23 Higher Integral of Composition

23.1 Formula of Higher Integral of Composition

Formula 23.1.1

Let $f=f(x)$, $g^{<n>}$ be the lineal higher primitive function of $g(f)$ and $h, h^{(k)}$ are the functions of f such that

$$h=\frac{dx}{df}=\frac{1}{f^{(1)}} \quad , \quad h^{(k)}=\frac{d^k h}{df^k} \qquad k=1,2,3,\cdots$$

Let S_k, M_k, r_k are the polynomials such that

$$S_k=\sum_{r_{k1}=0}^{m_k-1}\binom{-1}{r_{k1}}\sum_{r_{k2}=0}^{r_{k1}}\binom{r_{k1}}{r_{k2}}\sum_{r_{k3}=0}^{r_{k2}}\binom{r_{k2}}{r_{k3}}\cdots\sum_{r_{kk}=0}^{r_{kk-1}}\binom{r_{kk-1}}{r_{kk}} \qquad k=1,2,\cdots,n$$

$$M_k=(-1)^{m_k}\sum_{r_{k2}=0}^{m_k}\binom{m_k}{r_{k2}}\sum_{r_{k3}=0}^{r_{k2}}\binom{r_{k2}}{r_{k3}}\sum_{r_{k4}=0}^{r_{k3}}\binom{r_{k3}}{r_{k4}}\cdots\sum_{r_{kk}=0}^{r_{kk-1}}\binom{r_{kk-1}}{r_{kk}} \qquad k=2,3,\cdots,n$$

$$R_{jk}=\sum_{i=k}^{j}r_{ik} \qquad j,k=1,2,\cdots,n$$

And let a, f_a are the zeros of the lineal higher primitive functions of $g\{f(x)\}$, gh respectively. Then, the lineal higher integral with respect to x of the composition $g\{f(x)\}$ is expressed as follows.

$$\int_a^x\cdots\int_a^x\{g(f(x))\}dx^n=S_1S_2\cdots S_n\, g^{\langle n+R_{n1}\rangle}h^{(R_{n1}-R_{n2})}\cdots h^{(R_{nn-1}-R_{nn})}h^{(R_{nn})}+R_{m_1}^n \qquad \text{(n.1)}$$

$$R_{m_1}^n=(-1)^{m_1}\int_{f_a}^f\left(\int_{f_a}^f\left(\int_{f_a}^f\cdots\left(\int_{f_a}^f g^{\langle m_1\rangle}h^{(m_1)}df\right)\cdots hdf\right)hdf\right)hdf \qquad \text{(n-fold nest)}$$

$$+S_1M_2\int_{f_a}^f\left(\int_{f_a}^f\cdots\left(\int_{f_a}^f g^{\langle 1+R_{11}+m_2\rangle}h^{(R_{11}-R_{22}+m_2)}h^{(R_{22})}df\right)\cdots hdf\right)hdf$$

$$+S_1S_2M_3\int_{f_a}^f\cdots\left(\int_{f_a}^f g^{\langle 2+R_{21}+m_3\rangle}h^{(R_{21}-R_{32}+m_3)}h^{(R_{32}-R_{33})}h^{(R_{33})}df\right)\cdots hdf$$

$$\vdots$$

$$+S_1S_2\cdots S_{n-1}M_n\int_{f_a}^f g^{\langle n-1+R_{n-11}+m_n\rangle}h^{(R_{n-11}-R_{n2}+m_n)}h^{(R_{n2}-R_{n3})}\cdots h^{(R_{nn})}df \qquad \text{(n.r)}$$

Proof

$dx=\dfrac{df}{f^{(1)}}$ from $g=g(f)$, $f=f(x)$.

Then, considering the transformation of variabl such that $x\rightarrow f$, $[a\,,\,x]\rightarrow[f_a\,,\,f]$,

$$\int_a^x\{g(f(x))\}dx=\int_{f_a}^f\frac{g(f)}{f^{(1)}}df$$

Here, let

$$h^{(r)}=\frac{\partial^r}{\partial f^r}\left(\frac{1}{f^{(1)}}\right)\ \left\{=\frac{\partial^r}{\partial f^r}\left(\frac{dx}{df}\right)\right\} \qquad r=0,1,2,\cdots$$

Then,

$$\int_a^x \{g(f(x))\}dx = \int_{f_a}^f g(f)h(f)df$$

According to Formula 16.1.2 in **16.1** ,

$$\int_{f_a}^f g\,h\,df = \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} g^{\langle 1+r_{11}\rangle} h^{(r_{11})} + R_{m_1}^1$$

$$R_{m_1}^1 = (-1)^{m_1} \int_{f_a}^f g^{\langle m_1\rangle} h^{(m_1)} df$$

Using this,

$$\int_a^x \{g(f(x))\}dx = \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} g^{\langle 1+r_{11}\rangle} h^{(r_{11})} + R_{m_1}^1 \tag{1}$$

$$R_{m_1}^1 = (-1)^{m_1} \int_{f_a}^f g^{\langle m_1\rangle} h^{(m_1)} df \tag{1r}$$

Next, integrating both sides of (1) and (1r) with respect to x from a to x ,

$$\int_a^x\int_a^x \{g(f(x))\}dx^2 = \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} \int_a^x g^{\langle 1+r_{11}\rangle} h^{(r_{11})} dx + \int_a^x R_{m_1}^1 dx$$

$$\int_a^x R_{m_1}^1 dx = (-1)^{m_1} \int_a^x \left(\int_{f_a}^f g^{\langle m_1\rangle} h^{(m_1)} df \right) dx$$

Substituting $dx = h\,df$ for this and rewriting the 2nd term of the right side with $R_{m_1}^2$,

$$\int_a^x\int_a^x \{g(f(x))\}dx^2 = \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} \int_{f_a}^f g^{\langle 1+r_{11}\rangle} h^{(r_{11})} h\,df + R_{m_1}^2$$

$$R_{m_1}^2 = (-1)^{m_1} \int_{f_a}^f \left(\int_{f_a}^f g^{\langle m_1\rangle} h^{(m_1)} df \right) h\,df$$

Here, according to Formula 20.2.1 in **20.2** ,

$$\int_a^x f_1 f_2 f_3 dx = \sum_{r=0}^{m-1} \sum_{s=0}^{r} \binom{-1}{r} \binom{r}{s} f_1^{<1+r>} f_2^{(r-s)} f_3^{(s)} + R_m^1$$

$$R_m^1 = (-1)^m \sum_{s=0}^{m} \binom{m}{s} \int_a^x f_1^{<m>} f_2^{(m-s)} f_3^{(s)} dx$$

Then, replacing m with m_2 and using this,

$$\int_{f_a}^f g^{\langle 1+r_{11}\rangle} h^{(r_{11})} h\,df = \sum_{r_{21}=0}^{m_2-1} \sum_{r_{22}=0}^{r_{21}} \binom{-1}{r_{21}} \binom{r_{21}}{r_{22}} g^{\langle 2+r_{11}+r_{21}\rangle} h^{(r_{11}+r_{21}-r_{22})} h^{(r_{22})} + R_{m_2}^1$$

$$R_{m_2}^1 = (-1)^{m_2} \sum_{r_{22}=0}^{m_2} \binom{m_2}{r_{22}} \int_{f_a}^f g^{\langle 1+r_{11}+m_2\rangle} h^{(r_{11}-r_{22}+m_2)} h^{(r_{22})} df$$

Substituting this for the above,

$$\int_a^x\int_a^x \{g(f(x))\}dx^2$$

$$= \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} \sum_{r_{21}=0}^{m_2-1} \sum_{r_{22}=0}^{r_{21}} \binom{-1}{r_{21}} \binom{r_{21}}{r_{22}} g^{\langle 2+r_{11}+r_{21}\rangle} h^{(r_{11}+r_{21}-r_{22})} h^{(r_{22})}$$

$$+ \sum_{r_{11}=0}^{m_0-1} \binom{-1}{r_{11}} R_{m_1}^1 + R_{m_0}^2$$

$$R_{m_2}^1 = (-1)^{m_2}\sum_{r_{22}=0}^{m_2}\binom{m_2}{r_{22}}\int_{f_a}^{f} g^{\langle 1+r_{11}+m_2\rangle} h^{(r_{11}-r_{22}+m_2)} h^{(r_{22})} df$$

$$R_{m_1}^2 = (-1)^{m_1}\int_{f_a}^{f}\left(\int_{f_a}^{f} g^{\langle m_1\rangle} h^{(m_1)} df\right) hdf$$

Redefining $R_{m_1}^2$,

$$\int_a^x\int_a^x \{g(f(x))\} dx^2$$
$$= \sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}\sum_{r_{21}=0}^{m_2-1}\sum_{r_{22}=0}^{r_{21}}\binom{-1}{r_{21}}\binom{r_{21}}{r_{22}} g^{\langle 2+r_{11}+r_{21}\rangle} h^{(r_{11}+r_{21}-r_{22})} h^{(r_{22})} + R_{m_1}^2 \tag{2}$$

$$R_{m_1}^2 = \sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}(-1)^{m_2}\sum_{r_{22}=0}^{m_2}\binom{m_2}{r_{22}}\int_{f_a}^{f} g^{\langle 1+r_{11}+m_2\rangle} h^{(r_{11}-r_{22}+m_2)} h^{(r_{22})} df$$
$$+ (-1)^{m_1}\int_{f_a}^{f}\left(\int_{f_a}^{f} g^{\langle m_1\rangle} h^{(m_1)} df\right) hdf \tag{2r}$$

Here, let

$$S_1 = \sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}} \quad , \quad S_2 = \sum_{r_{21}=0}^{m_2-1}\binom{-1}{r_{21}}\sum_{r_{22}=0}^{r_{21}}\binom{r_{21}}{r_{22}} \quad , \quad M_2 = (-1)^{m_2}\sum_{r_{22}=0}^{m_2}\binom{m_2}{r_{22}}$$

$$R_{11} = r_{11} \quad , \quad R_{21} = r_{11}+r_{21} \quad , \quad R_{22} = r_{22}$$

Then, using these, (2) and (2r) can be expressed as follows.

$$\int_a^x\int_a^x \{g(f(x))\} dx^2 = S_1 S_2 g^{\langle 2+R_{21}\rangle} h^{(R_{21}-R_{22})} h^{(R_{22})} + R_{m_1}^2$$

$$R_{m_1}^2 = S_1 M_2 \int_{f_a}^{f} g^{\langle 1+R_{11}+m_2\rangle} h^{(R_{11}-R_{22}+m_2)} h^{(R_{22})} df$$
$$+ (-1)^{m_1}\int_{f_a}^{f}\left(\int_{f_a}^{f} g^{\langle m_1\rangle} h^{(m_1)} df\right) hdf$$

Next, integrating both sides of (2) and (2r) with respect to x from a to x,

$$\int_a^x\int_a^x\int_a^x \{g(f(x))\} dx^3 = \sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}\sum_{r_{21}=0}^{m_2-1}\sum_{r_{22}=0}^{r_{21}}\binom{-1}{r_{21}}\binom{r_{21}}{r_{22}}$$
$$\times \int_a^x g^{\langle 2+r_{11}+r_{21}\rangle} h^{(r_{11}+r_{21}-r_{22})} h^{(r_{22})} dx + \int_a^x R_{m_1}^2 dx$$

$$\int_a^x R_{m_1}^2 dx = \sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}(-1)^{m_2}\sum_{r_{22}=0}^{m_2}\binom{m_2}{r_{22}}\int_a^x\left(\int_{f_a}^{f} g^{\langle 1+r_{11}+m_2\rangle} h^{(r_{11}-r_{22}+m_2)} h^{(r_{22})} df\right) dx$$
$$+ (-1)^{m_1}\int_a^x\left(\int_{f_a}^{f}\left(\int_{f_a}^{f} g^{\langle m_1\rangle} h^{(m_1)} df\right) hdf\right) dx$$

Substituting $dx = h\,df$ for this and rewriting the 2nd term of the right side with $R_{m_1}^3$,

$$\int_a^x\int_a^x\int_a^x \{g(f(x))\} dx^3$$
$$= \sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}\sum_{r_{21}=0}^{m_2-1}\sum_{r_{22}=0}^{r_{21}}\binom{-1}{r_{21}}\binom{r_{21}}{r_{22}}\int_{f_a}^{f} g^{\langle 2+r_{11}+r_{21}\rangle} h^{(r_{11}+r_{21}-r_{22})} h^{(r_{22})} hdf + R_{m_1}^3$$

$$R_{m_1}^3 = \sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}(-1)^{m_2}\sum_{r_{22}=0}^{m_2}\binom{m_2}{r_{22}}\int_{f_a}^{f}\left(\int_{f_a}^{f} g^{\langle 1+r_{11}+m_2\rangle} h^{(r_{11}-r_{22}+m_2)} h^{(r_{22})} df\right) hdf$$

$$+(-1)^{m_1}\int_{f_a}^{f}\left(\int_{f_a}^{f}\left(\int_{f_a}^{f} g^{\langle m_1\rangle} h^{(m_1)} df\right) hdf\right) hdf$$

Here again, according to Formula 20.2.1 in **20.2** ,

$$\int_a^x f_1 f_2 f_3 f_4 dx = \sum_{r=0}^{m-1}\sum_{s=0}^{r}\sum_{t=0}^{s}\binom{-1}{r}\binom{r}{s}\binom{s}{t} f_1^{<1+r>} f_2^{(r-s)} f_3^{(s-t)} f_4^{(t)} + R_m^1$$

$$R_m^1 = (-1)^m \sum_{s=0}^{m}\sum_{t=0}^{s}\binom{m}{s}\binom{s}{t}\int_a^x f_1^{<m>} f_2^{(m-s)} f_3^{(s-t)} f_4^{(t)} dx$$

Then, replacing m with m_3 and using this,

$$\int_{f_a}^{f} g^{\langle 2+r_{11}+r_{21}\rangle} h^{(r_{11}+r_{21}-r_{22})} h^{(r_{22})} hdf = \sum_{r_{31}=0}^{m_3-1}\sum_{r_{32}=0}^{r_{31}}\sum_{r_{33}=0}^{r_{32}}\binom{-1}{r_{31}}\binom{r_{31}}{r_{32}}\binom{r_{32}}{r_{33}}$$

$$\times g^{\langle 3+r_{11}+r_{21}+r_{31}\rangle} h^{(r_{11}+r_{21}+r_{31}-r_{22}-r_{32})} h^{(r_{22}+r_{32}-r_{33})} h^{(r_{33})} + R_{m_3}^1$$

$$R_{m_3}^1 = (-1)^{m_3}\sum_{r_{32}=0}^{m_3}\sum_{r_{33}=0}^{r_{32}}\binom{m_3}{r_{32}}\binom{r_{32}}{r_{33}}$$

$$\times \int_{f_a}^{f} g^{\langle 2+r_{11}+r_{21}+m_3\rangle} h^{(r_{11}+r_{21}-r_{22}-r_{32}+m_3)} h^{(r_{22}+r_{32}-r_{33})} h^{(r_{33})} df$$

