18 Higher Derivative of the Product of Two Functions

18.1 Leibniz Rule about the Higher Order Differentiation

Theorem 18.1.1 (Leibniz)

When functions $f(x)$ and $g(x)$ are n times differentiable, the following expression holds.

$$\{f(x)g(x)\}^{(n)} = \sum_{r=0}^{n}\binom{n}{r} f^{(n-r)}(x)g^{(r)}(x) \tag{1.1}$$

Proof

Theorem 16.1.2 (2.1) in **16.1.2** was as follows.

$$\begin{aligned}\int_{a_n}^{x}\!\!\cdots\int_{a_1}^{x} f^{<0>}g^{(0)}dx^n &= \sum_{r=0}^{m-1}\binom{-n}{r} f^{<n+r>}g^{(r)} \\ &- \sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s} f_{a_{n-r}}^{<n-r+s>} g_{a_{n-r}}^{(s)}\int_{a_n}^{x}\!\!\cdots\int_{a_{n-r+1}}^{x} dx^r \\ &+ (-1)^m\sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1}\, f_{a_{n-r}}^{\langle m+n-r+s\rangle} g_{a_{n-r}}^{(m+s)}\int_{a_n}^{x}\!\!\cdots\int_{a_{n-r+1}}^{x} dx^r \\ &+ \frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_{a_n}^{x}\!\!\cdots\int_{a_1}^{x} f^{<m+k>}g^{(m+k)}dx^n\end{aligned}$$

Should be noted here is the next two.

i When $n=1$ $\sum\sum\sum$ of the 3rd line does not exist, when $n=0$ $\sum\sum$ of the 2nd line does not exist also.

ii When the binomial coefficient of the 4th line is generalized, the upper limit $n-1$ of $\sum$ can be replaced by ∞.

Since $1>0\ge -n$ at the time $n=0,1,2,\cdots$, if the index n of the integration operator is substituted for $-n$ in consideration of these, it becomes as follows.

$$\begin{aligned}\int_{a_n}^{x}\!\!\cdots\int_{a_1}^{x} f^{<0>}g^{(0)}dx^{-n} &= \sum_{r=0}^{m-1}\binom{n}{r} f^{<-n+r>}g^{(r)} \\ &+ \frac{(-1)^m}{B(-n,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{-n-1}{k}\int_{a_n}^{x}\!\!\cdots\int_{a_1}^{x} f^{<m+k>}g^{(m+k)}dx^{-n}\end{aligned}$$

Since m may be arbitrary integer, when $m=n+1$, it is as follows.

$$\begin{aligned}\int_{a_n}^{x}\!\!\cdots\int_{a_1}^{x} f^{<0>}g^{(0)}dx^{-n} &= \sum_{r=0}^{n}\binom{n}{r} f^{<-n+r>}g^{(r)} \\ &+ \frac{(-1)^{n+1}}{B(-n,n+1)}\sum_{k=0}^{\infty}\frac{1}{n+1+k}\binom{-n-1}{k}\int_{a_n}^{x}\!\!\cdots\int_{a_1}^{x} f^{<n+1+k>}g^{(n+1+k)}dx^{-n}\end{aligned}$$

However, since $B(-n,n+1)=\pm\infty$ $for\ n=0,1,2,\cdots$, the 2nd line disappears. That is,

$$\int_{a_n}^{x}\!\!\cdots\int_{a_1}^{x} fg\,dx^{-n} = \sum_{r=0}^{n}\binom{n}{r} f^{<-n+r>}g^{(r)} \qquad n=0,1,2,\cdots$$

Then, replacing the integration operators $dx^{-n}, <-n+r>$ with the differentiation operators $(n),(n-r)$ respectivly, we obtain the desired expression.

18.2 Higher Derivative of x^a f (x)

Formula 18.2.0

When $\Gamma(z)$ denotes the gamma function and $f(x)$ is n times differentiable continuous function, the following expressions hold for a natural number n.

(1)

$$\left\{x^{\alpha}f(x)\right\}^{(n)} = \sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}f^{(n-r)}(x) \tag{0.1}$$

Where, if $\alpha = -1,-2,-3,\cdots$, it shall read as follows.

$$\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} \longrightarrow (-1)^{-r}\frac{\Gamma(-\alpha+r)}{\Gamma(-\alpha)}$$

(2) Especially, when $\alpha = m = 0,1,2,\cdots$

$$\left\{x^{m}f(x)\right\}^{(n)} = \sum_{r=0}^{m}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}f^{(n-r)}(x) \tag{0.1'}$$

(3) When $\alpha \ne -1,-2,-3,\cdots$ & $\alpha - n \ne -1,-2,-3,\cdots$

$$\left\{x^{\alpha}f(x)\right\}^{(n)} = \sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n+r)}x^{\alpha-n+r}f^{(r)}(x) \tag{0.2}$$

Proof

When $g(x) = x^{\alpha}$ in Theorem 18.1.1 , since

$$\left(x^{\alpha}\right)^{(r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}$$

we obtain the following expression immediately.

$$\left\{x^{\alpha}f(x)\right\}^{(n)} = \sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}f^{(n-r)}(x) \tag{0.1}$$

Especially, when $\alpha = m = 0,1,2,\cdots$, (0.1) is as follows.

$$\left\{x^{m}f(x)\right\}^{(n)} = \sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}f^{(n-r)}(x) \qquad \frac{\Gamma(1+m)}{\Gamma(1+m-r)} = 0 \quad for\ \ m<r\leqq n$$

$$= \sum_{r=0}^{m}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}f^{(n-r)}(x) \qquad \binom{n}{r} = 0 \qquad for\ \ n<r\leqq m$$

We adopt the convenient latter for mathematical software.

When $\alpha = -1,-2,-3,\cdots$, from **1.1.5 (Properties of the Gamma Function)** (5.5),

$$\frac{\Gamma(-z)}{\Gamma(-z-n)} = (-1)^{-n}\frac{\Gamma(1+z+n)}{\Gamma(1+z)} \qquad (n \text{ is a non-negative integer })$$

Then substituting $-z = 1+\alpha$, $n = r$ for this, we obtain the proviso.

Last, replacing r with $n-r$ in (0.1), we obtain (0.2).

Below, substituting various functions f for Formula 18.2.0 , we obtain various formulas.
Although there are (1) and (2) in Formula 18.2.0, since (2) is almost meaningless in the case of higher differentiation, we adopt (1) in principle.

18.2.1 Higher Derivative of $(ax+b)^p(cx+d)^q$

Formula 18.2.1

The following expressions hold for $p>0$ and $n=1,2,3,\cdots$.

$$\left\{(ax+b)^p(cx+d)^q\right\}^{(n)} = \sum_{r=0}^{n}\binom{n}{r}\frac{(1/a)^{-n+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(1+p-n+r)\Gamma(1+q-r)}\frac{(ax+b)^{p-n+r}}{(cx+d)^{r-q}} \tag{1.1}$$

Especially, when $m=0,1,2,\cdots$

$$\left\{(ax+b)^p(cx+d)^m\right\}^{(n)} = \sum_{r=0}^{m}\binom{n}{r}\frac{(1/a)^{-n+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+m)}{\Gamma(1+p-n+r)\Gamma(1+m-r)}\frac{(ax+b)^{p-n+r}}{(cx+d)^{r-m}} \tag{1.1'}$$

Proof

Let $f(x)=(ax+b)^p$, $g(x)=(cx+d)^q$, then

$$f^{(n-r)}=\left\{(ax+b)^p\right\}^{(n-r)} = \left(\frac{1}{a}\right)^{-n+r}\frac{\Gamma(1+p)}{\Gamma(1+p-n+r)}(ax+b)^{p-n+r}$$

$$g^{(r)}=\left\{(cx+d)^q\right\}^{(r)}= \left(\frac{1}{c}\right)^{-r}\frac{\Gamma(1+q)}{\Gamma(1+q-r)}(cx+d)^{q-r} \qquad q\ne -1,-2,-3,\cdots$$

Substituting these for Theorem 18.1.1 , we obtain (1.1) .

And especially, when $q=m=0,1,2,\cdots$, from (1.1)

$$\left\{(ax+b)^p(cx+d)^m\right\}^{(n)} = \sum_{r=0}^{n}\binom{n}{r}\frac{(1/a)^{-n+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+m)}{\Gamma(1+p-n+r)\Gamma(1+m-r)}\frac{(ax+b)^{p-n+r}}{(cx+d)^{r-m}}$$

$$= \sum_{r=0}^{m}\binom{n}{r}\frac{(1/a)^{-n+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+m)}{\Gamma(1+p-n+r)\Gamma(1+m-r)}\frac{(ax+b)^{p-n+r}}{(cx+d)^{r-m}}$$

We adopt the latter expression as (1.1').

