29 Higher Derivative of Complex Function

Abstract

(1) Cauchy-Riemann equations can be generalized to the odd-order derivative of the 3rd or higher.

(2) Laplace's equations can be generalized to the even-order derivative of the 4th or higher.

(3) The higher derivative $f^{(n)}(z)$ of a holomorphic function $f(z)$ ($z=x+iy$) is represented by the higher derivatives with respect to y .

29.1 The 1~4th order Derivatives

Cauchy-Riemann equations and Laplace's equations were as follows respectively.

Theorem 29.1.1 (Cauchy-Riemann equations)

$f(z)=u(x,y)+i\,v(x,y)$ is holomorphic in the whole domain D if and only if u,v are totally differentiable in D and the following equations hold.

$$\frac{\partial}{\partial x}u(x,y) = \frac{\partial}{\partial y}v(x,y)$$

$$\frac{\partial}{\partial y}u(x,y) = -\frac{\partial}{\partial x}v(x,y)$$

Theorem 29.1.2 (Laplace's equations)

When $f(z)=u(x,y)+i\,v(x,y)$ is holomorphic in the whole domain D the following expressions hold.

$$\frac{\partial^2}{\partial x^2}u(x,y) = -\frac{\partial^2}{\partial y^2}u(x,y)$$

$$\frac{\partial^2}{\partial x^2}v(x,y) = -\frac{\partial^2}{\partial y^2}v(x,y)$$

The 1st order derivative of $f(z)$

Cauchy-Riemann equations are often abbreviated as follows.

$$u_x = v_y \quad , \quad v_x = -u_y \qquad \text{Cauchy-Riemann the 1st order equations} \qquad (1.10)$$

From this,

$$f^{(1)}(z) = u_x + i\,v_x = v_y - i\,u_y$$

The 2nd order derivative of $f(z)$

If (1.10) is differentiated with x , $u_{xx}=v_{yx} \;,\; v_{xx}=-u_{yx}$ (L.21)

If (1.10) is differentiated with y , $u_{xy}=v_{yy} \;,\; v_{xy}=-u_{yy}$ (L.22)

From these,

$$u_{xx}=-u_{yy} \quad , \quad v_{xx}=-v_{yy} \qquad \text{Laplace's the 2nd order equations} \qquad (1.20)$$

From this,

$$f^{(2)}(z) = u_{xx} + i\,v_{xx} = -u_{yy} - i\,v_{yy}$$

The 3rd order derivative of $f(z)$

If (L.21) is differentiated with x , $u_{xxx} = v_{yxx}$, $v_{xxx} = -u_{yxx}$ (C.31)

If (L.21) is differentiated with y , $u_{xxy} = v_{yxy}$, $v_{xxy} = -u_{yxy}$ (C.32)

If (L.22) is differentiated with x , $u_{xyx} = v_{yyx}$, $v_{xyx} = -u_{yyx}$ (C.33)

If (L.22) is differentiated with y , $u_{xyy} = v_{yyy}$, $v_{xyy} = -u_{yyy}$ (C.34)

From these,

$$u_{xxx} = v_{yxx} \ , \ v_{xxy} = -u_{yxy} \ , \ u_{xyy} = v_{yyy} \quad \Longrightarrow \quad u_{xxx} = -v_{yyy}$$

$$v_{xxx} = -u_{yxx} \ , \ u_{xyx} = v_{yyx} \ , \ v_{xyy} = -u_{yyy} \quad \Longrightarrow \quad v_{xxx} = u_{yyy}$$

$$u_{xxy} = v_{yyx} \ , \ v_{xxy} = -u_{yyx}$$

That is

$u_{xxx} = -v_{yyy}$, $v_{xxx} = u_{yyy}$ Cauchy-Riemann the 3nd order equations (1.30)

$u_{xxy} = v_{yyx}$, $v_{xxy} = -u_{yyx}$ Cauchy-Riemann the 3nd order equations

From (1.30) ,

$$f^{(3)}(z) = u_{xxx} + i\, v_{xxx} = -v_{yyy} + i\, u_{yyy}$$

The 4th order derivative of $f(z)$

If (C.31) is differentiated with x , $u_{xxxx} = v_{yxxx}$, $v_{xxxx} = -u_{yxxx}$