Substituting this for the above,

$$\int_a^x\int_a^x\int_a^x \{g(f(x))\} dx^3$$

$$= \sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}\sum_{r_{21}=0}^{m_2-1}\sum_{r_{22}=0}^{r_{21}}\binom{-1}{r_{21}}\binom{r_{21}}{r_{22}}\sum_{r_{31}=0}^{m_3-1}\sum_{r_{32}=0}^{r_{31}}\sum_{r_{33}=0}^{r_{32}}\binom{-1}{r_{31}}\binom{r_{31}}{r_{32}}\binom{r_{32}}{r_{33}}$$

$$\times g^{\langle 3+r_{11}+r_{21}+r_{31}\rangle} h^{(r_{11}+r_{21}+r_{31}-r_{22}-r_{32})} h^{(r_{22}+r_{32}-r_{33})} h^{(r_{33})}$$

$$+\sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}\sum_{r_{21}=0}^{m_2-1}\sum_{r_{22}=0}^{r_{21}}\binom{-1}{r_{21}}\binom{r_{21}}{r_{22}} R_{m_3}^1 + R_{m_1}^3$$

$$R_{m_3}^1 = (-1)^{m_3}\sum_{r_{32}=0}^{m_3}\sum_{r_{33}=0}^{r_{32}}\binom{m_2}{r_{32}}\binom{r_{32}}{r_{33}}$$

$$\times \int_{f_a}^{f} g^{\langle 2+r_{11}+r_{21}+m_3\rangle} h^{(r_{11}+r_{21}-r_{22}-r_{32}+m_3)} h^{(r_{22}+r_{32}-r_{33})} h^{(r_{33})} df$$

$$R_{m_1}^3 = \sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}(-1)^{m_2}\sum_{r_{22}=0}^{m_2}\binom{m_2}{r_{22}}\int_{f_a}^{f}\left(\int_{f_a}^{f} g^{\langle 1+r_{11}+m_2\rangle} h^{(r_{11}-r_{22}+m_2)} h^{(r_{22})} df\right) hdf$$

$$+(-1)^{m_1}\int_{f_a}^{f}\left(\int_{f_a}^{f}\left(\int_{f_a}^{f} g^{\langle m_1\rangle} h^{(m_1)} df\right) hdf\right) hdf$$

Redefining $R_{m_1}^3$,

$$\int_a^x\int_a^x\int_a^x \{g(f(x))\} dx^3$$

$$= \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} \sum_{r_{21}=0}^{m_2-1} \sum_{r_{22}=0}^{r_{21}} \binom{-1}{r_{21}} \binom{r_{21}}{r_{22}} \sum_{r_{31}=0}^{m_3-1} \sum_{r_{32}=0}^{r_{31}} \sum_{r_{33}=0}^{r_{32}} \binom{-1}{r_{31}} \binom{r_{31}}{r_{32}} \binom{r_{32}}{r_{33}}$$

$$\times g^{\langle 3+r_{11}+r_{21}+r_{31}\rangle} h^{(r_{11}+r_{21}+r_{31}-r_{22}-r_{32})} h^{(r_{22}+r_{32}-r_{33})} h^{(r_{33})} + R_{m_1}^3 \qquad (3)$$

$$R_{m_1}^3 = \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} \sum_{r_{21}=0}^{m_2-1} \sum_{r_{22}=0}^{r_{21}} \binom{-1}{r_{21}} \binom{r_{21}}{r_{22}} (-1)^{m_3} \sum_{r_{32}=0}^{m_3} \sum_{r_{33}=0}^{r_{32}} \binom{m_3}{r_{32}} \binom{r_{32}}{r_{33}}$$

$$\times \int_{f_a}^{f} g^{\langle 2+r_{11}+r_{21}+m_3\rangle} h^{(r_{11}+r_{21}-r_{22}-r_{32}+m_3)} h^{(r_{22}+r_{32}-r_{33})} h^{(r_{33})} df$$

$$+ \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} (-1)^{m_2} \sum_{r_{22}=0}^{m_2} \binom{m_2}{r_{22}} \int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle 1+r_{11}+m_2\rangle} h^{(r_{11}-r_{22}+m_2)} h^{(r_{22})} df \right) hdf$$

$$+ (-1)^{m_1} \int_{f_a}^{f} \left(\int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle m_1\rangle} h^{(m_1)} df \right) hdf \right) hdf \qquad (3r)$$

Further, (3) and (3r) are expressed as follows using S_k , M_k , R_{jk} .

$$\int_a^x \int_a^x \int_a^x \{g(f(x))\} dx^3 = S_1 S_2 S_3\, g^{\langle 3+R_{31}\rangle} h^{(R_{31}-R_{32})} h^{(R_{32}-R_{33})} h^{(R_{33})} + R_{m_1}^3$$

$$R_{m_1}^3 = S_1 S_2 M_3 \int_{f_a}^{f} g^{\langle 2+R_{21}+m_3\rangle} h^{(R_{21}-R_{32}+m_3)} h^{(R_{32}-R_{33})} h^{(R_{33})} df$$

$$+ S_1 M_2 \int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle 1+R_{11}+m_2\rangle} h^{(R_{11}-R_{22}+m_2)} h^{(R_{22})} df \right) hdf$$

$$+ (-1)^{m_1} \int_{f_a}^{f} \left(\int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle m_1\rangle} h^{(m_1)} df \right) hdf \right) hdf$$

Calculating the 4th order integral in a similar way,

$$\int_a^x \cdots \int_a^x \{g(f(x))\} dx^4 = S_1 S_2 S_3 S_4\, g^{\langle 4+R_{41}\rangle} h^{(R_{41}-R_{42})} h^{(R_{42}-R_{43})} h^{(R_{43}-R_{44})} h^{(R_{44})} + R_{m_1}^4$$

$$R_{m_1}^4 = S_1 S_2 S_3 M_4 \int_{f_a}^{f} g^{\langle 3+R_{31}+m_4\rangle} h^{(R_{31}-R_{42}+m_4)} h^{(R_{42}-R_{43})} h^{(R_{43}-R_{44})} h^{(R_{44})} df$$

$$+ S_1 S_2 M_3 \int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle 2+R_{21}+m_3\rangle} h^{(R_{21}-R_{32}+m_3)} h^{(R_{32}-R_{33})} h^{(R_{33})} df \right) hdf$$

$$+ S_1 M_2 \int_{f_a}^{f} \left(\int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle 1+R_{11}+m_2\rangle} h^{(R_{11}-R_{22}+m_2)} h^{(R_{22})} df \right) hdf \right) hdf$$

$$+ (-1)^{m_1} \int_{f_a}^{f} \left(\int_{f_a}^{f} \left(\int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle m_1\rangle} h^{(m_1)} df \right) hdf \right) hdf \right) hdf$$

Hereafter, by induction, we obtained the desired expression.

Higher Integral in case the core function is the 1st degree

Although the higher integration of a general composition is complicated in this way, If the core function $f(x)$ is the 1st degree, it becomes remarkably easy.

Formula 23.1.2

When $f(x)=cx+d$,

$$\int_a^x \cdots \int_a^x \{g(f(x))\}dx^n = \left(\frac{1}{c}\right)^n \int_{f_a}^f \cdots \int_{f_a}^f g(f)\,df^n$$

Proof

When $f(x) = cx+d$ in Formula 23.1.1 ,

$$h = \frac{dx}{df} = \frac{1}{f^{(1)}} = \frac{1}{c} \quad , \quad h^{(k)} = \frac{d^k h}{df^k} = 0 \qquad k = 1,2,3,\cdots$$

Then, the remainder term

$$R_{m_1}^n = (-1)^{m_1} \int_{f_a}^f \left(\int_{f_a}^f \left(\int_{f_a}^f \cdots \left(\int_{f_a}^f g^{\langle m_1 \rangle} h^{(m_1)} df \right) \cdots hdf \right) hdf \right) hdf \qquad \text{(n-fold nest)}$$

$$+ S_1 M_2 \int_{f_a}^f \left(\int_{f_a}^f \cdots \left(\int_{f_a}^f g^{\langle 1+R_{11}+m_2 \rangle} h^{(R_{11}-R_{22}+m_2)} h^{(R_{22})} df \right) \cdots hdf \right) hdf$$

$$+ S_1 S_2 M_3 \int_{f_a}^f \cdots \left(\int_{f_a}^f g^{\langle 2+R_{21}+m_3 \rangle} h^{(R_{21}-R_{32}+m_3)} h^{(R_{32}-R_{33})} h^{(R_{33})} df \right) \cdots hdf$$

$$\vdots$$

$$+ S_1 S_2 \cdots S_{n-1} M_n \int_{f_a}^f g^{\langle n-1+R_{n-11}+m_n \rangle} h^{(R_{n-11}-R_{n2}+m_n)} h^{(R_{n2}-R_{n3})} \cdots h^{(R_{nn})} df \qquad \text{(nr)}$$

becomes as follows.

1st line: Since $m_1 > 0$, $h^{(m_1)} = 0$. Then the 1st line = 0.

2nd line: When $R_{22} > 0$, $h^{(R_{22})} = 0$. Then, the 2nd line = 0.

When $R_{22} = 0$, $h^{(R_{11} - R_{22} + m_2)} = 0$. Then, the 2nd line = 0.

3rd line: When $R_{33} > 0$, $h^{(R_{33})} = 0$. Then, the 3rd line = 0.

When $R_{33} = 0$, if $R_{32} > 0$, $h^{(R_{32} - R_{33})} = 0$. Then, the 3rd line = 0.

When $R_{33} = R_{32} = 0$, $h^{(R_{21} - R_{32} + m_3)} = 0$. Then, the 3rd line = 0.

$\vdots$

n th line: When $R_{nn} > 0$, $h^{(R_{nn})} = 0$. Then, the n th line = 0.

When $R_{nn} = 0$, if at least one is positive in $R_{n\,n-1} \sim R_{n2}$, Then, the n th line = 0.

When $R_{nn} = R_{n\,n-1} = \cdots = R_{n2} = 0$, $h^{(R_{n-11} - R_{n2} + m_n)} = 0$. Then, the n th line = 0.

Thus, it is sure to become $R_{m_1}^n = 0$ under the assumption. Therefore,

$$\int_a^x \cdots \int_a^x \{g(f(x))\}dx^n = S_1 S_2 \cdots S_n\, g^{\langle n+R_{n1} \rangle} h^{(R_{n1}-R_{n2})} \cdots h^{(R_{n\,n-1}-R_{nn})} h^{(R_{nn})} \qquad \text{(n)}$$

Next, observing (n), we find out $R_{nn} = R_{n\,n-1} = \cdots = R_{n1} = 0$ immediately. If it is why, in order for (n) not to be 0, $R_{nn} = 0$ is required first. The 2nd, $R_{nn-1} = 0$ is required. The 3rd, $\cdots$. Hereafter, it advances one after another, finaly $R_{n1} = 0$ is required. Then,

$$h^{(R_{n1}-R_{n2})} \cdots h^{(R_{n\,n-1}-R_{nn})} h^{(R_{nn})} = \left\{h^{(0)}\right\}^n = \left(\frac{1}{c}\right)^n$$

$$S_1 S_2 \cdots S_n = 1$$

Thus, we obtain

$$\int_a^x \cdots \int_a^x \{g(f(x))\} dx^n = \left(\frac{1}{c}\right)^n g^{<n>} = \left(\frac{1}{c}\right)^n \int_{f_a}^f \cdots \int_{f_a}^f g(f) df^n$$

Q.E.D

Example When $g = log(ax+b)$

$$\int_{-\frac{b}{a}}^x \cdots \int_{-\frac{b}{a}}^x log(ax+b) dx^n = \left(\frac{1}{a}\right)^n \frac{(ax+b)^n}{n!} \left\{ log(ax+b) - \sum_{k=1}^n \frac{1}{k} \right\}$$

This is consistent with linear form in Formula4.3.3 in " **4 Higher Integral** " .

Compositions that Formula 23.1.1 is applicable

Even if restricted only to the combination of an elementary function, there are many kinds of composition. However, Formula 23.1.1 is not applicable all of these. The 1st, the inverse function of the core function have to be known. If it is not so, h cannot be expressed by the function of f . The 2nd, the higher primitive function of the enclosing function $g(f)$ must have the property such as $\lim_{n \to \infty} g^{<n>}(f) = c$. If it is not so, not the series but the remainder term becomes the prime part of the integration.

Considering these, there are not many compositions that Formula 23.1.1 is applicable. This decreases the usefulness of Formula 23.1.1. However, about some compositions that Formula 23.1.1 was applicable, a quite interesting result was obtained. I describe it in the following sections.

23.2 Higher Integral of (x^β - c)^α

Formula 23.2.1

Let $\Gamma(z)$ be the gamma function. Let S_k, r_k and $g\{f(x)\}$ are as follows respectively.

$$g\{f(x)\} = \left(x^{\beta}-c\right)^{\alpha} \quad , \quad \beta>0 \ \& \neq 1/k \ (k=1,2,\cdots) , \qquad \alpha, \beta, c>0$$

$$S_k = \sum_{r_{k1}=0}^{\infty}\binom{-1}{r_{k1}}\sum_{r_{k2}=0}^{r_{k1}}\binom{r_{k1}}{r_{k2}}\sum_{r_{k3}=0}^{r_{k2}}\binom{r_{k2}}{r_{k3}}\cdots\sum_{r_{kk}=0}^{r_{kk-1}}\binom{r_{kk-1}}{r_{kk}} \qquad k=1,2,\cdots,n$$

$$R_{jk} = \sum_{i=k}^{j} r_{ik} \qquad j,k=1,2,\cdots,n$$

Then, the following expressions hold for $x > \sqrt[\beta]{c}$.

$$\int_{\sqrt[\beta]{c}}^{x}\left(x^{\beta}-c\right)^{\alpha}dx = \left(x^{\beta}-c\right)^{\alpha}\frac{x}{\beta}\sum_{r=0}^{\infty}\binom{-1}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r)}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r)}\left(1-\frac{c}{x^{\beta}}\right)^{1+r} \tag{1.1}$$

$$\int_{\sqrt[\beta]{c}}^{x}\int_{\sqrt[\beta]{c}}^{x}\left(x^{\beta}-c\right)^{\alpha}dx^2 = \left(x^{\beta}-c\right)^{\alpha}\frac{x^2}{\beta^2}\sum_{r=0}^{\infty}\binom{-1}{r}\sum_{s=0}^{\infty}\sum_{t=0}^{s}\binom{-1}{s}\binom{s}{t}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+2+r+s)}$$
$$\times\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r-s+t)}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-t)}\left(1-\frac{c}{x^{\beta}}\right)^{2+r+s} \tag{1.2}$$

$$\int_{\sqrt[\beta]{c}}^{x}\cdots\int_{\sqrt[\beta]{c}}^{x}\left(x^{\beta}-c\right)^{\alpha}dx^3 = \left(x^{\beta}-c\right)^{\alpha}\frac{x^3}{\beta^3}$$
$$\times\sum_{r_{11}=0}^{\infty}\binom{-1}{r_{11}}\sum_{r_{21}=0}^{\infty}\sum_{r_{22}=0}^{r_{21}}\binom{-1}{r_{21}}\binom{r_{21}}{r_{22}}\sum_{r_{31}=0}^{\infty}\sum_{r_{32}=0}^{r_{31}}\sum_{r_{33}=0}^{r_{32}}\binom{-1}{r_{31}}\binom{r_{31}}{r_{32}}\binom{r_{32}}{r_{33}}$$
$$\times\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+3+r_{11}+r_{21}+r_{31})}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r_{11}-r_{21}-r_{31}+r_{22}+r_{32})}$$
$$\times\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r_{22}-r_{32}+r_{33})}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r_{33})}\left(1-\frac{c}{x^{\beta}}\right)^{3+r_{11}+r_{21}+r_{31}} \tag{1.3}$$