Example1 The 2nd order derivative of $\sqrt{x-2}\,\sqrt[3]{3x+4}$

Substituting $a=1, b=-2, p=1/2, c=3, d=4, q=1/3, n=2$ for (1.1) ,

$$\left(\sqrt{x-2}\,\sqrt[3]{3x+4}\right)^{(2)}=\sum_{r=0}^{2}\binom{2}{r}3^r\frac{\Gamma(3/2)\Gamma(4/3)}{\Gamma(r-1/2)\Gamma(4/3-r)}(x-2)^{r-\frac{3}{2}}(3x+4)^{\frac{1}{3}-r}$$

$$= -\sqrt{x-2}\,\sqrt[3]{3x+4}\left\{\frac{1}{4(x-2)^2}-\frac{1}{(x-2)(3x+4)}+\frac{2}{(3x+4)^2}\right\}$$

Example1' The 3rd order derivative of $\sqrt{x-2}\,(3x+4)^2$

Substituting $a=1, b=-2, p=1/2, c=3, d=4, m=2, n=3$ for (1.1') ,

$$\left\{\sqrt{x-2}\,(3x+4)^2\right\}^{(3)} = \sum_{r=0}^{2}\binom{3}{r}3^r\frac{\Gamma(3/2)\Gamma(3)}{\Gamma(-3/2+r)\Gamma(3-r)}\frac{(x-2)^{-\frac{5}{2}+r}}{(3x+4)^{r-2}}$$

$$=\sqrt{x-2}\,(3x+4)^2\left\{\frac{3^1}{8(x-2)^3}-\frac{3^2}{2(3x+4)(x-2)^2}+\frac{3^3}{(3x+4)^2(x-2)}\right\}$$

Example2 The 3rd order derivative of $\sqrt{x-2}\,/\,(3x+4)$

When $q=-1,-2,-3,\cdots$, (1.1) can be read as follows.

$$\left\{(ax+b)^p(cx+d)^q\right\}^{(n)}$$

$$=\sum_{r=0}^{n}\binom{n}{r}\frac{(1/a)^{-n+r}}{(-1/c)^r}\frac{\Gamma(1+p)\Gamma(-q+r)}{\Gamma(1+p-n+r)\Gamma(-q)}\frac{(ax+b)^{p-n+r}}{(cx+d)^{r-q}}$$

Substituting $a=1,\ b=-2,\ p=1/2,\ c=3,\ d=4,\ q=-1,\ n=3$ for this,

$$\left(\frac{\sqrt{x-2}}{3x+4}\right)^{(3)}=\sum_{r=0}^{3}\binom{3}{r}(-3)^r\frac{\Gamma(3/2)\Gamma(1+r)}{\Gamma(-3/2+r)\Gamma(1)}\frac{(x-2)^{-\frac{5}{2}+r}}{(3x+4)^{r+1}}$$

$$=\frac{\sqrt{x-2}}{3x+4}\left\{\frac{3}{8(x-2)^3}+\frac{9}{4(x-2)^2(3x+4)}+\frac{27}{(x-2)(3x+4)^2}-\frac{162}{(3x+4)^3}\right\}$$

18.2.2 Higher Derivative of $x^\alpha \log x$

Formula 18.2.2

$$\left(x^\alpha \log x\right)^{(n)}=-\sum_{r=0}^{n-1}(-1)^{n-r}\binom{n}{r}\frac{\Gamma(n-r)\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-n}+\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)}x^{\alpha-n}\log x \tag{2.1}$$

Especially, when $m=0,1,2,\cdots$

$$\left(x^m \log x\right)^{(n)}=-\sum_{r=0}^{n-1}(-1)^{n-r}\binom{n}{r}\frac{\Gamma(n-r)\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-n}+\frac{\Gamma(1+m)}{\Gamma(1+m-n)}x^{m-n}\log x \tag{2.1'}$$

Where, there shall be no 2nd term of the right side at the time of $m<n$.

Proof

Let $f(x)=\log x$. Then

$$(\log x)^{(n-r)} = -(-1)^{n-r}\Gamma(n-r)\,x^{-n+r} \qquad r=0,1,\cdots,n-1$$
$$= \log x \qquad r=n$$

Substituting these for (0.1) in Theorem 18.2.0 , we obtain (2.1).

When $m=0,1,2,\cdots$, applying (0.1') , we obtain the following.

$$\left(x^m \log x\right)^{(n)}=-\sum_{r=0}^{n-1}(-1)^{n-r}\binom{n}{r}\frac{\Gamma(n-r)\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-n}+\frac{\Gamma(1+m)}{\Gamma(1+m-n)}x^{m-n}\log x$$

$$+\sum_{r=n+1}^{m}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-n}(\log x)^{(n-r)}$$

Where, since r does not reach n at the time of $m<n$, the 2nd term does not exist.

Example1 The 3rd order derivaive of $\sqrt{x}\ log\,x$

Substituting $\alpha=1/2$, $n=3$ for (2.1) ,

$$\left(\sqrt{x}\,log\,x\right)^{(3)} = -\sum_{r=0}^{2}(-1)^{3-r}\binom{3}{r}\frac{\Gamma(2-r)\Gamma(3/2)}{\Gamma(3/2-r)}x^{-\frac{5}{2}}+\frac{\Gamma(3/2)}{\Gamma(-3/2)}x^{-\frac{5}{2}}log\,x$$

$$= -\left\{-\binom{3}{0}2!\frac{\Gamma(3/2)}{\Gamma(3/2)}+\binom{3}{1}1!\frac{\Gamma(3/2)}{\Gamma(1/2)}-\binom{3}{2}0!\frac{\Gamma(3/2)}{\Gamma(-1/2)}\right\}x^{-\frac{5}{2}}$$

$$+\frac{\Gamma(3/2)}{\Gamma(-3/2)}x^{-\frac{5}{2}}log\,x$$

$$=\left(2-\frac{3}{2}-\frac{3}{4}\right)x^{-\frac{5}{2}}+\frac{\Gamma(3/2)}{\Gamma(-3/2)}x^{-\frac{5}{2}}log\,x=\left(-\frac{1}{4}+\frac{3}{8}log\,x\right)x^{-\frac{5}{2}}$$

Example1' The 2nd order derivaive of $x^3 log\,x$

Substituting $m=3$, $n=2$ for (2.1') ,

$$\left(x^3 log\,x\right)^{(2)} = -\sum_{r=0}^{1}(-1)^{2-r}\binom{2}{r}\frac{\Gamma(2-r)\Gamma(4)}{\Gamma(4-r)}x^1+\frac{\Gamma(4)}{\Gamma(2)}x^1 log\,x$$

$$= -\left\{(-1)^2\binom{2}{0}\frac{1!\,3!}{3!}+(-1)^1\binom{2}{1}\frac{0!\,3!}{2!}\right\}x^1+\frac{3!}{1!}x^1 log\,x$$

$$=(5+6\,log\,x)x^1$$

Example1" The 3rd order derivaive of $x^2 log\,x$

Substituting $m=2$, $n=3$ for (2.1') ,

$$\left(x^2 log\,x\right)^{(3)} = -\sum_{r=0}^{2}\binom{3}{r}(-1)^{3-r}\frac{\Gamma(3-r)\Gamma(3)}{\Gamma(3-r)}x^{-1}$$

$$= -\left\{\binom{3}{0}(-1)^3 2!+\binom{3}{1}(-1)^2 2!+\binom{3}{2}(-1)^1 2!\right\}x^{-1}$$

$$=2\,x^{-1}$$

Example2 The 3rd order derivaive of $log\,x\,/\,x$

When $\alpha=-1,-2,-3,\cdots$, (2.1) can be read as follows.

$$\left(x^{\alpha}log\,x\right)^{(n)} = -(-1)^n\sum_{r=0}^{n-1}\binom{n}{r}\frac{\Gamma(n-r)\Gamma(-\alpha+r)}{\Gamma(-\alpha)}x^{\alpha-n}+(-1)^n\frac{\Gamma(-\alpha+n)}{\Gamma(-\alpha)}x^{\alpha-n}log\,x$$

Substituting $\alpha=-1$, $n=3$ for this ,

$$\left(\frac{log\,x}{x}\right)^{(3)} = -(-1)^{-3}\sum_{r=0}^{2}\binom{3}{r}\frac{\Gamma(3-r)\Gamma(1+r)}{\Gamma(1)}x^{-4}+(-1)^{-3}\frac{\Gamma(4)}{\Gamma(1)}x^{-4}log\,x$$

$$=x^{-4}\left\{\binom{3}{0}\Gamma(3)\Gamma(1)+\binom{3}{1}\Gamma(2)\Gamma(2)+\binom{3}{2}\Gamma(1)\Gamma(3)-6\,log\,x\right\}$$

$$=\frac{1}{x^4}(11-6\,log\,x)$$

18.2.3 Higher Derivatives of $x^{\alpha}\sin x$, $x^{\alpha}\cos x$

Formula 18.2.3

$$\left(x^{\alpha}\sin x\right)^{\{n\}}=\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}\sin\left\{x+\frac{(n-r)\pi}{2}\right\} \tag{3.1s}$$

$$\left(x^{\alpha}\cos x\right)^{\{n\}}=\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}\cos\left\{x+\frac{(n-r)\pi}{2}\right\} \tag{3.1c}$$

Especially, when $m=0,1,2,\cdots$

$$\left(x^{m}\sin x\right)^{\{n\}}=\sum_{r=0}^{m}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\sin\left\{x+\frac{(n-r)\pi}{2}\right\} \tag{3.1's}$$

$$\left(x^{m}\cos x\right)^{\{n\}}=\sum_{r=0}^{m}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\cos\left\{x+\frac{(n-r)\pi}{2}\right\} \tag{3.1'c}$$