If (C.31) is differentiated with y , $u_{xxxy} = v_{yxxy}$, $v_{xxxy} = -u_{yxxy}$

If (C.32) is differentiated with x , $u_{xxyx} = v_{yxyx}$, $v_{xxyx} = -u_{yxyx}$

If (C.32) is differentiated with y , $u_{xxyy} = v_{yxyy}$, $v_{xxyy} = -u_{yxyy}$

If (C.34) is differentiated with x , $u_{xyyx} = v_{yyyx}$, $v_{xyyx} = -u_{yyyx}$

If (C.34) is differentiated with y , $u_{xyyy} = v_{yyyy}$, $v_{xyyy} = -u_{yyyy}$

Note that the derivatives of (C33) are excluded because they overlap with those of (C32) .

From these,

$$u_{xxxx} = v_{yxxx} \ , v_{xxxy} = -u_{yxxy} \ , \ u_{xxyy} = v_{yxyy} \ , \ v_{xyyy} = -u_{yyyy} \quad \Longrightarrow \quad u_{xxxx} = u_{yyyy}$$

$$v_{xxxx} = -u_{yxxx} \ , \ u_{xxxy} = v_{yxxy} \ , v_{xyyx} = -u_{yyyx} \ , u_{xyyy} = v_{yyyy} \quad \Longrightarrow \quad v_{xxxx} = v_{yyyy}$$

$$u_{xxyx} = v_{yxyx} \ , \ v_{xxyy} = -u_{yxyy} \ , \ v_{xxyx} = -u_{yxyx} \ , \ u_{xyyx} = v_{yyyx}$$

$$\Longrightarrow \quad u_{xxxy} = -u_{yyyx} \ , \ v_{xxxy} = -v_{yyyx}$$

That is

$u_{xxxx} = u_{yyyy}$, $v_{xxxx} = v_{yyyy}$ Laplace's the 4th order equations (1.40)

$u_{xxxy} = -u_{yyyx}$, $v_{xxxy} = -v_{yyyx}$ Laplace's the 4th order equations

From (1.40) ,

$$f^{(4)}(z) = u_{xxxx} + i\, v_{xxxx} = u_{yyyy} + i\, v_{yyyy}$$

If the aboves are wrote together, it is as follows.

The 1~4th order derivatives of $f(z)$

$$f^{(1)}(z) = u_x + i\, v_x \quad = v_y - i\, u_y$$

$$f^{(2)}(z) = u_{xx} + i\, v_{xx} \quad = -u_{yy} - i\, v_{yy}$$

$$f^{(3)}(z) = u_{xxx} + i\, v_{xxx} \quad = -v_{yyy} + i\, u_{yyy}$$

$$f^{(4)}(z) = u_{xxxx} + i\, v_{xxxx} = u_{yyyy} + i\, v_{yyyy}$$

$$\vdots$$

Thus, Cauchy-Riemann the 3nd order equations and Laplace's the 4th order equations were obtained. However, It is difficult to calculate the 5th order and above in this way. Therefore, in the following sections, we will consider another way.

29.2 Series Expression of Higher Derivative

As first, we reprint Formula 14.1.2 in " **14 Taylor Expansion by Real Part & Imaginary Part** " (**A la Carte**) **.** And, we perform higher order differentiation of Taylor series by real and imaginary parts.

Formula 14.1.2 (Reprint)

Suppose that a complex function $f(z)$ $(z=x+iy)$ is expanded around a real number a into a Taylor series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(s)}(a)\frac{(z-a)^s}{s!} \tag{2.1}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$. Where, $0^0=1$.