⋮

$$\int_{\sqrt[\beta]{c}}^{x}\cdots\int_{\sqrt[\beta]{c}}^{x}\left(x^{\beta}-c\right)^{\alpha}dx^n = \left(x^{\beta}-c\right)^{\alpha}\frac{x^n}{\beta^n}S_1S_2\cdots S_n\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+R_{n1})}\left(1-\frac{1}{x^{\beta}}\right)^{n+R_{n1}}$$
$$\times\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{n1}+R_{n2})}\cdots\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{nn-1}+R_{nn})}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{nn})} \tag{1.n}$$

Proof

$$g\{f(x)\} = \left(x^{\beta}-c\right)^{\alpha}, \qquad h = \frac{dx}{df} = \frac{1}{\beta}(f+c)^{\frac{1}{\beta}-1}$$

From these,

$$g^{\langle n+R_{n1}\rangle} = \left(f^{\alpha}\right)^{\langle n+R_{n1}\rangle} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+R_{n1})}f^{\alpha+n+R_{n1}}$$

$$h^{(R_{n1}-R_{n2})} = \frac{1}{\beta}\left\{(f+c)^{\frac{1}{\beta}-1}\right\}^{(R_{n1}-R_{n2})}$$

$$= \frac{1}{\beta}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{n1}+R_{n2})}(f+c)^{\frac{1}{\beta}-1-R_{n1}+R_{n2}}$$

$$\vdots$$

$$h^{(R_{nn-1}-R_{nn})} = \frac{1}{\beta}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{nn-1}+R_{nn})}(f+c)^{\frac{1}{\beta}-1-R_{nn-1}+R_{nn}}$$

$$h^{(m_1)} = \frac{1}{\beta}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{nn})}(f+c)^{\frac{1}{\beta}-1-R_{nn}}$$

Substituting these for Formula 23.1.1 (n.1) ,

$$\int_{\sqrt[\beta]{c}}^{x}\cdots\int_{\sqrt[\beta]{c}}^{x}(x^\beta-c)^\alpha dx^n = S_1S_2\cdots S_n\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+R_{n1})}f^{\alpha+n+R_{n1}}$$

$$\times\frac{1}{\beta}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{n1}+R_{n2})}(f+c)^{\frac{1}{\beta}-1-R_{n1}+R_{n2}}$$

$$\vdots$$

$$\times\frac{1}{\beta}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{nn-1}+R_{nn})}(f+c)^{\frac{1}{\beta}-1-R_{nn-1}+R_{nn}}$$

$$\times\frac{1}{\beta}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{nn})}(f+c)^{\frac{1}{\beta}-1-R_{nn}} + R_{m_1}^n$$

$$= \frac{1}{\beta^n}S_1S_2\cdots S_n\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+R_{n1})}(x^\beta-c)^{\alpha+n+R_{n1}}(x^\beta)^{\frac{n}{\beta}-n-R_{n1}}$$

$$\times\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{n1}+R_{n2})}\cdots\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{nn-1}+R_{nn})}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{nn})} + R_{m_1}^n$$

i.e.

$$\int_{\sqrt[\beta]{c}}^{x}\cdots\int_{\sqrt[\beta]{c}}^{x}(x^\beta-c)^\alpha dx^n = (x^\beta-c)^\alpha\frac{x^n}{\beta^n}S_1S_2\cdots S_n\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+R_{n1})}\left(1-\frac{c}{x^\beta}\right)^{n+R_{n1}}$$

$$\times\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{n1}+R_{n2})}\cdots\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{nn-1}+R_{nn})}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{nn})} + R_{m_1}^n \qquad \text{(n)}$$

Since $R_{m_1}^n$ is too long, we omit it. If (n) is written about the 1st order ~ the 3rd order, it is as follows.

$$\int_{\sqrt[\beta]{c}}^{x}(x^\beta-c)^\alpha dx$$

$$= (x^\beta-c)^\alpha\frac{x}{\beta}\sum_{r=0}^{m_1-1}\binom{-1}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r)}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r)}\left(1-\frac{c}{x^\beta}\right)^{1+r} + R_{m_1}^1 \qquad (1)$$

$$R_{m_1}^1 = \frac{(-1)^{m_1}}{\beta}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m_1)}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-m_1)}\int_0^f f^{\alpha+m_1}(f+c)^{\frac{1}{\beta}-1-m_1}df \qquad \text{(1r)}$$

$$\int_{\sqrt[\beta]{c}}^{x}\int_{\sqrt[\beta]{c}}^{x}(x^\beta-c)^\alpha dx^2$$

$$
\begin{aligned}
&= \left(x^{\beta}-c\right)^{\alpha}\frac{x^{2}}{\beta^{2}}\sum_{r=0}^{m_1-1}\binom{-1}{r}\sum_{s=0}^{m_2-1}\sum_{t=0}^{s}\binom{-1}{s}\binom{s}{t}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+2+r+s)} \\
&\quad\times\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r-s+t)}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-t)}\left(1-\frac{c}{x^{\beta}}\right)^{2+r+s}+R_{m_1}^{2}
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
R_{m_1}^{2} &= \frac{(-1)^{m_2}}{\beta^{2}}\sum_{r=0}^{m_1-1}\binom{-1}{r}\sum_{t=0}^{m_2}\binom{m_2}{t}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r+m_2)}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r+t-m_2)} \\
&\quad\times\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-t)}\int_0^f f^{\alpha+1+r+m_2}(f+c)^{\frac{2}{\beta}-2-r-m_2}df \\
&\quad+\frac{(-1)^{m_1}}{\beta^{2}}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m_1)}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-m_1)} \\
&\quad\times\int_0^f\left(\int_0^f f^{\alpha+m_1}(f+c)^{\frac{1}{\beta}-1-m_1}df\right)(f+c)^{\frac{1}{\beta}-1}df
\end{aligned}
\tag{2r}
$$

$$
\begin{aligned}
&\int_{\sqrt[\beta]{c}}^{x}\cdots\int_{\sqrt[\beta]{c}}^{x}\left(x^{\beta}-c\right)^{\alpha}dx^{3} = \left(x^{\beta}-c\right)^{\alpha}\frac{x^{3}}{\beta^{3}} \\
&\quad\times\sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}\sum_{r_{21}=0}^{m_2-1}\sum_{r_{22}=0}^{r_{21}}\binom{-1}{r_{21}}\binom{r_{21}}{r_{22}}\sum_{r_{31}=0}^{m_3-1}\sum_{r_{32}=0}^{r_{31}}\sum_{r_{33}=0}^{r_{32}}\binom{-1}{r_{31}}\binom{r_{31}}{r_{32}}\binom{r_{32}}{r_{33}} \\
&\quad\times\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+3+r_{11}+r_{21}+r_{31})}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r_{11}-r_{21}-r_{31}+r_{22}+r_{32})} \\
&\quad\times\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r_{22}-r_{32}+r_{33})}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r_{33})}\left(1-\frac{c}{x^{\beta}}\right)^{3+r_{11}+r_{21}+r_{31}}+R_{m_1}^{3}
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
R_{m_1}^{3} &= \frac{(-1)^{m_3}}{\beta^{3}}\sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}\sum_{r_{21}=0}^{m_2-1}\sum_{r_{22}=0}^{r_{21}}\binom{-1}{r_{21}}\binom{r_{21}}{r_{22}}\sum_{r_{32}=0}^{m_3}\sum_{r_{33}=0}^{r_{32}}\binom{m_3}{r_{32}}\binom{r_{32}}{r_{33}} \\
&\quad\times\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+2+r_{11}+r_{21}+m_3)}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r_{11}-r_{21}+r_{22}+r_{32}-m_3)} \\
&\quad\times\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r_{22}-r_{32}+r_{33})}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r_{33})} \\
&\quad\times\int_0^f f^{\alpha+2+r_{11}+r_{21}+m_3}(f+c)^{\frac{3}{\beta}-3-r_{11}-r_{21}-m_3}df \\
&\quad+\frac{(-1)^{m_2}}{\beta^{3}}\sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}\sum_{r_{22}=0}^{m_2}\binom{m_2}{r_{22}} \\
&\quad\times\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r_{11}+m_2)}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r_{11}+r_{22}-m_2)}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-r_{22})} \\
&\quad\times\int_0^f\left(\int_0^f f^{\alpha+1+r_{11}+m_2}(f+c)^{\frac{2}{\beta}-2-r_{11}-m_2}df\right)(f+c)^{\frac{1}{\beta}-1}df \\
&\quad+\frac{(-1)^{m_1}}{\beta}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m_1)}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-m_1)}
\end{aligned}
$$

$$\times \int_0^f \left(\int_0^f \left(\int_0^f f^{\alpha+m_1} (f+c)^{\frac{1}{\beta}-1-m_1} df \right) (f+c)^{\frac{1}{\beta}-1} df \right) (f+c)^{\frac{1}{\beta}-1} df \qquad \text{(3r)}$$

$R^n_{m_1}$ also roughly looks like (3r). The integration included in these is more difficult than the original one (the left side), and solving these is unthinkable. However, If $\alpha>0, \beta>0, R^n_{m_1} \rightarrow 0$ at the time $m_i \rightarrow \infty \; i=1,2, \cdots$. That is shown as follows.

Now, cnsider the function $S(m)$ such that

$$S(m) = \frac{1}{\Gamma(1+m)\,\Gamma(1/b-m)}$$

Giving the values to b of this function parametrically and illustrating this, we find out the following. (Refer to the figure.)

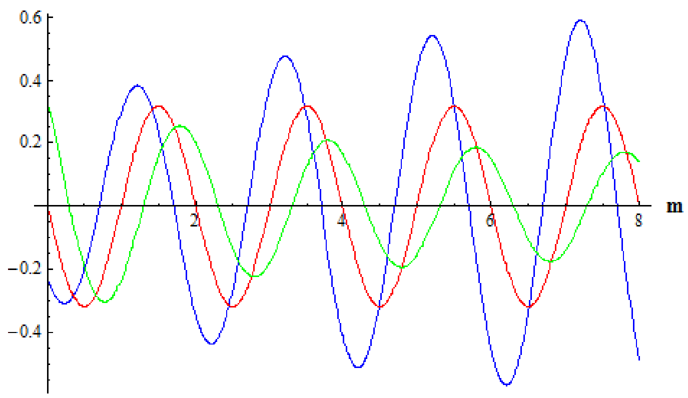
$b<0$: $S(m)$ diverges to $\pm\infty$.

$b=0$: $S(m)$ oscillates around 0.

$b>0$: $S(m)$ converges to 0 .

```
s[b_] := 1 / (Gamma[1 + m] Gamma[b - m])
```

Blue: b<0 , Red: b=0 , Green: b>0



Although the description was omitted previously, the last term of the remainder $R^n_{m_1}$ is as follows.

$$\frac{(-1)^{m_1}}{\beta} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m_1)} \frac{\Gamma(1/\beta)}{\Gamma(1/\beta-m_1)}$$

$$\times \int_0^f \left(\int_0^f \cdots \left(\int_0^f f^{\alpha+n-1+R_{n-11}+m_n} (f+c)^{\frac{n}{\beta}-n-R_{n-11}-m_n} df \right) \cdots (f+c)^{\frac{n}{\beta}-1} df \right) (f+c)^{\frac{n}{\beta}-1} df$$

And when $\alpha>0 \,, \beta>0$,

$$\lim_{m_1\rightarrow\infty} \frac{1}{\Gamma(1+\alpha+m_1)} \frac{1}{|\Gamma(1/\beta-m_1)|} < \lim_{m_1\rightarrow\infty} \frac{1}{\Gamma(1+m_1)} \frac{1}{|\Gamma(1/\beta-m_1)|} = 0$$

Moreover, the integrand is as follows.

$$f^{\alpha+n-1+R_{n-11}+m_n} (f+c)^{\frac{n}{\beta}-n-R_{n-11}-m_n}$$

$$= (x^\beta - c)^{\alpha+n-1+R_{n-11}+m_n} (x^\beta)^{\frac{n}{\beta}-n-R_{n-11}-m_n}$$

$$= \left(x^{\beta}-c\right)^{\alpha+n-1+R_{n-11}}\left(x^{\beta}\right)^{\frac{n}{\beta}-n-R_{n-11}}\left(1-\frac{c}{x^{\beta}}\right)^{m_n}$$

By assumption, $x>\sqrt[\beta]{c}$, $c>0$. Then,

$$\lim_{m_n\to\infty}\left(x^{\beta}-c\right)^{\alpha+n-1+R_{n-11}}\left(x^{\beta}\right)^{\frac{n}{\beta}-n-R_{n-11}}\left(1-\frac{c}{x^{\beta}}\right)^{m_n} = 0$$

Together with these, if $m_i\to\infty$ $i=1,2,3,\cdots$, the last term of the remainder $R^n_{m_1}$ have to converge to 0 .

. Next, terms except the last term of the remainder $R^n_{m_1}$ contain the products as follows.

$$\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+k-1+R_{k-11}+m_k)}\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{k-11}+R_{k2}-m_k)}$$

$$\times\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{k2}+R_{k3})}\cdots\frac{\Gamma(1/\beta)}{\Gamma(1/\beta-R_{kn})}$$

$$\times\int_0^f\left(\int_0^f\cdots\left(\int_0^f f^{\alpha+k-1+R_{k-11}+m_k}(f+c)^{\frac{k}{\beta}-k-R_{k-11}-m_k}df\right)\cdots(f+c)^{\frac{k}{\beta}-1}df\right)(f+c)^{\frac{k}{\beta}-1}df$$

The term that does not include m_k among these can be regarded as a constant. Since $k-1\geqq 0$,

The term that include m_k is as follows.

$$\lim_{m_k\to\infty}\frac{1}{\Gamma(1+\alpha+k-1+R_{k-11}+m_k)}\frac{1}{|\Gamma(1/\beta-R_{k-11}+R_{k2}-m_k)|}$$

$$<\lim_{m_k\to\infty}\frac{1}{\Gamma\{1+(R_{k-11}+m_k)\}}\frac{1}{|\Gamma\{1/\beta+R_{k2}-(R_{k-11}+m_k)\}|} = 0$$

And the integrand is as follows.

$$\lim_{m_k\to\infty} f^{\alpha+k-1+R_{k-11}+m_k}(f+c)^{\frac{k}{\beta}-k-R_{k-11}-m_k} = 0$$

Together with these, if $m_i\to\infty$ $i=1,2,3,\cdots$, the non-last term of the remainder $R^n_{m_1}$ have to also converge to 0 . Therefore, $\lim_{m_i\to\infty}R^n_{m_1} = 0$ $i=1,2,3,\cdots$. Thus, (1),(2),(3), $\cdots$,(n) reduce to (1.1),(1.2),(1.3), $\cdots$,(1.n) respectively.

