4 Higher Integral

4.1 Higher Primitive Function and Higher Integral

4.1.1 Higher Primitive Function

Definition 4.1.1

When $f^{<n>}(x)$ denotes the primitive function of $f^{<n-1>}(x)$ for $n=1,2,3,\cdots$,
we call $f^{<n>}(x)$ ***Higher Order Primitive Function of f(x)*** ,
or for short , ***Higher Primitive of f(x)*** .

Where, $f^{<n>}(x)$ might mean indefinite integral $\int f^{<n-1>}(x)\,dx + c_n$,
or might mean integral function $\int_{a_n}^{x} f^{<n-1>}(x)\,dx$.

The primitive function of $f(x)$ is denoted by a capital letter $F(x)$ in many cases. Then with what symbols should the primitive function of $F(x)$ and its further primitive functions be denoted ?
For example, $G(x), H(x), \cdots$ etc. can be considered. However, by these notation, it is difficult to denote the primitive function such as the 27 th order. So, applying the notation of the differentiation, I had denoted $F(x), G(x), H(x), \cdots$ by $f^{(-1)}(x), f^{(-2)}(x), f^{(-3)}(x), \cdots$. Soon, the minus sign became obstructive. Then I thought to use $[n]$ instead of $(-n)$, but this notation was as confusing as floor function. So I came to adopt the notation $f^{<1>}(x), f^{<2>}(x), f^{<3>}(x), \cdots$ using <> . The power of this notation will become clear gradually .

Example

$(\sin x)^{<1>} = -\cos x + c_1$ c_1 is an arbitrary constant.

$(\sin x)^{<2>} = -\sin x + c_1 x + c_2$ c_1, c_2 are arbitrary constants.

$(\sin x)^{<3>} = \cos x + c_1 x^2 + c_2 x + c_3$ c_1, c_2, c_3 are arbitrary constants.

4.1.2 Higher Integral

Definition 4.1.2

We call it ***Higher Integral*** to integrate a function f with respect to an independent variable x repeatedly.
And it is described as follows.

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} f(x)\,dx^n \quad \left\{ = \int_{a_n}^{x}\left(\cdots \int_{a_3}^{x}\left(\int_{a_2}^{x}\left(\int_{a_1}^{x} f(x)\,dx \right) dx \right) dx \cdots \right) dx \right\}$$

And

when $a_k = a \quad for\ k=1, \cdots, n$, we call it ***higher integral with a fixed lower limit***,

when $a_k \neq a \quad for\ some\ k$, we call it ***higher integral with variable lower limits***.

Difference between Multiple Integral and Higher Integral

Multiple integral is what integrated $f(x)$ with respect to $x_1, x_2, \cdots, x_n$. On the other hand, Higher integral is what integrated $f(x)$ with respect to x repeatedly. Then it is also called repeated integration.

By familiar expression, a triple integral is what integrates $f(x,y,z)$ with respect to length, width and height. On the other hand, the 3 rd order integral is what integrates $f(x)$ 3 times with respect to only length.
As a notation of the higher integral, it is denoted in many cases as follows using a notation of multiple integral.

$$\int_a^x \int_a^{t_n} \cdots \int_a^{t_2} f(t_1)\,dt_1 dt_2 \cdots dt_n$$

However, this notation is complicated and inconvenient. Hence in this paper, for these problem solutions and saving of variables, we replaced $t_1, t_2, \cdots, t_n$ with x and replaced $dt_1\, dt_2 \cdots dt_n$ with dx^n, as seen in the definition.

Example

$$\int_0^x \int_0^x e^x dx^2 = \int_0^x \left(\int_0^x e^x dx \right) dx = \int_0^x \left(e^x - 1 \right) dx = e^x - x - 1$$

$$\int_{-\infty}^x \int_{-\infty}^x e^x dx^2 = \int_{-\infty}^x \left(\int_{-\infty}^x e^x dx \right) dx = \int_{-\infty}^x e^x\, dx = e^x$$

$$\int_0^x \int_0^x \int_0^x \sin x\, dx^3 = \int_0^x \int_0^x (1 - \cos x)\, dx^2 = \int_0^x (x - \sin x)\, dx = \cos x + \frac{x^2}{2} - 1$$

$$\int_{\frac{3\pi}{2}}^x \int_{\frac{2\pi}{2}}^x \int_{\frac{1\pi}{2}}^x \sin x\, dx^3 = \int_{\frac{3\pi}{2}}^x \int_{\frac{2\pi}{2}}^x (-\cos x)\, dx^2 = \int_{\frac{3\pi}{2}}^x (-\sin x)\, dx = \cos x$$

4.1.3 Fundamental Theorem of Higher Integral

In the case of the first order integral, the relation between a primitive function and the integral is shown by the following theorem.

Fundamental Theorem of Calculus

Let f be a continuous function defined on a closed interval $[a, b]$. Let F be the arbitrary primitive function of f. Then the following equation holds for $x \in [a, b]$.

$$\int_a^x f(x)\, dx = F(x) - F(a)$$

Where, $F(a)$ is called ***Constant-of-integration***.

The following theorem holds also about the Higher Integrl.

Theorem 4.1.3

Let $f^{<r>}$ $r = 0, 1, \cdots, n$ be continuous functions defined on a closed interval I and $f^{<r+1>}$ be the arbitrary primitive function of $f^{<r>}$. Then the following expression holds for $a_r, x \in I$.

$$\int_{a_n}^x \cdots \int_{a_1}^x f(x)\, dx^n = f^{<n>}(x) - \sum_{r=0}^{n-1} f^{<n-r>}(a_{n-r}) \int_{a_n}^x \cdots \int_{a_{n-r+1}}^x dx^r \tag{1.1}$$

Especially, when $a_r = a \quad for \ \ r = 1, 2, \cdots, n$,

$$\int_a^x \cdots \int_a^x f(x)\, dx^n = f^{<n>}(x) - \sum_{r=0}^{n-1} f^{<n-r>}(a) \frac{(x-a)^r}{r!} \tag{1.2}$$

Proof

First, the following equations hold according to Fundamental Theorem of Calculus.

$$\int_{a_1}^{x} f^{<0>}dx = f^{<1>} - f_{a_1}^{<1>} \tag{01}$$

$$\int_{a_2}^{x} f^{<1>}dx = f^{<2>} - f_{a_2}^{<2>} \tag{12}$$

$$\int_{a_3}^{x} f^{<2>}dx = f^{<3>} - f_{a_3}^{<3>} \tag{23}$$

$$\int_{a_4}^{x} f^{<3>}dx = f^{<4>} - f_{a_4}^{<4>} \tag{34}$$

$$\vdots$$

Integrating both sides of (01) with respect to x from a_2 to x,

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>}dx^2 = \int_{a_2}^{x} f^{<1>}dx - f_{a_1}^{<1>}\int_{a_2}^{x}dx$$

Substituting (12) for this,

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>}dx^2 = f^{<2>} - f_{a_2}^{<2>} - f_{a_1}^{<1>}\int_{a_2}^{x}dx$$

Integrating both sides of this with respect to x from a_3 to x,

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f(x)dx^3 = \int_{a_3}^{x} f^{<2>}dx - f_{a_2}^{<2>}\int_{a_3}^{x}dx - f_{a_1}^{<1>}\int_{a_3}^{x}\int_{a_2}^{x}dx^2$$

Substituting (23) for this,

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f(x)dx^3 = f^{<3>} - f_{a_3}^{<3>} - f_{a_2}^{<2>}\int_{a_3}^{x}dx - f_{a_1}^{<1>}\int_{a_3}^{x}\int_{a_2}^{x}dx^2$$

Integrating both sides of this with respect to x from a_4 to x,

$$\int_{a_4}^{x}\cdots\int_{a_1}^{x} f(x)dx^4 = \int_{a_4}^{x} f^{<3>}dx - f_{a_3}^{<3>}\int_{a_4}^{x}dx - f_{a_2}^{<2>}\int_{a_4}^{x}\int_{a_3}^{x}dx^2 - f_{a_1}^{<1>}\int_{a_4}^{x}\int_{a_3}^{x}\int_{a_2}^{x}dx^3$$

Substituting (34) for this,

$$\int_{a_4}^{x}\cdots\int_{a_1}^{x} f(x)dx^4 = f^{<4>} - f_{a_4}^{<4>} - f_{a_3}^{<3>}\int_{a_4}^{x}dx - f_{a_2}^{<2>}\int_{a_4}^{x}\int_{a_3}^{x}dx^2 - f_{a_1}^{<1>}\int_{a_4}^{x}\int_{a_3}^{x}\int_{a_2}^{x}dx^3$$

Hereafter by induction, we obtain the following expression.

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} f(x)dx^n = f^{<n>} - f_{a_n}^{<n>} - \sum_{r=1}^{n-1} f_{a_{n-r}}^{<n-r>}\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x}dx^r$$

And pushing $f_{a_n}^{<n>}$ into $\sum$, we obtain (1.1) .

Especially, when $a_r = a \quad for \quad r=1,2, \cdots ,n$,

$$\int_{a}^{x}\cdots\int_{a}^{x}dx^r = \frac{(x-a)^r}{r!} \quad r=1,2, \cdots ,n-1$$

Then, using this, we obtain

$$\int_{a}^{x}\cdots\int_{a}^{x} f(x)dx^n = f^{<n>}(x) - \sum_{r=0}^{n-1} f^{<n-r>}(a)\frac{(x-a)^r}{r!} \tag{1.2}$$

Constant-of-integration Polynomial

We call $\sum_{r=0}^{n-1} f^{<n-r>}(a_{n-r}) \int_{a_n}^{x} \cdots \int_{a_{n-r+1}}^{x} dx^r$ and $\sum_{r=0}^{n-1} f^{<n-r>}(a) \frac{(x-a)^r}{r!}$ in the theorem

***Constant-of-integration Polynomial*.**

In the case of (1.1) , it is very difficult to express the higher integral of 1 by a polynomial. For reference, if this is calculated by force and the 4th order integral of $f(x)$ is expressed by a polynomial, it is as follows. The 5th order or more are unmanageable...

$$\int_{a_4}^{x} \cdots \int_{a_1}^{x} f(x)\, dx^4 = f^{<4>} - f_{a_4}^{<4>} - f_{a_3}^{<3>} \frac{(x-a_4)^1}{1!} - f_{a_2}^{<2>} \frac{(x-a_4)(x-2a_3+a_4)}{2!}$$
$$- f_{a_1}^{<1>} \frac{(x-a_4)\left(x^2+a_4x-3a_2x+6a_2a_3-3a_2a_4-3a_3^2+a_4^2\right)}{3!}$$

Since the remainder term of (1.2) is already expressed by the polynomial, we can find out easily that it is the (n-1)-th order. We can also express it explicitly as follows, though there is no merit not much.

$$\int_{a}^{x} \cdots \int_{a}^{x} f(x)\, dx^n = f^{<n>}(x) - \sum_{r=0}^{n-1} c_{r+1} x^r , \quad c_{r+1} = \sum_{s=r}^{n-1} (-1)^{s-r} \frac{f_a^{<n-s>}}{s!} {}_sC_{s-r}\, a^{s-r}$$

Example1

$$\int_{a_2}^{x}\int_{a_1}^{x} f(x)\, dx^2 = f^{<2>} - f_{a_2}^{<2>} - f_{a_1}^{<1>} \int_{a_2}^{x} dx = f^{<2>} - f_{a_2}^{<2>} - f_{a_1}^{<1>} \frac{(x-a_2)^1}{1!}$$

Let $f = e^x$, $(e^x)^{<1>} = e^x + c_1$, $(e^x)^{<2>} = e^x + c_1 x + c_2$

Substituting these for the above experession, we obtain as follows.