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{2.1u}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{2.1v}$$

Example

$$f(z) = (z-1)^2 e^{z-1}$$
$$u(x,y) = e^{x-1}\left\{\left(1-2x+x^2-y^2\right)\cos y - (2xy-2y)\sin y\right\}$$
$$v(x,y) = e^{x-1}\left\{\left(1-2x+x^2-y^2\right)\sin y + (2xy-2y)\cos y\right\}$$

Expanding $f(z)$ to the Taylor series around 1 and applying the above formula,

$$f(z) = \sum_{s=0}^{\infty} s(s-1)\frac{(z-1)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s)(2r+s-1)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+1)(2r+s)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

In fact, given $2+i$ to these functions and series, it is as follows.

```
N[{u[2, 1], u[2, 1, 30]}]
{-4.57471 , -4.57471}

N[{v[2, 1], v[2, 1, 30]}]
{2.93739, 2.93739}
```

29.2.1 Higher Derivative with respect to x

Formula 29.2.1

Suppose that a complex function $f(z)$ $(z=x+iy)$ is expanded around a real number a into a Taylor series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(s)}(a)\frac{(z-a)^s}{s!} \tag{2.1}$$

Then, the n - th order derivative and the real part $ux_n(x,y)$ & the imaginary part $vx_n(x,y)$ are as follows.

$$f^{(n)}(z) = \sum_{s=0}^{\infty} f^{(s+n)}(a)\frac{(z-a)^s}{s!}$$

$$ux_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+n)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$vx_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1+n)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$.

Proof

Differentiating (2.1) n - times with respect to z,

$$f^{(n)}(z) = \sum_{s=n}^{\infty} f^{(s)}(a)\frac{(z-a)^{s-n}}{(s-n)!} = \sum_{s=0}^{\infty} f^{(s+n)}(a)\frac{(z-a)^s}{s!}$$

Applying Formula 14.1.2 to this, we obtain the desired expressions.

Example

$$f(z) = (z-1)^2 e^{z-1} = \sum_{s=0}^{\infty} s(s-1)\frac{(z-1)^s}{s!}$$

Differentiating the both sides n - times with respect to z,

$$f^{(n)}(z) = \sum_{s=n}^{\infty} s(s-1)\frac{(z-1)^{s-n}}{(s-n)!} = \sum_{s=0}^{\infty}(s+n)(s-1+n)\frac{(z-1)^s}{s!}$$

Applying Formula 29.2.1 to this,

$$ux_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+n)(2r+s-1+n)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$vx_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+1+n)(2r+s+n)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Given $2+3i$ to $f^{(n)}(z)$, $ux_n(x,y)$, $vx_n(x,y)$, it is as follows.

```
f5[z_] = ∂z ∂z ∂z ∂z ∂z f[z];     f6[z_] = ∂z ∂z ∂z ∂z ∂z ∂z f[z];

N[{f5[2 + 3 i]              , ux5[2, 3, 30], vx5[2, 3, 30]}]
{-73.0135 - 88.4395 i  , -73.0135      , -88.4395}

N[{f6[2 + 3 i]              , ux6[2, 3, 30], vx6[2, 3, 30]}]
{-107.608 - 99.9828 i  , -107.608      , -99.9828}
```

Note

$$f^{(n)}(z) = ux_n(x,y) + i\, vx_n(x,y)$$

29.2.2 Higher Derivative with respect to y

Formula 29.2.2

Suppose that a complex function $f(z)$ $(z=x+iy)$ is expanded around a real number a into a Taylor series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(s)}(a)\frac{(z-a)^s}{s!} \tag{2.1}$$

Then, the n - th order derivatives of the real part $u(x,y)$ & the imaginary part $v(x,y)$ with respect to y are as follows.

$$uy_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r-n}}{(2r-n)!} \tag{2.2u}$$

$$vy_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1-n}}{(2r+1-n)!} \tag{2.2v}$$

Where, $0^0 = 1$.

Proof

Differentiating the right sides of (2.1u) and (2.1v) n - times with respect to z respectively, we obtain the desired expressions.