Example1 Area between the third hyperbola and x-axis

Let $\alpha=1/3$, $\beta=3$, $c=1$ in (1,1) . Then,

$$\int_1^x\sqrt[3]{x^3-1}\,dx = \sqrt[3]{x^3-1}\,\frac{x}{3}\sum_{r=0}^{\infty}\binom{-1}{r}\frac{\Gamma(1+1/3)}{\Gamma(1+1/3+1+r)}\frac{\Gamma(1/3)}{\Gamma(1/3-r)}\left(1-\frac{1}{x^3}\right)^{1+r}$$

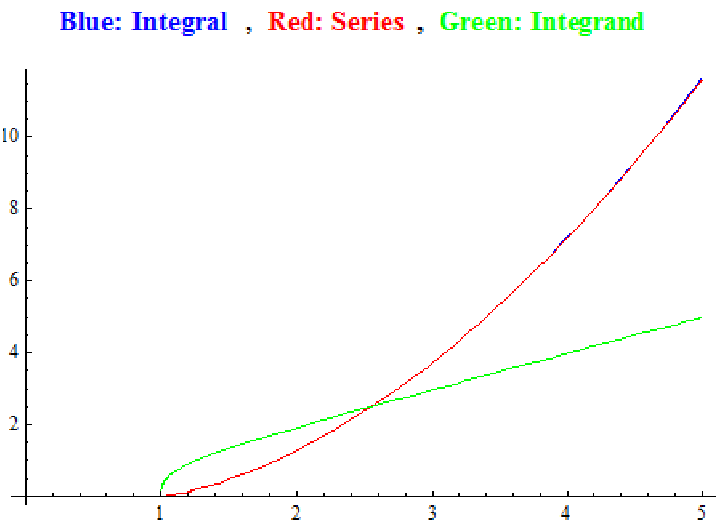
The function $y=\sqrt[3]{x^3-1}$ is a part of the third hyperbola $\frac{1}{x^3}-\frac{1}{y^3}=1$ in the 1st quadran.

Therefore, this integration means to ask for the function expressing the area between the hyperbola and x-axis in the 1st quadrant. Although even the elliptic integral is impossible to express this area, this example can express it with a series. In fact, when both sides are drawn in figure, it is as follows. As the result of the calculation until the 100th term in the right side, both sides almost overlapp.

```
a = 1 / 3; b = 3; c = 1; m = 100;
f1[x_] := ∫_{b√c}^{x} (t^b - c)^a dt
```

$$\texttt{fr[x\_] := }\left(x^{b}-c\right)^{a}\frac{x}{b}\sum_{r=0}^{m}\texttt{Binomial[-1, r]}\frac{\texttt{Gamma[1 + a]}}{\texttt{Gamma[1 + a + 1 + r]}}\frac{\texttt{Gamma[1 / b]}}{\texttt{Gamma[1 / b - r]}}\left(1-\frac{c}{x^{b}}\right)^{1+r}$$

$$\texttt{g[x\_] := }\left(x^{b}-c\right)^{a}$$



Example2 The 2nd integral of $\left(x^{e}-\pi\right)^{\log 2}$

Let $\alpha=\log 2$, $\beta=e$, $c=\pi$ in (1,2) . Then,

$$\int_{\sqrt[e]{\pi}}^{x}\int_{\sqrt[e]{\pi}}^{x}\left(x^{e}-\pi\right)^{\log 2}dx^{2}=\left(x^{e}-\pi\right)^{\log 2}\frac{x^{2}}{e^{2}}\sum_{r=0}^{\infty}\binom{-1}{r}\sum_{s=0}^{\infty}\sum_{t=0}^{s}\binom{-1}{s}\binom{s}{t}$$
$$\times\frac{\Gamma(1+\log 2)}{\Gamma(1+\log 2+2+r+s)}\frac{\Gamma(1/e)}{\Gamma(1/e-r-s+t)}\frac{\Gamma(1/e)}{\Gamma(1/e-t)}\left(1-\frac{\pi}{x^{e}}\right)^{2+r+s}$$

The values of the both sides on arbitrary point $x=3.5$ are as follows.

```
a = Log[2]; b = e; c = π; m = 100;
```

$$\texttt{f1[x\_] := }\int_{\sqrt[b]{c}}^{x}\left(\int_{\sqrt[b]{c}}^{u}\left(t^{b}-c\right)^{a}dt\right)du$$

$$\texttt{fr[x\_] := }\left(x^{b}-c\right)^{a}\frac{x^{2}}{b^{2}}\sum_{r=0}^{m}\texttt{Binomial[-1, r]}\sum_{s=0}^{s}\sum_{t=0}^{s}\texttt{Binomial[-1, s] Binomial[s, t]}$$
$$\times\frac{\texttt{Gamma[1 + a]}}{\texttt{Gamma[1 + a + 2 + r + s]}}\frac{\texttt{Gamma[1 / b]}}{\texttt{Gamma[1 / b - r - s + t]}}\frac{\texttt{Gamma[1 / b]}}{\texttt{Gamma[1 / b - t]}}\left(1-\frac{c}{x^{b}}\right)^{2+r+s}$$

```
N[f1[3.5]]
6.24841 + 6.44797 × 10^-15 i
```

```
N[fr[3.5]]
6.24841
```

23.3 Higher Integral of g(log x)

When the core function is $f=\log x$, the inverse function is $x=e^f$. Then $h^{(r)}=e^f$ $r=0,1,\cdots$.
Thus, the n th order integral of $g(\log x)$ is expressed with n-fold series.

Formula 23.3.1

When $g^{<n>}$ is the lineal higher primitive function of $g(f)$ and a , f_a are zeros of the lineal higher primitive functions of $g(\log x)$, $g(f)e^f$ respectively, the lineal higher integral of $g(\log x)$ is expressed as follows.

$$\int_a^x\cdots\int_a^x\{g(\log x)\}dx^n = x^n\sum_{r_1=0}^{m_1-1}\sum_{r_2=0}^{m_2-1}\cdots\sum_{r_n=0}^{m_n-1}(-1)^{\sum_{k=1}^n r_k}\prod_{k=1}^n k^{r_k}g^{\left\langle n+\sum_{k=1}^n r_k\right\rangle}+R_{m_1}^n \tag{1.n}$$

$$R_{m_1}^n=(-1)^{m_1}1^{m_1}\int_{f_a}^f\left(\int_{f_a}^f\left(\int_{f_a}^f\cdots\left(\int_{f_a}^f g^{\langle m_1\rangle}e^f df\right)\cdots e^f df\right)e^f df\right)e^f df \quad \text{(n-fold nest)}$$

$$+\sum_{r_1=0}^{m_1-1}(-1)^{r_1+m_2}1^{m_1}2^{m_2}\int_{f_a}^f\left(\int_{f_a}^f\cdots\left(\int_{f_a}^f g^{\langle 1+r_1+m_2\rangle}e^{2f}df\right)\cdots e^f df\right)e^f df$$

$$+\sum_{r_1=0}^{m_1-1}\sum_{r_2=0}^{m_2-1}(-1)^{r_1+r_2+m_3}1^{m_1}2^{m_2}3^{m_3}\int_{f_a}^f\cdots\left(\int_{f_a}^f g^{\langle 2+r_1+r_2+m_3\rangle}e^{3f}df\right)\cdots e^f df$$

$$\vdots$$

$$+\sum_{r_1=0}^{m_1-1}\sum_{r_2=0}^{m_2-1}\cdots\sum_{r_{n-1}=0}^{m_{n-1}-1}(-1)^{\sum_{k=1}^{n-1}r_k+m_n}\prod_{k=1}^n k^{m_k}\int_{f_a}^f g^{\left\langle n-1+\sum_{k=1}^{n-1}r_k+m_n\right\rangle}e^{nf}df \tag{1.nr}$$

Proof

Since $f=\log x$, $x=e^f$

$$h=\frac{dx}{df}=e^f=x \quad , \quad h^{(r)}=e^f=x \quad (r=1,2,3,\cdots,n)$$

Substituting these for (1), (1,r) in Formula 23.1.1 ,

$$\int_a^x\{g(\log x)\}dx = x\sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}g^{\langle 1+r_{11}\rangle}+R_{m_1}^1$$

$$R_{m_1}^1=(-1)^{m_1}\int_{f_a}^f g^{\langle m_1\rangle}e^f df$$

Replacing $\binom{-1}{r_{11}}$ with $(-1)^{r_{11}}$ and adding $1^{r_{11}}$, 1^{m_1} to this, we obtain

$$\int_a^x\{g(\log x)\}dx = x\sum_{r_{11}=0}^{m_1-1}(-1)^{r_{11}}1^{r_{11}}g^{\langle 1+r_{11}\rangle}+R_{m_1}^1 \tag{1.1}$$

$$R_{m_1}^1=(-1)^{m_1}1^{m_1}\int_{f_a}^f g^{\langle m_1\rangle}e^f df \tag{1.1r}$$

Next, substituting $h^{(r)}=e^f=x$ $(r=0,1,2,\cdots,n)$ for (2), (2,r) in Formula 23.1.1 ,

$$\int_a^x\int_a^x\{g(\log x)\}dx^2=x^2\sum_{r_{11}=0}^{m_1-1}\binom{-1}{r_{11}}\sum_{r_{21}=0}^{m_2-1}\sum_{r_{22}=0}^{r_{21}}\binom{-1}{r_{21}}\binom{r_{21}}{r_{22}}g^{\langle 2+r_{11}+r_{21}\rangle}+R_{m_1}^2$$

$$R_{m_1}^2 = \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} (-1)^{m_2} \sum_{r_{22}=0}^{m_2} \binom{m_2}{r_{22}} \int_{f_a}^{f} g^{\langle 1+r_{11}+m_2 \rangle} e^{2f} df$$
$$+ (-1)^{m_1} \int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle m_1 \rangle} e^f df \right) e^f df$$

Here,

$$\binom{-1}{r_{11}} = (-1)^{r_{11}} \quad , \quad \binom{-1}{r_{21}} = (-1)^{r_{21}}$$

$$\sum_{r_{22}=0}^{r_{21}} \binom{r_{21}}{r_{22}} = 2^{r_{21}} \quad , \quad \sum_{r_{22}=0}^{m_2} \binom{m_2}{r_{22}} = 2^{m_2}$$

Substituting these for the above, and adding $1^{r_{11}}, 1^{m_1}$ to this, we obtain

$$\int_a^x \int_a^x \{g(\log x)\} dx^2 = x^2 \sum_{r_{11}=0}^{m_1-1} \sum_{r_{21}=0}^{m_2-1} (-1)^{r_{11}+r_{21}} 1^{r_{11}} 2^{r_{21}} g^{\langle 2+r_{11}+r_{21} \rangle} + R_{m_1}^2 \tag{1.2}$$

$$R_{m_1}^2 = \sum_{r_{11}=0}^{m_1-1} (-1)^{r_{11}+m_2} 1^{m_1} 2^{m_2} \int_{f_a}^{f} g^{\langle 1+r_{11}+m_2 \rangle} e^{2f} df$$
$$+ (-1)^{m_1} 1^{m_1} \int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle m_1 \rangle} e^f df \right) e^f df \tag{1.2r}$$

Next, substituting $h^{(r)} = e^f = x \quad (r=0,1,2,\cdots,n)$ for (3), (3,r) in Formula 23.1.1 ,

$$\int_a^x \int_a^x \int_a^x \{g(\log x)\} dx^3$$
$$= x^3 \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} \sum_{r_{21}=0}^{m_2-1} \sum_{r_{22}=0}^{r_{21}} \binom{-1}{r_{21}} \binom{r_{21}}{r_{22}} \sum_{r_{31}=0}^{m_3-1} \sum_{r_{32}=0}^{r_{31}} \sum_{r_{33}=0}^{r_{32}} \binom{-1}{r_{31}} \binom{r_{31}}{r_{32}} \binom{r_{32}}{r_{33}}$$
$$\times g^{\langle 3+r_{11}+r_{21}+r_{31} \rangle} + R_{m_1}^3$$

$$R_{m_1}^3 = \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} \sum_{r_{21}=0}^{m_2-1} \sum_{r_{22}=0}^{r_{21}} \binom{-1}{r_{21}} \binom{r_{21}}{r_{22}} (-1)^{m_3} \sum_{r_{32}=0}^{m_3} \sum_{r_{33}=0}^{r_{32}} \binom{m_3}{r_{32}} \binom{r_{32}}{r_{33}}$$
$$\times \int_{f_a}^{f} g^{\langle 2+r_{11}+r_{21}+m_3 \rangle} e^{3f} df$$
$$+ \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} (-1)^{m_2} \sum_{r_{22}=0}^{m_2} \binom{m_2}{r_{22}} \int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle 1+r_{11}+m_2 \rangle} e^{2f} df \right) e^f df$$
$$+ (-1)^{m_1} \int_{f_a}^{f} \left(\int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle m_1 \rangle} e^f df \right) e^f df \right) e^f df$$

Here,

$$\binom{-1}{r_{11}} = (-1)^{r_{11}} \quad , \quad \binom{-1}{r_{21}} = (-1)^{r_{21}} \quad , \quad \binom{-1}{r_{31}} = (-1)^{r_{31}}$$

$$\sum_{r_{22}=0}^{r_{21}} \binom{r_{21}}{r_{22}} = 2^{r_{21}} \quad , \quad \sum_{r_{32}=0}^{r_{31}} \sum_{r_{33}=0}^{r_{32}} \binom{r_{31}}{r_{32}} \binom{r_{32}}{r_{33}} = 3^{r_{31}}$$

$$\sum_{r_{22}=0}^{m_2} \binom{m_2}{r_{22}} = 2^{m_2} \quad , \quad \sum_{r_{32}=0}^{m_3} \sum_{r_{33}=0}^{r_{32}} \binom{m_3}{r_{32}} \binom{r_{32}}{r_{33}} = 3^{m_3}$$

Substituting these for the above, and adding $1^{r_{11}}$, 1^{m_1} to this, we obtain

$$\int_a^x\int_a^x\int_a^x \{g(\log x)\}dx^3$$

$$= x^3\sum_{r_{11}=0}^{m_1-1}\sum_{r_{21}=0}^{m_2-1}\sum_{r_{31}=0}^{m_3-1}(-1)^{r_{11}+r_{21}+r_{31}}1^{r_{11}}2^{r_{21}}3^{r_{31}}g^{\langle 3+r_{11}+r_{21}+r_{31}\rangle} + R_{m_1}^3 \quad (1.3)$$

$$R_{m_1}^3 = \sum_{r_{11}=0}^{m_1-1}\sum_{r_{21}=0}^{m_2-1}(-1)^{r_{11}+r_{21}+m_3}1^{m_1}2^{m_2}3^{m_3}\int_{f_a}^f g^{\langle 2+r_{11}+r_{21}+m_3\rangle}e^{3f}df$$

$$+\sum_{r_{11}=0}^{m_1-1}(-1)^{r_{11}+m_2}1^{m_1}2^{m_2}\int_{f_a}^f\left(\int_{f_a}^f g^{\langle 1+r_{11}+m_2\rangle}e^{2f}df\right)e^f df$$

$$+(-1)^{m_1}1^{m_1}\int_{f_a}^f\left(\int_{f_a}^f\left(\int_{f_a}^f g^{\langle m_1\rangle}e^f df\right)e^f df\right)e^f df \quad (1.3r)$$

