22 Higher Derivative of Composition

22.1 Formulas of Higher Derivative of Composition

22.1.1 Faà di Bruno's Formula

About the formula of the higher derivative of composition, the one by a mathematician *Faà di Bruno* in Italy of about 150 years ago seems to be the beginning. And it is called *Faà di Bruno's Formula*.

Formula 22.1.1 (Faà di Bruno)

Let $j_1, j_2, \cdots, j_n$ are non-negative integers. Let $g^{(n)}$, f_n are derivative functions and $B_{n,r}(f_1, f_2, \ldots)$ are ***Bell polynomials*** such that

$$g^{(n)} = g^{(n)}(f)\ , \quad f_n = f^{(n)}(x) \qquad (n=1,2,3,\cdots)$$

$$B_{n,r}(f_1, f_2, \ldots, f_n) = \sum \frac{n!}{j_1!\ j_2! \ldots j_n!}\left(\frac{f_1}{1!}\right)^{j_1}\left(\frac{f_2}{2!}\right)^{j_2}\cdots\left(\frac{f_n}{n!}\right)^{j_n}$$

$$\left(j_1 + j_2 + \cdots + j_n = r \quad \& \quad j_1 + 2j_2 + \cdots + nj_n = n\right)$$

Then, the higher derivative function with respect to x of the composition $g\{f(x)\}$ is expressed as follows.

$$\{g\{f(x)\}\}^{(n)} = \sum_{r=1}^{n} g^{(r)} B_{n,r}(f_1, f_2, \ldots, f_n) \tag{1.1}$$

Example $\{g\{f(x)\}\}^{(4)}$

First, (j_1, j_2, j_3, j_4) such that $j_1+j_2+j_3+j_4 = 1$, $j_1+2j_2+3j_3+4j_4 = 4$ are

$$(j_1, j_2, j_3, j_4) = (0, 0, 0, 1)$$

Then,

$$B_{4,1}(f_1 \cdots f_4) = \frac{4!}{0!\,0!\,0!\,1!}\left(\frac{f_1}{1!}\right)^0\left(\frac{f_2}{2!}\right)^0\left(\frac{f_3}{3!}\right)^0\left(\frac{f_4}{4!}\right)^1 = f_4^1$$

2nd, (j_1, j_2, j_3, j_4) such that $j_1+j_2+j_3+j_4 = 2$, $j_1+2j_2+3j_3+4j_4 = 4$ are

$$(j_1, j_2, j_3, j_4) = (1, 0, 1, 0)$$

$$(j_1, j_2, j_3, j_4) = (0, 2, 0, 0)$$

Then,

$$B_{4,2}(f_1 \cdots f_4) = \frac{4!}{1!0!1!0!}\left(\frac{f_1}{1!}\right)^1\left(\frac{f_2}{2!}\right)^0\left(\frac{f_3}{3!}\right)^1\left(\frac{f_4}{4!}\right)^0$$

$$+ \frac{4!}{0!2!0!0!}\left(\frac{f_1}{1!}\right)^0\left(\frac{f_2}{2!}\right)^2\left(\frac{f_3}{3!}\right)^0\left(\frac{f_4}{4!}\right)^0 = 4f_1^1 f_3^1 + 3f_2^2$$

3rd, (j_1, j_2, j_3, j_4) such that $j_1+j_2+j_3+j_4 = 3$, $j_1+2j_2+3j_3+4j_4 = 4$ are

$$(j_1, j_2, j_3, j_4) = (2, 1, 0, 0)$$

Then,

$$B_{4,3}(f_1 \cdots f_4) = \frac{4!}{2!\,1!\,0!\,0!}\left(\frac{f_1}{1!}\right)^2\left(\frac{f_2}{2!}\right)^1\left(\frac{f_3}{3!}\right)^0\left(\frac{f_4}{4!}\right)^0 = 6f_1^2 f_2^1$$

Last, (j_1, j_2, j_3, j_4) such that $j_1+j_2+j_3+j_4 = 4$, $j_1+2j_2+3j_3+4j_4 = 4$ are

$$(j_1, j_2, j_3, j_4) = (4, 0, 0, 0)$$

Then,

$$B_{4,4}(f_1 \cdots f_4) = \frac{4!}{4!\,0!\,0!\,0!}\left(\frac{f_1}{1!}\right)^4\left(\frac{f_2}{2!}\right)^0\left(\frac{f_3}{3!}\right)^0\left(\frac{f_4}{4!}\right)^0 = f_1^4$$

Thus,

$$\begin{aligned}\{g\{f(x)\}\}^{(4)} = \; & g^{(1)}B_{4,1}(f_1,f_2,f_3,f_4) + g^{(2)}B_{4,2}(f_1,f_2,f_3,f_4) \\ & + g^{(3)}B_{4,3}(f_1,f_2,f_3,f_4) + g^{(4)}B_{4,4}(f_1,f_2,f_3,f_4) \\ = \; & g^{(1)}f_4^1 + g^{(2)}\left(4f_1^1 f_3^1 + 3f_2^2\right) + 6\,g^{(3)}f_1^2 f_2^1 + g^{(4)}f_1^4\end{aligned}$$

As understood from this example, obtaining (j_1, j_2, j_3, j_4) such that

$$j_1 + j_2 + \cdots + j_n = k \quad , \quad j_1 + 2j_2 + 3j_3 + \cdots + nj_n = n \quad , \quad j_k \geqq 0 \quad k=1,2,\cdots,n$$

is equivalent with solving the following indeterminate equation. It is not easy.

$$1j_2 + 2j_3 + \cdots + (n-1)j_n = n-k \quad , \quad j_k \geqq 0 \quad k=2,3,\cdots,n$$

Althogh the condition is $j_k \geqq 0 \quad k=1,2,\cdots,n$ in Formula 22.1.1 , if $j_k > 0 \quad k=1,2,\cdots$ is adopted, Formula 22.1.1 can be expressed as follows.