Example

$$f(z) = (z-1)^2 e^{z-1} = \sum_{s=0}^{\infty} s(s-1)\frac{(z-1)^s}{s!}$$

The real and imaginary parts of this are

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s)(2r+s-1)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+1)(2r+s)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Differentiating these n - times with respect to y ,

$$uy_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s)(2r+s-1)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r-n}}{(2r-n)!}$$

$$vy_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+1)(2r+s)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r+1-n}}{(2r+1-n)!}$$

Given $2-i$ to the direct derivatives of $u(x,y), v(x,y)$ and $uy_n(x,y)$, $vy_n(x,y)$, it is as follows.

```
uy5[x_, y_] = ∂y∂y∂y∂y∂y u[x, y];
N[{uy5[2, -1], uy5[2, -1, 30]}]
      {86.245, 86.245}

vy6[x_, y_] = ∂y∂y∂y∂y∂y∂y v[x, y];
N[{vy6[2, -1], vy6[2, -1, 30]}]
      {116.631, 116.631}
```

Note

$$f^{(n)}(z) \neq uy_n(x,y) + i\, vy_n(x,y)$$

29.3 Cauchy-Riemann & Laplace Equations

In this section, using the Taylor series obtained in the previous section, we generalize Cauchy-Riemann equations and Laplace's equations to the 3rd order and above. The advantage of this method is that the formula can be visually confirmed.

Formula 29.3.1 **(Cauchy-Riemann higher order partial differential equations)**

When a complex function $f(z)=u(x,y)+i\,v(x,y)$ is holomorphic in the whole domain D , the following expressions hold

$$\frac{\partial^{2n-1}}{\partial x^{2n-1}}u(x,y) = (-1)^{n-1}\frac{\partial^{2n-1}}{\partial y^{2n-1}}v(x,y) \qquad n=1,2,3,\cdots \tag{3.1u}$$

$$\frac{\partial^{2n-1}}{\partial x^{2n-1}}v(x,y) = (-1)^{n}\frac{\partial^{2n-1}}{\partial y^{2n-1}}u(x,y) \qquad n=1,2,3,\cdots \tag{3.1v}$$

Proof

From Formula 29.2.1 ,

$$\frac{\partial^n}{\partial x^n}u(x,y) = ux_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+n)}(a)\frac{(x-a)^s}{s\,!}\frac{(-1)^r y^{2r}}{(2r)\,!}$$

$$\frac{\partial^n}{\partial x^n}v(x,y) = vx_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1+n)}(a)\frac{(x-a)^s}{s\,!}\frac{(-1)^r y^{2r+1}}{(2r+1)\,!}$$

Replacing n with $2n-1$ in both expressions,

$$\frac{\partial^{2n-1}}{\partial x^{2n-1}}u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2n-1)}(a)\frac{(x-a)^s}{s\,!}\frac{(-1)^r y^{2r}}{(2r)\,!} \tag{ux1}$$

$$\frac{\partial^{2n-1}}{\partial x^{2n-1}}v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2n)}(a)\frac{(x-a)^s}{s\,!}\frac{(-1)^r y^{2r+1}}{(2r+1)\,!} \tag{vx1}$$

On the other hand, from Formula 29.2.2 ,

$$\frac{\partial^n}{\partial y^n}u(x,y) = uy_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a)\frac{(x-a)^s}{s\,!}\frac{(-1)^r y^{2r-n}}{(2r-n)\,!} \tag{2.2u}$$

$$\frac{\partial^n}{\partial y^n}v(x,y) = vy_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s\,!}\frac{(-1)^r y^{2r+1-n}}{(2r+1-n)\,!} \tag{2.2v}$$

Replacing n with $2n-1$ in both expressions,

$$uy_{2n-1}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a)\frac{(x-a)^s}{s\,!}\frac{(-1)^r y^{2r-2n+1}}{(2r-2n+1)\,!}$$

$$= \sum_{r=n}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a)\frac{(x-a)^s}{s\,!}\frac{(-1)^r y^{2r-2n+1}}{(2r-2n+1)\,!}$$

$$vy_{2n-1}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s\,!}\frac{(-1)^r y^{2r-2n+2}}{(2r-2n+2)\,!}$$

$$= \sum_{r=n-1}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s\,!}\frac{(-1)^r y^{2r-2n+2}}{(2r-2n+2)\,!}$$

$$\left(\because\ 1/(2r-2n+1)! = 0 \ \ for\ r < n \ \ ,\ \ 1/(2r-2n+2)! = 0 \ \ for\ r < n-1\right)$$