Left: $\int_{a_2}^{x}\int_{a_1}^{x} e^x dx^2 = e^x - e^{a_2} - e^{a_1} \frac{(x-a_2)^1}{1!}$

Right: $(e^x)^{<2>} - (e^x)_{a_2}^{<2>} - (e^x)_{a_1}^{<1>} \frac{(x-a_2)^1}{1!}$

$$= e^x + c_1 x + c_2 - e^{a_2} - c_1 a_2 - c_2 - \left(e^{a_1} + c_1\right) \frac{(x-a_2)^1}{1!}$$
$$= e^x - e^{a_2} - e^{a_1} \frac{(x-a_2)^1}{1!}$$

Example2

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f(x)\, dx^3 = f^{<3>} - f_{a_3}^{<3>} - f_{a_2}^{<2>} \int_{a_3}^{x} dx - f_{a_1}^{<1>} \int_{a_3}^{x}\int_{a_2}^{x} dx^2$$
$$= f^{<3>} - f_{a_3}^{<3>} - f_{a_2}^{<2>} \frac{(x-a_3)^1}{1!} - f_{a_1}^{<1>} \frac{(x-a_3)(x-2a_2+a_3)}{2!}$$

Let $f = \sin x$, $(\sin x)^{<1>} = -\cos x$, $(\sin x)^{<2>} = -\sin x$, $(\sin x)^{<3>} = \cos x$

Substituting these for the above experession, we obtain as follows.

Left: $\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} \sin x\, dx^3$

$$= \cos x - \cos a_3 + \frac{\left(x^2 - a_3^2\right)\cos a_1}{2!} + (x - a_3)\sin a_2 - a_2(x - a_3)\cos a_1$$

Right: $\cos x - \cos a_3 + \sin a_2 \frac{(x - a_3)^1}{1!} + \cos a_1 \frac{(x - a_3)(x - 2a_2 + a_3)}{2!}$

$$= \cos x - \cos a_3 + (x - a_3)\sin a_2 + \frac{\left(x^2 - a_3^2\right)\cos a_1}{2!} - a_2(x - a_3)\cos a_1$$

4.1.4 Lineal and Collateral

Definition 4.1.4

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} f(x)\,dx^n = f^{<n>}(x) - \sum_{r=0}^{n-1} f^{<n-r>}(a_{n-r}) \int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r \qquad (1.1)$$

In this expression,

when Constant-of-integration Polynomial is 0,

we call $\int_{a_n}^{x}\cdots\int_{a_1}^{x} f(x)\,dx^n$ ***Lineal Higher Integral*** and

we call the function equal to this ***Lineal Higher Primitive Function***.

(or for short , ***Lineal Higher Primitive.***)

when Constant-of-integration Polynomial is not 0,

we call $\int_{a_n}^{x}\cdots\int_{a_1}^{x} f(x)\,dx^n$ ***Collateral Higher Integral*** and

we call the function equal to this ***Collateral Higher Primitive Function***.

(or for short , ***Collateral Higher Primitive.***)

These are the same also in (1.2).

In short , ***Lineal Higher Primitive Function*** is what integrated $f(x)$ with respect to x repeatedly **without considering the constant of the integration**.

Example

Left: Collateral Higher Integral Right: Collateral Higher Primitive

$$\int_{3}^{x}\int_{2}^{x}\int_{1}^{x} \sin x\, dx^3 = \cos x - \cos 3 + \sin 2 \cdot \frac{(x-3)^1}{1!} + \cos 1 \cdot \frac{(x-3)(x-2\cdot 2+3)}{2!}$$

$$\int_{0}^{x}\int_{0}^{x} e^x dx^2 = e^x - 1 - x$$

Left: Lineal Higher Integral Right: Lineal Higher Primitive

$$\int_{\frac{3\pi}{2}}^{x}\int_{\frac{2\pi}{2}}^{x}\int_{\frac{1\pi}{2}}^{x} \sin x\, dx^3 = \cos x$$

$$\int_{-\infty}^{x}\int_{-\infty}^{x} e^x dx^2 = e^x$$

4.1.5 The Conditions for the Higher Integral being Lineal

In Theorem 4.1.3 (1.1), since the higher integral of 1 can be arbitrary value, in order for Constant-of-integration Polynomial to be 0, we find out that it must be $f^{<r>}(a_r) = 0 \; for \; r = 1,2,\cdots,n$. Since this becomes important later, we state here as a theorem.

Theorem 4.1.5

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} f(x)dx^n = f^{<n>}(x) - \sum_{r=0}^{n-1} f^{<n-r>}(a_{n-r})\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r \tag{1.1}$$

$$\int_{a}^{x}\cdots\int_{a}^{x} f(x)dx^n = f^{<n>}(x) - \sum_{r=0}^{n-1} f^{<n-r>}(a)\frac{(x-a)^r}{r!} \tag{1.2}$$

In these expressions, the necessary conditions for higher integrals being lineal are as follows respectively

$$f^{<r>}(a_r) = 0 \qquad r = 1,2,\cdots,n$$

$$f^{<r>}(a) = 0 \qquad r = 1,2,\cdots,n$$

That is, **The necessary condition for the higher integral being lineal is** that **a_r or a is a zero of the higher primitive function $f^{<r>}$ for each r from 1 to n.**

Note

These are necessary conditions and are not sufficient conditions. For example, let

$$f^{<r>}(x) = \int_{a_r}^{x}\cdots\int_{a_1}^{x} f(x)dx^r \qquad r = 1,2,\cdots,n$$

Then, whether or not this higher integral is lineal, certainly $f^{<r>}(a_r) = 0 \quad r=1,2,\cdots,n$

In the case of $f(x)=e^x$, let

$$f^{<r>}(x) = \int_{0}^{x}\cdots\int_{0}^{x} e^x dx^r \qquad r = 1,2$$

Then, $f^{<1>}(x) = e^x - 1$, $f^{<2>}(x) = e^x - 1 - x$ and the lower limit 0 is a zero of these primitive functions. Nevertheless, substituting these for (1.2), we obtain

$$\int_{0}^{x}\int_{0}^{x} e^x dx^2 = e^x - 1 - x - \left(e^0 - 1 - 0\right)\frac{x^0}{0!} - \left(e^0 - 1\right)\frac{x^1}{1!} = e^x - 1 - x$$

This is not lineal higher integral.

4.2 Reimann-Liouville Integral and Higher Integral

When $a_k = a\ ,\ k=1,2,\cdots,n$ the higher ingegral $\int_{a_n}^{x}\cdots\int_{a_1}^{x} f(x)\,dx^n$ can result in the single integral called *Reimann-Liouville Integral*.

Theorem 4.2.1 (Cauchy formula for repeated integration)

When $f(x)$ is continuously integrable function and $\Gamma(z)$ denotes the gamma function, the following equation holds.

$$\int_a^x\cdots\int_a^x f(x)\,dx^n = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)\,dt \tag{2.1}$$

Proof

First, Reimann-Liouville Integral can be expanded as follows.

$$\frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)\,dt = f^{<n>}(x) - \sum_{r=0}^{n-1}\frac{(x-a)^r}{r!} f^{<n-r>}(a) \tag{2.2}$$

Because,

$$\begin{aligned}
\frac{1}{\Gamma(1)}\int_a^x (x-t)^0 f(t)\,dt &= \int_a^x f(t)\,dt = \left[f^{<1>}(t)\right]_a^x \\
&= f^{<1>}(x) - f^{<1>}(a) \\
\frac{1}{\Gamma(2)}\int_a^x (x-t)^1 f(t)\,dt &= \frac{1}{1!}\left\{\left[(x-t)^1 f^{<1>}(t)\right]_a^x + \int_a^x f^{<1>}(t)\,dt\right\} \\
&= -(x-a)^1 f^{<1>}(a) + f^{<2>}(x) - f^{<2>}(a) \\
\frac{1}{\Gamma(3)}\int_a^x (x-t)^2 f(t)\,dt &= \frac{1}{2!}\left\{\left[(x-t)^2 f^{<1>}(t)\right]_a^x + 2\int_a^x (x-t)^1 f^{<1>}(t)\,dt\right\} \\
&= -\frac{1}{2!}(x-a)^2 f^{<1>}(a) + \left[(x-t)^1 f^{<2>}(t)\right]_a^x + \int_a^x f^{<2>}(t)\,dt \\
&= -\frac{(x-a)^2}{2!} f^{<1>}(a) - \frac{(x-a)^1}{1!} f^{<2>}(a) + f^{<3>}(x) - f^{<3>}(a) \\
&\ \ \vdots
\end{aligned}$$

$$\frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)\,dt = f^{<n>}(x) - \sum_{r=0}^{n-1}\frac{(x-a)^r}{r!} f^{<n-r>}(a) \tag{2.2}$$

On the other hand, Theorem 4.1.3 (1.2) was as follows.

$$\int_a^x\cdots\int_a^x f(x)\,dx^n = f^{<n>}(x) - \sum_{r=0}^{n-1} f^{<n-r>}(a)\frac{(x-a)^r}{r!} \tag{1.2}$$

From these, we obtain the desired expressin.

Note

This theorem shows that the higher integral with fixed lower limit is equivalent to Riemann-Liouville integral. Therefore, we can examine the higher integral in left side by Riemann-Liouville integral in right side. Moreover, this theorem shows that we can not apply Riemann-Liouville Integral to the higher integral with variable lower limits. So, we can not apply Riemann-Liouville integral to the higher integral of $\sin x$ etc.. Strictly speaking, if we apply it by force, then we obtain the collateral higher integral inevitably.

Reimann-Liouville Integral and Taylor's Theorem

The right side of (2.2) is rewritten as follows.

$$f^{<n>}(x)=\sum_{r=0}^{n-1} f^{<n-r>}(a)\frac{(x-a)^r}{r!}+\frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)\,dt \tag{2.2}$$

Then we notice that this is Taylor expansion of $f^{<n>}(x)$ around a and Reimann-Liouville Integral is the remainder term.

Actually, from this, we can derive usual Taylor's theorem. Let us shift by $-n$ the index of the integration operator of $f(x)$. Then

$$f^{<0>}(x)=\sum_{r=0}^{n-1} f^{(r)}(a)\frac{(x-a)^r}{r!}+\frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f^{(n)}(t)\,dt \tag{2.2$_0$}$$

Just, this is the Taylor expansion of $f(x)$ around a . And Reimann-Liouville Integral of $f^{(n)}$ is the remainder term called *Bernoulli form .*

And according to the first mean value theorem for integration, when $g(t)$ is a continuous real valued function on the $[a,b]$ and $\phi(t)$ is an integrable function that does not change the sign on the (a,b) , there exists a $\xi \in (a,b)$ such that

$$\int_a^b g(t)\phi(t)\,dt = g(\xi)\int_a^b \phi(t)\,dt$$

Applying this to (2.2$_0$) , we obtain the following.

$$\frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f^{(n)}(t)\,dt = \frac{f^{(n)}(\xi)}{(n-1)!}\int_a^x (x-t)^{n-1}\,dt$$

$$= \frac{f^{(n)}(\xi)}{n!}\left[-(x-t)^n\right]_a^x = \frac{(x-a)^n}{n!} f^{(n)}(\xi)$$

This is the remainder term called *Lagrange form* .

4.3 Higher Integrals of Power Function etc.

Formula 4.3.1 : Higher Integral of Power Function

When $\Gamma(z)$ denotes a gamma function, the following expressions hold.