Formula 22.1.1'

Let $\alpha, \beta, \cdots, \varepsilon$ and n are natural numbers, $g^{(n)}$, f_n are $g^{(n)} = g^{(n)}(f)$, $f_n = f^{(n)}(x)$

Then, the higher derivative function with respect to x of the composition $g\{f(x)\}$ is expressed as follows.

$$\{g\{f(x)\}\}^{(n)} = \sum g^{(\alpha+\beta+\cdots+\varepsilon)} \frac{n!\, f_p^{\alpha} f_q^{\beta} \cdots f_t^{\varepsilon}}{p!^{\alpha} q!^{\beta} \cdots t!^{\varepsilon}\, \alpha!\, \beta! \cdots \varepsilon!} \qquad p\alpha + q\beta + \cdots + t\varepsilon = n \tag{1.1'}$$

Algorithm of calculation of the Formula 22.1.1'

If the formula of a composition is expressed in this way, (1.1') can be calculated with a easier algorithm. This algorithm of author design is suitable for the computer. However, since it is easy, manual calculation can also be performed.

(1) Obtain all the products $f_p^{\alpha} f_q^{\beta} \cdots f_t^{\varepsilon}$ such that $p\alpha + q\beta + \cdots + t\varepsilon = n$ as follows.

Step1 Put n on the 1st row 1st column, and arrange 0 from the 2nd column to the n th column .

Step2 Look for the first number that is smaller than the number of the 1st column as for 2 or more sequentially from the 2 nd column.

Ⅰ When such a number exists in the k th column, make a new sequence (new row) as follows.

i From the 2 nd column to the k th column, arrange the number that is 1 greater than the number in the k th column of the old sequence (the row just above) .

ii From the $k+1$ th column to the n th column, copy the numbers of old sequence (row just above)

iii To the 1 st column, write the complement that the total of each column becomes n .

Ⅱ When such a number does not exist, finish calculating and go to **Step4** .

Step3 Repeat **Step2** for the new sequence (new row) .

Step4 Let the number in the each sequence be the order p and the repetition frequency be the degree α . And generate the product $f_p^{\alpha} f_q^{\beta} \cdots f_t^{\varepsilon}$.

For example, the combination of the 7 th order is generated as follows.

Seq.	**1**	**2**	**3**	**4**	**5**	**6**	**7**		
	7	0	0	0	0	0	0	f_7^1	Red is smaller than the 1st column by 2 or more.
	6	1	0	0	0	0	0	$f_6^1 f_1^1$	Blue is a complement.
	5	2	0	0	0	0	0	$f_5^1 f_2^1$	
	4	3	0	0	0	0	0	$f_4^1 f_3^1$	
	5	1	1	0	0	0	0	$f_5^1 f_1^2$	
	4	2	1	0	0	0	0	$f_4^1 f_2^1 f_1^1$	
	3	3	1	0	0	0	0	$f_3^2 f_1^1$	
	3	2	2	0	0	0	0	$f_3^1 f_2^2$	
	4	1	1	1	0	0	0	$f_4^1 f_1^3$	
	3	2	1	1	0	0	0	$f_3^1 f_2^1 f_1^2$	
	2	2	2	1	0	0	0	$f_2^3 f_1^1$	
	3	1	1	1	1	0	0	$f_3^1 f_1^4$	
	2	2	1	1	1	0	0	$f_2^2 f_1^3$	
	2	1	1	1	1	1	0	$f_2^1 f_1^5$	
	1	1	1	1	1	1	1	f_1^7	

(2) The coefficient of each term $f_p^{\alpha} f_q^{\beta} \cdots f_t^{\varepsilon}$ is calculated as follows, for example.

Coefficient of $f_3^1 f_2^1 f_1^2$: $\dfrac{7\,!}{3!^1 2!^1 1!^2 1!\,1!\,2!} = 210$

Coefficient of $f_2^3 f_1^1$: $\dfrac{7\,!}{2!^3 1!^1 3!\,1!} = 105$

(3) The multiplier $g^{(s)}$ of each term $f_p^{\alpha} f_q^{\beta} \cdots f_t^{\varepsilon}$ is calculated as follows, for example.

Multiplier of $f_3^1 f_2^1 f_1^2$: $g^{(s)} = g^{(1+1+2)} = g^{(4)}$

(4) Therefore, the 7th order derivative z_7 is as follows, for example.