Replacing r with $r+n$ in the former and replacing r with $r+n-1$ in the latter,

$$uy_{2n-1}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2n)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^{r+n}y^{2r+1}}{(2r+1)!}$$

$$vy_{2n-1}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2n-1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^{r+n-1}y^{2r}}{(2r)!}$$

i.e.

$$\frac{\partial^{2n-1}}{\partial y^{2n-1}}u(x,y) = (-1)^n \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2n)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad \text{(uy1)}$$

$$\frac{\partial^{2n-1}}{\partial y^{2n-1}}v(x,y) = (-1)^{n-1} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2n-1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!} \qquad \text{(vy1)}$$

From (vy1) and (ux1) ,

$$\frac{\partial^{2n-1}}{\partial y^{2n-1}}v(x,y) = (-1)^{n-1}\frac{\partial^{2n-1}}{\partial x^{2n-1}}u(x,y)$$

From (uy1) and (vx1) ,

$$\frac{\partial^{2n-1}}{\partial y^{2n-1}}u(x,y) = (-1)^{n}\frac{\partial^{2n-1}}{\partial x^{2n-1}}v(x,y)$$

Swapping these signs left and right, we obtain the desired expressions.

Example 1

$$f(z) = (z-1)^2 e^{z-1}$$

$$u(x,y) = e^{x-1}\left\{\left(1-2x+x^2-y^2\right)\cos y - (2xy-2y)\sin y\right\}$$

$$v(x,y) = e^{x-1}\left\{\left(1-2x+x^2-y^2\right)\sin y + (2xy-2y)\cos y\right\}$$

Expanding these into Taylor series around 1 ,

$$f(z) = \sum_{s=0}^{\infty} s(s-1)\frac{(z-1)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s)(2r+s-1)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+1)(2r+s)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Odd-order partial derivative of $u(x,y)$, $v(x,y)$ are

$$uxo_n(x,y) = \frac{\partial^{2n-1}}{\partial x^{2n-1}}u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+2n-1)(2r+s+2n-2)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$vxo_n(x,y) = \frac{\partial^{2n-1}}{\partial x^{2n-1}}v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+2n)(2r+s+2n-1)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

$$uyo_n(x,y) = \frac{\partial^{2n-1}}{\partial y^{2n-1}} u(x,y) = (-1)^n \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (2r+s+2n)(2r+s+2n-1) \frac{(x-1)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

$$vyo_n(x,y) = \frac{\partial^{2n-1}}{\partial y^{2n-1}} v(x,y) = (-1)^{n-1} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (2r+s+2n-1)(2r+s+2n-2) \frac{(x-1)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

Given $2-i$ to these for verification, it is as follows.

```
ux3[x_, y_] = ∂x ∂x ∂x u[x, y];
N[{ux3[2, -1], uxo2[2, -1, 30]}]
  {-0.674515, -0.674515}

vx3[x_, y_] = ∂x ∂x ∂x v[x, y];
N[{vx3[2, -1], vxo2[2, -1, 30]}]
  {-39.1978, -39.1978}

uy3[x_, y_] = ∂y ∂y ∂y u[x, y];
N[{uy3[2, -1], uyo2[2, -1, 30]}]
  {-39.1978, -39.1978}

vy3[x_, y_] = ∂y ∂y ∂y v[x, y];
N[{vy3[2, -1], vyo2[2, -1, 30]}]
  {0.674515, 0.674515}
```