(1) Basic form

$$\int_0^x \cdots \int_0^x x^{\alpha} dx^n = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n)} x^{\alpha+n} \qquad (\alpha \geqq 0)$$

$$\int_{\infty}^x \cdots \int_{\infty}^x x^{\alpha} dx^n = (-1)^n \frac{\Gamma(-\alpha-n)}{\Gamma(-\alpha)} x^{\alpha+n} \qquad (\alpha < -n)$$

(2) Linear form

$$\int_{-\frac{b}{a}}^x \cdots \int_{-\frac{b}{a}}^x (ax+b)^{\alpha} dx^n = \left(\frac{1}{a}\right)^n \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n)} (ax+b)^{\alpha+n} \qquad (\alpha \geqq 0)$$

$$\int_{\infty}^x \cdots \int_{\infty}^x (ax+b)^{\alpha} dx^n = \left(-\frac{1}{a}\right)^n \frac{\Gamma(-\alpha-n)}{\Gamma(-\alpha)} (ax+b)^{\alpha+n} \qquad (\alpha < -n)$$

Proof

When $\alpha \geqq 0$, the zero of the higher primitive functions of $(ax+b)^{\alpha}$ is $-b/a$. Hence

$$\int_{-\frac{b}{a}}^x (ax+b)^{\alpha} dx = \frac{1}{a} \frac{1}{\alpha+1} (ax+b)^{\alpha+1}$$

$$\int_{-\frac{b}{a}}^x \int_{-\frac{b}{a}}^x (ax+b)^{\alpha} dx^2 = \left(\frac{1}{a}\right)^2 \frac{1}{(\alpha+1)(\alpha+2)} (ax+b)^{\alpha+2}$$

$$\vdots$$

$$\int_{-\frac{b}{a}}^x \cdots \int_{-\frac{b}{a}}^x (ax+b)^{\alpha} dx^n = \left(\frac{1}{a}\right)^n \frac{1}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)} (ax+b)^{\alpha+n}$$

Substituting $\dfrac{1}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)} = \dfrac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n)}$ for this, we obtain (2).

Furthermore, substituting $a=1, b=0$ for this, we obtain (1).

When $\alpha < -n$, if we replace α with $-\beta$ then $(ax+b)^{\alpha} = (ax+b)^{-\beta}$. And the zero of higher primitive functions of $(ax+b)^{-\beta}$ is ∞. Therfore,

$$\int_{\infty}^x (ax+b)^{-\beta} dx = \frac{1}{a} \frac{1}{-\beta+1} (ax+b)^{-\beta+1}$$

$$\int_{\infty}^x \int_{\infty}^x (ax+b)^{-\beta} dx^2 = \left(\frac{1}{a}\right)^2 \frac{1}{(-\beta+1)(-\beta+2)} (ax+b)^{-\beta+2}$$

$$\vdots$$

$$\int_{\infty}^x \cdots \int_{\infty}^x (ax+b)^{-\beta} dx^n = \left(\frac{1}{a}\right)^n \frac{1}{(-\beta+1)(-\beta+2)\cdots(-\beta+n)} (ax+b)^{-\beta+n}$$

Since

$$\frac{1}{(-\beta+1)(-\beta+2)\cdots(-\beta+n)} = \frac{(-1)^n}{(\beta-1)(\beta-2)\cdots(\beta-n)}$$

$$= (-1)^n \frac{\Gamma[1+\{\beta-(n+1)\}]}{\Gamma\{1+(\beta-1)\}} = (-1)^n \frac{\Gamma(\beta-n)}{\Gamma(\beta)}$$

we can substitute this for the above as follows.

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x}(ax+b)^{-\beta}dx^n = \left(-\frac{1}{a}\right)^n \frac{\Gamma(\beta-n)}{\Gamma(\beta)}(ax+b)^{-\beta+n}$$

Then replacing β with $-\alpha$ we botain (2), furthermore, substituting $a=1, b=0$ for this, we obtain (1).

Note

When $-n \leqq \alpha < 0$, we do not dare define the higher integral of the power function. It is because the lower limit of the integral changes irregularly in this case. For example as follows.

$$\int_0^x\int_1^x\int_{\infty}^x x^{-2}dx^3 = -x(\log x - 1) \quad , \quad \int_0^x\int_{\infty}^x\int_{\infty}^x x^{-\frac{5}{2}}dx^3 = \frac{8}{3}\sqrt{x}$$

Examples

3rd order integral of x ^ 5/2

```
a = 5 / 2; Clear t;
HI[n_] := Gamma[1 + a] / Gamma[1 + a + n] x^(a+n)
RL[n_] := 1 / Gamma[n] ∫_0^x (x - t)^(n-1) t^a dt
```

HI[3]	RL[3]
$\frac{8x^{11/2}}{693}$	$\frac{8x^{11/2}}{693}$

2nd order integral of x ^ -3

```
a = -3; Clear t;
HI[n_] := (-1)^n Gamma[-a - n] / Gamma[-a] x^(a+n)
RL[n_] := 1 / Gamma[n] ∫_∞^x (x - t)^(n-1) t^a dt
```

HI[2]	RL[2]
$\frac{1}{2x}$	$\frac{1}{2x}$, Im[x] ≠ 0 || x > 0

Formula 4.3.2 : Higher Integral of Exponential Function

(1) Basic form

$$\int_{\mp\infty}^{x}\cdots\int_{\mp\infty}^{x} e^{\pm x}dx^n = (\pm 1)^n e^{\pm x}$$

(2) Linear form

$$\int_{\mp\infty}^{x}\cdots\int_{\mp\infty}^{x} e^{ax+b}dx^n = \left(\frac{1}{a}\right)^n e^{ax+b} \qquad \begin{pmatrix} a>0 : - \\ a<0 : + \end{pmatrix}$$

Proof

When $a>0$ the zero of higher primitive functions of e^{ax+b} is $-\infty$, and when $a<0$ the zero of same one is $+\infty$. Hence we obtain (2). Furthermore, substituting $a=\pm 1, b=0$ for this, we obtain (1).

Formula 4.3.3 : Higher Integral of Logarithmic Function

(1) Basic form

$$\int_0^x \cdots \int_0^x \log x \, dx^n = \frac{x^n}{n!}\left(\log x - \sum_{k=1}^{n}\frac{1}{k}\right)$$

(2) Linear form

$$\int_{-\frac{b}{a}}^x \cdots \int_{-\frac{b}{a}}^x \log(ax+b) dx^n = \frac{1}{n!}\left(x+\frac{b}{a}\right)^n \left\{\log(ax+b) - \sum_{k=1}^{n}\frac{1}{k}\right\}$$

Proof

$$\int_0^x x^m \log x \, dx = \frac{x^{m+1}}{m+1}\log x - \frac{x^{m+1}}{(m+1)^2}$$

Therfore,

$$\int_0^x \log x \, dx = x(\log x - 1) = \frac{x^1}{1!}\log x - \frac{x^1}{1!}\left(\frac{1}{1}\right) = \frac{x^1}{1!}\left(\log x - \frac{1}{1}\right)$$

$$\int_0^x\int_0^x \log x \, dx^2 = \int_0^x x(\log x - 1) dx = \int_0^x x \log x \, dx - \int_0^x x \, dx$$

$$= \frac{1}{1!}\left(\frac{x^2}{2}\log x - \frac{x^2}{2^2}\right) - \frac{x^2}{2} = \frac{x^2}{2!}\log x - \frac{x^2}{2!2} - \frac{x^2}{2!}$$

$$= \frac{x^2}{2!}\log x - \frac{x^2}{2!}\left(1+\frac{1}{2}\right) = \frac{x^2}{2!}\left\{\log x - \left(1+\frac{1}{2}\right)\right\}$$

$$\int_0^x\int_0^x\int_0^x \log x \, dx^3 = \frac{1}{2!}\int_0^x x^2 \log x \, dx - \frac{1}{2!}\left(1+\frac{1}{2}\right)\int_0^x x^2 dx$$

$$= \frac{1}{2!}\left(\frac{x^3}{3}\log x - \frac{x^3}{3^2}\right) - \frac{1}{3!}\left(1+\frac{1}{2}\right)x^3$$

$$= \frac{x^3}{3!}\log x - \frac{x^3}{3!3} - \frac{1}{3!}\left(1+\frac{1}{2}\right)x^3$$

$$= \frac{x^3}{3!}\log x - \frac{x^3}{3!}\left(1+\frac{1}{2}+\frac{1}{3}\right) = \frac{x^3}{3!}\left\{\log x - \left(1+\frac{1}{2}+\frac{1}{3}\right)\right\}$$

Hereafter, we obtain (1) by induction. And replacing x with $ax+b$, we obtain (2) by similar way.

Example: 3rd order integral of log x

```
Clear t;
HI[n_] := x^n/n! (Log[x] - HarmonicNumber[n])
RL[n_] := 1/Gamma[n] ∫_0^x (x - t)^(n-1) Log[t] dt
```

```
HI[3]
1/6 x^3 (-11/6 + Log[x])
```

```
RL[3]
1/36 x^3 (-11 + 6 Log[x])
```

Formula 4.3.4 : Higher Integrals of sin x , cos x

(1) Basic form

$$\int_{\frac{n\pi}{2}}^{x}\cdots\int_{\frac{2\pi}{2}}^{x}\int_{\frac{1\pi}{2}}^{x}\sin x\,dx^{n} = \sin\left(x-\frac{n\pi}{2}\right)$$

$$\int_{\frac{(n-1)\pi}{2}}^{x}\cdots\int_{\frac{1\pi}{2}}^{x}\int_{\frac{0\pi}{2}}^{x}\cos x\,dx^{n} = \cos\left(x-\frac{n\pi}{2}\right)$$

(2) Linear form

$$\int_{\frac{n\pi}{2a}-\frac{b}{a}}^{x}\cdots\int_{\frac{1\pi}{2a}-\frac{b}{a}}^{x}\sin(ax+b)\,dx^{n} = \left(\frac{1}{a}\right)^{n}\sin\left(ax+b-\frac{n\pi}{2}\right)$$

$$\int_{\frac{(n-1)\pi}{2a}-\frac{b}{a}}^{x}\cdots\int_{\frac{0\pi}{2a}-\frac{b}{a}}^{x}\cos(ax+b)\,dx^{n} = \left(\frac{1}{a}\right)^{n}\cos\left(ax+b-\frac{n\pi}{2}\right)$$

Proof

Lineal higher primitive function of $f(x)=\sin x$ and its zeros are as follows.

$$f^{<1>}(x) = -\cos x = \sin\left(x-\frac{1\pi}{2}\right) \quad , \quad x=\frac{1\pi}{2}$$

$$f^{<2>}(x) = -\sin x = \sin\left(x-\frac{2\pi}{2}\right) \quad , \quad x=\frac{2\pi}{2}$$

$$f^{<3>}(x) = \cos x = \sin\left(x-\frac{3\pi}{2}\right) \quad , \quad x=\frac{3\pi}{2}$$

$$f^{<4>}(x) = \sin x = \sin\left(x-\frac{4\pi}{2}\right) \quad , \quad x=\frac{4\pi}{2}$$

$$\vdots$$

Thus we obtain (1). And replacing x with $ax+b$, we obtain (2) by similar way. For $\cos x$, it is obtained by deducting $\pi/2$ from the lower limit of the higher integral in the left side and replacing $\sin$ with $\cos$ in the right hand.