$$z_7 = \frac{7!\, f_7^1}{7!^1 1!} g^{(1)} + 7! \left(\frac{f_6^1 f_1^1}{6!^1 1!^1 1!\,1!} + \frac{f_5^1 f_2^1}{5!^1 2!^1 1!\,1!} + \frac{f_4^1 f_3^1}{4!^1 3!^1 1!\,1!} \right) g^{(2)}$$

$$+\ 7! \left(\frac{f_5^1 f_1^2}{5!^1 1!^2 1!\,2!} + \frac{f_4^1 f_2^1 f_1^1}{4!^1 2!^1 1!^1 1!\,1!\,1!} + \frac{f_3^2 f_1^1}{3!^2 1!^1 2!\,1!} + \frac{f_3^1 f_2^2}{3!^1 2!^2 1!\,2!} \right) g^{(3)}$$

$$+\ 7! \left(\frac{f_4^1 f_1^3}{4!^1 1!^1 1!\,3!} + \frac{f_3^1 f_2^1 f_1^2}{3!^1 2!^1 1!^2 1!\,1!\,2!} + \frac{f_2^3 f_1^1}{2!^3 1!^1 3!\,1!} \right) g^{(4)}$$

$$+ 7!\left(\frac{f_3^1 f_1^4}{3!^1 1!^1 1!\,4!} + \frac{f_2^2 f_1^3}{2!^2 1!^3 2!\,3!}\right) g^{(5)}$$

$$+ \frac{7!\ f_2^1 f_1^5}{2!^1 1!^5 1!\,5!} g^{(6)} + \frac{7!\ f_1^7}{1!^1 7!} g^{(7)}$$

$$= g^{(1)} f_7 + g^{(2)}(7 f_6 f_1 + 21 f_5 f_2 + 35 f_4 f_3)$$
$$+ g^{(3)}\left(21 f_5 f_1^2 + 105 f_4 f_2 f_1 + 70 f_3^2 f_1 + 105 f_3 f_2^2\right)$$
$$+ g^{(4)}\left(35 f_4 f_1^3 + 210 f_3 f_2 f_1^2 + 105 f_2^3 f_1\right)$$
$$+ g^{(5)}\left(35 f_3 f_1^4 + 105 f_2^2 f_1^3\right) + 21 g^{(6)} f_2 f_1^5 + g^{(7)} f_1^7$$

Higher differentiation up to the 8th order

If the higher derivative z_n up to the 8th order of composition $z = g\{f(x)\}$ is calculatedby such a way, it becomes the following.

$$z_1 = g^{(1)} f_1$$
$$z_2 = g^{(1)} f_2 + g^{(2)} f_1^2$$
$$z_3 = g^{(1)} f_3 + 3 g^{(2)} f_2 f_1 + g^{(3)} f_1^3$$
$$z_4 = g^{(1)} f_4 + g^{(2)}\left(4 f_3 f_1 + 3 f_2^2\right) + 6 g^{(3)} f_2 f_1^2 + g^{(4)} f_1^4$$
$$z_5 = g^{(1)} f_5 + g^{(2)}(5 f_4 f_1 + 10 f_3 f_2) + g^{(3)}\left(10 f_3 f_1^2 + 15 f_2^2 f_1\right) + 10 g^{(4)} f_2 f_1^3 + g^{(5)} f_1^5$$
$$z_6 = g^{(1)} f_6 + g^{(2)}\left(6 f_5 f_1 + 15 f_4 f_2 + 10 f_3^2\right) + g^{(3)}\left(15 f_4 f_1^2 + 60 f_3 f_2 f_1 + 15 f_2^3\right)$$
$$+ g^{(4)}\left(20 f_3 f_1^3 + 45 f_2^2 f_1^2\right) + 15 g^{(5)} f_2 f_1^4 + g^{(6)} f_1^6$$
$$z_7 = g^{(1)} f_7 + g^{(2)}(7 f_6 f_1 + 21 f_5 f_2 + 35 f_4 f_3)$$
$$+ g^{(3)}\left(21 f_5 f_1^2 + 105 f_4 f_2 f_1 + 70 f_3^2 f_1 + 105 f_3 f_2^2\right)$$
$$+ g^{(4)}\left(35 f_4 f_1^3 + 210 f_3 f_2 f_1^2 + 105 f_2^3 f_1\right)$$
$$+ g^{(5)}\left(35 f_3 f_1^4 + 105 f_2^2 f_1^3\right) + 21 g^{(6)} f_2 f_1^5 + g^{(7)} f_1^7$$
$$z_8 = g^{(1)} f_8 + g^{(2)}\left(8 f_7 f_1 + 28 f_6 f_2 + 56 f_5 f_3 + 35 f_4^2\right)$$
$$+ g^{(3)}\left(28 f_6 f_1^2 + 168 f_5 f_2 f_1 + 280 f_4 f_3 f_1 + 210 f_4 f_2^2 + 280 f_3^2 f_2\right)$$
$$+ g^{(4)}\left(56 f_5 f_1^3 + 420 f_4 f_2 f_1^2 + 280 f_3^2 f_1^2 + 840 f_3 f_2^2 f_1 + 105 f_2^4\right)$$
$$+ g^{(5)}\left(70 f_4 f_1^4 + 560 f_3 f_2 f_1^3 + 420 f_2^3 f_1^2\right)$$
$$+ g^{(6)}\left(56 f_3 f_1^5 + 210 f_2^2 f_1^4\right) + 28 g^{(7)} f_2 f_1^6 + g^{(8)} f_1^8$$

Example When $z = g\{f(x)\} = sin\left(x^3\right)$

$$g^{(1)} = cos\, f \quad , \quad g^{(2)} = -sin\, f \quad , \quad g^{(3)} = -cos\, f$$
$$f_1 = 3x^2 \quad , \quad f_2 = 6x \quad , \quad f_3 = 6$$

Then,

$$z_1 = cos(x^3) \cdot 3x^2$$

$$z_2 = cos(x^3) \cdot 6x - sin(x^3) \cdot (3x^2)^2$$

$$z_3 = cos(x^3) \cdot 6 - 3sin(x^3) \cdot 3x^2 \cdot 6x - cos(x^3) \cdot (3x^2)^3$$

22.1.2 Hoppe's Formula

Afterwards, it was discovered that Formula 22.1.1 is expressed with a double series of $g^{(r)}(f)$ and $f^{(s)}(x)$. The finished type is *Hoppe's formula* mentioned as follows.