Next,

$$uxo_n(x,y),\ vyo_n(x,y) \Longrightarrow \quad (3.1u)$$

$$vxo_n(x,y),\ uyo_n(x,y) \Longrightarrow \quad (3.1v)$$

Formula 29.3.2 **(Laplace's higher order partial differential equations)**

When a complex function $f(z) = u(x,y) + i\,v(x,y)$ is holomorphic in the whole domain D the following expressions hold

$$\frac{\partial^{2n}}{\partial x^{2n}} u(x,y) = (-1)^n \frac{\partial^{2n}}{\partial y^{2n}} u(x,y) \qquad n=1,2,3,\cdots \qquad (3.2u)$$

$$\frac{\partial^{2n}}{\partial x^{2n}} v(x,y) = (-1)^n \frac{\partial^{2n}}{\partial y^{2n}} v(x,y) \qquad n=1,2,3,\cdots \qquad (3.2v)$$

Proof

From Formula 29.2.1 ,

$$\frac{\partial^n}{\partial x^n} u(x,y) = ux_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+n)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$\frac{\partial^n}{\partial x^n} v(x,y) = vx_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1+n)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Replacing n with $2n$ in both expressions,

$$\frac{\partial^{2n}}{\partial x^{2n}} u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2n)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!} \qquad (ux2)$$

$$\frac{\partial^{2n}}{\partial x^{2n}} v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2n+1)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad (vx2)$$

On the other hand, from Formula 29.2.2 ,

$$\frac{\partial^n}{\partial y^n} u(x,y) = uy_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r-n}}{(2r-n)!} \qquad (2.2u)$$

$$\frac{\partial^n}{\partial y^n}v(x,y) = vy_n(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1-n}}{(2r+1-n)!} \tag{2.2v}$$

Replacing n with $2n$ in both expressions,

$$\begin{aligned} uy_{2n}(x,y) &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r-2n}}{(2r-2n)!} \\ &= \sum_{r=n}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r-2n}}{(2r-2n)!} \end{aligned}$$

$$\begin{aligned} vy_{2n}(x,y) &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r-2n+1}}{(2r-2n+1)!} \\ &= \sum_{r=n}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r-2n+1}}{(2r-2n+1)!} \end{aligned}$$

$$\left(\because \; 1/(2r-2n)! = 0 \quad for\ r<n \quad , \quad 1/(2r-2n+1)! = 0 \quad for\ r<n\right)$$

Replacing r with $r+n$ in both expressions,

$$uy_{2n}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2n)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^{r+n} y^{2r}}{(2r)!}$$

$$vy_{2n}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2n+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^{r+n} y^{2r+1}}{(2r+1)!}$$

i.e.

$$\frac{\partial^{2n}}{\partial y^{2n}}u(x,y) = (-1)^n\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2n)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{uy2}$$

$$\frac{\partial^{2n}}{\partial y^{2n}}v(x,y) = (-1)^n\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2n+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{vy2}$$

From (uy2) and (ux2)

$$\frac{\partial^{2n}}{\partial y^{2n}}u(x,y) = (-1)^n\frac{\partial^{2n}}{\partial x^{2n}}u(x,y)$$

From (vy2) and (vx2)

$$\frac{\partial^{2n}}{\partial y^{2n}}v(x,y) = (-1)^n\frac{\partial^{2n}}{\partial x^{2n}}v(x,y)$$

Multiplying both side by $(-1)^n$ respectively, we obtain the desired expressions.

Example 2

It is assumed that $f(z)$, $u(x,y)$, $v(x,y)$ are all the same as in Example 1 .

Even-order partial derivatives of $u(x,y)$, $v(x,y)$ are

$$uxe_n(x,y) = \frac{\partial^{2n}}{\partial x^{2n}}u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+2n)(2r+s+2n-1)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$vxe_n(x,y) = \frac{\partial^{2n}}{\partial x^{2n}}v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+2n+1)(2r+s+2n)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