Termwise higher integrals of $\sin x$, $\cos x$

If common value 0 is adopted as lower limits of higher integrals of $\sin x, \cos x$, we obtain the following termwise higher integral. These are **collateral higher integrals** as understood from the constant-of-integration polynomial in the right side

$$\int_{0}^{x}\cdots\int_{0}^{x}\sin x\,dx^{n} = \sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2k+1+n)!}x^{2k+1+n}$$

$$= \sin\left(x-\frac{n\pi}{2}\right) + \sum_{k=1}^{n}\frac{x^{n-k}}{(n-k)!}\sin\frac{k\pi}{2}$$

$$\int_{0}^{x}\cdots\int_{0}^{x}\cos x\,dx^{n} = \sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2k+n)!}x^{2k+n}$$

$$= cos\left(x - \frac{n\pi}{2}\right) - \sum_{k=1}^{n} \frac{x^{n-k}}{(n-k)!} cos\frac{k\pi}{2}$$

If these are written without using $\sum$, it is as follows.

$$\int_0^x sinx\,dx = -cosx + 1 \quad , \quad \int_0^x cosx\,dx = sinx$$

$$\int_0^x\int_0^x sinx\,dx^2 = -sinx + \frac{x^1}{1!} \quad , \quad \int_0^x\int_0^x cosx\,dx^2 = -cosx + 1$$

$$\int_0^x\int_0^x\int_0^x sinx\,dx^3 = cosx + \frac{x^2}{2!} - 1 \quad , \quad \int_0^x\int_0^x\int_0^x cosx\,dx^3 = -sinx + \frac{x^1}{1!}$$

$$\int_0^x\cdots\int_0^x sinx\,dx^4 = sinx - \frac{x^1}{1!} + \frac{x^3}{3!} \quad , \quad \int_0^x\cdots\int_0^x cosx\,dx^4 = cosx + \frac{x^2}{2!} - 1$$

$$\vdots \qquad\qquad\qquad \vdots$$

Formula 4.3.5 : Higher Integral of $sinhx$, $coshx$

(1) Basic form

$$\int_{\frac{n\pi i}{2}}^x \cdots \int_{\frac{2\pi i}{2}}^x \int_{\frac{1\pi i}{2}}^x sinhx\,dx^n = i^n sinh\left(x - \frac{n\pi i}{2}\right) = \frac{e^x - (-1)^n e^{-x}}{2}$$

$$\int_{\frac{(n-1)\pi i}{2}}^x \cdots \int_{\frac{1\pi i}{2}}^x \int_{\frac{0\pi i}{2}}^x coshx\,dx^n = i^n cosh\left(x - \frac{n\pi i}{2}\right) = \frac{e^x + (-1)^n e^{-x}}{2}$$

(2) Linear form

$$\int_{\frac{n\pi i}{2a}-\frac{b}{a}}^x \cdots \int_{\frac{1\pi i}{2a}-\frac{b}{a}}^x sinh(ax+b)\,dx^n = \left(\frac{i}{a}\right)^n sinh\left(ax+b - \frac{n\pi i}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a}\right)^n \left\{e^{ax+b} - (-1)^n e^{-(ax+b)}\right\}$$

$$\int_{\frac{(n-1)\pi i}{2a}-\frac{b}{a}}^x \cdots \int_{\frac{0\pi i}{2a}-\frac{b}{a}}^x cosh(ax+b)\,dx^n = \left(\frac{i}{a}\right)^n cosh\left(ax+b - \frac{n\pi i}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a}\right)^n \left\{e^{ax+b} + (-1)^n e^{-(ax+b)}\right\}$$

Proof

Lineal higher primitive function of $f(x) = sinhx$ and its zeros are as follows.

$$f^{<1>}(x) = coshx = i\,sinh\left(x - \frac{1\pi i}{2}\right), \quad x = \frac{1\pi i}{2}$$

$$f^{<2>}(x) = sinhx = i^2 sinh\left(x - \frac{2\pi i}{2}\right) \quad , \quad x = \frac{2\pi i}{2}$$

$$f^{<3>}(x) = \cosh x \quad = i^3 \sinh\left(x - \frac{3\pi i}{2}\right) \quad , \quad x = \frac{3\pi i}{2}$$

$$f^{<4>}(x) = \sinh x \quad = i^4 \sinh\left(x - \frac{4\pi i}{2}\right) \quad , \quad x = \frac{4\pi i}{2}$$

$$\vdots$$

Thus we obtain the hyperbolic form of (1). And replacing x with $ax+b$, we obtain the hyperbolic form of (2) by similar way. For $\cosh x$, it is obtained by deducting $\pi i/2$ from the lower limit of the higher integral in the left side and replacing $\sinh$ with $\cosh$ in the right side.

Next,

$$\sinh\left(ax+b-\frac{n\pi i}{2}\right) = \frac{1}{2}\left\{e^{ax+b-\frac{n\pi i}{2}} - e^{-\left(ax+b-\frac{n\pi i}{2}\right)}\right\}$$

$$= \frac{e^{-\frac{n\pi i}{2}}}{2}\left\{e^{ax+b} - e^{-(ax+b-n\pi i)}\right\} = \frac{\left(e^{\frac{\pi i}{2}}\right)^{-n}}{2}\left\{e^{ax+b} - \left(e^{\pi i}\right)^n e^{-(ax+b)}\right\}$$

$$= \frac{i^{-n}}{2}\left\{e^{ax+b} - (-1)^n e^{-(ax+b)}\right\}$$

$$\cosh\left(ax+b-\frac{n\pi i}{2}\right) = \frac{1}{2}\left\{e^{ax+b-\frac{n\pi i}{2}} + e^{-\left(ax+b-\frac{n\pi i}{2}\right)}\right\}$$

$$= \frac{i^{-n}}{2}\left\{e^{ax+b} + (-1)^n e^{-(ax+b)}\right\}$$

Therefore, using these we botain exponential forms of (1), (2).

Termwise higher integrals of $\sinh x$, $\cosh x$

If common value 0 is adopted as lower limits of higher integrals of $\sinh x, \cosh x$, we obtain the following termwise higher integral. These are **collateral higher integrals** as understood from the constant-of-integration polynomial in the right side

$$\int_0^x \cdots \int_0^x \sinh x \, dx^n = \sum_{k=0}^{\infty} \frac{1}{(2k+1+n)!} x^{2k+1+n}$$

$$= i^n \sinh\left(x - \frac{n\pi i}{2}\right) - \sum_{k=1}^{n} \frac{x^{n-k}}{(n-k)!} i^{-k} \sinh\frac{k\pi i}{2}$$

$$\int_0^x \cdots \int_0^x \cosh x \, dx^n = \sum_{k=0}^{\infty} \frac{1}{(2k+n)!} x^{2k+n}$$

$$= i^n \cosh\left(x - \frac{n\pi i}{2}\right) - \sum_{k=1}^{n} \frac{x^{n-k}}{(n-k)!} i^{-k} \cosh\frac{k\pi i}{2}$$

4.4 Higher Integrals of Inverse Trigonometric Functions

Formula 4.4.1 : Higher Integrals of $tan^{-1}x$, $cot^{-1}x$

When $\downarrow$, $\uparrow$, $\psi(x)$ denote floor function, ceiling function and digamma function respectively, the following expressions hold for a natural number $n \geqq 2$.

$$\int_0^x \cdots \int_0^x tan^{-1}x\, dx^n = \frac{tan^{-1}x}{n!}\sum_{k=0}^{n/2\downarrow}(-1)^k {}_nC_{n-2k}\, x^{n-2k}$$
$$+ \frac{log(1+x^2)}{2\cdot n!}\sum_{k=1}^{n/2\uparrow}(-1)^k {}_nC_{n+1-2k}\, x^{n+1-2k}$$
$$- \frac{1}{n!}\sum_{r=1}^{n/2\downarrow}(-1)^r {}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\}x^{n+1-2r}$$

$$\int_0^x \cdots \int_0^x cot^{-1}x\, dx^n = \frac{x^n}{n!}cot^{-1}x - \frac{tan^{-1}x}{n!}\sum_{k=1}^{n/2\downarrow}(-1)^k {}_nC_{n-2k}\, x^{n-2k}$$
$$- \frac{log(1+x^2)}{2\cdot n!}\sum_{k=1}^{n/2\uparrow}(-1)^k {}_nC_{n+1-2k}\, x^{n+1-2k}$$
$$+ \frac{1}{n!}\sum_{r=1}^{n/2\downarrow}(-1)^r {}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\}x^{n+1-2r}$$

Proof

$$\int_0^x tan^{-1}x\, dx = x\,tan^{-1}x - \frac{1}{2}log(1+x^2)$$
$$= \frac{1}{1!}\,{}_1C_1\, x\,tan^{-1}x - \frac{1}{2\cdot 1!}\,{}_1C_0\, x^0 log(1+x^2)$$

$$\int_0^x\int_0^x tan^{-1}x\, dx^2 = \left(\frac{x^2}{2}-\frac{1}{2}\right)tan^{-1}x - \frac{x}{2}log(1+x^2) + \frac{x}{2}$$
$$= \frac{1}{2!}\left({}_2C_2\, x^2 - {}_2C_0\right) tan^{-1}x - \frac{1}{2\cdot 2!}\,{}_2C_1\, x^1 log(1+x^2) + \frac{1}{1!}\left(\frac{1}{2}\right)\frac{x}{1!}$$

$$\int_0^x\int_0^x\int_0^x tan^{-1}x\, dx^3 = \left(\frac{x^3}{6}-\frac{x}{2}\right)tan^{-1}x - \left(\frac{x^2}{4}-\frac{1}{12}\right)log(1+x^2) + \frac{5x^2}{12}$$
$$= \frac{1}{3!}\left({}_3C_3\, x^3 - {}_3C_1\, x\right) tan^{-1}x$$
$$- \frac{1}{2\cdot 3!}\left({}_3C_2\, x^2 - {}_3C_0\, x^0\right) log(1+x^2) + \frac{1}{1!}\left(\frac{1}{2}+\frac{1}{3}\right)\frac{x^2}{2!}$$

$$\int_0^x \cdots \int_0^x tan^{-1}x\, dx^4 = \left(\frac{x^4}{24}-\frac{x^2}{4}+\frac{1}{24}\right)tan^{-1}x - \left(\frac{x^3}{12}-\frac{x}{12}\right)log(1+x^2) + \frac{13x^3}{72} - \frac{x}{24}$$
$$= \frac{1}{4!}\left({}_4C_4\, x^4 - {}_4C_2\, x^2 + {}_4C_0\right) tan^{-1}x - \frac{1}{2\cdot 4!}\left({}_4C_3\, x^3 - {}_4C_1\, x^1\right) log(1+x^2)$$
$$+ \frac{1}{1!}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{x^3}{3!} - \frac{1}{3!}\left(\frac{1}{4}\right)\frac{x}{1!}$$

$$\int_0^x\cdots\int_0^x \tan^{-1}x\,dx^5=\left(\frac{x^5}{120}-\frac{x^3}{12}+\frac{x}{24}\right)\tan^{-1}x-\left(\frac{x^4}{48}-\frac{x^2}{24}+\frac{1}{240}\right)\log\left(1+x^2\right)+\frac{77x^4}{1440}-\frac{3x^2}{80}$$

$$=\frac{1}{5!}\left({}_5C_5x^5-{}_5C_3x^3+{}_5C_1x\right)\tan^{-1}x-\frac{1}{2\cdot 5!}\left({}_5C_4x^4-{}_5C_2x^2+{}_5C_0x^0\right)\log\left(1+x^2\right)$$

$$+\frac{1}{1!}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)\frac{x^4}{4!}-\frac{1}{3!}\left(\frac{1}{4}+\frac{1}{5}\right)\frac{x^2}{2!}$$

Hereafter, by induction, we obtain the desired expression.

The desired expression for $\cot^{-1}x$ is also obtained in a similar way.