Example1 The 2nd order derivative of $\sqrt[3]{x}\sin x$

Substituting $\alpha=1/3$, $n=2$ for (3.1s) ,

$$\left(\sqrt[3]{x}\sin x\right)^{(2)}=\sum_{r=0}^{2}\binom{2}{r}\frac{\Gamma(4/3)}{\Gamma(4/3-r)}x^{\frac{1}{3}-r}\sin\left\{x+\frac{(2-r)\pi}{2}\right\}$$

$$=\binom{2}{0}\frac{\Gamma(4/3)}{\Gamma(4/3)}x^{\frac{1}{3}}\sin(x+\pi)+\binom{2}{1}\frac{\Gamma(4/3)}{\Gamma(1/3)}x^{-\frac{2}{3}}\sin\left(x+\frac{\pi}{2}\right)+\binom{2}{2}\frac{\Gamma(4/3)}{\Gamma(-2/3)}x^{-\frac{5}{3}}\sin x$$

$$=-x^{\frac{1}{3}}\sin x+\frac{2}{3}x^{-\frac{2}{3}}\cos x-\frac{2}{9}x^{-\frac{5}{3}}\sin x$$

Example1' The 3rd order derivative of $x^{2}\sin x$

Substituting $m=2$, $n=3$ for (3.1's) ,

$$\left(x^{2}\sin x\right)^{\{3\}}=\sum_{r=0}^{2}\binom{3}{r}\frac{\Gamma(3)}{\Gamma(3-r)}x^{2-r}\sin\left\{x+\frac{(3-r)\pi}{2}\right\}$$

$$=\binom{3}{0}\frac{\Gamma(3)}{\Gamma(3)}x^{2}\sin\left(x+\frac{3\pi}{2}\right)+\binom{3}{1}\frac{\Gamma(3)}{\Gamma(2)}x^{1}\sin\left(x+\frac{2\pi}{2}\right)$$

$$+\binom{3}{2}\frac{\Gamma(3)}{\Gamma(1)}x^{0}\sin\left(x+\frac{\pi}{2}\right)$$

$$=-x^{2}\cos x-6x\sin x+6\cos x$$

18.2.4 Higher Derivatives of $x^{\alpha}\sinh x$, $x^{\alpha}\cosh x$

Formula 18.2.4

$$\left(x^{\alpha}\sinh x\right)^{(n)}=\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}\frac{e^{x}-(-1)^{-(n-r)}e^{-x}}{2} \tag{4.1s}$$

$$\left(x^{\alpha}\cosh x\right)^{(n)}=\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}\frac{e^{x}+(-1)^{-(n-r)}e^{-x}}{2} \tag{4.1c}$$

Especially, when $m = 0, 1, 2, \cdots$

$$\left(x^m sinh x\right)^{(n)} = \sum_{r=0}^{m} \binom{n}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \frac{e^x - (-1)^{-(n-r)} e^{-x}}{2} \tag{4.1's}$$

$$\left(x^m cosh x\right)^{(n)} = \sum_{r=0}^{m} \binom{n}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \frac{e^x + (-1)^{-(n-r)} e^{-x}}{2} \tag{4.1'c}$$

Example1 The 2nd order derivaive of $\sqrt[3]{x}\, sinh x$

Substituting $\alpha = 1/3$, $n = 2$ for (4.1s) ,

$$\left(\sqrt[3]{x}\, sinh x\right)^{(2)} = \sum_{r=0}^{2} \binom{2}{r} \frac{\Gamma(4/3)}{\Gamma(4/3-r)} x^{\frac{1}{3}-r} \frac{e^x - (-1)^{-(2-r)} e^{-x}}{2}$$

$$= \binom{2}{0} \frac{\Gamma(4/3)}{\Gamma(4/3)} x^{\frac{1}{3}} sinh x + \binom{2}{1} \frac{\Gamma(4/3)}{\Gamma(1/3)} x^{-\frac{2}{3}} cosh x + \binom{2}{2} \frac{\Gamma(4/3)}{\Gamma(-2/3)} x^{-\frac{5}{3}} sinh x$$

$$= x^{\frac{1}{3}} sinh x + \frac{2}{3} x^{-\frac{2}{3}} cosh x - \frac{2}{9} x^{-\frac{5}{3}} sinh x$$

Example1' The 3rd order derivative of $x^2 sinh x$

Substituting $m = 2$, $n = 3$ for (4.1's) ,

$$\left(x^2 sinh x\right)^{\{3\}} = \sum_{r=0}^{2} \binom{3}{r} \frac{\Gamma(3)}{\Gamma(3-r)} x^{2-r} \frac{e^x - (-1)^{-(3-r)} e^{-x}}{2}$$

$$= \binom{3}{0} \frac{\Gamma(3)}{\Gamma(3)} x^2 cosh x + \binom{3}{1} \frac{\Gamma(3)}{\Gamma(2)} x^1 sinh x + \binom{3}{2} \frac{\Gamma(3)}{\Gamma(1)} x^0 cosh x$$

$$= x^2 cosh x + 6\, x\, sinh x + 6\, cosh x$$

18.3 Higher Derivative of log x f (x)

18.3.1 Higher Derivative of $(\log x)^2$

Formula 18.3.1

$$\left(\log^2 x\right)^{(n)} = \frac{(-1)^{n-1}}{x^n}\left\{2\Gamma(n)\log x - \sum_{r=1}^{n-1}\binom{n}{r}\Gamma(n-r)\Gamma(r)\right\} \tag{1.1}$$

Proof

Let $f(x)=g(x)=\log x$. Then

$$(\log x)^{(n-r)} = (-1)^{n-r-1}\frac{\Gamma(n-r)}{x^{n-r}} \ , \quad (\log x)^{(r)} = (-1)^{r-1}\frac{\Gamma(r)}{x^r} \qquad n=1,2,\cdots$$

Substituting these for Theorem 18.1.1 ,

$$\begin{aligned}\left(\log^2 x\right)^{(n)} &= \sum_{r=0}^{n}\binom{n}{r}(\log x)^{(n-r)}(\log x)^{(r)} \\ &= \binom{n}{0}(\log x)^{(n)}(\log x)^{(0)} + \sum_{r=1}^{n-1}\binom{n}{r}(\log x)^{(n-r)}(\log x)^{(r)} \\ &\qquad + \binom{n}{n}(\log x)^{(n-n)}(\log x)^{(n)} \\ &= (-1)^{n-1}\frac{2}{x^n}\Gamma(n)\log x + \frac{(-1)^n}{x^n}\sum_{r=1}^{n-1}\binom{n}{r}\Gamma(n-r)\Gamma(r) \\ &= \frac{(-1)^{n-1}}{x^n}\left\{2\Gamma(n)\log x - \sum_{r=1}^{n-1}\binom{n}{r}\Gamma(n-r)\Gamma(r)\right\}\end{aligned}$$

Example The 3rd order derivative of $(\log x)^2$

$$\begin{aligned}\left(\log^2 x\right)^{(3)} &= \frac{(-1)^{3-1}}{x^3}\left\{2\Gamma(3)\log x - \sum_{r=1}^{2}\binom{3}{r}\Gamma(3-r)\Gamma(r)\right\} \\ &= \frac{1}{x^3}\left\{2\cdot 2\log x - \binom{3}{1}\Gamma(2)\Gamma(1) - \binom{3}{2}\Gamma(1)\Gamma(2)\right\} \\ &= \frac{1}{x^3}(4\log x - 6)\end{aligned}$$

18.3.2 Higher Derivatives of $\log x\cdot\sin x$, $\log x\cdot\cos x$

Formula 18.3.2

$$\begin{aligned}(\log x\cdot\sin x)^{(n)} &= \log x\cdot\sin\left(x+\frac{n\pi}{2}\right) \\ &\qquad + \sum_{r=1}^{n}(-1)^{r-1}\binom{n}{r}\frac{\Gamma(r)}{x^r}\sin\left\{x+\frac{(n-r)\pi}{2}\right\}\end{aligned} \tag{2.0s}$$

$$(log\,x\cdot cos\,x)^{(n)} = log\,x\cdot cos\left(x+\frac{n\pi}{2}\right)$$

$$+\sum_{r=1}^{n}(-1)^{r-1}\binom{n}{r}\frac{\Gamma(r)}{x^r}cos\left\{x+\frac{(n-r)\pi}{2}\right\} \qquad (2.0c)$$

Example **The 3rd order derivative of** $log\,x\cdot sin\,x$

$$(log\,x\cdot sin\,x)^{(3)} = log\,x\cdot sin\left(x+\frac{3\pi}{2}\right)+\sum_{r=1}^{3}(-1)^{r-1}\binom{3}{r}\frac{\Gamma(r)}{x^r}sin\left\{x+\frac{(3-r)\pi}{2}\right\}$$

$$= -log\,x\cdot cos\,x + \binom{3}{1}\frac{\Gamma(1)}{x^1}sin\left(x+\frac{2\pi}{2}\right) - \binom{3}{2}\frac{\Gamma(2)}{x^2}sin\left(x+\frac{1\pi}{2}\right)$$

$$+\binom{3}{3}\frac{\Gamma(3)}{x^3}sin\left(x+\frac{0\pi}{2}\right)$$

$$= -log\,x\cdot cos\,x - \frac{3}{x^1}sin\,x - \frac{3}{x^2}cos\,x + \frac{2}{x^3}sin\,x$$