Formula 22.1.2 (Reinhold Hoppe)

Let n be a natural number, $g^{(n)}$, f_n are $g^{(n)} = g^{(n)}(f)$, $f_n = f^{(n)}(x)$. Then, the higher derivative function with respect to x of the composition $g\{f(x)\}$ is expressed as follows.

$$\{g(f(x))\}^{(n)} = \sum_{r=0}^{n} \frac{g^{(r)}}{r!} \sum_{s=0}^{r} \binom{r}{s} (-f)^{r-s} (f^s)^{(n)} \tag{1.2}$$

Although this formula is beautiful, the calculation is not easy. When the differentiation is actually performed according to (1.2), it is as follows.

$$\{g(f(x))\}^{(2)} = \sum_{r=0}^{2} \frac{g^{(r)}}{r!} \sum_{s=0}^{r} \binom{r}{s} (-f)^{r-s} (f^s)^{(2)}$$

$$= \frac{g^{(0)}}{0!} \binom{0}{0} (-f)^0 (f^0)^{(2)} + \frac{g^{(1)}}{1!} \sum_{s=0}^{1} \binom{1}{s} (-f)^{1-s} (f^s)^{(2)} + \frac{g^{(2)}}{2!} \sum_{s=0}^{2} \binom{2}{s} (-f)^{2-s} (f^s)^{(2)}$$

$$= \frac{g^{(1)}}{1!} \left\{ \binom{1}{0} (-f)^1 (f^0)^{(2)} + \binom{1}{1} (-f)^0 (f^1)^{(2)} \right\} + \frac{g^{(2)}}{2!} \left\{ \binom{2}{0} (-f)^2 (f^0)^{(2)} + \binom{2}{1} (-f)^1 (f^1)^{(2)} + \binom{2}{2} (-f)^0 (f^2)^{(2)} \right\}$$

$$= \frac{g^{(1)}}{1!} \binom{1}{1} f_2^1 + \frac{g^{(2)}}{2!} \left\{ \binom{2}{1} (-f)^1 f_2^1 + \binom{2}{2} 2(f_1 f_1 + f f_2) \right\}$$

$$= \frac{g^{(1)} f_2^1}{1!} + \frac{g^{(2)}}{2!} \{ -2f f_2 + 2(f_1 f_1 + f f_2) \} = g^{(1)} f_2 + g^{(2)} f_1^2$$

22.1.3 Higher order differentiation using Mathematica

A function $BellY[n, r, \{x_1, x_2, \cdots, x_{n-r+1}\}]$ is implemented in a formula manipulation software ***Mathematica*** since 2010, By using this, we can easily generate the Bell polynomials $B_{n,r}(f_1, f_2, \ldots, f_n)$. For example, $B_{8,3}(f_1, f_2, \ldots, f_8)$ is as follows.

```
BellY[8, 3, {f1, f2, f3, f4, f5, f6, f7, f8}]
```

$$280\, f_2 f_3^2 + 210\, f_2^2 f_4 + 280\, f_1 f_3 f_4 + 168\, f_1 f_2 f_5 + 28\, f_1^2 f_6$$

Further, replacing the table $\{x_1, x_2, \cdots, x_{n-r+1}\}$ with a function, we can generate Formula 22.1.1 itself. For example, let

```
Tblf[n_] := Table[f_k, {k, 1, n}]
z[n_] := Sum_{r=1}^{n} g_r BellY[n, r, Tblf[n]]
```

Then, the above z_8 is easily generated as follows.

```
z[8]
```

$$
\begin{aligned}
&f_8 g_1 + (35 f_4^2 + 56 f_3 f_5 + 28 f_2 f_6 + 8 f_1 f_7) g_2 \\
&\quad + (280 f_2 f_3^2 + 210 f_2^2 f_4 + 280 f_1 f_3 f_4 + 168 f_1 f_2 f_5 + 28 f_1^2 f_6) g_3 \\
&\quad + (105 f_2^4 + 840 f_1 f_2^2 f_3 + 280 f_1^2 f_3^2 + 420 f_1^2 f_2 f_4 + 56 f_1^3 f_5) g_4 \\
&\quad + (420 f_1^2 f_2^3 + 560 f_1^3 f_2 f_3 + 70 f_1^4 f_4) g_5 \\
&\quad + (210 f_1^4 f_2^2 + 56 f_1^5 f_3) g_6 + 28 f_1^6 f_2 g_7 + f_1^8 g_8
\end{aligned}
$$

Of course, if a specific function is given to g_r or f_k, a desired higher order derivative function can be obtained. The examples are shown in the following chapters.

22.1.4 Higher Derivative in case the core function is the 1st degree

Although the higher differentiation of a general composition is complicated in this way, If the core function $f(x)$ is the 1st degree, it becomes remarkably easy.