Example: 4th order integral of $\tan^{-1}x$

```
Clear [t]

HI[n_] := ArcTan[x]/n! Sum[(-1)^k Binomial[n, n - 2 k] x^(n-2 k), {k, 0, Floor[n/2]}] +
    Log[1 + x^2]/(2 n!) Sum[(-1)^k Binomial[n, n + 1 - 2 k] x^(n+1-2 k), {k, 1, Ceiling[n/2]}] -
    1/n! Sum[(-1)^r Binomial[n, n + 1 - 2 r] (PolyGamma[1 + n] - PolyGamma[2 r]) x^(n+1-2 r), {r, 1, Floor[n/2]}]

RL[n_] := 1/Gamma[n] Integrate[(x - t)^(n-1) ArcTan[t], {t, 0, x}]

Simplify[HI[4]]      1/72 (3 (1 - 6 x^2 + x^4) ArcTan[x] + x (-3 + 13 x^2 - 6 (-1 + x^2) Log[1 + x^2]))

RL[4]                1/72 (3 (1 - 6 x^2 + x^4) ArcTan[x] + x (-3 + 13 x^2 - 6 (-1 + x^2) Log[1 + x^2]))
```

Formula 4.4.2 : Higher Integrals of $\sin^{-1}x$, $\cos^{-1}x$

When $\downarrow$, $\uparrow$ denote floor function and ceiling function respectively, the following expressions hold for a natural number n.

$$\int_{a_n}^x\cdots\int_{a_1}^x \sin^{-1}x\,dx^n=\sum_{r=0}^{n/2\downarrow}\frac{x^{n-2r}}{(2r)!!^2(n-2r)!}\sin^{-1}x$$

$$+\sum_{r=1}^{n/2\uparrow}\sum_{s=0}^{n-2r+1}(-1)^s{}_{n-2r+1}C_s\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r+1}}{(n-2r+1)!}\sqrt{1-x^2}$$

Where $a_1=i\cdot 1.5088\cdots$, $a_2=0$, $a_3=-i\cdot 0.4758\cdots$, $a_4=0$, $\cdots$

$$\int_{a_n}^x\cdots\int_{a_1}^x \cos^{-1}x\,dx^n=\frac{x^n}{n!}\cos^{-1}x-\sum_{r=1}^{n/2\downarrow}\frac{x^{n-2r}}{(2r)!!^2(n-2r)!}\sin^{-1}x$$

$$-\sum_{r=1}^{n/2\uparrow}\sum_{s=0}^{n-2r+1}(-1)^s{}_{n-2r+1}C_s\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r+1}}{(n-2r+1)!}\sqrt{1-x^2}$$

Where $a_1=1$, $a_2=0$, $a_3=$ a complex number, $a_4=0$, $\cdots$

Proof

$$\int_{a_1}^{x} sin^{-1}x\,dx = x\,sin^{-1}x + \sqrt{1-x^2}$$

$$= \frac{x^1}{0!!^2 \cdot 1!} sin^{-1}x + \frac{1\cdot(-1)!!^2}{1!!^2}\frac{x^0}{0!}\sqrt{1-x^2}$$

$$\int_{a_2}^{x}\int_{a_1}^{x} sin^{-1}x\,dx^2 = \left(\frac{x^2}{2}+\frac{1}{4}\right) sin^{-1}x + \frac{3x}{4}\sqrt{1-x^2}$$

$$= \left(\frac{x^2}{0!!^2\cdot 2!}+\frac{x^0}{2!!^2\cdot 0!}\right) sin^{-1}x + \left(\frac{1\cdot(-1)!!^2}{1!!^2}-\frac{1\cdot 0!!^2}{2!!^2}\right)\frac{x^1}{1!}\sqrt{1-x^2}$$

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} sin^{-1}x\,dx^3 = \left(\frac{x^3}{6}+\frac{x}{4}\right) sin^{-1}x + \left(\frac{11x^2}{36}+\frac{1}{9}\right)\sqrt{1-x^2}$$

$$= \left(\frac{x^3}{0!!^2\cdot 3!}+\frac{x^1}{2!!^2\cdot 1!}\right) sin^{-1}x$$

$$+ \left(\left(\frac{1\cdot(-1)!!^2}{1!!^2}-\frac{2\cdot 0!!^2}{2!!^2}+\frac{1\cdot 1!!^2}{3!!^2}\right)\frac{x^2}{2!}+\frac{1\cdot(-1)!!^2}{3!!^2}\frac{x^0}{0!}\right)\sqrt{1-x^2}$$

$$\int_{a_4}^{x}\cdots\int_{a_1}^{x} sin^{-1}x\,dx^4 = \left(\frac{x^4}{24}+\frac{x^2}{8}+\frac{1}{64}\right) sin^{-1}x + \left(\frac{25x^3}{288}+\frac{55\,x}{576}\right)\sqrt{1-x^2}$$

$$= \left(\frac{x^4}{0!!^2\cdot 4!}+\frac{x^2}{2!!^2\cdot 2!}+\frac{x^0}{4!!^2\cdot 0!}\right) sin^{-1}x$$

$$+ \left(\left(\frac{1\cdot(-1)!!^2}{1!!^2}-\frac{3\cdot 0!!^2}{2!!^2}+\frac{3\cdot 1!!^2}{3!!^2}-\frac{1\cdot 2!!^2}{4!!^2}\right)\frac{x^3}{3!}+\left(\frac{1\cdot(-1)!!^2}{3!!^2}-\frac{1\cdot 0!!^2}{4!!^2}\right)\frac{x^1}{1!}\right)$$
$$\times\sqrt{1-x^2}$$

Hereafter, by induction, we obtain the desired expression.
The desired expression for $cos^{-1}x$ is also obtained in a similar way.

Note

Integration lower limits $a_1, a_2, \cdots, a_n$ are the solutions of the transcendental equations which made the formal right-hand sides equal to 0. For example, in the case of $sin^{-1}x$, a_1, a_2 are the solutions of the following transcendental equation respectively.

$$x\,sin^{-1}x + \sqrt{1-x^2} = 0 \quad , \quad \left(\frac{x^2}{2}+\frac{1}{4}\right) sin^{-1}x + \frac{3x}{4}\sqrt{1-x^2} = 0$$

On the other hand, if the integration lower limit is replaced by a fixed lower limit 0 in this formula, the following collateral high integrals are obtained. The difference among both is only the existence of a *constant-of-integration polynomial* .

Formula 4.4.2' : Collateral Higher Integrals of $sin^{-1}x$, $cos^{-1}x$

When $\downarrow, \uparrow$ denote floor function and ceiling function respectively, the following expressions hold for a natural number n .

$$\int_0^x \cdots \int_0^x \sin^{-1}x\, dx^n = \sum_{r=0}^{n/2\downarrow} \frac{x^{n-2r}}{(2r)!!^2 (n-2r)!} \sin^{-1}x$$

$$+ \sum_{r=1}^{n/2\uparrow} \sum_{s=0}^{n-2r+1} (-1)^s {}_{n-2r+1}C_s \frac{(s-1)!!^2}{(s+2r-1)!!^2} \frac{x^{n-2r+1}}{(n-2r+1)!} \sqrt{1-x^2}$$

$$+ \sum_{r=1}^{n/2\uparrow} \frac{x^{n-2r+1}}{(2r-1)!!^2 (n-2r+1)!}$$

$$\int_0^x \cdots \int_0^x \cos^{-1}x\, dx^n = \frac{x^n}{n!} \cos^{-1}x - \sum_{r=1}^{n/2\downarrow} \frac{x^{n-2r}}{(2r)!!^2 (n-2r)!} \sin^{-1}x$$

$$- \sum_{r=1}^{n/2\uparrow} \sum_{s=0}^{n-2r+1} (-1)^s {}_{n-2r+1}C_s \frac{(s-1)!!^2}{(s+2r-1)!!^2} \frac{x^{n-2r+1}}{(n-2r+1)!} \sqrt{1-x^2}$$

$$+ \sum_{r=1}^{n/2\uparrow} \frac{x^{n-2r+1}}{(2r-1)!!^2 (n-2r+1)!}$$

Example: Collaterral the 3rd order integral of $\cos^{-1}x$

```
Clear [t]
```

$$\text{CI}[n\_] := \frac{x^n}{n!} \text{ArcCos}[x] - \sum_{r=1}^{\text{Floor}[\frac{n}{2}]} \frac{x^{n-2r}}{((2r)!!)^2 (n-2r)!} \text{ArcSin}[x] - \sum_{r=1}^{\text{Ceiling}[\frac{n}{2}]} \sum_{s=0}^{n-2r+1} (-1)^s \text{Binomial}[n-2r+1, s] \frac{((s-1)!!)^2}{((s+2r-1)!!)^2} \times \frac{x^{n-2r+1}}{(n-2r+1)!} \sqrt{1-x^2} + \sum_{r=1}^{\text{Ceiling}[\frac{n}{2}]} \frac{x^{n-2r+1}}{((2r-1)!!)^2 (n-2r+1)!}$$

$$\text{RL}[n\_] := \frac{1}{\text{Gamma}[n]} \int_0^x (x-t)^{n-1} \text{ArcCos}[t]\, dt$$

$$\text{Expand [CI[3]]} \quad \frac{1}{9} + \frac{x^2}{2} - \frac{\sqrt{1-x^2}}{9} - \frac{11}{36} x^2 \sqrt{1-x^2} + \frac{1}{6} x^3 \text{ArcCos}[x] - \frac{1}{4} x\, \text{ArcSin}[x]$$

$$\text{Expand [RL[3]]} \quad \frac{1}{9} + \frac{x^2}{2} - \frac{\sqrt{1-x^2}}{9} - \frac{11}{36} x^2 \sqrt{1-x^2} + \frac{1}{6} x^3 \text{ArcCos}[x] - \frac{1}{4} x\, \text{ArcSin}[x]$$

Formula 4.4.3 : Higher Integrals of $\sec^{-1}x$, $\csc^{-1}x$

When $\downarrow$, $\uparrow$ denote floor function and ceiling function respectively, the following expressions hold for a natural number $n \geqq 2$.

$$\int_1^x \cdots \int_1^x \sec^{-1}x\, dx^n = \frac{x^n}{n!} \sec^{-1}x$$

$$- \sum_{r=0}^{(n-1)/2\downarrow} \frac{(2r-1)!!}{(2r)!!\,(2r+1)!} \frac{x^{n-2r-1}}{(n-2r-1)!} \log\left(x+\sqrt{x^2-1}\right)$$

$$+\sum_{r=1}^{n/2\downarrow}\sum_{s=0}^{n-2r}(-1)^s\frac{{}_{n-2r}C_s}{2r+s}\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}\sqrt{x^2-1}$$

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x}csc^{-1}x\,dx^n=\frac{x^n}{n!}csc^{-1}x$$

$$+\sum_{r=0}^{(n-1)/2\downarrow}\frac{(2r-1)!!}{(2r)!!(2r+1)!}\frac{x^{n-2r-1}}{(n-2r-1)!}log\left(x+\sqrt{x^2-1}\right)$$

$$-\sum_{r=1}^{n/2\downarrow}\sum_{s=0}^{n-2r}(-1)^s\frac{{}_{n-2r}C_s}{2r+s}\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}\sqrt{x^2-1}$$

Where $a_1, a_2, \cdots, a_n$ are all complex numbers.

Proof

$$\int_1^x sec^{-1}x\,dx=x\,sec^{-1}x-log\left(x+\sqrt{x^2-1}\right)$$

$$=\frac{x^1}{1!}sec^{-1}x-\frac{(-1)!!}{0!!1!}\frac{x^0}{0!}log\left(x+\sqrt{x^2-1}\right)$$

$$\int_1^x\int_1^x sec^{-1}x\,dx^2=\frac{x^2}{2}sec^{-1}x-x\,log\left(x+\sqrt{x^2-1}\right)+\frac{1}{2}\sqrt{x^2-1}$$

$$=\frac{x^2}{2!}sec^{-1}x-\frac{(-1)!!}{0!!1!}\frac{x^1}{1!}log\left(x+\sqrt{x^2-1}\right)+\frac{x^0}{0!}\frac{(-1)!!^2}{2\cdot1!!^2}\sqrt{x^2-1}$$

$$\int_1^x\int_1^x\int_1^x sec^{-1}x\,dx^3=\frac{x^3}{6}sec^{-1}x-\left(\frac{x^2}{2}+\frac{1}{12}\right)log\left(x+\sqrt{x^2-1}\right)+\frac{5x}{12}\sqrt{x^2-1}$$

$$=\frac{x^3}{3!}sec^{-1}x-\left(\frac{(-1)!!}{0!!1!}\frac{x^2}{2!}+\frac{1!!}{2!!3!}\frac{x^0}{0!}\right)log\left(x+\sqrt{x^2-1}\right)$$

$$+\frac{x^1}{1!}\left(\frac{(-1)!!^2}{2\cdot1!!^2}-\frac{0!!^2}{3\cdot2!!^2}\right)\sqrt{x^2-1}$$

$$\int_1^x\cdots\int_1^x sec^{-1}x\,dx^4=\frac{x^4}{24}sec^{-1}x-\left(\frac{x^3}{6}+\frac{x}{12}\right)log\left(x+\sqrt{x^2-1}\right)+\left(\frac{13x^2}{72}+\frac{1}{36}\right)\sqrt{x^2-1}$$

$$=\frac{x^4}{4!}sec^{-1}x-\left(\frac{(-1)!!}{0!!1!}\frac{x^3}{3!}+\frac{1!!}{2!!3!}\frac{x^1}{1!}\right)log\left(x+\sqrt{x^2-1}\right)$$

$$+\left(\frac{x^2}{2!}\left(\frac{(-1)!!^2}{2\cdot1!!^2}-\frac{2\cdot0!!^2}{3\cdot2!!^2}+\frac{1!!^2}{4\cdot3!!^2}\right)+\frac{x^0}{0!}\frac{(-1)!!^2}{4\cdot3!!^2}\right)\sqrt{x^2-1}$$

Hereafter, by induction, we obtain the desired expression.