18.3.3 Higher Derivatives of $log\,x\cdot sinh\,x$, $log\,x\cdot cosh\,x$

Formula 18.3.3

$$(log\,x\cdot sinh\,x)^{(n)} = log\,x\cdot\frac{e^x-(-1)^{-n}e^{-x}}{2}$$

$$+\sum_{r=1}^{n}(-1)^{r-1}\binom{n}{r}\frac{\Gamma(r)}{x^r}\frac{e^x-(-1)^{r-n}e^{-x}}{2} \qquad (3.0s)$$

$$(log\,x\cdot cosh\,x)^{(n)} = log\,x\cdot\frac{e^x+(-1)^{-n}e^{-x}}{2}$$

$$+\sum_{r=1}^{n}(-1)^{r-1}\binom{n}{r}\frac{\Gamma(r)}{x^r}\frac{e^x+(-1)^{r-n}e^{-x}}{2} \qquad (3.0c)$$

Example **The 4th order derivative of** $log\,x\cdot cosh\,x$

$$(log\,x\cdot cosh\,x)^{(4)} = log\,x\cdot\frac{e^x+(-1)^{-4}e^{-x}}{2}+\sum_{r=1}^{4}(-1)^{r-1}\binom{4}{r}\frac{\Gamma(r)}{x^r}\frac{e^x+(-1)^{r-4}e^{-x}}{2}$$

$$= log\,x\cdot cosh\,x + \binom{4}{1}\frac{\Gamma(1)}{x^1}sinh\,x - \binom{4}{2}\frac{\Gamma(2)}{x^2}cosh\,x$$

$$+\binom{4}{3}\frac{\Gamma(3)}{x^3}sinh\,x - \binom{4}{4}\frac{\Gamma(4)}{x^4}cosh\,x$$

$$= log\,x\cdot cos\,x+\frac{4}{x^1}sinh\,x-\frac{6}{x^2}cosh\,x+\frac{8}{x^3}sinh\,x-\frac{6}{x^4}cosh\,x$$

18.4 Higher Derivative of e^x f (x)

18.4.1 Higher Derivative of $e^x x^\alpha$

Formula 18.4.1

$$\left(e^x x^\alpha\right)^{(n)} = e^x \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} x^{\alpha-r} \qquad \text{for } \alpha \neq -1,-2,-3,\cdots \tag{1.1}$$

$$= e^x \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{\Gamma(-\alpha+r)}{\Gamma(-\alpha)} x^{\alpha-r} \qquad \text{for } \alpha = -1,-2,-3,\cdots \tag{1.2}$$

Especially, when $m = 0,1,2,\cdots$

$$\left(e^x x^m\right)^{(n)} = e^x \sum_{r=0}^{m} \binom{n}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \tag{1.1'}$$

Proof

Substite $f(x)=e^x$ for Theorem 18.2.0 . Then since $\left(e^x\right)^{(n-r)} = e^x$, we obtain the desired expression immediately.

Example1 The 2nd order derivative of $e^x\sqrt{x}$

$$\left(e^x\sqrt{x}\right)^{(2)} = e^x \sum_{r=0}^{2} \binom{2}{r} \frac{\Gamma(3/2)}{\Gamma(3/2-r)} x^{\frac{1}{2}-r}$$

$$= e^x \left\{ \binom{2}{0} \frac{\Gamma(3/2)}{\Gamma(3/2)} x^{\frac{1}{2}} + \binom{2}{1} \frac{\Gamma(3/2)}{\Gamma(1/2)} x^{-\frac{1}{2}} + \binom{2}{2} \frac{\Gamma(3/2)}{\Gamma(-1/2)} x^{-\frac{3}{2}} \right\}$$

$$= e^x \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}} - \frac{1}{4} x^{-\frac{3}{2}} \right) = e^x\sqrt{x} \left(1 + \frac{1}{x} - \frac{1}{4x^2} \right)$$

Example2 The 2nd order derivative of e^x/x

$$\left(\frac{e^x}{x}\right)^{(2)} = e^x \sum_{r=0}^{2} (-1)^r \binom{2}{r} \frac{\Gamma(1+r)}{\Gamma(1)} x^{-1-r}$$

$$= e^x \left\{ \binom{2}{0} \frac{\Gamma(1)}{\Gamma(1)} x^{-1} - \binom{2}{1} \frac{\Gamma(2)}{\Gamma(1)} x^{-2} + \binom{2}{2} \frac{\Gamma(3)}{\Gamma(1)} x^{-3} \right\}$$

$$= \frac{e^x}{x} \left(1 - \frac{2}{x} + \frac{2}{x^2} \right)$$

18.4.2 Higher Derivative of $e^x \log x$

Formula 18.4.2

$$\left(e^x \log x\right)^{(n)} = e^x \log x + e^x \sum_{r=1}^{n} (-1)^{r-1} \binom{n}{r} \frac{\Gamma(r)}{x^r} \tag{2.1}$$

Proof

Let $f(x)=e^x$, $g(x)=\log x$. Then

$$(\log x)^{(r)} = (-1)^{r-1}\frac{\Gamma(r)}{x^r} \qquad r = 1,2,3,\cdots$$

Substituting this for Theorem 18.1.1 ,

$$\left(e^x \log x\right)^{(n)} = \sum_{r=0}^{n}\binom{n}{r} e^x (\log x)^{(r)} = \binom{n}{0} e^x (\log x)^{(0)} + \sum_{r=1}^{n} e^x (\log x)^{(r)}$$

$$= e^x \log x + e^x \sum_{r=1}^{n}(-1)^{r-1}\binom{n}{r}\frac{\Gamma(r)}{x^r}$$

Example The 4th order derivative of $e^x \log x$

$$\left(e^x \log x\right)^{(4)} = e^x \log x + e^x \sum_{r=1}^{4}(-1)^{r-1}\binom{4}{r}\frac{\Gamma(r)}{x^r}$$

$$= e^x \log x + e^x\left\{\binom{4}{1}\frac{\Gamma(1)}{x^1} - \binom{4}{2}\frac{\Gamma(2)}{x^2} + \binom{4}{3}\frac{\Gamma(3)}{x^3} - \binom{4}{4}\frac{\Gamma(4)}{x^4}\right\}$$

$$= e^x \log x + e^x\left(\frac{4}{x^1} - \frac{6}{x^2} + \frac{8}{x^3} - \frac{6}{x^4}\right)$$

18.4.3 Higher Derivatives of $e^x \sin x$, $e^x \cos x$

Formula 18.4.3

$$\left(e^x \sin x\right)^{(n)} = \left(\sin\frac{\pi}{4}\right)^{-n} e^x \sin\left(x+\frac{n\pi}{4}\right) \tag{3.0s}$$

$$\left(e^x \cos x\right)^{(n)} = \left(\sin\frac{\pi}{4}\right)^{-n} e^x \cos\left(x+\frac{n\pi}{4}\right) \tag{3.0c}$$

Proof

″共立 数学公式″ p187 was posted as it was.

Example

$$\left(e^x \sin x\right)^{(2)} = \left(\sin\frac{\pi}{4}\right)^{-2} e^x \sin\left(x+\frac{2\pi}{4}\right) = 2e^x \cos x$$

$$\left(e^x \cos x\right)^{(3)} = \left(\sin\frac{\pi}{4}\right)^{-3} e^x \cos\left(x+\frac{3\pi}{4}\right) = -2e^x(\sin x + \cos x)$$

Higher Derivatives of $e^x \sin x$, $e^x \cos x$ end now. There is no necessity for Theorem 18.1.1. However, daring use Theorem 18.1.1, we obtain an interesting result.

Trigonometric Polynomial

Formula 18.4.3'

$$\sum_{r=0}^{n}\binom{n}{r}\sin\left(x+\frac{r\pi}{2}\right) = \left(\sin\frac{\pi}{4}\right)^{-n}\sin\left(x+\frac{n\pi}{4}\right) \tag{3.1s}$$

$$\sum_{r=0}^{n}\binom{n}{r}\cos\left(x+\frac{r\pi}{2}\right) = \left(\sin\frac{\pi}{4}\right)^{-n}\cos\left(x+\frac{n\pi}{4}\right) \tag{3.1c}$$

Especially, when $x=0$

$$\sum_{r=0}^{n}\binom{n}{r}\sin\frac{r\pi}{2} = \left(\sin\frac{\pi}{4}\right)^{-n}\sin\frac{n\pi}{4} \tag{3.1's}$$

$$\sum_{r=0}^{n}\binom{n}{r}\cos\frac{r\pi}{2} = \left(\sin\frac{\pi}{4}\right)^{-n}\cos\frac{n\pi}{4} \tag{3.1'c}$$

Proof

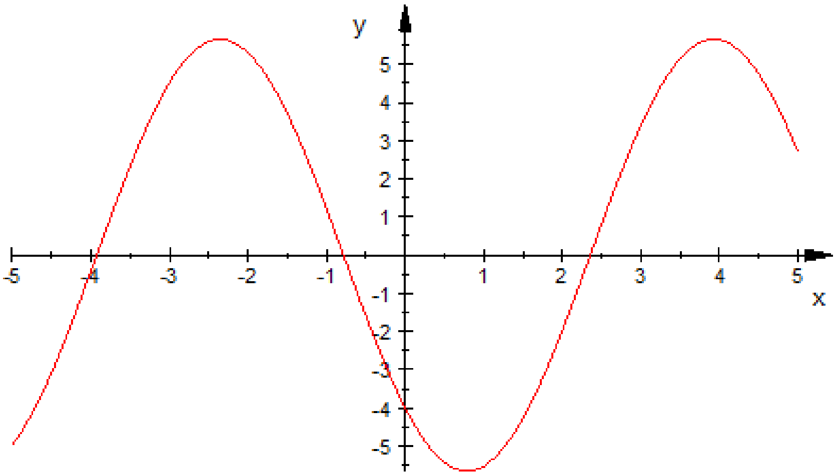
Substituting $f(x)=e^x$, $g(x)=\sin x, \cos x$ for Theorem 18.1.1 ,

$$\left(e^x \sin x\right)^{(n)} = e^x\sum_{r=0}^{n}\binom{n}{r}\sin\left(x+\frac{r\pi}{2}\right)$$

$$\left(e^x \cos x\right)^{(n)} = e^x\sum_{r=0}^{n}\binom{n}{r}\cos\left(x+\frac{r\pi}{2}\right)$$

And comparing these with Formula 18.4.3 , we obtain the desired expressions.