Formula 22.1.4

When $g^{(n)} = g^{(n)}(f)$, $f_n = f^{(n)}(x)$ $(n=1,2,3,\cdots)$, if $f(x)$ is the 1st degree, the higher derivative function with respect to x of the composition $g\{f(x)\}$ is expressed as follows.

$$\{g\{f(x)\}\}^{(n)} = g^{(n)} f_1^n \tag{1.4}$$

Proof

In Formula 22.1.1 , when $f(x)$ is the 1st degree, $f_2 = f_3 = f_4 = \cdots = 0$. Then, the Bell polynomials $B_{n,r}(f_1, f_2, \dots)$ are as follows.

$$B_{n,r}(f_1, f_2, \cdots, f_n) = \sum \frac{n!}{j_1!\, j_2! \dots j_n!} \left(\frac{f_1}{1!}\right)^{j_1} \left(\frac{f_2}{2!}\right)^{j_2} \cdots \left(\frac{f_n}{n!}\right)^{j_n} = 0$$

$$\left(j_1 + j_2 + \cdots + j_n = r < n \quad \& \quad j_1 + 2j_2 + \cdots + nj_n = n\right)$$

$$B_{n,n}(f_1, f_2, \cdots, f_n) = \sum \frac{n!}{n!\, 0! \dots 0!} \left(\frac{f_1}{1!}\right)^{n} \left(\frac{f_2}{2!}\right)^{0} \cdots \left(\frac{f_n}{n!}\right)^{0} = f_1^n$$

Therefore,

$$
\begin{aligned}
\{g\{f(x)\}\}^{(n)} &= \sum_{r=1}^{n} g^{(r)} B_{n,r}(f_1, f_2, \cdots, f_n) \\
&= \sum_{r=1}^{n-1} g^{(r)} B_{n,r}(f_1, f_2, \cdots, f_n) + g^{(n)} B_{n,n}(f_1, f_2, \cdots, f_n) \\
&= g^{(n)} f_1^n
\end{aligned}
$$

Example When $z = sin(ax+b)$

$$z_1 = \ cos(ax+b) \cdot a^1 = a^1 sin\left(ax+b+\frac{1\pi}{2} \right)$$

$$z_2 = -sin(ax+b) \cdot a^2 = a^2 sin\left(ax+b+\frac{2\pi}{2} \right)$$

$$\vdots$$

$$z_n = a^n sin\left(ax+b+\frac{n\pi}{2} \right)$$

The last expression is consistent with linear form in Formula 9.2.1 in " **9.2** Higher Derivative ".

22.2 Higher Derivative of Some Composition

There are quite a lot of combinations even if the composition is limited to the elementary function. We cannot calculate these one by one. So, in this section, we calculate only an easy and interesting thing.

22.2.1 Higher Derivative of $e^{\pm f(x)}$

Placing with $g = e^{\pm f(x)}$ in Formula 22.1.1 , we obtain the following formulas immediately.

$$\left\{e^{f(x)}\right\}^{(n)} = e^{f(x)} \sum_{k=1}^{n} B_{n,k}(f_1, f_2, \cdots, f_n) \qquad (2.1_+)$$

$$\left\{e^{-f(x)}\right\}^{(n)} = e^{-f(x)} \sum_{k=1}^{n} (-1)^r B_{n,k}(f_1, f_2, \cdots, f_n) \qquad (2.1_-)$$

If (2.1_+) is written down to the 5th order, it is as follows.

$$\left\{e^{f(x)}\right\}^{(1)} = e^{f(x)} f_1$$

$$\left\{e^{f(x)}\right\}^{(2)} = e^{f(x)}\left(f_2 + f_1^2\right)$$

$$\left\{e^{f(x)}\right\}^{(3)} = e^{f(x)}\left(f_3 + 3f_2 f_1 + f_1^3\right)$$

$$\left\{e^{f(x)}\right\}^{(4)} = e^{f(x)}\left\{f_4 + \left(4f_3 f_1 + 3f_2^2\right) + 6f_2 f_1^2 + f_1^4\right\}$$

$$\left\{e^{f(x)}\right\}^{(5)} = e^{f(x)}\left\{f_5 + \left(5f_4 f_1 + 10f_3 f_2\right) + \left(10f_3 f_1^2 + 15f_2^2 f_1\right) + 10f_2 f_1^3 + f_1^5\right\}$$

Example1 $\left(e^{-x^2}\right)^{(3)}$

$$f = -x^2, \quad f_1 = -2x, \quad f_2 = -2, \quad f_3 = 0$$

Then,

$$\left(e^{-x^2}\right)^{(3)} = e^{-x^2}\left\{0 + 3(-2)(-2x) + (-2x)^3\right\} = 4e^{-x^2} x\left(3 - 2x^2\right)$$

Example2 $\left(e^{\sin x}\right)^{(4)}$

$$f = \sin x, \quad f_1 = \cos x, \quad f_2 = -\sin x, \quad f_3 = -\cos x, \quad f_4 = \sin x$$

Then,

$$\left(e^{\sin x}\right)^{(4)} = e^{\sin x}\left\{\sin x + \left(-4\cos^2 x + 3\sin^2 x\right) - 6\sin x\cos^2 x + \cos^4 x\right\}$$

Example3 $\left(e^{-\cos x}\right)^{(8)}$

We calculate using a formula manipulation software ***Mathematica*** according to (2.1_-) .

$$f = \cos x, \quad f_n = \cos\left(x + \frac{n\pi}{2}\right) \qquad n = 1, 2, 3, \cdots$$