The desired expression for $csc^{-1}x$ is also obtained in a similar way.

Example: The 5th order integral of $sec^{-1}x$

```
Higher integral of arcsec x
• HI := n-> x^n/n!*arcsec(x) - ln(x+sqrt(x^2-1))
          * sum((2*r-1)!!/((2*r)!!*(2*r+1)!)
          *(x^(n-2*r-1)/(n-2*r-1)!), r=0..floor((n-1)/2))
     + sqrt(x^2-1) * sum(sum((-1)^s*binomial(n-2*r,s)/(2*r
```

```
*((s-1)!!^2/(s+2*r-1)!!^2), s=0..n-2*r)
*(x^(n-2*r)/(n-2*r)!), r=1..floor(n/2))
```

$$n \to \frac{x^n}{n!} \cdot \operatorname{arcsec}(x) - \ln\left(x+\sqrt{x^2-1}\right) \cdot \left(\sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{2 \cdot r - 1!!}{2 \cdot r!! \cdot (2 \cdot r + 1)!} \cdot \frac{x^{n-2 \cdot r-1}}{(n-2 \cdot r-1)!} \right)$$
$$+ \sqrt{x^2-1} \cdot \left(\sum_{r=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left(\sum_{s=0}^{n-2 \cdot r} \frac{(-1)^s \cdot \binom{n-2 \cdot r}{s}}{2 \cdot r + s} \cdot \frac{s-1!!^2}{s+2 \cdot r-1!!^2} \right) \cdot \frac{x^{n-2 \cdot r}}{(n-2 \cdot r)!} \right)$$

- `HI(5)`

$$\left(\frac{77 \cdot x^3}{1440} + \frac{71 \cdot x}{2880} \right) \cdot \sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right) \cdot \left(\frac{x^4}{24} + \frac{x^2}{24} + \frac{1}{320} \right) + \frac{x^5 \cdot \arccos\left(\frac{1}{x}\right)}{120}$$

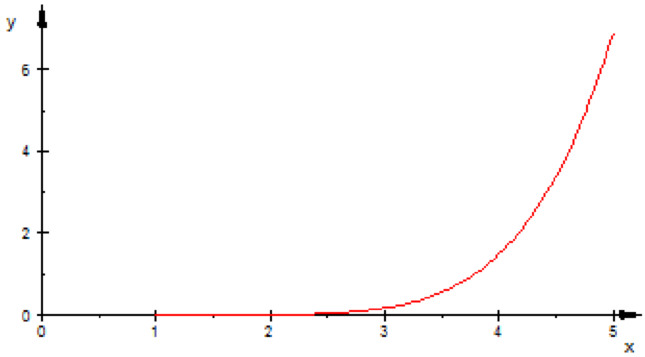
Riemann-Liouville Integral

- `RL := n-> 1/gamma(n)*int((x-t)^(n-1)*arcsec(t),t=1..x)`

$$n \to \frac{1}{\Gamma(n)} \cdot \int_1^x (x-t)^{n-1} \cdot \operatorname{arcsec}(t) \, dt$$

Blue: Riemann-Liouville Integral , Red: Higer Integral

- `plotfunc2d(RL(5),HI(5), x=0..5)`



Since both sides overlapped exactly, the blue (Riemann-Liouville Integral) can not be seen.

On the other hand, if the integration lower limit is replaced by a fixed lower limit 1 in this formula, the following collateral high integral is obtained. The difference among both is only the existence of a constant-of-integration polynomial.

Formula 4.4.3' : Collateral Higher Integral of $\csc^{-1}x$

When $\downarrow$, $\uparrow$ denote floor function and ceiling function respectively, the following expressions hold for a natural number $n \geqq 2$.

$$\int_1^x \cdots \int_1^x \csc^{-1}x \, dx^n = \frac{x^n}{n!} \csc^{-1}x$$
$$+ \sum_{r=0}^{(n-1)/2\downarrow} \frac{(2r-1)!!}{(2r)!!\,(2r+1)!} \frac{x^{n-2r-1}}{(n-2r-1)!} \log\left(x+\sqrt{x^2-1}\right)$$
$$- \sum_{r=1}^{n/2\downarrow} \sum_{s=0}^{n-2r} (-1)^s \frac{{}_{n-2r}C_s}{2r+s} \frac{(s-1)!!^2}{(s+2r-1)!!^2} \frac{x^{n-2r}}{(n-2r)!} \sqrt{x^2-1}$$
$$- \frac{\pi}{2 \cdot n!} \sum_{r=0}^{n-1} {}_nC_r (x-1)^r$$

4.5 Higher Integrals of Inverse Hyperbolic Functions

Formula 4.5.1 : Higher Integrals of $tanh^{-1}x$, $coth^{-1}x$

When $\downarrow, \uparrow, \psi(x)$ denote floor function, ceiling function and digamma function respectively, the following expressions hold for a natural number $n \geqq 2$.

$$\int_0^x \cdots \int_0^x tanh^{-1}x\, dx^n = \frac{tanh^{-1}x}{n!} \sum_{k=0}^{n/2\downarrow} {}_nC_{n-2k}\, x^{n-2k}$$

$$+ \frac{log(1-x^2)}{2\cdot n!} \sum_{k=1}^{n/2\uparrow} {}_nC_{n+1-2k}\, x^{n+1-2k}$$

$$- \frac{1}{n!} \sum_{r=1}^{n/2\downarrow} {}_nC_{n+1-2r} \{\psi(1+n) - \psi(2r)\} x^{n+1-2r}$$

$$\int_0^x \cdots \int_0^x coth^{-1}x\, dx^n = \frac{x^n}{n!} coth^{-1}x + \frac{tanh^{-1}x}{n!} \sum_{k=1}^{n/2\downarrow} {}_nC_{n-2k}\, x^{n-2k}$$

$$+ \frac{log(1-x^2)}{2\cdot n!} \sum_{k=1}^{n/2\uparrow} {}_nC_{n+1-2k}\, x^{n+1-2k}$$

$$- \frac{1}{n!} \sum_{r=1}^{n/2\downarrow} {}_nC_{n+1-2r} \{\psi(1+n) - \psi(2r)\} x^{n+1-2r} \qquad |x|<1$$

Proof

The integral up to the 4th order of $tanh^{-1}x$ are as follows.

$$\int_0^x tanh^{-1}x\, dx = x\,tanh^{-1}x + \frac{1}{2} log(1-x^2)$$

$$\int_0^x \int_0^x tanh^{-1}x\, dx^2 = \left(\frac{x^2}{2} + \frac{1}{2}\right) tanh^{-1}x + \frac{x}{2} log(1-x^2) - \frac{x}{2}$$

$$\int_0^x \int_0^x \int_0^x tanh^{-1}x\, dx^3 = \left(\frac{x^3}{6} + \frac{x}{2}\right) tanh^{-1}x + \left(\frac{x^2}{4} + \frac{1}{12}\right) log(1-x^2) - \frac{5x^2}{12}$$

$$\int_0^x \cdots \int_0^x tanh^{-1}x\, dx^4 = \left(\frac{x^4}{24} + \frac{x^2}{4} + \frac{1}{24}\right) tanh^{-1}x + \left(\frac{x^3}{12} + \frac{x}{12}\right) log(1-x^2) - \frac{13x^3}{72} - \frac{x}{24}$$

The absolute values of these coefficients are all consistent with the absolute values in Formula 4.4.1 . Therefore, these coefficients can be easily obtained by changing the signs of the coefficients in Formula 4.4.1. This is the same also for $coth^{-1}x$.

Example: The 3rd order integral of $coth^{-1}x$

```
Clear [t]

HI[n_] := x^n/n! ArcCoth[x] + ArcTanh[x]/n! Sum[Binomial[n, n - 2 k] x^(n-2 k), {k, 1, Floor[n/2]}] +
   Log[1 - x^2]/(2 n!) Sum[Binomial[n, n + 1 - 2 k] x^(n+1-2 k), {k, 1, Ceiling[n/2]}] -
   1/n! Sum[Binomial[n, n + 1 - 2 r] (PolyGamma[1 + n] - PolyGamma[2 r]) x^(n+1-2 r), {r, 1, Floor[n/2]}]
```

```
RL[n_] := 1/Gamma[n] ∫_0^x (x - t)^(n-1) ArcCoth[t] dt
```

```
Simplify[HI[3]]   1/12 (-5 x^2 + 2 x^3 ArcCoth[x] + 6 x ArcTanh[x] + (1 + 3 x^2) Log[1 - x^2])

RL[3]   1/12 (-5 x^2 + 2 x^3 ArcCoth[x] + 6 x ArcTanh[x] + (1 + 3 x^2) Log[1 - x]
             + Log[1 + x] + 3 x^2 Log[1 + x])
```

Formula 4.5.2 : Higher Integrals of $sinh^{-1}x$, $cosh^{-1}x$

When ↓, ↑ denote floor function and ceiling function respectively, the following expressions hold for a natural number n .

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} sinh^{-1}x\, dx^n = \sum_{r=0}^{n/2\downarrow}\frac{(-1)^r x^{n-2r}}{(2r)!!^2(n-2r)!} sinh^{-1}x$$
$$+\sum_{r=1}^{n/2\uparrow}\sum_{s=0}^{n-2r+1}(-1)^{r+s}{}_{n-2r+1}C_s\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r+1}}{(n-2r+1)!}\sqrt{1+x^2}$$

Where $a_1 = 1.5088\cdots$, $a_2 = 0$, $a_3 = -0.4758\cdots$, $a_4 = 0$, $\cdots$

$$\int_{1}^{x}\cdots\int_{1}^{x} cosh^{-1}x\, dx^n = \sum_{r=0}^{n/2\downarrow}\frac{x^{n-2r}}{(2r)!!^2(n-2r)!} cosh^{-1}x$$
$$-\sum_{r=1}^{n/2\uparrow}\sum_{s=0}^{n-2r+1}(-1)^{s}{}_{n-2r+1}C_s\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r+1}}{(n-2r+1)!}\sqrt{x^2-1}$$

Proof

The integral up to the 4th order of $sinh^{-1}x$ are as follows.

$$\int_{a_1}^{x} sinh^{-1}x\, dx = x sinh^{-1}x - \sqrt{1+x^2}$$

$$\int_{a_2}^{x}\int_{a_1}^{x} sinh^{-1}x\, dx^2 = \left(\frac{x^2}{2}-\frac{1}{4}\right)sinh^{-1}x - \frac{3x}{4}\sqrt{1+x^2}$$

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} sinh^{-1}x\, dx^3 = \left(\frac{x^3}{6}-\frac{x}{4}\right)sinh^{-1}x - \left(\frac{11x^2}{36}-\frac{1}{9}\right)\sqrt{1+x^2}$$

$$\int_{a_4}^{x}\cdots\int_{a_1}^{x} sinh^{-1}x\, dx^4 = \left(\frac{x^4}{24}-\frac{x^2}{8}+\frac{1}{64}\right)sinh^{-1}x - \left(\frac{25x^3}{288}-\frac{55\,x}{576}\right)\sqrt{1+x^2}$$

The absolute values of these coefficients are all consistent with the absolute values in Formula 4.4.2 . Therefore, these coefficients can be easily obtained by changing the signs of the coefficients in Formula 4.4.2. This is the same also for $cosh^{-1}x$.