$$uye_n(x,y) = \frac{\partial^{2n}}{\partial y^{2n}}u(x,y) = (-1)^n\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+2n)(2r+s+2n-1)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$vye_n(x,y) = \frac{\partial^{2n}}{\partial y^{2n}}v(x,y) = (-1)^n\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+2n+1)(2r+s+2n)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Given $1-3i$ to these for verification, it is as follows.

```
ux4[x_, y_] = ∂x∂x∂x∂x u[x, y];
N[{ux4[1, -3], uxe2[1, -3, 30]}]
    {-6.35686, -6.35686}
```

```
vx4[x_, y_] = ∂x∂x∂x∂x v[x, y];
N[{vx4[1, -3], vxe2[1, -3, 30]}]
    {23.3365, 23.3365}
```

```
uy4[x_, y_] = ∂y∂y∂y∂y u[x, y];
N[{uy4[1, -3], uye2[1, -3, 30]}]
    {-6.35686, -6.35686}
```

```
vy4[x_, y_] = ∂y∂y∂y∂y v[x, y];
N[{vy4[1, -3], vye2[1, -3, 30]}]
    {23.3365, 23.3365}
```

Next,

$uxe_n(x,y)\ ,\ uye_n(x,y) \Longrightarrow$ (3.2u)

$vxe_n(x,y)\ ,\ vye_n(x,y) \Longrightarrow$ (3.2v)

The above two formulas immediately lead to the following.

Formula 29.3.3 **(Higher Derivative with respect to y)**

If a complex function $f(z)=u(x,y)+i\,v(x,y)$ is holomorphic in the whole domain D the following expressions hold for a natural number n .

$$f^{(2n-1)}(z) = (-1)^{n-1}\frac{\partial^{2n-1}}{\partial y^{2n-1}}v(x,y) + i\,(-1)^n\frac{\partial^{2n-1}}{\partial y^{2n-1}}u(x,y) \tag{3.3C}$$

$$f^{(2n)}(z) = (-1)^n\frac{\partial^{2n}}{\partial y^{2n}}u(x,y) + i\,(-1)^n\frac{\partial^{2n}}{\partial y^{2n}}v(x,y) \tag{3.3L}$$

Example 3

It is assumed that $f(z)$, $u(x,y)$, $v(x,y)$ are all the same as in Example 1 .

The n - th order partial derivatives of $u(x,y)$, $v(x,y)$ are

$$uyo_n(x,y) = \frac{\partial^{2n-1}}{\partial y^{2n-1}}u(x,y) = (-1)^n\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+2n)(2r+s+2n-1)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

$$vyo_n(x,y) = \frac{\partial^{2n-1}}{\partial y^{2n-1}}v(x,y) = (-1)^{n-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+2n-1)(2r+s+2n-2)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$uye_n(x,y) = \frac{\partial^{2n}}{\partial y^{2n}}u(x,y) = (-1)^n\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+2n)(2r+s+2n-1)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$vye_n(x,y) = \frac{\partial^{2n}}{\partial y^{2n}}v(x,y) = (-1)^n\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+2n+1)(2r+s+2n)\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Then, (3.3C) , (3.3L) become as follows respectively.

$$f^{(2n-1)}(z) = (-1)^{n-1} vyo_n(x,y) + i (-1)^n uyo_n(x,y)$$
$$f^{(2n)}(z) = (-1)^n uye_n(x,y) + i (-1)^n vye_n(x,y)$$

When $n=3$, given $2+3i$ to these,

```
f5[z_] = ∂z∂z∂z∂z∂z f[z];
N[{f5[2 + 3 ⅈ] , (-1)^2 vyo3[2, 3, 30] + ⅈ (-1)^3 uyo3[2, 3, 30]}]
{-73.0135 - 88.4395 ⅈ , -73.0135 - 88.4395 ⅈ}
```

```
f6[z_] = ∂z∂z∂z∂z∂z∂z f[z];
N[{f6[2 + 3 ⅈ] , (-1)^3 uye3[2, 3, 30] + ⅈ (-1)^3 vye3[2, 3, 30]}]
{-107.608 - 99.9828 ⅈ , -107.608 - 99.9828 ⅈ}
```

c.f.

$$f^{(n)}(z) = \frac{\partial^n}{\partial x^n} u(x,y) + i \frac{\partial^n}{\partial x^n} v(x,y) \qquad n=1,2,3,\cdots$$

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Kano Kono

Alien's Mathematics