So,

```
Tblf[n_] := Table[Cos[x + kπ/2], {k, 1, n}]

z[n_] := e^-Cos[x] Σ_{r=1}^{n} (-1)^r BellY[n, r, Tblf[n]]

z[8]
```

$$e^{-\text{Cos}[x]}\left(-\text{Cos}[x]+63\,\text{Cos}[x]^2-210\,\text{Cos}[x]^3+105\,\text{Cos}[x]^4\right.$$
$$-64\,\text{Sin}[x]^2+756\,\text{Cos}[x]\,\text{Sin}[x]^2-1260\,\text{Cos}[x]^2\,\text{Sin}[x]^2$$
$$+420\,\text{Cos}[x]^3\,\text{Sin}[x]^2+336\,\text{Sin}[x]^4-630\,\text{Cos}[x]\,\text{Sin}[x]^4$$
$$\left.+210\,\text{Cos}[x]^2\,\text{Sin}[x]^4-56\,\text{Sin}[x]^6+28\,\text{Cos}[x]\,\text{Sin}[x]^6+\text{Sin}[x]^8\right)$$

22.2.2 Higher Derivative of $g\left(e^{\pm x}\right)$

Placing with $f=e^{\pm x}$ in Formula 22.1.1 , we obtain the following formulas.

$$\left\{g\left(e^{x}\right)\right\}^{(n)}=\sum_{r=1}^{n} g^{(r)} B_{n,r}\left(e^{x}, \ldots, e^{x}\right)$$

$$\left\{g\left(e^{-x}\right)\right\}^{(n)}=(-1)^{n} \sum_{r=1}^{n} g^{(r)} B_{n,r}\left(e^{-x}, \ldots, e^{-x}\right)$$

If the first expression is written down to the 5th order, it is as follows.

$$\left\{g\left(e^{x}\right)\right\}^{(1)}=g^{(1)} e^{x}$$
$$\left\{g\left(e^{x}\right)\right\}^{(2)}=g^{(1)} e^{x}+g^{(2)} e^{2x}$$
$$\left\{g\left(e^{x}\right)\right\}^{(3)}=g^{(1)} e^{x}+3 g^{(2)} e^{2x}+g^{(3)} e^{3x}$$
$$\left\{g\left(e^{x}\right)\right\}^{(4)}=g^{(1)} e^{x}+7 g^{(2)} e^{2x}+6 g^{(3)} e^{3x}+g^{(4)} e^{4x}$$
$$\left\{g\left(e^{x}\right)\right\}^{(5)}=g^{(1)} e^{x}+15 g^{(2)} e^{2x}+25 g^{(3)} e^{3x}+10 g^{(4)} e^{4x}+g^{(5)} e^{5x}$$

Coefficients $(1),(1,1),(1,3,1),(1,7,6,1),(1,15,25,10,1), \cdots$ of these right sides are called ***Stirling numbers of the 2nd kind*** and is given by the following expression.

$$S(n,r)=\frac{1}{r!} \sum_{s=0}^{r}(-1)^{s}\binom{r}{s}(r-s)^{n} \tag{2.s}$$

Using this notation, the above formulas are more briefly expressed as follows

$$\left\{g\left(e^{x}\right)\right\}^{(n)}=\sum_{r=1}^{n} S(n,r) g^{(r)} e^{rx} \tag{2.1+}$$

$$\left\{g\left(e^{-x}\right)\right\}^{(n)}=(-1)^{n} \sum_{r=1}^{n} S(n,r) g^{(r)} e^{-rx} \tag{2.1-}$$

Example1 $\left(\tan e^{x}\right)^{(3)}$

$$g^{(1)}=\tan^{2} e^{x}+1\ ,\quad g^{(2)}=2\tan^{3} e^{x}+2\tan e^{x}\ ,\quad g^{(3)}=6\tan^{4} e^{x}+8\tan^{2} e^{x}+2$$

Then,

$$\left(\tan e^{x}\right)^{(3)}=\left(\tan^{2} e^{x}+1\right) e^{x}+3\left(2\tan^{3} e^{x}+2\tan e^{x}\right) e^{2x}+\left(6\tan^{4} e^{x}+8\tan^{2} e^{x}+2\right) e^{3x}$$
$$=e^{x}\left(\tan^{2} e^{x}+1\right)+6 e^{2x}\left(\tan^{3} e^{x}+\tan e^{x}\right)+2 e^{3x}\left(3\tan^{4} e^{x}+4\tan^{2} e^{x}+1\right)$$

Example2 $\left(e^{e^{-x}}\right)^{(5)}$

$$g=e^{f}\ ,\quad g^{(n)}=e^{f}=e^{e^{-x}} \quad (n=1,2,3,\cdots)$$

So,

$$\left(e^{e^{-x}}\right)^{(5)} = -e^{e^{-x}}\left(e^{-x} + 15e^{-2x} + 25e^{-3x} + 10e^{-4x} + e^{-5x}\right)$$

Note

The horizontal total of Stirling numbers of the 2nd kind is called ***Bell Number*** . That is

$$B_n = \sum_{r=1}^{n} S(n,r)$$

The first few Bell numbers for $n=1,2,3,\cdots$ are $1,\ 2,\ 5,\ 15,\ 52,\ 203,\ 877,\cdots$.