Hereafter, by induction, we obtain

$$\int_a^x\cdots\int_a^x\{g(\log x)\}dx^n = x^n\sum_{r_{11}=0}^{m_1-1}\sum_{r_{21}=0}^{m_2-1}\cdots\sum_{r_{n1}=0}^{m_n-1}(-1)^{\sum_{k=1}^n r_{k1}}\prod_{k=1}^n k^{r_{k1}}g^{\left\langle n+\sum_{k=1}^n r_{k1}\right\rangle} + R_{m_1}^n \quad (1.n)$$

$$R_{m_1}^n = (-1)^{m_1}1^{m_1}\int_{f_a}^f\left(\int_{f_a}^f\left(\int_{f_a}^f\cdots\left(\int_{f_a}^f g^{\langle m_1\rangle}e^f df\right)\cdots e^f df\right)e^f df\right)e^f df \quad \text{(n-fold nest)}$$

$$+\sum_{r_{11}=0}^{m_1-1}(-1)^{r_{11}+m_2}1^{m_1}2^{m_2}\int_{f_a}^f\left(\int_{f_a}^f\cdots\left(\int_{f_a}^f g^{\langle 1+r_{11}+m_2\rangle}e^{2f}df\right)\cdots e^f df\right)e^f df$$

$$+\sum_{r_{11}=0}^{m_1-1}\sum_{r_{21}=0}^{m_2-1}(-1)^{r_{11}+r_{21}+m_3}1^{m_1}2^{m_2}3^{m_3}\int_{f_a}^f\cdots\left(\int_{f_a}^f g^{\langle 2+r_{11}+r_{21}+m_3\rangle}e^{3f}df\right)\cdots e^f df$$

$$\vdots$$

$$+\sum_{r_{11}=0}^{m_1-1}\sum_{r_{21}=0}^{m_2-1}\cdots\sum_{r_{n-1\,1}=0}^{m_{n-1}-1}(-1)^{\sum_{k=1}^{n-1}r_{k1}+m_n}\prod_{k=1}^n k^{m_k}\int_{f_a}^f g^{\left\langle n-1+\sum_{k=1}^{n-1}r_{k1}+m_n\right\rangle}e^{nf}df \quad (1.nr)$$

And replacing r_{k1} with r_k, we obtain the desired expression.

Q.E.D.

Although f_a is determined by the function $g(f)$ in Formula 23.3.1 , in many cases, $f_a = -\infty$ that is a zero of e^f. The calculation at the time of applying a concrete function to $g(f)$ should be a part of this section. However, since these are very long, these are described in the following independent sections.

23.4 Higher Integrals of (log x)^α , (log x)^m

23.4.1 Higher Integral of $(log\,x)^{\alpha}$

Lemma1

When $\Gamma(a,x)=\int_x^{\infty} t^{a-1}e^{-t}dt$ is the incomplete gamma function, the following expressions hold for $\lambda>0$ and $\alpha\neq -1,-2,-3,\cdots$.

(1) When $x\leqq 0$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} x^{\alpha}e^{\lambda x}dx^{n} = \frac{x^{\alpha}}{(-x)^{\alpha}}\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha,\ -\lambda x)(\lambda x)^{r}}{\lambda^{\alpha+n}(n-1)!}$$

(2) When $x>0$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} x^{\alpha}e^{\lambda x}dx^{n} = \frac{x^{\alpha}}{(-x)^{\alpha}}\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha,\ -\lambda x)(\lambda x)^{r}}{\lambda^{\alpha+n}(n-1)!}$$
$$+\ 2i\sin\alpha\pi\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha)(\lambda x)^{r}}{\lambda^{\alpha+n}(n-1)!}$$

Proof

Let $g(f)=f^{\alpha}e^{f}$, $f=\lambda x$ in Formula 23.1.2 . Then,

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x}(\lambda x)^{\alpha}e^{\lambda x}dx^{n} = \left(\frac{1}{\lambda}\right)^{n}\int_{-\infty}^{f}\cdots\int_{-\infty}^{f} f^{\alpha}e^{\alpha}df^{n}$$

Here, substitute Formula 16.5.1 in **16.5** for this right side. Then, when $x>0$,

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x}(\lambda x)^{\alpha}e^{\lambda x}dx^{n} = \left(\frac{1}{\lambda}\right)^{n}\frac{f^{\alpha}}{(-f)^{\alpha}}\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha,\ -f)f^{r}}{(n-1)!}$$
$$+\left(\frac{1}{\lambda}\right)^{n}2i\sin\alpha\pi\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha)f^{r}}{(n-1)!}$$
$$=\left(\frac{1}{\lambda}\right)^{n}\frac{(\lambda x)^{\alpha}}{(-\lambda x)^{\alpha}}\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha,\ -\lambda x)(\lambda x)^{r}}{(n-1)!}$$
$$+\left(\frac{1}{\lambda}\right)^{n}2i\sin\alpha\pi\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha)(\lambda x)^{r}}{(n-1)!}$$

i.e.

$$\lambda^{\alpha}\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} x^{\alpha}e^{\lambda x}dx^{n} = \frac{x^{\alpha}}{(-x)^{\alpha}}\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha,\ -\lambda x)(\lambda x)^{r}}{\lambda^{n}(n-1)!}$$
$$+\ 2i\sin\alpha\pi\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha)(\lambda x)^{r}}{\lambda^{n}(n-1)!}$$

From this, we obtain the desired expressions.

Formula 23.4.1

When $\Gamma(z)$ is the gamma function, the following expressions hold for a real number $\alpha \neq -1,-2,-3,\cdots$.

$$\int_0^x (\log x)^\alpha dx = x^1 \sum_{r=0}^{\infty} (-1)^r 1^r \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r)} (\log x)^{\alpha+1+r} + (-1)^\alpha \frac{\Gamma(1+\alpha)}{1!} {}_1C_1 \frac{x^0}{1^\alpha} \tag{1.1}$$

$$\int_0^x\int_0^x (\log x)^\alpha dx^2 = x^2 \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-1)^{r+s} 1^r 2^s \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+2+r+s)} (\log x)^{\alpha+2+r+s} + (-1)^\alpha \frac{\Gamma(1+\alpha)}{2!} \left({}_2C_1 \frac{x^1}{1^\alpha} - {}_2C_2 \frac{x^0}{2^\alpha} \right) \tag{1.2}$$

$$\int_0^x\int_0^x\int_0^x (\log x)^\alpha dx^3 = x^3 \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} (-1)^{r+s+t} 1^r 2^s 3^t \times \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+3+r+s+t)} (\log x)^{\alpha+3+r+s+t} + (-1)^\alpha \frac{\Gamma(1+\alpha)}{3!} \left({}_3C_1 \frac{x^2}{1^\alpha} - {}_3C_2 \frac{x^1}{2^\alpha} + {}_3C_3 \frac{x^0}{3^\alpha} \right) \tag{1.3}$$

$\vdots$

$$\int_0^x \cdots \int_0^x (\log x)^\alpha dx^n = x^n \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} (-1)^{\sum_{k=1}^n r_k} \prod_{k=1}^n k^{r_k} \times \frac{\Gamma(1+\alpha)}{\Gamma\left(1+\alpha+n+\sum_{k=1}^n r_k\right)} (\log x)^{\alpha+n+\sum_{k=1}^n r_k} + (-1)^\alpha \frac{\Gamma(1+\alpha)}{n!} \sum_{k=1}^n (-1)^{k-1} {}_nC_k \frac{x^{n-k}}{k^\alpha} \tag{1.n}$$

Proof

When the 1st order, substituting $g = f^\alpha$, $f = \log x$ for (1.1), (1.1r) in Formula 23.3.1 ,

$$\int_0^x (\log x)^\alpha dx = x \sum_{r=0}^{m-1} (-1)^r 1^r \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r)} (\log x)^{\alpha+1+r} + R_m^1 \tag{1}$$

$$R_m^1 = (-1)^m 1^m \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m)} \int_{-\infty}^f f^{\alpha+m} e^f df \tag{1r}$$

If $f \leqq 0$, from Lemma1,

$$\int_{-\infty}^f f^{\alpha+m} e^f df = \frac{f^{\alpha+m}}{(-f)^{\alpha+m}} \Gamma(1+\alpha+m, -f)$$

Substituting this for (1r),

$$R_m^1 = (-1)^m 1^m \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m)} \frac{f^{\alpha+m}}{(-f)^{\alpha+m}} \Gamma(1+\alpha+m, -f)$$

$$= (-1)^m(-1)^{\alpha+m}1^m \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m)}\Gamma(1+\alpha+m\ ,\ -f) \qquad \left\{\because\ \frac{f^{\alpha}}{(-f)^{\alpha}} = (-1)^{\alpha}\right\}$$

i.e.

$$R_m^1 = (-1)^{\alpha}\Gamma(1+\alpha)\cdot\frac{\Gamma(1+\alpha+m\ ,\ -f)}{\Gamma(1+\alpha+m)}$$

Let $m\to\infty$. Then,

$$\lim_{m\to\infty}\frac{\Gamma(1+\alpha+m\ ,\ -f)}{\Gamma(1+\alpha+m)} = 1$$

Therefore,

$$R_{\infty}^1 = (-1)^{\alpha}\Gamma(1+\alpha)$$

If $f>0$, from Lemma1,

$$\int_{-\infty}^{f} f^{\alpha+m}e^f df = \frac{f^{\alpha+m}}{(-f)^{\alpha+m}}\Gamma(1+\alpha+m\ ,\ -f)+2i\,sin\{(\alpha+m)\pi\}\,\Gamma(1+\alpha+m)$$

Substituting this for (1r) ,

$$R_m^1 = (-1)^m1^m\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m)}\int_{-\infty}^{f} f^{\alpha+m}e^f df$$

$$= (-1)^m\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m)}\left\{\frac{f^{\alpha+m}}{(-f)^{\alpha+m}}\Gamma(1+\alpha+m,\ -f)+2i\,sin(\alpha\pi+m\pi)\Gamma(1+\alpha+m)\right\}$$

$$= (-1)^m\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m)}\left\{(-1)^{-\alpha-m}\Gamma(1+\alpha+m,\ -f)+2i\,sin(\alpha\pi+m\pi)\Gamma(1+\alpha+m)\right\}$$

$$= \frac{\Gamma(1+\alpha)}{(-1)^{\alpha}}\frac{\Gamma(1+\alpha+m,\ -f)}{\Gamma(1+\alpha+m)} + 2i\,\Gamma(1+\alpha)(-1)^m sin(\alpha\pi+m\pi)$$

i.e.

$$R_m^1 = \frac{\Gamma(1+\alpha)}{(-1)^{\alpha}}\frac{\Gamma(1+\alpha+m,\ -f)}{\Gamma(1+\alpha+m)} + 2i\,\Gamma(1+\alpha)sin\alpha\pi$$

Let $m\to\infty$. Then,

$$R_{\infty}^1 = \Gamma(1+\alpha)\left\{\frac{1}{(-1)^{\alpha}} + 2i\,sin\alpha\pi\right\}$$

Here,

$$\frac{1}{(-1)^{\alpha}} + 2i\,sin\alpha\pi = (-1)^{\alpha}$$

Therefore $R_{\infty}^1 = (-1)^{\alpha}\Gamma(1+\alpha)$. This is the same result as the time of $f\leqq 0$.

Thus, (1) becomes as follows.

$$\int_0^x (log\,x)^{\alpha}dx = x^1\sum_{r=0}^{\infty}(-1)^r1^r\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r)}(log\,x)^{\alpha+1+r}$$

$$+ (-1)^{\alpha}\frac{\Gamma(1+\alpha)}{1!}{}_1C_1\frac{x^0}{1^{\alpha}} \qquad (1.1)$$

When the 2nd order, substituting $g=f^{\alpha}$, $f=log\,x$ for (1.2), (1.2r) in Formula 23.3.1 ,

$$\int_0^x\!\!\int_0^x (log\,x)^{\alpha}dx^2 = x^2\sum_{r=0}^{m_1-1}\sum_{s=0}^{m_2-1}(-1)^{r+s}1^r2^s\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+2+r+s)}(log\,x)^{\alpha+2+r+s} + R_{m_1}^2 \qquad (2)$$

$$R^{n}_{m_1} = (-1)^{m_1}1^{m_1}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m_1)}\int_{-\infty}^{f}\left(\int_{-\infty}^{f} f^{\alpha+m_1}e^{f}df\right)e^{f}df$$

$$+\sum_{r=0}^{m_1-1}(-1)^{r+m_2}1^{m_1}2^{m_2}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r+m_2)}\int_{-\infty}^{f} f^{\alpha+1+r+m_2}e^{2f}df \qquad (2r)$$

If $f \leqq 0$, from Lemma1,

$$\int_{-\infty}^{f} f^{\alpha+m_1}e^{f}df = \frac{f^{\alpha+m_1}}{(-f)^{\alpha+m_1}}\Gamma(1+\alpha+m_1,\ -f)$$

$$\int_{-\infty}^{f} x^{\alpha+1+r+m_2}e^{2f}df = \frac{f^{\alpha+1+r+m_2}}{(-f)^{\alpha+1+r+m_2}}\frac{\Gamma(1+\alpha+1+r+m_2,\ -2f)}{2^{\alpha+1+r+m_2+1}}$$

Substituting these for (2r),

$$R^{2}_{m_1} = (-1)^{m_1}1^{m_1}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m_1)}\int_{-\infty}^{f}\left\{\frac{f^{\alpha+m_1}}{(-f)^{\alpha+m_1}}\Gamma(1+\alpha+m_1,\ -f)\right\}e^{f}df$$

$$+\sum_{r=0}^{m_1-1}(-1)^{r+m_2}1^{m_1}2^{m_2}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r+m_2)}\frac{f^{\alpha+1+r+m_2}}{(-f)^{\alpha+1+r+m_2}}\frac{\Gamma(1+\alpha+1+r+m_2,\ -2f)}{2^{\alpha+1+r+m_2+1}}$$

$$= (-1)^{m_1}(-1)^{\alpha+m_1}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m_1)}\left\{e^{f}\Gamma(1+\alpha+m_1,\ -f) - \frac{\Gamma(1+\alpha+m_1,\ -2f)}{2^{\alpha+m_1+1}}\right\}$$

$$+\sum_{r=0}^{m_1-1}(-1)^{r+m_2}(-1)^{\alpha+1+r+m_2}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r+m_2)}\frac{\Gamma(1+\alpha+1+r+m_2,\ -2f)}{2^{\alpha+1+r+1}}$$

$$= (-1)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m_1)}\left\{e^{f}\Gamma(1+\alpha+m_1,\ -f) - \frac{\Gamma(1+\alpha+m_1,\ -2f)}{2^{\alpha+m_1+1}}\right\}$$

$$+\sum_{r=0}^{m_1-1}(-1)^{\alpha+1}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r+m_2)}\frac{\Gamma(1+\alpha+1+r+m_2,\ -2f)}{2^{\alpha+1+r+1}}$$

i.e.

