tan^{-1}x\right)^{(n)}$ (Formula 9.3.1). Then we obtain the desired expression.

Formula 9.4.2 : Higher Derivatives of arcsinh x, arccosh x

When ↓ denots floor function, the following expressions hold for a natural number n.

$$\left(\sinh^{-1}x\right)^{(n)} = (-1)^{n-1} \sum_{r=0}^{n/2\downarrow} (-1)^r \binom{n-1}{n-1-2r} \frac{(2r-1)!!\,(2n-3-2r)!!\,x^{n-1-2r}}{\left(x^2+1\right)^{n-r-\frac{1}{2}}}$$

$$\left(\cosh^{-1}x\right)^{(n)} = (-1)^{n-1} \sum_{r=0}^{n/2\downarrow} (-1)^r \binom{n-1}{n-1-2r} \frac{(2r-1)!!\,(2n-3-2r)!!\,x^{n-1-2r}}{\left(x^2-1\right)^{n-r-\frac{1}{2}}}$$

Proof

If we differentiate arcsinh x with respect to x one by one, these coefficients become the same as the ones of $\left(\sin^{-1}x\right)^{(n)}$ (Formula 9.3.2). Then we obtain the desired expression.

Formula 9.4.2 can also be expressed as follows, if partial fraction decomposition is applied to the right side.

Formula 9.4.2'

$$\left(\sinh^{-1}x\right)^{(n)} = \left(\frac{i}{2}\right)^{n-1} \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} (2r-1)!!\,(2n-2r-3)!!\,(1+ix)^{-\frac{1}{2}-r}(1-ix)^{\frac{1}{2}+r-n}$$

$$\left(\cosh^{-1}x\right)^{(n)} = \left(-\frac{1}{2}\right)^{n-1} \sum_{r=0}^{n-1} \binom{n-1}{r} (2r-1)!!\,(2n-2r-3)!!\,(x+1)^{-\frac{1}{2}-r}(x-1)^{\frac{1}{2}+r-n}$$

Example: the 5th order derivative of $\sinh^{-1}x$

Repeated differentiation

```
• g := diff(arcsinh(x),x,x,x,x,x):simplify(%)
```

$$\frac{24\cdot x^4 - 72\cdot x^2 + 9}{\left(x^2+1\right)^{\frac{9}{2}}}$$

Formula

```
• asinh := n-> (I/2)^(n-1)*
               sum((-1)^r*binomial(n-1,r)*(2*r-1)!!*(2*n-2*r-3)!!
                  *(1+I*x)^(-1/2-r)*(1-I*x)^(1/2+r-n), r=0..n-1)
```

$$n \rightarrow \left(\frac{i}{2}\right)^{n-1} \cdot \left(\sum_{r=0}^{n-1} (-1)^r \cdot \binom{n-1}{r} \cdot 2\cdot r - 1!! \cdot 2\cdot n - 2\cdot r - 3!! \cdot (1+i\cdot x)^{-\frac{1}{2}-r} \cdot (1-i\cdot x)^{\frac{1}{2}+r-n}\right)$$

- asinh(5):simplify(%)

$$\frac{24\cdot x^4-72\cdot x^2+9}{(1-\mathrm{i}\cdot x)^{\frac{9}{2}}\cdot(\mathrm{i}\cdot x+1)^{\frac{9}{2}}}$$

Formula 9.4.3 : Higher Derivatives of arcsech x , arccsah x

The following expressions hold for a natural number n.

$$\left(sech^{-1}x\right)^{(n)}=(-1)^{n-1}\sum_{r=1}^{n}\frac{(-1)^r\,{}_nA_r}{x^{n-1+2r}\left(x^{-2}-1\right)^{r-\frac{1}{2}}}$$

$$\left(csch^{-1}x\right)^{(n)}=(-1)^{n-1}\sum_{r=1}^{n}\frac{(-1)^r\,{}_nA_r}{x^{n-1+2r}\left(x^{-2}+1\right)^{r-\frac{1}{2}}}$$

where ${}_nA_r$ are same as ones in Formula 9.3.3 .

9.5 Termwise Higher Derivative of a Logarithmic Function

Since the higher derivative of a logarithmic function has already shown in the previous section, there is no necessity of differentiating this termwise. However, interesting results are obtained if the termwise higher derivative is compared with the higher integral.

Formula 9.5.1

$$\sum_{k=0}^{\infty}(-1)^k \frac{\{k+(n-1)\}!}{k!}x^k = \frac{(n-1)!}{(1+x)^n} \qquad -1<x<1 \tag{1.1}$$

$$\sum_{k=0}^{\infty}\binom{-n}{k}x^k = (1+x)^{-n} \qquad -1<x<1 \tag{1.1'}$$

Proof

$$\frac{x-1}{1}-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+-\cdots = \log x$$

Differentiating both sides of this with respect to x,

$$1-(x-1)+(x-1)^2-(x-1)^3+-\cdots = \frac{0!}{x} \qquad 0<x<2$$

Further differentiating both sides of this with respect to x,

$$1-2(x-1)+3(x-1)^2-4(x-1)^3+-\cdots = \frac{1!}{x^2} \qquad 0<x<2$$

Furthermore differentiating both sides of this with respect to x,

$$1\cdot 2-2\cdot 3(x-1)+3\cdot 4(x-1)^2-4\cdot 5(x-1)^3+-\cdots = \frac{2!}{x^3} \qquad 0<x<2$$

Thus generally,

$$\sum_{k=0}^{\infty}(-1)^k \frac{\{k+(n-1)\}!}{k!}(x-1)^k = \frac{(n-1)!}{x^n} \qquad 0<x<2$$

Replacing x with 1+ x , we obtain

$$\sum_{k=0}^{\infty}(-1)^k \frac{\{k+(n-1)\}!}{k!}x^k = \frac{(n-1)!}{(1+x)^n} \qquad -1<x<1 \tag{1.1}$$

Next, if we divide both sides by $(n-1)!$, the left side becomes as follows.

$$\sum_{k=0}^{\infty}(-1)^k \frac{\{k+(n-1)\}!}{k!(n-1)!}x^k = \sum_{k=0}^{\infty}(-1)^k\binom{n-1+k}{k}x^k = \sum_{k=0}^{\infty}\binom{-n}{k}x^k$$

Therefore we obtain

$$\sum_{k=0}^{\infty}\binom{-n}{k}x^k = (1+x)^{-n} \qquad -1<x<1 \tag{1.1'}$$

c.f.

In (1.1'), n may be the real number already. Here, let $\alpha = -n$, then

$$\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k = (1+x)^{\alpha} \qquad -1<x<1$$

This is Newton's generalized binomial theorem.

Special Values

In fact, Formula 9.5.1 holds also on x=1. Therfore, giving x=1 to (1.1) without considering convergence conditions, we obtain the following special values.

$$1 - 1 + 1 - 1 +- \cdots = \frac{0!}{2^1}$$

$$1 - 2 + 3 - 4 +- \cdots = \frac{1!}{2^2}$$

$$1\cdot 2 - 2\cdot 3 + 3\cdot 4 - 4\cdot 5 +- \cdots = \frac{2!}{2^3}$$

$$1\cdot 2\cdot 3 - 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 - 4\cdot 5\cdot 6 +- \cdots = \frac{3!}{2^4}$$

$$\vdots$$

$$\sum_{k=0}^{\infty}(-1)^k \frac{\{k+(n-1)\}!}{k!} = \frac{(n-1)!}{2^n}$$

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