Example: The 4th order integral of $cosh^{-1}x$

```
Clear[t]

HI[n_] := Sum[x^(n-2 r) / (((2 r)!!)^2 (n - 2 r)!) ArcCosh[x], {r, 0, Floor[n/2]}]
```

$$-\sum_{r=1}^{\text{Ceiling}[\frac{n}{2}]}\sum_{s=0}^{n-2r+1}(-1)^s \text{Binomial}[n-2r+1,s]\frac{((s-1)!!)^2}{((s+2r-1)!!)^2}\times\frac{x^{n-2r+1}}{(n-2r+1)!}\sqrt{x^2-1}$$

$$\text{RL}[n\_] := \frac{1}{\text{Gamma}[n]}\int_1^x (x-t)^{n-1}\,\text{ArcCosh}[t]\,dt$$

$$\text{Simplify}[\text{HI}[4]] \quad \frac{1}{576}\left(-5x\sqrt{-1+x^2}\left(11+10x^2\right)+3\left(3+24x^2+8x^4\right)\text{ArcCosh}[x]\right)$$

$$\text{RL}[4] \quad \frac{1}{576}\left(-5\sqrt{-1+x}\;x\sqrt{1+x}\left(11+10x^2\right)+3\left(3+24x^2+8x^4\right)\text{ArcCosh}[x]\right)$$

On the other hand, if the integration lower limit is replaced by a fixed lower limit 0 in this formula, the following collateral high integral is obtained. The difference among both is only the existence of a constant-of-integration polynomial.

Formula 4.5.2' : Collateral Higher Integral of $sinh^{-1}x$

When $\downarrow$, $\uparrow$ denote floor function and ceiling function respectively, the following expressions hold for a natural number n .

$$\int_0^x\cdots\int_0^x sinh^{-1}x\,dx^n = \sum_{r=0}^{n/2\downarrow}\frac{(-1)^r x^{n-2r}}{(2r)!!^2(n-2r)!}sinh^{-1}x$$

$$+\sum_{r=1}^{n/2\uparrow}\sum_{s=0}^{n-2r+1}(-1)^{r+s}{}_{n-2r+1}C_s\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r+1}}{(n-2r+1)!}\sqrt{1+x^2}$$

$$+\sum_{r=1}^{n/2\uparrow}\frac{(-1)^{r-1}x^{n-2r+1}}{(2r-1)!!^2(n-2r+1)!}$$

Formula 4.5.3 : Higher Integrals of $sech^{-1}x$, $csch^{-1}x$

When $\downarrow$, $\uparrow$ denote floor function and ceiling function respectively, the following expressions hold for a natural number $n \geqq 2$.

$$\int_{a_n}^x\cdots\int_{a_1}^x sech^{-1}x\,dx^n = \frac{x^n}{n!}sech^{-1}x + \sum_{r=0}^{(n-1)/2\downarrow}\frac{(2r-1)!!}{(2r)!!\,(2r+1)!}\frac{x^{n-2r-1}}{(n-2r-1)!}sin^{-1}x$$

$$+\sum_{r=1}^{n/2\downarrow}\sum_{s=0}^{n-2r}(-1)^s\frac{{}_{n-2r}C_s}{2r+s}\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}\sqrt{1-x^2}$$

Where $a_1 = a_3 = a_5 = \cdots = 0$, $a_2, a_4, a_6, \cdots$ are complex numbers

$$\int_{a_n}^x\cdots\int_{a_1}^x csch^{-1}x\,dx^n = \frac{x^n}{n!}csch^{-1}x + \sum_{r=0}^{(n-1)/2\downarrow}\frac{(-1)^r(2r-1)!!}{(2r)!!\,(2r+1)!}\frac{x^{n-2r-1}}{(n-2r-1)!}sinh^{-1}x$$

$$+\sum_{r=1}^{n/2\downarrow}\sum_{s=0}^{n-2r}(-1)^{r+s}\frac{{}_{n-2r}C_s}{2r+s}\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}\sqrt{x^2+1}$$

Where $a_1 = 0$, $a_2 = 0.6079\cdots$, $a_3 = 0$, $a_4 = 1.5539$, $\cdots$

Proof

The integral up to the 4th order of $sech^{-1}x$ are as follows.

$$\int_{a_1}^{x} sech^{-1}x\,dx = x\,sech^{-1}x + sin^{-1}x$$

$$\int_{a_2}^{x}\int_{a_1}^{x} sech^{-1}x\,dx^2 = \frac{x^2}{2}sech^{-1}x + x\,sin^{-1}x + \frac{1}{2}\sqrt{1-x^2} - \frac{1}{2}$$

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} sech^{-1}x\,dx^3 = \frac{x^3}{6}sech^{-1}x + \left(\frac{x^2}{2}+\frac{1}{12}\right)sin^{-1}x + \frac{5x}{12}\sqrt{1-x^2}$$

$$\int_{a_4}^{x}\cdots\int_{a_1}^{x} sech^{-1}x\,dx^4 = \frac{x^4}{24}sech^{-1}x + \left(\frac{x^3}{6}+\frac{x}{12}\right)sin^{-1}x + \left(\frac{13x^2}{72}+\frac{1}{36}\right)\sqrt{1-x^2}$$

The absolute values of these coefficients are all consistent with the absolute values in Formula 4.4.3. Therefore, these coefficients can be easily obtained by changing the signs of the coefficients in Formula 4.4.3. This is the same also for $csch^{-1}x$.

On the other hand, if the integration lower limit is replaced by fixed lower limit 0 in this formula, the following collateral high integrals are obtained. The difference among both is only the existence of a constant-of-integration polynomial.

Formula 4.5.3' : Collateral Higher Integrals of $sech^{-1}x$, $csch^{-1}x$

When $\downarrow$, $\uparrow$ denote floor function and ceiling function respectively, the following expressions hold for a natural number $n \geqq 2$.

$$\int_0^{x}\cdots\int_0^{x} sech^{-1}x\,dx^n = \frac{x^n}{n!}sech^{-1}x + \sum_{r=0}^{(n-1)/2\downarrow}\frac{(2r-1)!!}{(2r)!!\,(2r+1)!}\frac{x^{n-2r-1}}{(n-2r-1)!}sin^{-1}x$$
$$+\sum_{r=1}^{n/2\downarrow}\sum_{s=0}^{n-2r}(-1)^s\frac{{}_{n-2r}C_s}{2r+s}\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}\sqrt{1-x^2}$$
$$-\sum_{r=1}^{n/2\downarrow}\frac{1}{2r(2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}$$

$$\int_0^{x}\cdots\int_0^{x} csch^{-1}x\,dx^n = \frac{x^n}{n!}csch^{-1}x + \sum_{r=0}^{(n-1)/2\downarrow}\frac{(-1)^r(2r-1)!!}{(2r)!!\,(2r+1)!}\frac{x^{n-2r-1}}{(n-2r-1)!}sinh^{-1}x$$
$$+\sum_{r=1}^{n/2\downarrow}\sum_{s=0}^{n-2r}(-1)^{r+s}\frac{{}_{n-2r}C_s}{2r+s}\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}\sqrt{x^2+1}$$
$$-\sum_{r=1}^{n/2\downarrow}\frac{(-1)^r}{2r(2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}$$

Example: Collateral the 4th order integral of $sech^{-1}x$

Collateral higher integral of arcsech

```
• CI := n-> x^n/n!*arcsech(x)
         + arcsin(x)*sum((2*r-1)!!/((2*r)!!*(2*r+1)!)
                   *(x^(n-2*r-1)/(n-2*r-1)!), r=0..floor((n-1)/2
         + sqrt(1-x^2)*sum(sum((-1)^s*binomial(n-2*r,s)/(2*r+s)
                     *((s-1)!!^2/(s+2*r-1)!!^2), s=0..n-2*r)
                     *(x^(n-2*r)/(n-2*r)!), r=1..floor(n/2))
```

```
    - sum(1/(2*r*(2*r-1)!!^2)*(x^(n-2*r)/(n-2*r)!),
                                        r=1..floor(n/2))
```

$$n \to \frac{x^n}{n!} \cdot \operatorname{arcsech}(x) + \arcsin(x) \cdot \left(\sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{2 \cdot r - 1!!}{2 \cdot r!! \cdot (2 \cdot r + 1)!} \cdot \frac{x^{n-2 \cdot r-1}}{(n-2 \cdot r-1)!} \right) + \sqrt{1-x^2} \cdot \left(\sum_{r=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left(\sum_{s=0}^{n-2 \cdot r} \frac{(-1)^s \cdot \binom{n-2 \cdot r}{s}}{2 \cdot r + s} \cdot \frac{s-1!!^2}{s+2 \cdot r - 1!!^2} \right) \cdot \frac{x^{n-2 \cdot r}}{(n-2 \cdot r)!} \right) - \left(\sum_{r=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{2 \cdot r \cdot 2 \cdot r - 1!!^2} \cdot \frac{x^{n-2 \cdot r}}{(n-2 \cdot r)!} \right)$$

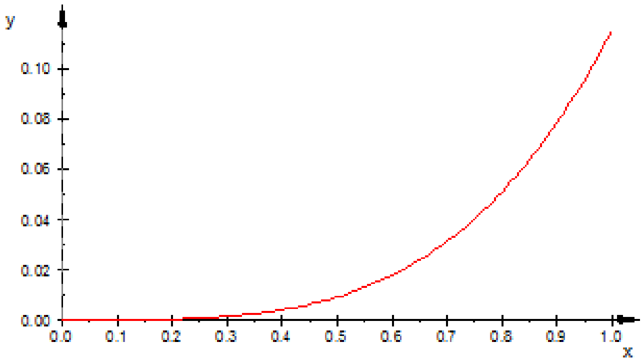
Riemann-Liouville Integral

- `RL := n-> 1/gamma(n)*int((x-t)^(n-1)*arcsech(t),t=0.000001..x)`

$$n \to \frac{1}{\Gamma(n)} \cdot \int_{0.000001}^{x} (x-t)^{n-1} \cdot \operatorname{arcsech}(t) \, dt$$

Blue: Riemann-Liouville Integral , Red: Collateral higer integral

- `plotfunc2d(RL(4),CI(4), x=0..1)`



Since both sides overlapped exactly, blue (Riemann-Liouville Integral) can not be seen.

This result also shows that Formula 4.5.3 is right.

4.6 Termwise Higher Integral and Taylor series of higher primitive function

4.6.1 Theorem of the Termwise Higher Integral with a fixed lower limit

Theorem 4.6.1

Let $f^{<r>}$ $r=0,1,\cdots,n$ be continuous functions defined on $[a,b]$ and $f^{<r+1>}$ be the arbi-trary primitive function of $f^{<r>}$. At this time, if $f(x)$ can be expanded to the Taylor series around a, the following expressions hold for $x\in[a,b]$.

$$f^{<n>}(x) = \sum_{r=0}^{n-1} f^{<n-r>}(a)\frac{(x-a)^r}{r!} + \sum_{r=0}^{\infty} f^{(r)}(a)\frac{(x-a)^{n+r}}{(n+r)!} \tag{1.1}$$

$$= \sum_{r=0}^{\infty} f^{<n-r>}(a)\frac{(x-a)^r}{r!} \tag{1.2}$$

Proof

The following expression holds according to Theorem 4.1.3 .

$$f^{<n>}(x) = \sum_{r=0}^{n-1} f^{<n-r>}(a)\frac{(x-a)^r}{r!} + \int_a^x\cdots\int_a^x f(x)\,dx^n$$

On the other hand, if $f(x)$ can be expanded to the Taylor series around a,

$$f(x) = \sum_{r=0}^{\infty} f^{(r)}(a)\frac{(x-a)^r}{r!}$$

This can be integrated termwise as follows.

$$\int_a^x\cdots\int_a^x f(x)\,dx^n = \sum_{r=0}^{\infty} f^{(r)}(a)\frac{(x-a)^{n+r}}{(n+r)!}$$

Substituting this for the above,

$$f^{<n>}(x) = \sum_{r=0}^{n-1} f^{<n-r>}(a)\frac{(x-a)^r}{r!} + \sum_{r=0}^{\infty} f^{(r)}(a)\frac{(x-a)^{n+r}}{(n+r)!} \tag{1.1}$$

Here,

$$\sum_{r=0}^{\infty} f^{(r)}(a)\frac{(x-a)^{n+r}}{(n+r)!} = \sum_{r=n}^{\infty} f^{(r-n)}(a)\frac{(x-a)^r}{r!} = \sum_{r=n}^{\infty} f^{<n-r>}(a)\frac{(x-a)^r}{r!}$$

Using this for (1.1),

$$f^{<n>}(x) = \sum_{r=0}^{n-1} f^{<n-r>}(a)\frac{(x-a)^r}{r!} + \sum_{r=n}^{\infty} f^{<n-r>}(a)\frac{(x-a)^r}{r!}$$

$$= \sum_{r=0}^{\infty} f^{<n-r>}(a)\frac{(x-a)^r}{r!} \tag{1.2}$$

Remark

(1.1) shows that

$f^{<n>}(x)$ consists of the termwise integral of $f(x)$ and the constant-of-integration-polynomial.