$$R^{2}_{m_1} = (-1)^{\alpha}\Gamma(1+\alpha)\left\{\frac{e^{f}\Gamma(1+\alpha+m_1,\ -f)}{\Gamma(1+\alpha+m_1)} - \frac{\Gamma(1+\alpha+m_1,\ -2f)}{2^{\alpha+m_1+1}\Gamma(1+\alpha+m_1)}\right\}$$

$$-(-1)^{\alpha}\Gamma(1+\alpha)\sum_{r=0}^{m_1-1}\frac{\Gamma(1+\alpha+1+r+m_2,\ -2f)}{2^{\alpha+2+r}\Gamma(1+\alpha+1+r+m_2)}$$

Let $m_1, m_2 \to \infty$. Then,

$$R^{2}_{\infty} = (-1)^{\alpha}\Gamma(1+\alpha)e^{f} - (-1)^{\alpha}\Gamma(1+\alpha)\sum_{r=0}^{\infty}\frac{1}{2^{\alpha+2+r}}$$

$$= (-1)^{\alpha}\Gamma(1+\alpha)\left(e^{\log x} - \frac{1}{2^{\alpha+1}}\right)$$

$$= (-1)^{\alpha}\frac{\Gamma(1+\alpha)}{2!}\left(\frac{2x^{1}}{1^{\alpha}} - \frac{x^{0}}{2^{\alpha}}\right)$$

If $f > 0$, from Lemma1,

$$\int_{-\infty}^{f} f^{\alpha+m_1} e^f df = \frac{x^{\alpha+m_1}}{(-f)^{\alpha+m_1}} \Gamma(1+\alpha+m_1, -f) + 2i \sin\{(\alpha+m_1)\pi\} \Gamma(1+\alpha+m_1)$$

$$\int_{-\infty}^{f} x^{\alpha+1+r+m_2} e^{2f} df = \frac{f^{\alpha+1+r+m_2}}{(-f)^{\alpha+1+r+m_2}} \frac{\Gamma(1+\alpha+1+r+m_2, -2f)}{2^{\alpha+1+r+m_2+1}}$$

$$+ 2i \sin(\alpha+1+r+m_2)\pi \frac{\Gamma(1+\alpha+1+r+m_2)}{2^{\alpha+1+r+m_2+1}}$$

Substituting these for (2r) ,

$$R_{m_1}^2 = (-1)^{m_1} 1^{m_1} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m_1)} \int_{-\infty}^{f} \left\{ \frac{f^{\alpha+m_1}}{(-f)^{\alpha+m_1}} \Gamma(1+\alpha+m_1, -f) \right\} e^f df$$

$$+ (-1)^{m_1} 1^{m_1} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m_1)} \int_{-\infty}^{f} \{2i \sin\{(\alpha+m_1)\pi\} \Gamma(1+\alpha+m_1)\} e^f df$$

$$+ \sum_{r=0}^{m_1-1} (-1)^{r+m_2} 1^{m_1} 2^{m_2} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r+m_2)} \frac{f^{\alpha+1+r+m_2}}{(-f)^{\alpha+1+r+m_2}} \frac{\Gamma(1+\alpha+1+r+m_2, -2f)}{2^{\alpha+1+r+m_2+1}}$$

$$+ \sum_{r=0}^{m_1-1} (-1)^{r+m_2} 1^{m_1} 2^{m_2} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r+m_2)} \cdot 2i \sin(\alpha+1+r+m_2)\pi \frac{\Gamma(1+\alpha+1+r+m_2)}{2^{\alpha+1+r+m_2+1}}$$

$$= \frac{(-1)^{m_1}}{(-1)^{\alpha+m_1}} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m_1)} \left\{ e^f \Gamma(1+\alpha+m_1, -f) - \frac{\Gamma(1+\alpha+m_1, -2f)}{2^{\alpha+m_1+1}} \right\}$$

$$+ (-1)^{m_1} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m_1)} \{2i \sin\{(\alpha+m_1)\pi\} \Gamma(1+\alpha+m_1)\} e^f$$

$$+ \sum_{r=0}^{m_1-1} \frac{(-1)^{r+m_2}}{(-1)^{\alpha+1+r+m_2}} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r+m_2)} \frac{\Gamma(1+\alpha+1+r+m_2, -2f)}{2^{\alpha+1+r+1}}$$

$$+ \sum_{r=0}^{m_1-1} (-1)^{r+m_2} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1+r+m_2)} \cdot 2i \sin\{(\alpha+1+r+m_2)\pi\} \frac{\Gamma(1+\alpha+1+r+m_2)}{2^{\alpha+1+r+1}}$$

i.e.

$$R_{m_1}^2 = \frac{\Gamma(1+\alpha)}{(-1)^{\alpha}} \left\{ \frac{e^f \Gamma(1+\alpha+m_1, -f)}{\Gamma(1+\alpha+m_1)} - \frac{\Gamma(1+\alpha+m_1, -2f)}{2^{\alpha+m_1+1} \Gamma(1+\alpha+m_1)} \right\} + 2i \sin\alpha\pi\, \Gamma(1+\alpha) e^f$$

$$+ \frac{1}{(-1)^{\alpha+1}} \frac{\Gamma(1+\alpha)}{2^{\alpha+2}} \sum_{r=0}^{m_1-1} \frac{\Gamma(1+\alpha+1+r+m_2, -2f)}{2^r \Gamma(1+\alpha+1+r+m_2)}$$

$$+ 2i \sin\{(\alpha+1)\pi\} \frac{\Gamma(1+\alpha)}{2^{\alpha+2}} \sum_{r=0}^{m_1-1} \frac{\Gamma(1+\alpha+1+r+m_2)}{2^r \Gamma(1+\alpha+1+r+m_2)}$$

Let $m_1, m_2 \to \infty$. Then,

$$R_{\infty}^2 = \frac{\Gamma(1+\alpha) e^f}{(-1)^{\alpha}} + 2i \sin\alpha\pi\, \Gamma(1+\alpha) e^f$$

$$+ \frac{1}{(-1)^{\alpha+1}} \frac{\Gamma(1+\alpha)}{2^{\alpha+2}} \sum_{r=0}^{\infty} \frac{1}{2^r} + 2i \sin\{(\alpha+1)\pi\} \frac{\Gamma(1+\alpha)}{2^{\alpha+2}} \sum_{r=0}^{\infty} \frac{1}{2^r}$$

$$= \left\{ \frac{1}{(-1)^{\alpha}} + 2i\sin\alpha\pi \right\} \Gamma(1+\alpha)\, e^{\log x}$$
$$+ \left\{ \frac{1}{(-1)^{\alpha+1}} + 2i\sin\{(\alpha+1)\pi\} \right\} \frac{\Gamma(1+\alpha)}{2^{\alpha+1}}$$

Here,

$$\frac{1}{(-1)^{\alpha}} + 2i\sin\alpha\pi = (-1)^{\alpha}$$

Therefore,

$$R_{\infty}^{2} = (-1)^{\alpha}\Gamma(1+\alpha)\,x + (-1)^{\alpha+1}\frac{\Gamma(1+\alpha)}{2^{\alpha+1}} = (-1)^{\alpha}\frac{\Gamma(1+\alpha)}{2}\left(\frac{2x^{1}}{1^{\alpha}} - \frac{x^{0}}{2^{\alpha}} \right)$$

This is the same result as the time of $f \leqq 0$.
Thus, (2) becomes as follows.

$$\int_0^x\!\!\int_0^x (\log x)^{\alpha} dx^2 = x^2 \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-1)^{r+s} 1^r 2^s \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+2+r+s)} (\log x)^{\alpha+2+r+s}$$
$$+ (-1)^{\alpha}\frac{\Gamma(1+\alpha)}{2!}\left({}_2C_1\frac{x^1}{1^{\alpha}} - {}_2C_2\frac{x^0}{2^{\alpha}} \right) \qquad (1.2)$$

Hereafter, by induction, we obtain the general expression.

Example The 3rd integral of $(\log x)^{5/2}$

When the left side of (1.3) is replaced with Riemann-Liouville Integral and the one arbitrary point $x=2$ is given, the values of the both sides are as follows.

```
a = 5 / 2; n = 3; m = 10;
fl[x_] := 1/Gamma[n] ∫_0^x (x - t)^(n-1) (Log[t])^a dt
fr[x_] := x^3 Σ_{r=0}^m Σ_{s=0}^m Σ_{t=0}^m (-1)^(r+s+t) 1^r 2^s 3^t Gamma[1 + a]/Gamma[1 + a + 3 + r + s + t] (Log[x])^(a+3+r+s+t) +
   (-1)^a Gamma[1 + a]/3! (Binomial[3, 1] x^2/1^a - Binomial[3, 2] x^1/2^a + Binomial[3, 3] x^0/3^a)

N[fl[2]]                              N[fr[2]]
0.00675076  + 6.09474 i               0.00675076  + 6.09474 i
```

23.4.2 Higher Integrals of $(\log x)^m$

Lemma2

Let λ be a positive number, $\Gamma(z)$ be the gamma function and m, n are non-negative integers.

$$\int_{-\infty}^{x}\!\cdots\!\int_{-\infty}^{x} e^{\lambda x} x^m dx^n = \frac{e^{\lambda x}}{\lambda^{m+n}} \sum_{r=0}^{m} \binom{-n}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} (\lambda x)^{m-r}$$

Proof

Let $g(f) = f^m e^f$, $f = \lambda x$ in Formula 23.1.2 . Then,

$$\int_{-\infty}^{x}\!\cdots\!\int_{-\infty}^{x} (\lambda x)^m e^{\lambda x} dx^n = \left(\frac{1}{\lambda}\right)^n \int_{-\infty}^{f}\!\cdots\!\int_{-\infty}^{f} f^m e^f df^n$$

Here, substitute Formula 16.5.1' in **16.5** for this right side. Then,

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x}(\lambda x)^m e^{\lambda x}dx^n = \left(\frac{1}{\lambda}\right)^n e^f\sum_{r=0}^{m}\binom{-n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}f^{m-r}$$

i.e.

$$\lambda^m\int_{-\infty}^{x}\cdots\int_{-\infty}^{x}e^{\lambda x}x^m dx^n = \frac{e^{\lambda x}}{\lambda^n}\sum_{r=0}^{m}\binom{-n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}(\lambda x)^{m-r}$$

From this, we obtain the desired expression.

Formula 23.4.2

When $m=1,2,3,\cdots$,

$$\int_0^x(\log x)^m dx = \frac{m!}{1}x^1\sum_{r=0}^{m}\frac{(-1)^r}{1^r}\frac{(\log x)^{m-r}}{(m-r)!} \tag{2.1}$$

$$\int_0^x\int_0^x(\log x)^m dx^2 = -\frac{m!}{2}x^2\sum_{r=1}^{m+1}\sum_{s=0}^{m+1-r}\frac{(-1)^{r+s}}{1^r 2^s}\frac{(\log x)^{m+1-r-s}}{(m+1-r-s)!} \tag{2.2}$$

$$\int_0^x\int_0^x\int_0^x(\log x)^m dx^3 = \frac{m!}{3}x^3\sum_{r=1}^{m+2}\sum_{s=1}^{m+2-r}\sum_{t=0}^{m+2-r-s}\frac{(-1)^{r+s+t}}{1^r 2^s 3^t}\frac{(\log x)^{m+2-r-s-t}}{(m+2-r-s-t)!} \tag{2.3}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x(\log x)^m dx^n = (-1)^n\frac{m!}{n}x^n\sum_{r_1=1}^{m+n-1}\sum_{r_2=1}^{m+n-1-r_1}\cdots\sum_{r_{n-1}=1}^{m+n-1-\sum_{k=1}^{n-2}r_k}\sum_{r_n=0}^{m+n-1-\sum_{k=1}^{n-1}r_k}$$

$$\times\frac{(-1)^{\sum_{k=1}^{n}r_k}}{\prod_{k=1}^{n}k^{r_k}}\frac{(\log x)^{m+n-1-\sum_{k=1}^{n}r_k}}{\left(m+n-1-\sum_{k=1}^{n}r_k\right)!} \tag{2.n}$$

Proof

When the 1st order, substituting $g=f^{\alpha}$, $f=\log x$ for (1.1), (1.1r) in Formula 23.3.1 ,

$$\int_0^x(\log x)^m dx = x\sum_{r=0}^{m_1-1}(-1)^r 1^r\frac{\Gamma(1+m)}{\Gamma(1+m+1+r)}(\log x)^{m+1+r} + R_{m_1}^1$$

$$R_{m_1}^1 = (-1)^{m_1}1^{m_1}\frac{\Gamma(1+m)}{\Gamma(1+m+m_1)}\int_{-\infty}^{f}f^{m+m_1}e^f df$$

On the other hand, from Lemma2 ,

$$\int_{-\infty}^{f}e^f x^{m+m_1}dx^n = e^f\sum_{r=0}^{m+m_1}\binom{-1}{r}\frac{\Gamma(1+m+m_1)}{\Gamma(1+m+m_1-r)}f^{m+m_1-r}$$

Substituting this for $R_{m_1}^1$,

$$R_{m_1}^1 = (-1)^{m_1}1^{m_1}\frac{\Gamma(1+m)}{\Gamma(1+m+m_1)}e^f\sum_{r=0}^{m+m_1}\binom{-1}{r}\frac{\Gamma(1+m+m_1)}{\Gamma(1+m+m_1-r)}f^{m+m_1-r}$$

$$= (-1)^{m_1}1^{m_1}e^{\log x}\sum_{r=0}^{m+m_1}\binom{-1}{r}\frac{\Gamma(1+m)}{\Gamma(1+m+m_1-r)}(\log x)^{m+m_1-r}$$

Therefore,

$$\int_0^x (\log x)^m dx = x\sum_{r=0}^{m_1-1}(-1)^r 1^r \frac{\Gamma(1+m)}{\Gamma(1+m+1+r)}(\log x)^{m+1+r}$$

$$+ (-1)^{m_1}1^{m_1} x\sum_{r=0}^{m+m_1}(-1)^r \frac{\Gamma(1+m)}{\Gamma(1+m+m_1-r)}(\log x)^{m+m_1-r}$$

Here, let $m_1 = 1$. Then,

$$\int_0^x (\log x)^m dx = x\frac{\Gamma(1+m)}{\Gamma(1+m+1)}(\log x)^{m+1}$$

$$- x\sum_{r=0}^{m+1}(-1)^r \frac{\Gamma(1+m)}{\Gamma(1+m+1-r)}(\log x)^{m+1-r}$$

$$= x\frac{\Gamma(1+m)}{\Gamma(1+m+1)}(\log x)^{m+1} - x\frac{\Gamma(1+m)}{\Gamma(1+m+1)}(\log x)^{m+1}$$

$$- x\sum_{r=1}^{m+1}(-1)^r \frac{\Gamma(1+m)}{\Gamma(1+m+1-r)}(\log x)^{m+1-r}$$

$$= x\sum_{r=0}^{m}(-1)^r \frac{\Gamma(1+m)}{\Gamma(1+m-r)}(\log x)^{m-r}$$

i.e.