When $n=5$, if both sides of (3.1s) are illustrated, it is as follows. Both overlap exactly and blue (left) can not be seen.



Alternative Binomial Polynomial

Removing $\sin\frac{r\pi}{2}$, $\cos\frac{r}{2}$ from (3.1's), (3.1'c), we obtain the following interesting polynomial.

Formula 18.4.3"

When $\downarrow$ denotes the floor function, the following expressions hold.

$$\sum_{r=0}^{(n-1)/2\downarrow}(-1)^r\binom{n}{2r+1} = 2^{\frac{n}{2}}\sin\frac{n\pi}{4} \tag{3.2s}$$

$$\sum_{r=0}^{n/2\downarrow}(-1)^r\binom{n}{2r} = 2^{\frac{n}{2}}\cos\frac{n\pi}{4} \tag{3.2c}$$

Proof

Since the odd-numbered terms of the left side in (3.1's) are all 0,

$$\sum_{r=0}^{n}\binom{n}{r}\sin\frac{r\pi}{2} = \binom{n}{0}\sin\frac{0\pi}{2}+\binom{n}{1}\sin\frac{1\pi}{2}+\binom{n}{2}\sin\frac{2\pi}{2}+\cdots+\binom{n}{n}\sin\frac{n\pi}{2}$$

$$= \binom{n}{1}\sin\frac{1\pi}{2}+\binom{n}{3}\sin\frac{3\pi}{2}+\binom{n}{5}\sin\frac{5\pi}{2}+\cdots\pm\binom{n}{\frac{n-1}{2}\downarrow}\sin\frac{\frac{n-1}{2}\downarrow\pi}{2}$$

$$= \binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots \pm \binom{n}{\frac{n-1}{2}\downarrow} = \sum_{r=0}^{(n-1)/2\downarrow}(-1)^r\binom{n}{2r+1}$$

Also, since $\{\sin(\pi/4)\}^{-n} = 2^{n/2}$ in the right side in (3.1's) ,

$$\therefore \qquad \sum_{r=0}^{(n-1)/2\downarrow}(-1)^r\binom{n}{2r+1} = 2^{\frac{n}{2}}\sin\frac{n\pi}{4} \tag{3.2s}$$

Next, since the even-numbered terms of the left side in (3.1'c) are all 0,

$$\sum_{r=0}^{n}\binom{n}{r}\cos\frac{r\pi}{2} = \binom{n}{0}\cos\frac{0\pi}{2}+\binom{n}{1}\cos\frac{1\pi}{2}+\binom{n}{2}\cos\frac{2\pi}{2}+\cdots+\binom{n}{n}\cos\frac{n\pi}{2}$$

$$= \binom{n}{0}\cos\frac{0\pi}{2}+\binom{n}{2}\cos\frac{2\pi}{2}+\binom{n}{4}\cos\frac{4\pi}{2}+\cdots\pm\binom{n}{n/2\downarrow}\cos\frac{n/2\downarrow\pi}{2}$$

$$= \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots \pm \binom{n}{n/2\downarrow} = \sum_{r=0}^{n/2\downarrow}(-1)^r\binom{n}{2r}$$

Also, since $\{\sin(\pi/4)\}^{-n} = 2^{n/2}$ in the right side in (3.1'c) ,

$$\therefore \qquad \sum_{r=0}^{n/2\downarrow}(-1)^r\binom{n}{2r} = 2^{\frac{n}{2}}\cos\frac{n\pi}{4} \tag{3.2c}$$

In addition, this formula is known. (See ″岩波　数学公式Ⅱ″ p11)

Note

When $n = 4k-3$, $k=1,2,3,\cdots$

$$\sum_{r=0}^{(n-1)/2\downarrow}(-1)^r\binom{n}{2r+1} = \sum_{r=0}^{n/2\downarrow}(-1)^r\binom{n}{2r}$$

Example

$$\binom{5}{1} - \binom{5}{3} + \binom{5}{5} = 5 - 10 + 1 = 2^{\frac{5}{2}}\sin\frac{5\pi}{4} = -4$$

$$\binom{5}{0} - \binom{5}{2} + \binom{5}{4} = 1 - 10 + 5 = 2^{\frac{5}{2}}\cos\frac{5\pi}{4} = -4$$

18.4.4 Higher Derivatives of $e^x \sinh x$, $e^x \cosh x$

Formula 18.4.4

$$\left(e^x \sinh x\right)^{(n)} = e^x\sum_{r=0}^{n}\binom{n}{r}\frac{e^x-(-1)^{-r}e^{-x}}{2} \tag{4.0s}$$

$$\left(e^x \cosh x\right)^{(n)} = e^x \sum_{r=0}^{n} \binom{n}{r} \frac{e^x + (-1)^{-r} e^{-x}}{2} \tag{4.0c}$$

Example

$$\left(e^x \sinh x\right)^{(0)} = e^x \sum_{r=0}^{0} \binom{0}{r} \frac{e^x - (-1)^{-r} e^{-x}}{2} = e^x \sinh x$$

$$\begin{aligned}\left(e^x \cosh x\right)^{(3)} &= e^x \sum_{r=0}^{3} \binom{3}{r} \frac{e^x + (-1)^{-r} e^{-x}}{2} \\ &= e^x \left\{ \binom{3}{0} \sinh x + \binom{3}{1} \cosh x + \binom{3}{2} \sinh x + \binom{3}{3} \cosh x \right\} \\ &= 4e^x (\sinh x + \cosh x)\end{aligned}$$

Note

The following formula is known for a natural number n .

$$\left(e^x \sinh x\right)^{(n)} = \left(e^x \cosh x\right)^{(n)} = 2^{n-1} e^x (\sinh x + \cosh x)$$

However, this formula does not hold for $n=0$. That is, in this formula, the natural number n is inextensible to the real number p . So, this is insufficient as a general formula.

18.5 Higher Derivative of f (x) / e^x

18.5.1 Higher Derivative of $e^{-x}x^{\alpha}$

Formula 18.5.1

$$\left(e^{-x}x^{\alpha}\right)^{(n)} = e^{-x}\sum_{r=0}^{n}(-1)^{-(n-r)}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r} \quad for\ \alpha \ne -1,-2,-3,\cdots \tag{1.1}$$

$$= (-1)^{-n}e^{-x}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(-\alpha+r)}{\Gamma(-\alpha)}x^{\alpha-r} \quad for\ \alpha = -1,-2,-3,\cdots \tag{1.2}$$

Especially, when $m = 0,1,2,\cdots$

$$\left(e^{-x}x^{m}\right)^{(n)} = e^{-x}\sum_{r=0}^{m}(-1)^{-(n-r)}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r} \tag{1.1'}$$

Proof

Substite $f(x)=e^{-x}$ for Theorem 18.2.0 . Then since $\left(e^{-x}\right)^{(n-r)} = (-1)^{-(n-r)}e^{-x}$, we obtain the desired expression immediately.