(1.2) shows that these constitute the Taylor series of $f^{<n>}(x)$.

Conclusion

(1) A termwise higher integral with a fixed lower limit is collateral generally.

(2) It is the following case that a termwise higher integral with a fixed lower limit is lineal.

$$f^{<r>}(a)=0 \quad for \ r=1,2,\cdots,n \quad \& \quad f^{(s)}(a)\neq 0,\pm\infty \quad for\ at\ least\ one\ \ s\geqq 0$$

Example 1: Termwise higher integral of $f(x)=e^x$

Let $f^{<n>}(x)$ be the lineal hegher primitive function of e^x. Then,

$$f^{<n>}(x)=e^x \quad , \quad f^{<n-r>}(a)=e^a \quad , \quad f^{(n)}(a)=e^a \qquad for \ a\neq -\infty$$

Substituting these for the theorem,

$$\left(e^x\right)^{<n>} = \sum_{r=0}^{n-1} e^a \frac{(x-a)^r}{r!} + \sum_{r=0}^{\infty} e^a \frac{(x-a)^{n+r}}{(n+r)!} = \sum_{r=0}^{\infty} e^a \frac{(x-a)^r}{r!}$$

The 1st term (constant-of-integration-polynomial) is not 0 for $a\neq -\infty$.

On the other hand, $f(x)=e^x$ is expanded to the Taylor series around a as follows.

$$e^x = \sum_{r=0}^{\infty} e^a \frac{(x-a)^r}{r!}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain

$$\int_a^x\cdots\int_a^x e^x dx^n = \sum_{r=0}^{\infty} e^a \frac{(x-a)^{n+r}}{(n+r)!}$$

This termwise higher integral is a part of the Taylor series of $\left(e^x\right)^{<n>}$. Therefore, this is a collateral haigher integral.

Example 2: Termwise higher integral of $f(x)=tan^{-1}x$

Let $f^{<n>}(x)$ be the lineal higher primitive function of $tan^{-1}x$. Then, $a=0$ is a zero of these functions according to Formula 4.4.1 . That is, $f^{<n-r>}(0)=0 \ \ for \ r=0,2,\cdots,n-1$.

Therefore, the constant-of-integration-polynomial becomes as follows.

$$\sum_{r=0}^{n-1} f^{<n-r>}(0)\frac{(x-0)^r}{r!} = 0 \tag{w1}$$

On the other hand, from Formula 9.2.7 (later **9.2**),

$$\left(tan^{-1}x\right)^{(n)} = (-1)^n \frac{(n-1)!}{\left(x^2+1\right)^n} \sum_{k=1}^{n/2\uparrow} (-1)^k {}_nC_{n+1-2k}\, x^{n+1-2k}$$

From this

$$f^{(n)}(0) = (-1)^n (n-1)! \sum_{k=1}^{n/2\uparrow} (-1)^k {}_nC_{n+1-2k}\, 0^{n+1-2k}$$

When n is even, since $n+1-2k \neq 0 \ \ k=1,\cdots,n/2\uparrow$, $f^{(n)}(0)=0$.

When n is odd,

$$\begin{aligned} f^{(2r+1)}(0) &= (-1)^{2r+1}(2r)! \sum_{k=1}^{r+1} (-1)^k {}_{2r+1}C_{2r+2-2k}\, 0^{2r+2-2k} \\ &= (-1)^{2r+1}(2r)! \left\{ \sum_{k=1}^{r} (-1)^k {}_{2r+1}C_{2r+2-2k}\, 0^{2r+2-2k} + (-1)^{r+1} {}_{2r+1}C_0 0^0 \right\} \\ &= (-1)^r (2r)! \end{aligned}$$

Thus, the Taylor series around 0 of $f(x)=tan^{-1}x$ becoms as follows.

$$tan^{-1}x = \sum_{r=0}^{\infty}(-1)^r \frac{(2r)!}{(2r+1)!}x^{2r+1}$$

And we obtain the following termwise higher integral.

$$\int_0^x \cdots \int_0^x tan^{-1}x\,dx^n = \sum_{r=0}^{\infty}(-1)^r \frac{(2r)!}{(n+2r+1)!}x^{n+2r+1} \tag{w2}$$

Substituting (w1) and (w2) for the thorem,

$$\left(tan^{-1}x\right)^{<n>} = \sum_{r=0}^{\infty}(-1)^r \frac{(2r)!}{(n+2r+1)!}x^{n+2r+1}$$

The termwise higher integral (w2) is all of Taylor series of $\left(tan^{-1}x\right)^{<n>}$. Therefore, this is the lineal higher integral.

4.6.2 Taylor expansion of the higher integral of $log\,x$

The higher primitive function of $log\,x$ has already shown in the previous section. If this and the Taylor series around 1 of this is compared, interesting results are obtained.

Formula 4.6.2

When a harmonic number is denoted by $H_k = \sum_{s=1}^{k}\frac{1}{s} = \psi(1+k)+\gamma$, the following expressions hold.

$$\int_0^x \cdots \int_0^x log\,x\,dx^n = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}(x-1)^{n+r}}{r(r+1)\cdots(r+n)} - \frac{1}{n!}\sum_{r=0}^{n-1}{}_nC_r H_{n-r}(x-1)^r \tag{2.1}$$

$$\sum_{r=1}^{\infty}\frac{1}{r(r+1)\cdots(r+n)} = \frac{(-1)^{n-1}}{n!}\sum_{r=0}^{n-1}(-1)^r{}_nC_r H_{n-r} \tag{2.2}$$

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r(r+1)\cdots(r+n)} = \frac{2^n}{n!}\left(log\,2 - H_n\right) + \frac{1}{n!}\sum_{r=0}^{n-1}{}_nC_r H_{n-r} \tag{2.3}$$

Proof

From Formula 4.3.3 , the lineal higher primitive function of $f(x)=log\,x$ as follows.

$$(log\,x)^{<r>} = \frac{x^r}{r!}\left(log\,x - \sum_{s=1}^{r}\frac{1}{s}\right) \qquad r=1,2,\cdots,n$$

From Formula 9.2.3 (later **9.2**), the higher derivative function of this as follows.

$$(log\,x)^{(r)} = (-1)^{r-1}\frac{(r-1)!}{x^r} \qquad r=1,2,3,\cdots$$

Substituting these for Thorem 4.6.1 (1.1) ,

$$\frac{x^n}{n!}\left(log\,x - \sum_{s=1}^{n}\frac{1}{s}\right)$$

$$= \sum_{r=0}^{n-1}\frac{a^{n-r}}{(n-r)!}\left(log\,a - \sum_{s=1}^{n-r}\frac{1}{s}\right)\frac{(x-a)^r}{r!} + \sum_{r=0}^{\infty}(-1)^{r-1}\frac{(r-1)!}{a^r}\frac{(x-a)^{n+r}}{(n+r)!}$$

Substitute $a=1$, $H_k = \sum_{s=1}^{k}\frac{1}{s}$ for the right side, then

$$\frac{x^n}{n!}\left(log\,x - \sum_{s=1}^{n}\frac{1}{s}\right) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}(x-1)^{n+r}}{r(r+1)\cdots(r+n)} - \frac{1}{n!}\sum_{r=0}^{n-1}{}_nC_r H_{n-r}(x-1)^r \tag{2.1}$$

Substitute $x=0$, $x=2$ for this respectively, then

$$\frac{0^n}{n!}\left(\log 0-\sum_{s=1}^{n}\frac{1}{s}\right)=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}(-1)^{n+r}}{r(r+1)\cdots(r+n)}-\frac{1}{n!}\sum_{r=0}^{n-1}{}_nC_r H_{n-r}(-1)^r$$

$$\frac{2^n}{n!}\left(\log 2-\sum_{s=1}^{n}\frac{1}{s}\right)=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}1^{n+r}}{r(r+1)\cdots(r+n)}-\frac{1}{n!}\sum_{r=0}^{n-1}{}_nC_r H_{n-r}1^r$$

From these, we obtain (2.2), (2.3) .

Example: $n=3$

$$\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{2\cdot3\cdot4\cdot5}+\frac{1}{3\cdot4\cdot5\cdot6}+\frac{1}{4\cdot5\cdot6\cdot7}+\cdots$$

$$=\frac{(-1)^2}{3!}\left\{\binom{3}{0}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\binom{3}{1}\left(1+\frac{1}{2}\right)+\binom{3}{2}\right\}=\frac{1}{18}$$

$$\frac{1}{1\cdot2\cdot3\cdot4}-\frac{1}{2\cdot3\cdot4\cdot5}+\frac{1}{3\cdot4\cdot5\cdot6}-\frac{1}{4\cdot5\cdot6\cdot7}+-\cdots$$

$$=\frac{2^3}{3!}\left\{\log 2-\left(1+\frac{1}{2}+\frac{1}{3}\right)\right\}+\frac{1}{3!}\left\{\binom{3}{0}\left(1+\frac{1}{2}+\frac{1}{3}\right)+\binom{3}{1}\left(1+\frac{1}{2}\right)+\binom{3}{2}\right\}$$

$$=\frac{4}{3}\log 2-\frac{8}{9}$$

By-product

The following equations are known about (2.2) and (2.3).

$$\sum_{r=1}^{\infty}\frac{1}{r(r+1)\cdots(r+n)}=\frac{1}{n\cdot n!} \tag{2.2'}$$

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r(r+1)\cdots(r+n)}=\frac{1}{n!}\int_0^1\frac{(1-t)^n}{1+t}dt \tag{2.3'}$$

Therefore , the following expressions are derived from (2.2),(2.2') and (2.3),(2.3')

$$\sum_{r=0}^{n-1}(-1)^r{}_nC_r H_{n-r}=\frac{(-1)^{n-1}}{n} \qquad \{H_r=\psi(1+r)+\gamma\} \tag{2.4}$$

$$\int_0^1\frac{(1-x)^n}{1+x}dx=2^n(\log 2-H_n)+\sum_{r=1}^{n}{}_nC_r H_r \qquad \{H_r=\psi(1+r)+\gamma\} \tag{2.5}$$

(2.5) can be generalized further, and the following expression holds for $p>0$.

$$\int_0^1\frac{(1-x)^p}{1+x}dx=2^p\{\log 2-\psi(1+p)-\gamma\}+\sum_{r=1}^{\infty}\binom{p}{r}\{\psi(1+r)+\gamma\} \tag{2.5'}$$

Example1

(-1)^rnCr Hn-r

```
f1[n_] := Sum[(-1)^r Binomial[n, r] HarmonicNumber[n - r], {r, 0, n-1}]
fr[n_] := (-1)^(n-1)/n
f1[6]    -1/6                fr[6]    -1/6
```

Example2

A certain definite integral

$$\text{f1}[p\_] := \int_0^1 \frac{(1-x)^p}{1+x}\, dx$$

$$\text{fr}[p\_] := 2^p\,(\text{Log}[2] - \text{PolyGamma}[1+p] - \text{EulerGamma}) + \sum_{r=1}^{700} \text{Binomial}[p, r]\,(\text{PolyGamma}[1+r] + \text{EulerGamma})$$

N[f1[3 / 2]]	N[fr[3 / 2]]
0.319135	0.319135

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