$$\int_0^x (\log x)^m dx = \frac{m!}{1}x^1\sum_{r=0}^{m}\frac{(-1)^r}{1^r}\frac{(\log x)^{m-r}}{(m-r)!} \tag{2.1}$$

When the 2nd order, substituting $g = f^{\alpha}$, $f = \log x$ for (1.2), (1.2r) in Formula 23.3.1 ,

$$\int_0^x\int_0^x (\log x)^m dx^2 = x^2\sum_{r=0}^{m_1-1}\sum_{s=0}^{m_2-1}(-1)^{r+s}1^r 2^s \frac{\Gamma(1+m)}{\Gamma(1+m+2+r+s)}(\log x)^{m+2+r+s} + R_{m_1}^2$$

$$R_{m_1}^2 = (-1)^{m_1}1^{m_1}\frac{\Gamma(1+m)}{\Gamma(1+m+m_1)}\int_{-\infty}^f\left(\int_{-\infty}^f f^{m+m_1}e^f df\right)e^f df$$

$$+\sum_{r=0}^{m_1-1}(-1)^{r+m_2}1^{m_1}2^{m_2}\frac{\Gamma(1+m)}{\Gamma(1+m+1+r+m_2)}\int_{-\infty}^f f^{m+1+r+m_2}e^{2f}df$$

On the other hand, from Lemma2 ,

$$\int_{-\infty}^f e^f f^{m+m_1}dx^n = e^f\sum_{r=0}^{m+m_1}\binom{-1}{r}\frac{\Gamma(1+m+m_1)}{\Gamma(1+m+m_1-r)}f^{m+m_1-r}$$

$$\int_{-\infty}^f e^{2f}f^{m+1+r+m_2}dx = \frac{e^{2f}}{2^{m+1+r+m_2+1}}$$

$$\times\sum_{s=0}^{m+1+r+m_2}\binom{-1}{s}\frac{\Gamma(1+m+1+r+m_2)}{\Gamma(1+m+1+r+m_2-s)}(2f)^{m+1+r+m_2-s}$$

$$\int_{-\infty}^f e^{2f}f^{m+m_1-r}df = \frac{e^{2f}}{2^{m+m_1-r+1}}\sum_{s=0}^{m+m_1-r}\binom{-1}{s}\frac{\Gamma(1+m+m_1-r)}{\Gamma(1+m+m_1-r-s)}(2f)^{m+m_1-r-s}$$

Substituting these for $R_{m_1}^2$,

$$R_{m_1}^2 = (-1)^{m_1}1^{m_1}\frac{\Gamma(1+m)}{\Gamma(1+m+m_1)}\int_{-\infty}^f\left\{e^f\sum_{r=0}^{m+m_1}\binom{-1}{r}\frac{\Gamma(1+m+m_1)}{\Gamma(1+m+m_1-r)}f^{m+m_1-r}\right\}e^f df$$

$$+\sum_{r=0}^{m_1-1}(-1)^{r+m_2}1^{m_1}2^{m_2}\frac{\Gamma(1+m)}{\Gamma(1+m+1+r+m_2)}\frac{e^{2f}}{2^{m+1+r+m_2+1}}$$

$$\times\sum_{s=0}^{m+1+r+m_2}\binom{-1}{s}\frac{\Gamma(1+m+1+r+m_2)}{\Gamma(1+m+1+r+m_2-s)}(2f)^{m+1+r+m_2-s}$$

$$=(-1)^{m_1}1^{m_1}\sum_{r=0}^{m+m_1}\binom{-1}{r}\frac{\Gamma(1+m)}{\Gamma(1+m+m_1-r)}\int_{-\infty}^{f}e^{2f}f^{m+m_1-r}df$$

$$+\sum_{r=0}^{m_1-1}(-1)^{r+m_2}1^{m_1}2^{m_2}\frac{e^{2f}}{2^1}\sum_{s=0}^{m+1+r+m_2}\binom{-1}{s}\frac{\Gamma(1+m)}{\Gamma(1+m+1+r+m_2-s)}\frac{f^{m+1+r+m_2-s}}{2^s}$$

$$=(-1)^{m_1}1^{m_1}\sum_{r=0}^{m+m_1}\binom{-1}{r}\frac{\Gamma(1+m)}{\Gamma(1+m+m_1-r)}\frac{e^{2\log x}}{2^{m+m_1-r+1}}$$

$$\times\sum_{s=0}^{m+m_1-r}\binom{-1}{s}\frac{\Gamma(1+m+m_1-r)}{\Gamma(1+m+m_1-r-s)}(2f)^{m+m_1-r-s}$$

$$+\sum_{r=0}^{m_1-1}(-1)^{r+m_2}1^{m_1}2^{m_2}\frac{e^{2\log x}}{2}\sum_{s=0}^{m+1+r+m_2}\binom{-1}{s}\frac{\Gamma(1+m)}{\Gamma(1+m+1+r+m_2-s)}\frac{f^{m+1+r+m_2-s}}{2^s}$$

$$=(-1)^{m_1}1^{m_1}\frac{x^2}{2}\sum_{r=0}^{m+m_1}(-1)^r\sum_{s=0}^{m+m_1-r}(-1)^s\frac{\Gamma(1+m)}{\Gamma(1+m+m_1-r-s)}\frac{(\log x)^{m+m_1-r-s}}{2^s}$$

$$+\sum_{r=0}^{m_1-1}(-1)^{r+m_2}1^{m_1}2^{m_2}\frac{x^2}{2}\sum_{s=0}^{m+1+r+m_2}(-1)^s\frac{\Gamma(1+m)}{\Gamma(1+m+1+r+m_2-s)}\frac{(\log x)^{m+1+r+m_2-s}}{2^s}$$

Therefore,

$$\int_0^x\int_0^x(\log x)^m dx^2=x^2\sum_{r=0}^{m_1-1}\sum_{s=0}^{m_2-1}(-1)^{r+s}1^r2^s\frac{\Gamma(1+m)}{\Gamma(1+m+2+r+s)}(\log x)^{m+2+r+s}$$

$$+(-1)^{m_1}1^{m_1}\frac{x^2}{2}\sum_{r=0}^{m+m_1}\sum_{s=0}^{m+m_1-r}(-1)^{r+s}\frac{\Gamma(1+m)}{\Gamma(1+m+m_1-r-s)}\frac{(\log x)^{m+m_1-r-s}}{2^s}$$

$$+\sum_{r=0}^{m_1-1}(-1)^{r+m_2}1^{m_1}2^{m_2}\frac{x^2}{2}\sum_{s=0}^{m+1+r+m_2}(-1)^s\frac{\Gamma(1+m)}{\Gamma(1+m+1+r+m_2-s)}\frac{(\log x)^{m+1+r+m_2-s}}{2^s}$$

Here, let $m_1=m_2=1$, Then,

$$\int_0^x\int_0^x(\log x)^m dx^2= x^2\frac{\Gamma(1+m)}{\Gamma(1+m+2)}(\log x)^{m+2}$$

$$-\frac{x^2}{2}\sum_{r=0}^{m+1}\sum_{s=0}^{m+1-r}(-1)^{r+s}\frac{\Gamma(1+m)}{\Gamma(1+m+1-r-s)}\frac{(\log x)^{m+1-r-s}}{2^s}$$

$$+x^2\sum_{s=0}^{m+2}(-1)^{s+1}\frac{\Gamma(1+m)}{\Gamma(1+m+2-s)}\frac{(\log x)^{m+2-s}}{2^s}$$

$$=x^2\frac{\Gamma(1+m)}{\Gamma(1+m+2)}(\log x)^{m+2}-x^2\frac{\Gamma(1+m)}{\Gamma(1+m+2)}\frac{(\log x)^{m+2}}{2^0}$$

$$-x^2\sum_{s=0}^{m+1}(-1)^s\frac{\Gamma(1+m)}{\Gamma(1+m+1-s)}\frac{(\log x)^{m+1-s}}{2^{s+1}}$$

$$-\frac{x^2}{2}\sum_{r=1}^{m+1}\sum_{s=0}^{m+1-r}(-1)^{r+s}\frac{\Gamma(1+m)}{\Gamma(1+m+1-r-s)}\frac{(\log x)^{m+1-r-s}}{2^s}$$

$$+x^2\sum_{s=1}^{m+2}(-1)^{s+1}\frac{\Gamma(1+m)}{\Gamma(1+m+2-s)}\frac{(\log x)^{m+2-s}}{2^s}$$

$$=-\frac{x^2}{2}\sum_{r=1}^{m+1}\sum_{s=0}^{m+1-r}(-1)^{r+s}\frac{\Gamma(1+m)}{\Gamma(1+m+1-r-s)}\frac{(\log x)^{m+1-r-s}}{2^s}$$

i.e.

$$\int_0^x\int_0^x(\log x)^m dx^2=-\frac{m!}{2}x^2\sum_{r=1}^{m+1}\sum_{s=0}^{m+1-r}\frac{(-1)^{r+s}}{1^r 2^s}\frac{(\log x)^{m+1-r-s}}{(m+1-r-s)!} \tag{2.2}$$

Hereafter, by induction, we obtain the general expression.

Example The 3rd integral of $(\log x)^4$

When both the sides are calculated in (2.3) at $m=4$, it is as follows.

```
m = 4;
```

$$\text{f1}[x\_] := \int_0^x\int_0^v\int_0^u(\text{Log}[t])^m\,dt\,du\,dv$$

$$\text{fr}[x\_] := \frac{m!}{3}x^3\sum_{r=1}^{m+2}\sum_{s=1}^{m+2-r}\sum_{t=0}^{m+2-r-s}\frac{(-1)^{r+s+t}}{1^r 2^s 3^t}\frac{(\text{Log}[x])^{m+2-r-s-t}}{(m+2-r-s-t)!}$$

```
Expand[f1[x]]
```

$$\frac{3661x^3}{324}-\frac{575}{54}x^3\,\text{Log}[x]+\frac{85}{18}x^3\,\text{Log}[x]^2-\frac{11}{9}x^3\,\text{Log}[x]^3+\frac{1}{6}x^3\,\text{Log}[x]^4$$

```
Expand[fr[x]]
```

$$\frac{3661x^3}{324}-\frac{575}{54}x^3\,\text{Log}[x]+\frac{85}{18}x^3\,\text{Log}[x]^2-\frac{11}{9}x^3\,\text{Log}[x]^3+\frac{1}{6}x^3\,\text{Log}[x]^4$$

23.5 Higher Integral of 1 / log x

Formula 23.5.1

When $\Gamma(z), \psi(z), \gamma$ denote the gamma function, the digamma function, Euler-Mascheroni-Constant (= 0.57721566...) respectively, the following expressions hold for $x \geqq 0$.

$$\int_0^x \frac{1}{\log x}dx = x\sum_{r=0}^{\infty}(-1)^r 1^r \frac{\log|\log x| - \psi(1+r) - \gamma}{\Gamma(1+r)}(\log x)^r + \frac{\gamma+\log 1}{0!} \tag{1.1}$$

$$\int_0^x\int_0^x \frac{1}{\log x}dx^2 = x^2\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}1^r 2^s \frac{\log|\log x| - \psi(2+r+s) - \gamma}{\Gamma(2+r+s)}(\log x)^{1+r+s}$$

$$+\frac{1}{1!}\{(\gamma+\log 1)x - (\gamma+\log 2)\} \tag{1.2}$$

$$\int_0^x\int_0^x\int_0^x \frac{1}{\log x}dx^3 = x^3\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}1^r 2^s 3^t$$

$$\times\frac{\log|\log x| - \psi(3+r+s+t) - \gamma}{\Gamma(3+r+s+t)}(\log x)^{2+r+s+t}$$

$$+\frac{1}{2!}\{(\gamma+\log 1)x^2 - 2(\gamma+\log 2)x + (\gamma+\log 3)\} \tag{1.3}$$

$\vdots$

$$\int_0^x\cdots\int_0^x \frac{1}{\log x}dx^n = x^n\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\prod_{k=1}^{n} k^{r_k}$$

$$\times\frac{\log|\log x| - \psi\left(n+\sum_{k=1}^{n} r_k\right) - \gamma}{\Gamma\left(n+\sum_{k=1}^{n} r_k\right)}(\log x)^{n-1+\sum_{k=1}^{n} r_k}$$

$$+\frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r {}_{n-1}C_r\{\gamma+\log(r+1)\}x^{n-1-r} \tag{1.n}$$

Calculation

Let $g = 1/f$. Then, for $f > 0$,

$$g^{<k>} = (\log f)^{<k-1>} = \frac{\log f - \psi(k) - \gamma}{\Gamma(k)}f^{k-1} \qquad k=1,2,3,\cdots$$

Therefore,

$$g^{\left\langle n+\sum_{k=1}^{n} r_k\right\rangle} = \frac{\log f - \psi\left(n+\sum_{k=1}^{n} r_k\right) - \gamma}{\Gamma\left(n+\sum_{k=1}^{n} r_k\right)}f^{n+\sum_{k=1}^{n} r_k - 1}$$

$$g^{\langle m_1\rangle} = \frac{\log f - \psi(m_1) - \gamma}{\Gamma(m_1)}f^{m_1-1}$$

$$g^{\langle 1+r_1+m_2\rangle} = \frac{\log f - \psi(1+r_1+m_2) - \gamma}{\Gamma(1+r_1+m_2)}f^{1+r_1+m_2-1}$$

$\vdots$

$$g^{\left\langle n-1+\sum_{k=1}^{n-1} r_k+m_n \right\rangle} = \frac{\log f - \psi\left(n-1+\sum_{k=1}^{n-1} r_k+m_n\right) - \gamma}{\Gamma\left(n-1+\sum_{k=1}^{n-1} r_k+m_n\right)} f^{\,n-1+\sum_{k=1}^{n-1} r_k+m_n-1}$$

Substitute these for Formula 23.3.1 and returning f to x in the non-remainder terms, since $[-\infty, f] \rightarrow [0, x]$,

$$\int_0^x \cdots \int_0^x \frac{1}{\log x} dx^n = x^n \sum_{r_1=0}^{m_1-1} \sum_{r_2=0}^{m_2-1} \cdots \sum_{r_n=0}^{m_n-1} (-1)^{\sum_{k=1}^{n} r_k} \prod_{k=1}^{n} k^{r_k}$$
$$\times \frac{\log|\log x| - \psi\left(n+\sum_{k=1}^{n} r_k\right) - \gamma}{\Gamma\left(n+\sum_{k=1}^{n} r_k\right)} (\log x)^{n+\sum_{k=1}^{n} r_k - 1} + R_{m_1}^{n} \qquad \text{(n)}$$

$$R_{m_1}^{n} = (-1)^{m_1} 1^{m_1} \int_{-\infty}^{f} \left(\int_{-\infty}^{f} \left(\int_{-\infty}^{f} \cdots \left(\int_{-\infty}^{f} \frac{\log f - \psi(m_1) - \gamma}{\Gamma(m_1)} f^{\,m_1-1} e^f df \right) \cdots e^f df \right) e^f df \right) e^f df$$
$$+ \sum_{r_1=0}^{m_1-1} (-1)^{r_1+m_2} 1^{m_1} 2^{m_2}$$
$$\times \int_{-\infty}^{f} \left(\int_{-\infty}^{f} \cdots \left(\int_{-\infty}^{f} \frac{\log f - \psi(1+r_1+m_2) - \gamma}{\Gamma(1+r_1+m_2)} f^{\,r_1+m_2} e^{2f} df \right) \cdots e^f df \right) e^f df$$
$$\vdots$$
$$+ \sum_{r_1=0}^{m_1-1} \sum_{r_2=0}^{m_2-1} \cdots \sum_{r_{n-1}=0}^{m_{n-1}-1} (-1)^{\sum_{k=1}^{n-1} r_k+m_n} \prod_{k=1}^{n} k^{m_k}$$
$$\times \int_{-\infty}^{f} \frac{\log f - \psi\left(n-1+\sum_{k=1}^{n-1} r_k+m_n\right) - \gamma}{\Gamma\left(n-1+\sum_{k=1}^{n-1} r_k+m_n\right)} f^{\,n-1+\sum_{k=1}^{n-1} r_k+m_n-1} e^{nf} df \qquad \text{(n,r)}$$