This Bell number is given by the following expression that is called ***Dobinski's formula***, too.

$$B_n = \frac{1}{e}\sum_{r=0}^{\infty}\frac{r^n}{r!} = \sum_{r=1}^{n}\frac{r^n}{r!}\sum_{s=0}^{n-r}\frac{(-1)^s}{s!}$$

22.2.3 Higher Derivative of $\log f(x)$

$g^{(r)} = (-1)^{r-1}(r-1)!\,f^{-r}$ from $g = \log f$. Substituting this for Formula 22.1.1 ,

$$\{\log f(x)\}^{(n)} = \sum_{r=1}^{n}(-1)^{r-1}(r-1)!\,B_{n,r}(f_1, f_2, \cdots, f_n)\,f^{-r} \tag{2.3}$$

If these are written down to the 5th order, it is as follows.

$$\{\log f(x)\}^{(1)} = 0!\,f_1 f^{-1}$$

$$\{\log f(x)\}^{(2)} = 0!\,f_2 f^{-1} - 1!\,f_1^2 f^{-2}$$

$$\{\log f(x)\}^{(3)} = 0!\,f_3 f^{-1} - 1!\,3f_2 f_1 f^{-2} + 2!\,f_1^3 f^{-3}$$

$$\{\log f(x)\}^{(4)} = 0!\,f_4 f^{-1} - 1!\left(4f_3 f_1 + 3f_2^2\right)f^{-2} + 2!\,6f_2 f_1^2 f^{-3} - 3!\,f_1^4 f^{-4}$$

$$\{\log f(x)\}^{(5)} = 0!\,f_5 f^{-1} - 1!\left(5f_4 f_1 + 10f_3 f_2\right)f^{-2} + 2!\left(10f_3 f_1^2 + 15f_2^2 f_1\right)f^{-3} - 3!\,10f_2 f_1^3 f^{-4} + 4!\,f_1^5 f^{-5}$$

Example $(\log\sin x)^{(3)}$

$$f = \sin x \quad , \quad f_1 = \cos x \quad , \quad f_2 = -\sin x \quad , \quad f_3 = -\cos x \ ,$$

So,

$$(\log\sin x)^{(3)} = -0!\,\frac{\cos x}{\sin x} + 1!\,3\frac{\sin x\cos x}{(\sin x)^2} + 2!\,\frac{(\cos x)^3}{(\sin x)^2} = 2\cot x + 2(\cot x)^3$$

22.2.4 Higher Derivative of $g(\log x)$

$f^{(r)} = (-1)^{r-1}(r-1)!\,x^{-r}$ from $f = \log x$. Substituting this for Formula 22.1.1 ,

$$\{g(\log x)\}^{(n)} = \sum_{r=1}^{n} g^{(r)} B_{n,r}\left(\frac{0!}{x}, -\frac{1!}{x^2}, \cdots, (-1)^{n-1}\frac{(n-1)!}{x^n}\right) \tag{2.4}$$

If these are written down to the 4th order, it is as follows.

$$\{g(\log x)\}^{(1)} = \frac{1}{x}g^{(1)}0!$$

$$\{g(\log x)\}^{(2)} = -\frac{1}{x^2}\left\{g^{(1)}1! - g^{(2)}(0!)^2\right\}$$

$$\{g(\log x)\}^{(3)} = \frac{1}{x^3}\{g^{(1)}2! - 3g^{(2)}1!\,0! + g^{(3)}(0!)^3\}$$

$$\{g(\log x)\}^{(4)} = -\frac{1}{x^4}\{g^{(1)}3! - g^{(2)}(4\cdot 2!\,0! + 3(1!)^2) + 6g^{(3)}1!\,(0!)^2 - g^{(4)}(0!)^4\}$$

Example1 $(\cos\log x)^{(4)}$

Since $g^{(1)} = -\sin f$, $g^{(2)} = -\cos f$, $g^{(3)} = \sin f$, $g^{(4)} = \cos f$,

$$(\cos\log x)^{(4)} = -\frac{1}{x^4}\left(6g^{(1)} - 11g^{(2)} + 6g^{(3)} - g^{(4)}\right)$$

$$= -\frac{1}{x^4}(-6\sin f + 11\cos f + 6\sin f - \cos f) = -\frac{10\cos\log x}{x^4}$$

Example2 $(1/\log x)^{(3)}$

Since $g^{(1)} = -\frac{1!}{f^2}$, $g^{(2)} = \frac{2!}{f^3}$, $g^{(3)} = -\frac{3!}{f^4}$,

$$(1/\log x)^{(3)} = \frac{1}{x^3}\left(2g^{(1)} - 3g^{(2)} + g^{(3)}\right) = \frac{1}{x^3}\left(-\frac{2\cdot 1!}{f^2} - \frac{3\cdot 2!}{f^3} - \frac{3!}{f^4}\right)$$

$$= -\frac{1}{x^3}\left\{\frac{2}{(\log x)^2} + \frac{6}{(\log x)^3} + \frac{6}{(\log x)^4}\right\}$$

Example3 $(\log\log x)^{(3)}$

Since $g^{(1)} = \frac{0!}{f}$, $g^{(2)} = -\frac{1!}{f^2}$, $g^{(3)} = \frac{2!}{f^3}$,

$$(\log\log x)^{(3)} = \frac{1}{x^3}\left(2g^{(1)} - 3g^{(2)} + g^{(3)}\right) = \frac{1}{x^3}\left(\frac{2\cdot 0!}{f} + \frac{3\cdot 1!}{f^2} + \frac{2!}{f^3}\right)$$

$$= \frac{1}{x^3}\left\{\frac{2}{\log x} + \frac{3}{(\log x)^2} + \frac{2}{(\log x)^3}\right\}$$

22.3 Higher Derivative of Gamma Function & the Reciprocal

Formula 22.3.1 (Masayuki Ui)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,r}(f_1, f_2, \dots)$ are Bell polynomials , the following expressions hold.