Example1 The 3rd order derivative of e^{-x}/x

$$\left(\frac{e^{-x}}{x}\right)^{(3)} = (-1)^{-3}e^{-x}\sum_{r=0}^{3}\binom{3}{r}\frac{\Gamma(1+r)}{\Gamma(1)}x^{-1-r}$$

$$= -e^{-x}\left\{\binom{3}{0}\frac{\Gamma(1)}{\Gamma(1)}x^{-1}+\binom{3}{1}\frac{\Gamma(2)}{\Gamma(1)}x^{-2}+\binom{3}{2}\frac{\Gamma(3)}{\Gamma(1)}x^{-3}+\binom{3}{3}\frac{\Gamma(4)}{\Gamma(1)}x^{-4}\right\}$$

$$= -\frac{e^{-x}}{x}\left(1+\frac{3}{x}+\frac{6}{x^2}+\frac{6}{x^3}\right)$$

Example1' The 3rd order derivative of $e^{-x}x^7$

$$\left(e^{-x}x^{7}\right)^{(3)} = e^{-x}\sum_{r=0}^{7}(-1)^{-(3-r)}\binom{3}{r}\frac{\Gamma(8)}{\Gamma(8-r)}x^{7-r}$$

$$= e^{-x}\left\{-\binom{3}{0}\frac{\Gamma(8)}{\Gamma(8)}x^{7}+\binom{3}{1}\frac{\Gamma(8)}{\Gamma(7)}x^{6}-\binom{3}{2}\frac{\Gamma(8)}{\Gamma(6)}x^{5}+\binom{3}{3}\frac{\Gamma(8)}{\Gamma(5)}x^{4}\right\}$$

$$= e^{-x}x^{4}\left(-x^{3}+21x^{2}-126x^{1}+210\right)$$

18.5.2 Higher Derivative of $e^{-x}\log x$

Formula 18.5.2

$$\left(e^{-x}\log x\right)^{(n)} = \frac{(-1)^{-n}}{e^{x}}\left\{\log x - \sum_{r=1}^{n}\binom{n}{r}\frac{\Gamma(r)}{x^{r}}\right\} \tag{2.1}$$

Proof

Let $f(x)=e^{-x}$, $g(x)=\log x$. Then

$$\left(e^{-x}\right)^{(n-r)} = (-1)^{-n+r} e^{-x}$$

$$(\log x)^{(r)} = (-1)^{r-1} \frac{\Gamma(r)}{x^r} \qquad r = 1, 2, 3, \cdots$$

Substituting these for Theorem 18.1.1 ,

$$\begin{aligned}\left(e^{-x} \log x\right)^{(n)} &= \sum_{r=0}^{n} \binom{n}{r} (-1)^{-n+r} e^{-x} (\log x)^{(r)} \\ &= \binom{n}{0} (-1)^{-n} e^{-x} (\log x)^{(0)} + \sum_{r=1}^{n} \binom{n}{r} (-1)^{-n+r} e^{-x} (\log x)^{(r)} \\ &= \frac{(-1)^{-n}}{e^x} \left\{ \log x - \sum_{r=1}^{n} \binom{n}{r} \frac{\Gamma(r)}{x^r} \right\}\end{aligned}$$

Example The 4th order derivative of $e^{-x} \log x$

$$\begin{aligned}\left(e^{x} \log x\right)^{(4)} &= \frac{(-1)^{-4}}{e^x} \left\{ \log x - \sum_{r=1}^{4} \binom{4}{r} \frac{\Gamma(r)}{x^r} \right\} \\ &= \frac{\log x}{e^x} - \frac{1}{e^x} \left\{ \binom{4}{1} \frac{\Gamma(1)}{x^1} + \binom{4}{2} \frac{\Gamma(2)}{x^2} + \binom{4}{3} \frac{\Gamma(3)}{x^3} + \binom{4}{r} \frac{\Gamma(4)}{x^4} \right\} \\ &= \frac{\log x}{e^x} - \frac{1}{e^x} \left(\frac{4}{x^1} + \frac{6}{x^2} + \frac{8}{x^3} + \frac{6}{x^4} \right)\end{aligned}$$

18.5.3 Higher Derivatives of $e^{-x} \sin x$, $e^{-x} \cos x$

Formula 18.5.3

$$\left(e^{-x} \sin x\right)^{(n)} = \left(-\sin\frac{\pi}{4}\right)^{-n} e^{-x} \sin\left(x - \frac{n\pi}{4}\right) \tag{3.0s}$$

$$\left(e^{-x} \cos x\right)^{(n)} = \left(-\sin\frac{\pi}{4}\right)^{-n} e^{-x} \cos\left(x - \frac{n\pi}{4}\right) \tag{3.0c}$$

Proof

Replacing x with $-x$ in Formula 18.4.3 , we obtain the desired expressions.

Example

$$\left(e^{-x} \sin x\right)^{(2)} = \left(-\sin\frac{\pi}{4}\right)^{-2} e^{-x} \sin\left(x - \frac{2\pi}{4}\right) = -2e^{-x} \cos x$$

$$\left(e^{-x} \cos x\right)^{(3)} = \left(-\sin\frac{\pi}{4}\right)^{-3} e^{-x} \cos\left(x - \frac{3\pi}{4}\right) = -2e^{-x} (\sin x - \cos x)$$

Higher Derivatives of $e^{-x} \sinh x, e^{-x} \cosh x$ end now. There is no necessity for Theorem 18.1.1. Daring use Theorem 18.1.1, we obtain the following expression first.

$$\left(e^{-x} \sin x\right)^{(n)} = e^{-x} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \sin\left(x + \frac{r\pi}{2}\right)$$

And from this and (3.0s) , we obtain

$$\sum_{r=0}^{n}(-1)^{-r}\binom{n}{r}\sin\left(x+\frac{r\pi}{2}\right) = \left(\sin\frac{\pi}{4}\right)^{-n}\sin\left(x-\frac{n\pi}{4}\right)$$

A similar expression is obtained about $e^{-x}\cos x$ too. Then removing $\sin\frac{r\pi}{2}, \cos\frac{r}{2}$ from these, we obtain the completely same results as Formula 18.4.3" .

18.5.4 Higher Derivatives of $e^{-x}\sinh x$, $e^{-x}\cosh x$

Formula 18.5.4

$$\left(e^{-x}\sinh x\right)^{(n)} = e^{-x}\sum_{r=0}^{n}(-1)^{-n+r}\binom{n}{r}\frac{e^{x}-(-1)^{-r}e^{-x}}{2} \tag{4.0s}$$

$$\left(e^{-x}\cosh x\right)^{(n)} = e^{-x}\sum_{r=0}^{n}(-1)^{-n+r}\binom{n}{r}\frac{e^{x}+(-1)^{-r}e^{-x}}{2} \tag{4.0c}$$

Proof

Substituting $f(x)=e^{-x}$, $g(x)=\sinh x$, $\cosh x$ for Theorem 18.1.1 , we obtain the dsired expressions.

Example

$$\left(e^{-x}\sinh x\right)^{(0)} = e^{-x}\sum_{r=0}^{0}(-1)^{-0+r}\binom{0}{r}\frac{e^{x}-(-1)^{-r}e^{-x}}{2} = e^{-x}\sinh x$$

$$\begin{aligned}\left(e^{-x}\cosh x\right)^{(3)} &= e^{-x}\sum_{r=0}^{3}(-1)^{-3+r}\binom{3}{r}\frac{e^{x}+(-1)^{-r}e^{-x}}{2}\\ &= e^{-x}\left\{-\binom{3}{0}\cosh x+\binom{3}{1}\sinh x-\binom{3}{2}\cosh x+\binom{3}{3}\sinh x\right\}\\ &= -4e^{-x}(\cosh x-\sinh x)\end{aligned}$$

Note

The following formula is known for a natural number n .

$$\left(e^{-x}\sinh x\right)^{(n)} = \left(e^{-x}\cosh x\right)^{(n)} = (-2)^{n-1}e^{x}(\cosh x-\sinh x)$$

However, this formula does not hold for $n=0$. That is, in this formula, the natural number n is inextensible to the real number p . So, this is insufficient as a general formula.

18.6 Higher Derivatives of sin x f (x), cos x f (x)

18.6.1 Higher Derivatives of sin^2x, cos^2x

Formula 18.6.1

$$\left(sin^2x\right)^{(n)} = -2^{n-1}cos\left(2x+\frac{n\pi}{2}\right) \tag{1.0s}$$

$$\left(cos^2x\right)^{(n)} = 2^{n-1}cos\left(2x+\frac{n\pi}{2}\right) \tag{1.0c}$$

Proof

From Formula 18.6.1' mentioned next ,

$$\left(cos^2x\right)^{(n)} = \sum_{r=0}^{n}\binom{n}{r}cos\left\{x+\frac{(n-r)\pi}{2}\right\}cos\left(x+\frac{r\pi}{2}\right)$$

Here

$$cosA\,cosB = \frac{1}{2^1}\{cos(A+B)+cos(A-B)\}$$

Using this,

$$\left(cos^2x\right)^{(n)} = \frac{1}{2^1}\sum_{r=0}^{n}\binom{n}{r}\left\{cos\left(2x+\frac{n\pi}{2}\right)+cos\left(\frac{n\pi}{2}-r\pi\right)\right\}$$

$$= \frac{1}{2^1}cos\left(2x+\frac{n\pi}{2}\right)\sum_{r=0}^{n}\binom{n}{r} + \frac{1}{2^1}cos\frac{n\pi}{2}\sum_{r=0}^{n}(-1)^r\binom{n}{r}$$

And since

$$\sum_{r=0}^{n}\binom{n}{r} = 2^n \ , \quad \sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0$$

substituting these for the above, we obtain (1.0c). (1.0s) is also obtained in a similar way.

Example

$$\left(sin^2x\right)^{(2)} = -2^{2-1}cos\left(2x+\frac{2\pi}{2}\right) = 2\,cos2x = 2\left(cos^2x-sin^2x\right)$$

$$\left(cos^2x\right)^{(3)} = 2^{3-1}cos\left(2x+\frac{3\pi}{2}\right) = 4\,sin2x = 8\,sinx\,cosx$$

Formula 18.6.1'

$$\left(sin^2x\right)^{(n)} = \sum_{r=0}^{n}\binom{n}{r}sin\left\{x+\frac{(n-r)\pi}{2}\right\}sin\left(x+\frac{r\pi}{2}\right) \tag{1.1s}$$

$$\left(cos^2x\right)^{(n)} = \sum_{r=0}^{n}\binom{n}{r}cos\left\{x+\frac{(n-r)\pi}{2}\right\}cos\left(x+\frac{r\pi}{2}\right) \tag{1.1c}$$

Proof

Substituting $f(x)=g(x)=sin\,x$ for Theorem 18.1.1 , we obtain (1.1s). (1.1c) is also obtained in a similar way.