$$\frac{d^n}{dz^n}\Gamma(z) = \Gamma(z)\sum_{k=1}^{n} B_{n,k}\left(\psi_0(z), \psi_1(z), \dots, \psi_{n-1}(z)\right) \tag{3.1$_+$}$$

$$\frac{d^n}{dz^n}\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z)}\sum_{k=1}^{n}(-1)^k B_{n,k}\left(\psi_0(z), \psi_1(z), \dots, \psi_{n-1}(z)\right) \tag{3.1$_-$}$$

Proof

When $f(z) = \log\Gamma(z)$,

$$f_1 = \frac{d}{dz}\log\Gamma(z) = \psi_0(z)$$

$$f_2 = \frac{d}{dz}\psi_0(z) = \psi_1(z)$$

$$\vdots$$

$$f_n = \frac{d}{dz}\psi_{n-2}(z) = \psi_{n-1}(z)$$

Substituting these for (2.1$_+$) , (2.1$_-$) in Formula 22.2.1 , we obtain

$$\left\{e^{\log\Gamma(z)}\right\}^{(n)} = e^{\log\Gamma(z)}\sum_{r=1}^{n} B_{n,r}\left(\psi_0(z), \psi_1(z), \dots, \psi_{n-1}(z)\right) \tag{3.1$_+$}$$

$$\left\{e^{-\log\Gamma(z)}\right\}^{(n)} = e^{-\log\Gamma(z)}\sum_{r=1}^{n}(-1)^r B_{n,r}\left(\psi_0(z), \psi_1(z), \dots, \psi_{n-1}(z)\right) \tag{3.1$_-$}$$

Example1 $\Gamma^{(4)}(z)$

We calculate using a formula manipulation software ***Mathematica*** according to (3.1$_+$) . When this table definition is adopted, the result is slightly hard to see, but the differential coefficient is calculable.

```
Tblψ[n_, z_] := Table[PolyGamma[k, z], {k, 0, n - 1}]

dΓ[n_, z_] := Gamma[z] Sum[BellY[n, r, Tblψ[n, z]], {r, 1, n}]

dΓ[4, z]

Gamma[z] (PolyGamma[0, z]^4
      + 6 PolyGamma[0, z]^2 PolyGamma[1, z] + 3 PolyGamma[1, z]^2
      + 4 PolyGamma[0, z] PolyGamma[2, z] + PolyGamma[3, z])

N[dΓ[4, 1]]
23.5615
```

Example2 $\{1/\Gamma(z)\}^{(4)}$

We calculate using a formula manipulation software ***Mathematica*** according to (3.1$_-$) . When this table definition is adopted, the result is easy to see, but the differential coefficient is incalculable.

```
Tblψ[n_, z_] := Table[ψ_k[z], {k, 0, n - 1}]
```

$$df[n\_, z\_] := \frac{1}{Gamma[z]} \sum_{r=1}^{n} (-1)^r \, BellY[n, r, Tbl\psi[n, z]]$$

```
df[4, z]
```

$$\frac{\psi_0[z]^4 - 6\,\psi_0[z]^2\,\psi_1[z] + 3\,\psi_1[z]^2 + 4\,\psi_0[z]\,\psi_2[z] - \psi_3[z]}{Gamma[z]}$$

```
df[4, 1]
```

$$\psi_0[1]^4 - 6\,\psi_0[1]^2\,\psi_1[1] + 3\,\psi_1[1]^2 + 4\,\psi_0[1]\,\psi_2[1] - \psi_3[1]$$

Note

On December 9, 2016, I received a mail from Mr. Ui living in Yokohama city. In the mail, it was written that the coefficients of the Bell polynomials appear in the higher order derivative of the gamma function. I was vary surprised. Because, it means that the gamma function is a composite function. In fact, it was a too simple composite function. (Mr. Ui seems to have noticed it soon, but I needed 3 days to notice it.)

As far as I get to know, the discoverer of Formula 22.3.1 is Mr. Ui. This section is what I added a simple proof with the consent of Mr. Ui.

22.4 Possibility to super differentiation of composition

Faà di Bruno's Formula was as follows.

$$\{g\{f(x)\}\}^{(n)} = \sum_{r=1}^{n} g^{(r)} B_{n,r}(f_1, f_2, \dots, f_n) \tag{1.1}$$

Hoppe's Formula was as follows.

$$\{g(f(x))\}^{(n)} = \sum_{r=0}^{n} \frac{g^{(r)}}{r!} \sum_{s=0}^{r} \binom{r}{s} (-f)^{r-s} (f^s)^{(n)} \tag{1.2}$$

We cannot make the upper limit n of $\sum$ ∞ in both formulas. It is because the number of terms in the right side cannot become larger than the number of orders in the left side. So, it is hopeless to extend the domain of (1.1) or (1.2) from the natural number n to the real number p . That is, the super differentiation of the general composition $\{g\{f(x)\}\}$ is impossible now.

However, when the core function $f(x)$ is the 1st degree, according to Formula 22.1.4 ,

$$\{g\{f(x)\}\}^{(n)} = g^{(n)} f_1^n \tag{1.4}$$

So, there is no obstacle in extending this domain from the natural number n to real number p .
Thus, the following expression holds for a real number $p>0$.

$$\{g\{f(x)\}\}^{(p)} = g^{(p)} f_1^p \tag{4.4}$$

This is the grounds for which we have used "Linear form" since " **12 Super Derivative** " as a fait accompli .

2011.01.06
2016.12.26 updated.

K. Kono

Alien's Mathematics