Formula 18.6.1"

When $\downarrow$ denotes the floor function, the following expressions hold.

$$\sum_{r=0}^{n/2\downarrow}\binom{n}{2r} = 2^{n-1} \tag{1.2e}$$

$$\sum_{r=0}^{(n-1)/2\downarrow}\binom{n}{2r+1} = 2^{n-1} \tag{1.2o}$$

Proof

(1.1s) is transformed as follows.

$$\begin{aligned}(\sin^2 x)^{(n)} &= \sum_{r=0}^{n}\binom{n}{r}\sin\left\{x+\frac{(n-r)\pi}{2}\right\}\sin\left(x+\frac{r\pi}{2}\right)\\ &= \sum_{r=0}^{n/2\downarrow}\binom{n}{2r}\sin\left\{x+\frac{(n-2r)\pi}{2}\right\}\sin\left(x+\frac{2r\pi}{2}\right)\\ &\quad+\sum_{r=0}^{(n-1)/2\downarrow}\binom{n}{2r+1}\sin\left\{x+\frac{(n-2r-1)\pi}{2}\right\}\sin\left\{x+\frac{(2r+1)\pi}{2}\right\}\\ &= \sum_{r=0}^{n/2\downarrow}(-1)^r\binom{n}{2r}\sin\left\{x+\frac{(n-2r)\pi}{2}\right\}\sin x\\ &\quad+\sum_{r=0}^{(n-1)/2\downarrow}(-1)^r\binom{n}{2r+1}\sin\left\{x+\frac{(n-2r-1)\pi}{2}\right\}\cos x\end{aligned}$$

i.e.

$$(\sin^2 x)^{(n)} = \sin\left(x+\frac{n\pi}{2}\right)\sin x\sum_{r=0}^{n/2\downarrow}\binom{n}{2r} - \cos\left(x+\frac{n\pi}{2}\right)\cos x\sum_{r=0}^{(n-1)/2\downarrow}\binom{n}{2r+1}$$

On the other hand, (1.0s) is transformed as follows too.

$$(\sin^2 x)^{(n)} = 2^{n-1}\sin\left(x+\frac{n\pi}{2}\right)\sin x - 2^{n-1}\cos\left(x+\frac{n\pi}{2}\right)\cos x$$

From these, the following expression follows .

$$\sin\left(x+\frac{n\pi}{2}\right)\sin x\left\{\sum_{r=0}^{n/2\downarrow}\binom{n}{2r} - 2^{n-1}\right\} = \cos\left(x+\frac{n\pi}{2}\right)\cos x\left\{\sum_{r=0}^{(n-1)/2\downarrow}\binom{n}{2r+1} - 2^{n-1}\right\}$$

In order to hold this equation for arbitrary x , the followings are necessary.

$$\sum_{r=0}^{n/2\downarrow}\binom{n}{2r} - 2^{n-1} = 0 \quad , \quad \sum_{r=0}^{(n-1)/2\downarrow}\binom{n}{2r+1} - 2^{n-1} = 0$$

In addition, this formula is known. (See ″岩波 数学公式Ⅱ″ p11)

18.6.2 Higher Derivatives of $\sin^3 x, \cos^3 x$

Formula 18.6.2

$$(\sin^3 x)^{(n)} = \frac{3}{4}\sin\left(x+\frac{n\pi}{2}\right) - \frac{3^n}{4}\sin\left(3x+\frac{n\pi}{2}\right) \tag{2.0s}$$

$$(\cos^3 x)^{(n)} = \frac{3}{4}\cos\left(x+\frac{n\pi}{2}\right) + \frac{3^n}{4}\cos\left(3x+\frac{n\pi}{2}\right) \tag{2.0c}$$

Proof

From Formula 18.6.2' mentioned next, it is obtained in a similar way in the case of the 2nd degree. However, it is not so easy as the case of the 2nd degree. (See **20.1.3**)

Example

$$\left(\sin^3 x\right)^{(2)} = \frac{3}{4}\sin\left(x+\frac{2\pi}{2}\right) - \frac{3^2}{4}\sin\left(3x+\frac{2\pi}{2}\right) = -\frac{3}{4}\sin x + \frac{9}{4}\sin 3x$$

$$\left(\cos^3 x\right)^{(3)} = \frac{3}{4}\cos\left(x+\frac{3\pi}{2}\right) + \frac{3^3}{4}\cos\left(3x+\frac{3\pi}{2}\right) = \frac{3}{4}\sin x + \frac{27}{4}\sin 3x$$

Formula 18.6.2'

$$\left(\sin^3 x\right)^{(n)} = \left(\sin^2 x\right)^{(0)}\sin\left(x+\frac{n\pi}{2}\right) - \sum_{r=1}^{n}\binom{n}{r}2^{r-1}\cos\left(2x+\frac{r\pi}{2}\right)\sin\left\{x+\frac{(n-r)\pi}{2}\right\} \qquad (2.1s)$$

$$\left(\cos^3 x\right)^{(n)} = \left(\cos^2 x\right)^{(0)}\cos\left(x+\frac{n\pi}{2}\right) + \sum_{r=1}^{n}\binom{n}{r}2^{r-1}\cos\left(2x+\frac{r\pi}{2}\right)\cos\left\{x+\frac{(n-r)\pi}{2}\right\} \qquad (2.1c)$$

Proof

Substituting $f(x)=\sin^2 x$, $g(x)=\sin x$ for Theorem 18.1.1, we obtain (2.1s). (2.1c) is also obtained in a similar way.

Formula 18.6.2''

$$\sum_{r=0}^{n/2\downarrow}2^{2r-1}\binom{n}{2r} = \frac{3^n+(-1)^n}{4} \qquad (2.2e)$$

$$\sum_{r=0}^{(n-1)/2\downarrow}2^{2r}\binom{n}{2r+1} = \frac{3^n-(-1)^n}{4} \qquad (2.2o)$$

Proof

From (2.1s)

$$\left(\sin^3 x\right)^{(n)} = \left(\sin^2 x\right)^{(0)}\sin\left(x+\frac{n\pi}{2}\right) - \sum_{r=1}^{n}\binom{n}{r}2^{r-1}\cos\left(2x+\frac{r\pi}{2}\right)\sin\left\{x+\frac{(n-r)\pi}{2}\right\}$$

$$= \left(\sin^2 x\right)^{(0)}\sin\left(x+\frac{n\pi}{2}\right)$$

$$- \sum_{r=0}^{(n-1)/2\downarrow}\binom{n}{2r+1}2^{2r}\cos\left\{2x+\frac{(2r+1)\pi}{2}\right\}\sin\left\{x+\frac{(n-2r-1)\pi}{2}\right\}$$

$$- \sum_{r=1}^{n/2\downarrow}\binom{n}{2r}2^{2r-1}\cos\left(2x+\frac{2r\pi}{2}\right)\sin\left\{x+\frac{(n-2r)\pi}{2}\right\}$$

$$= \left(\frac{1}{2} - 2^{-1}\cos 2x \right) \sin\left(x + \frac{n\pi}{2} \right)$$

$$+ \sum_{r=0}^{(n-1)/2\downarrow} (-1)^r \binom{n}{2r+1} 2^{2r} \sin 2x \sin\left\{ x + \frac{(n-2r-1)\pi}{2} \right\}$$

$$- \sum_{r=1}^{n/2\downarrow} (-1)^r \binom{n}{2r} 2^{2r-1} \cos 2x \sin\left\{ x + \frac{(n-2r)\pi}{2} \right\}$$

$$= \frac{1}{2}\sin\left(x + \frac{n\pi}{2} \right) - \sum_{r=0}^{(n-1)/2\downarrow} \binom{n}{2r+1} 2^{2r} \sin 2x \cos\left(x + \frac{n\pi}{2} \right)$$

$$- \binom{n}{0} 2^{-1}\cos 2x \sin\left(x + \frac{n\pi}{2} \right) - \sum_{r=1}^{n/2\downarrow} \binom{n}{2r} 2^{2r-1} \cos 2x \sin\left(x + \frac{n\pi}{2} \right)$$

$$= \frac{1}{2}\sin\left(x + \frac{n\pi}{2} \right) - \sum_{r=0}^{(n-1)/2\downarrow} \binom{n}{2r+1} 2^{2r} \sin 2x \cos\left(x + \frac{n\pi}{2} \right)$$

$$- \sum_{r=0}^{n/2\downarrow} \binom{n}{2r} 2^{2r-1} \cos 2x \sin\left(x + \frac{n\pi}{2} \right)$$

When $n=1$,

$$\left(\sin^3 x \right)^{(1)}$$

$$= \frac{1}{2}\sin\left(x + \frac{1\pi}{2} \right) - \cos 2x \sin\left(x + \frac{1\pi}{2} \right) 2^{-1}\binom{1}{0} - \sin 2x \cos\left(x + \frac{1\pi}{2} \right) 2^{0}\binom{1}{1}$$

$$= \frac{1}{2}\sin\left(x + \frac{1\pi}{2} \right) - \frac{1}{2}\cos 2x \cos x + 1 \sin 2x \sin x$$

$$= \frac{1}{2}\sin\left(x + \frac{1\pi}{2} \right) + \frac{1}{2}\sin 2x \sin x - \frac{1}{2}\left(\cos 2x \cos x - \sin 2x \sin x \right)$$

$$= \frac{1}{2}\sin\left(x + \frac{1\pi}{2} \right) + \frac{1}{2}\sin 2x \sin x - \frac{1}{2}\cos 3x$$

$$= \frac{1}{2}\sin\left(x + \frac{1\pi}{2} \right) - \frac{1}{2}\,\frac{1}{2}\left(\cos 3x - \cos x \right) - \frac{1}{2}\cos 3x$$

$$= \frac{3}{4}\sin\left(x + \frac{1\pi}{2} \right) - \frac{3^1}{4}\sin\left(3x + \frac{1\pi}{2} \right)$$

$$\therefore \sum_{r=0}^{1/2\downarrow} 2^{2r-1}\binom{1}{2r} = \frac{3^1-1}{4} = \frac{1}{2} = \frac{3^1+(-1)^1}{4}$$

$$\therefore \sum_{r=0}^{(1-1)/2\downarrow} 2^{2r}\binom{1}{2r+1} = \frac{3^1-1}{4} + \frac{1}{2} = \frac{3^1+1}{4} = 1 = \frac{3^1-(-1)^1}{4}$$

When $n=2$,

$$\left(\sin^3 x \right)^{(2)} = \frac{1}{2}\sin\left(x + \frac{2\pi}{2} \right) - \cos 2x \sin\left(x + \frac{2\pi}{2} \right) \left\{ 2^{-1}\binom{2}{0} + 2^{1}\binom{2}{2} \right\}$$

$$- \sin 2x \cos\left(x + \frac{2\pi}{2} \right) 2^{0}\binom{2}{1}$$

$$= \frac{1}{2}\sin\left(x+\frac{2\pi}{2}\right) + \frac{5}{2}\cos 2x \sin x + 2\sin 2x\cos x$$

$$= \frac{1}{2}\sin\left(x+\frac{2\pi}{2}\right) + \frac{1}{2}\frac{1}{2}(\sin 3x - \sin x) + 2\sin 3x$$

$$= \frac{3}{4}\sin\left(x+\frac{2\pi}{2}\right) + \frac{9}{4}\sin 3x$$

$$= \frac{3}{4}\sin\left(x+\frac{2\pi}{2}\right) - \frac{3^2}{4}\sin\left(3x+\frac{2\pi}{2}\right)$$

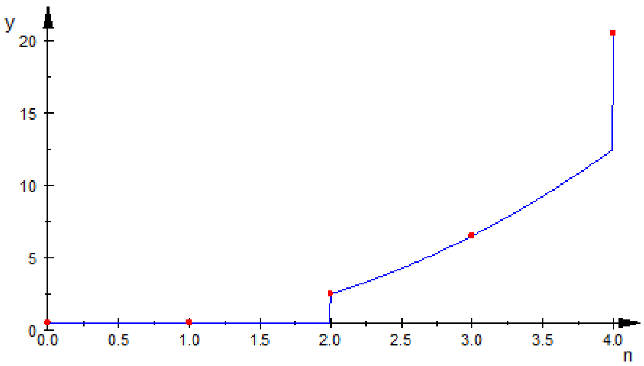
$$\therefore \sum_{r=0}^{2/2\downarrow} 2^{2r-1}\binom{2}{2r} = \frac{3^2-1}{4}+\frac{1}{2} = \frac{5}{2} = \frac{3^2+1}{4} = \frac{3^2+(-1)^2}{4}$$

$$\therefore \sum_{r=0}^{(2-1)/2\downarrow} 2^{2r}\binom{2}{2r+1} = \frac{3^2-1}{4} = 2 = \frac{3^2-(-1)^2}{4}$$

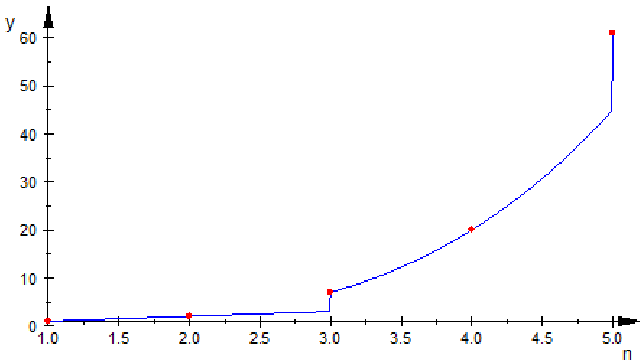
Hereafter, by induction, we obtain the desired expressions.

If both sides of Formula18.6.2" are illustrated, it is as follows. The left side is blue line and the right side is red point.

- $Se := n \rightarrow \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} 2^{2\cdot r-1}\cdot\binom{n}{2\cdot r}$ • $Ser := n \rightarrow \frac{3^n+(-1)^n}{4}$



- $So := n \rightarrow \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{2\cdot r}\cdot\binom{n}{2\cdot r+1}$ • $Sor := n \rightarrow \frac{3^n-(-1)^n}{4}$



18.6.3 Higher Derivatives of the product of trigonometric and hyperbolic functions

Formula 18.6.3

$$(\sin x\cdot\sinh x)^{(n)}=\sum_{r=0}^{n}\binom{n}{r}\sin\left\{x+\frac{(n-r)\pi}{2}\right\}\frac{e^{x}-(-1)^{-r}e^{-x}}{2} \tag{3.1}$$

$$(\sin x\cdot\cosh x)^{(n)}=\sum_{r=0}^{n}\binom{n}{r}\sin\left\{x+\frac{(n-r)\pi}{2}\right\}\frac{e^{x}+(-1)^{-r}e^{-x}}{2} \tag{3.2}$$

$$(\cos x\cdot\sinh x)^{(n)}=\sum_{r=0}^{n}\binom{n}{r}\cos\left\{x+\frac{(n-r)\pi}{2}\right\}\frac{e^{x}-(-1)^{-r}e^{-x}}{2} \tag{3.3}$$

$$(\cos x\cdot\cosh x)^{(n)}=\sum_{r=0}^{n}\binom{n}{r}\cos\left\{x+\frac{(n-r)\pi}{2}\right\}\frac{e^{x}+(-1)^{-r}e^{-x}}{2} \tag{3.4}$$

Proof

Substituting $f(x)=\sin x$, $g(x)=\sinh x$ for Theorem 18.1.1, we obtain (3.1).
The others are also obtained in a similar way.

Example

$$(\sin x\cdot\cosh x)^{(0)}=\sum_{r=0}^{0}\binom{0}{r}\sin\left\{x+\frac{(0-r)\pi}{2}\right\}\frac{e^{x}+(-1)^{-0}e^{-x}}{2}=\sin x\cdot\cosh x$$

$$(\cos x\cdot\cosh x)^{(2)}=\sum_{r=0}^{2}\binom{2}{r}\cos\left\{x+\frac{(2-r)\pi}{2}\right\}\frac{e^{x}+(-1)^{-r}e^{-x}}{2}$$

$$=\cos\left(x+\frac{2\pi}{2}\right)\cosh x+2\cos\left(x+\frac{1\pi}{2}\right)\sinh x+\cos\left(x+\frac{0\pi}{2}\right)\cosh x$$

$$=-\cos x\cdot\cosh x-2\sin x\cdot\sinh x+\cos x\cdot\cosh x=-2\sin x\cdot\sinh x$$

18.7 Higher Derivatives of sinh x f (x), cosh x f (x)

18.7.1 Higher Derivatives of $sinh^2 x$, $cosh^2 x$

Formula 18.7.1

$$\left(sinh^2 x\right)^{(n)} = \sum_{r=0}^{n} \binom{n}{r} \frac{e^x - (-1)^{-n+r} e^{-x}}{2} \frac{e^x - (-1)^{-r} e^{-x}}{2} \tag{1.1s}$$

$$\left(cosh^2 x\right)^{(n)} = \sum_{r=0}^{n} \binom{n}{r} \frac{e^x + (-1)^{-n+r} e^{-x}}{2} \frac{e^x + (-1)^{-r} e^{-x}}{2} \tag{1.1c}$$

Proof

Substituting $f(x) = g(x) = sinh\,x$ for Theorem 18.1.1, we obtain (1.1s). (1.1c) is also obtained in a similar way.

Example

$$\left(sinh^2 x\right)^{(0)} = \sum_{r=0}^{0} \binom{0}{r} \frac{e^x - (-1)^{-0+r} e^{-x}}{2} \frac{e^x - (-1)^{-r} e^{-x}}{2} = sinh^2 x$$

$$\left(cosh^2 x\right)^{(3)} = \sum_{r=0}^{3} \binom{3}{r} \frac{e^x + (-1)^{-3+r} e^{-x}}{2} \frac{e^x + (-1)^{-r} e^{-x}}{2}$$

$$= \binom{3}{0} sinhx \cdot coshx + \binom{3}{1} coshx \cdot sinhx + \binom{3}{2} sinhx \cdot coshx + \binom{3}{3} coshx \cdot sinhx$$

$$= 8 sinhx \cdot coshx = 4 sinh(2x)$$

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Alien's Mathematics