When the 1st order,

$$\int_0^x \frac{1}{\log x} dx = x \sum_{r=0}^{m_1-1} (-1)^r 1^r \frac{\log|\log x| - \psi(1+r) - \gamma}{\Gamma(1+r)} (\log x)^r + R_{m_1}^{1}$$

$$R_{m_1}^{1} = (-1)^{m_1} 1^{m_1} \int_{-\infty}^{f} \frac{\log f - \psi(m_1) - \gamma}{\Gamma(m_1)} f^{\,m_1-1} e^f df$$

Here, surprisingly, when $m_1 \rightarrow \infty$,

$$\lim_{m_1\to\infty} R_{m_1}^{1} = \lim_{m_1\to\infty} (-1)^{m_1} 1^{m_1} \int_{-\infty}^{f} \frac{\log f - \psi(m_1) - \gamma}{\Gamma(m_1)} f^{\,m_1-1} e^f df = \gamma \qquad \text{(1r)}$$

Therefore,

$$\int_0^x \frac{1}{\log x} dx = x \sum_{r=0}^{\infty} (-1)^r 1^r \frac{\log|\log x| - \psi(1+r) - \gamma}{\Gamma(1+r)} (\log x)^r + \gamma \qquad \text{(1.1)}$$

When the 2nd order,

$$\int_0^x \int_0^x \frac{1}{\log x} dx^2 = x^2 \sum_{r=0}^{m_1-1} \sum_{s=0}^{m_2-1} (-1)^{r+s} 1^r 2^s \frac{\log|\log x| - \psi(2+r+s) - \gamma}{\Gamma(2+r+s)} (\log x)^{2+r+s-1}$$
$$+ R_{m_1}^{2}$$

$$R_{m_1}^2 = \sum_{r=0}^{m_1-1} (-1)^{r+m_2} 1^{m_1} 2^{m_2} \int_{-\infty}^{f} \frac{\log f - \psi(1+r+m_2) - \gamma}{\Gamma(1+r+m_2)} f^{1+r+m_2-1} e^{2f} df$$

$$+ (-1)^{m_1} 1^{m_1} \int_{-\infty}^{f} \left(\int_{-\infty}^{f} \frac{\log f - \psi(m_1) - \gamma}{\Gamma(m_1)} f^{m_1-1} e^f df \right) e^f df$$

Here, surprisingly again, when $m_1, m_2 \to \infty$,

$$\lim_{m_i \to \infty} R_{m_1}^2 = \gamma x - (\gamma + \log 2) \qquad i=1,2 \tag{2r}$$

Therefore,

$$\int_0^x \int_0^x \frac{1}{\log x} dx^2 = x^2 \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{r+s} 1^r 2^s \frac{\log|\log x| - \psi(2+r+s) - \gamma}{\Gamma(2+r+s)} (\log x)^{1+r+s}$$
$$+ (\gamma + \log 1)\, x - (\gamma + \log 2) \tag{1.2}$$

Also, when the 3rd order,

$$\int_0^x \int_0^x \int_0^x \frac{1}{\log x} dx^3 = x^3 \sum_{r_1=0}^{m_1-1} \sum_{r_2=0}^{m_2-1} \sum_{r_3=0}^{m_3-1} (-1)^{r_1+r_2+r_3} 1^{r_1} 2^{r_2} 3^{r_3}$$
$$\times \frac{\log f - \psi(3+r_1+r_2+r_3) - \gamma}{\Gamma(3+r_1+r_2+r_3)} f^{2+r_1+r_2+r_3} + R_{m_1}^3$$

$$R_{m_1}^3 = \sum_{r_1=0}^{m_1-1} \sum_{r_2=0}^{m_2-1} (-1)^{r_1+r_2+m_3} 1^{m_1} 2^{m_2} 3^{m_3} \int_{-\infty}^{f} \frac{\log f - \psi(2+r_1+r_2+m_3) - \gamma}{\Gamma(2+r_1+r_2+m_3)} f^{1+r_1+r_2+m_3} e^{3f} df$$
$$+ \sum_{r_1=0}^{m_1-1} (-1)^{r_1+m_2} 1^{m_1} 2^{m_2} \int_{-\infty}^{f} \left(\int_{-\infty}^{f} \frac{\log f - \psi(1+r_1+m_2) - \gamma}{\Gamma(1+r_1+m_2)} f^{r_1+m_2} e^{2f} df \right) e^f df$$
$$+ (-1)^{m_1} 1^{m_1} \int_{-\infty}^{f} \left(\int_{-\infty}^{f} \left(\int_{-\infty}^{f} \frac{\log f - \psi(m_1) - \gamma}{\Gamma(m_1)} f^{m_1-1} e^f df \right) e^f df \right) e^f df$$

And

$$\lim_{m_i \to \infty} R_{m_1}^3 = \frac{\gamma}{2!} x^2 - \frac{\gamma + \log 2}{1!} x + \frac{\gamma + \log 3}{2!} \qquad i=1,2,3 \tag{3r}$$

Therefore,

$$\int_0^x \int_0^x \int_0^x \frac{1}{\log x} dx^3 = x^3 \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^{r+s+t} 1^r 2^s 3^t$$
$$\times \frac{\log|\log x| - \psi(3+r+s+t) - \gamma}{\Gamma(3+r+s+t)} (\log x)^{2+r+s+t}$$
$$+ \frac{1}{2!} \left\{ (\gamma + \log 1) x^2 - 2(\gamma + \log 2) x + (\gamma + \log 3) \right\} \tag{1.3}$$

Hereafter, by induction, we obtain the general expression.

Although the problems are (1r), (2r), (3r), ⋯ , these proofs are difficult. However, surely they hold on the numerical computation

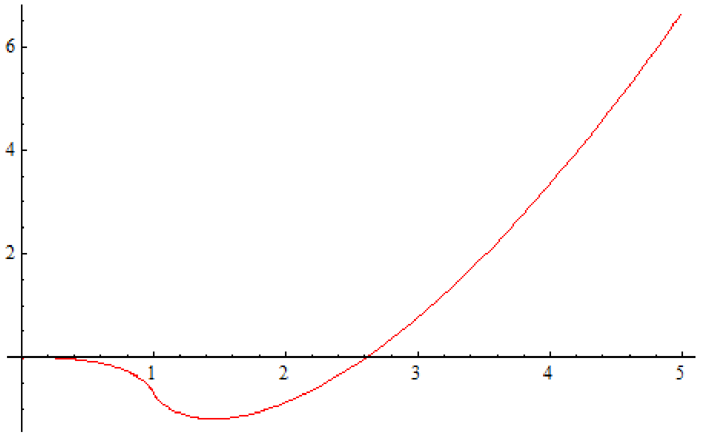
Example The 2nd integral of $1/\log x$

When both sides of (1.2) are illustrated, it is as follows. Since both sides overlapp exactly, the left side (blue) is not visible.

```
m = 15;
f1[x_] = ∫_0^x ∫_0^u 1/Log[t] dt du;
```

$$\texttt{fr[x\_]} = \mathbf{x}^2 \sum_{r=0}^{m}\sum_{s=0}^{m} (-1)^{r+s}\, 1^r\, 2^s \,\frac{\texttt{Log[Abs[Log[x]]] - PolyGamma[2 + r + s] - EulerGamma}}{\texttt{Gamma[2 + r + s]}}\,(\texttt{Log[x]})^{1+r+s}$$

$$+\ \texttt{(EulerGamma + Log[1]) x - (EulerGamma + Log[2]);}$$



Note

Therefore, the higher integral of the logarithmic integral $li(x)\left(=\int_0^x \frac{1}{\log t}dt\right)$ can be expressed as follows.

$$\int_0^x\!\!\cdots\!\int_0^x li(x)\,dx^n = x^{n+1}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\prod_{k=1}^{n+1}k^{r_k}$$

$$\times\frac{\log|\log x| - \psi\left(1+n+\sum_{k=1}^{n+1} r_k\right) - \gamma}{\Gamma\left(1+n+\sum_{k=1}^{n+1} r_k\right)}(\log x)^{n+\sum_{k=1}^{n+1} r_k}$$

$$+\frac{1}{n!}\sum_{r=0}^{n}(-1)^r{}_nC_r\{\gamma+\log(r+1)\}x^{n-r} \qquad (1.n)$$

Formula14.4.3 (**14.4**) using exponential integral $Ei(x)$ is easier as the formula of the higher integral of $li(x)$, In contrast, Formula 23.5.1 looks truly complicated and not useful. However, both have a decisive difference. That is well understood if (1.n) is rewritten using a harmonic number H_n and a factorial $!$ as follows.

$$\int_0^x\!\!\cdots\!\int_0^x li(x)\,dx^n = x^{n+1}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\prod_{k=1}^{n+1}k^{r_k}\frac{\log|\log x| - H_{n+\sum_{k=1}^{n+1} r_k}}{\left(n+\sum_{k=1}^{n+1} r_k\right)!}(\log x)^{n+\sum_{k=1}^{n+1} r_k}$$

$$+\frac{1}{n!}\sum_{r=0}^{n}(-1)^r{}_nC_r\{\gamma+\log(r+1)\}x^{n-r} \qquad (1.n)$$

That is, (1.n) is expressed by the elementary functions while Formula 14.4.3 is expressed by non-elementary functions. Here is the meaning of Formula 23.5.1 .

Now, Formula 23.5.1 can be simplified a little more. For the purpose, the following two lemmas are necessary.

Lemma3

$$\sum_{r=0}^{\infty}(-1)^r\frac{1^r}{\Gamma(1+r)}x^r = \frac{1}{0!}\frac{1}{e^x}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{1^r 2^s}{\Gamma(2+r+s)}x^{1+r+s} = \frac{1}{1!}\left(\frac{1}{e^x}-\frac{1}{e^{2x}}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r 2^s 3^t}{\Gamma(3+r+s+t)}x^{2+r+s+t} = \frac{1}{2!}\left(\frac{1}{e^x}-\frac{2}{e^{2x}}+\frac{1}{e^{3x}}\right)$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{\prod_{k=1}^{n}k^{r_k}}{\Gamma\left(n+\sum_{k=1}^{n}r_k\right)}x^{n-1+\sum_{k=1}^{n}r_k} = \frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r\frac{{}_{n-1}C_r}{e^{1+r}}$$

Proof

By induction. Details are discussed in one chapter of "**À la carte**".

Replacing x with $\log x$ in Lemma3 , we obtain the following Lemma immediately.

Lemma3'

$$x^1\sum_{r=0}^{\infty}(-1)^r\frac{1^r}{\Gamma(1+r)}(\log x)^r = \frac{x^0}{0!}$$

$$x^2\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{1^r 2^s}{\Gamma(2+r+s)}(\log x)^{1+r+s} = \frac{(x-1)^1}{1!}$$

$$x^3\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r 2^s 3^t}{\Gamma(3+r+s+t)}(\log x)^{2+r+s+t} = \frac{(x-1)^2}{2!}$$

$$\vdots$$

$$x^n\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{\prod_{k=1}^{n}k^{r_k}}{\Gamma\left(n+\sum_{k=1}^{n}r_k\right)}(\log x)^{n-1+\sum_{k=1}^{n}r_k} = \frac{(x-1)^{n-1}}{(n-1)!} \qquad (3'.n)$$

Using this Lemma3' , Formula 23.5.1 can be simplified as follows.

Formula 23.5.1'

When $\Gamma(z), \psi(z)$ denote the gamma function and the digamma function respectively, the following expressions hold for $x \geqq 0$.

$$\int_0^x \frac{1}{\log x}dx = -x\sum_{r=0}^{\infty}(-1)^r 1^r\frac{\psi(1+r)}{\Gamma(1+r)}(\log x)^r + \frac{1}{0!}\left\{(x-1)^0\log|\log x| + x^0\log 1\right\} \qquad (1.1')$$

$$\int_0^x\int_0^x \frac{1}{\log x}dx^2 = -x^2\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}1^r 2^s\frac{\psi(2+r+s)}{\Gamma(2+r+s)}(\log x)^{1+r+s} + \frac{1}{1!}\left\{(x-1)^1\log|\log x| + x^1\log 1 - x^0\log 2\right\} \qquad (1.2')$$

$$\int_0^x\int_0^x\int_0^x \frac{1}{\log x}dx^3 = -x^3\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}1^r 2^s 3^t \times \frac{\psi(3+r+s+t)}{\Gamma(3+r+s+t)}(\log x)^{2+r+s+t} + \frac{1}{2!}\left\{(x-1)^2\log|\log x| + x^2\log 1 - 2x^1\log 2 + x^0\log 3\right\} \tag{1.3}$$

⋮

$$\int_0^x\cdots\int_0^x \frac{1}{\log x}dx^n = -x^n\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^n r_k}\prod_{k=1}^n k^{r_k}\frac{\psi\left(n+\sum_{k=1}^n r_k\right)}{\Gamma\left(n+\sum_{k=1}^n r_k\right)}(\log x)^{n-1+\sum_{k=1}^n r_k} + \frac{1}{(n-1)!}\left\{(x-1)^{n-1}\log|\log x| - \sum_{r=0}^{n-1}(-1)^{n-r}{}_{n-1}C_r x^r\log(n-r)\right\}$$

(1.n')

Proof

Substituting Lemma3' (3'.n) for Formula 23.5.1 (1.n) ,

$$\int_0^x\cdots\int_0^x \frac{1}{\log x}dx^n = -x^n\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^n r_k}\prod_{k=1}^n k^{r_k}\frac{\psi\left(n+\sum_{k=1}^n r_k\right)}{\Gamma\left(n+\sum_{k=1}^n r_k\right)}(\log x)^{n-1+\sum_{k=1}^n r_k}$$

$$+(\log|\log x|-\gamma)x^n\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^n r_k}\frac{\prod_{k=1}^n k^{r_k}}{\Gamma\left(n+\sum_{k=1}^n r_k\right)}(\log x)^{n-1+\sum_{k=1}^n r_k}$$

$$+\frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\{\gamma+\log(r+1)\}x^{n-1-r}$$

$$= -x^n\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^n r_k}\prod_{k=1}^n k^{r_k}\frac{\psi\left(n+\sum_{k=1}^n r_k\right)}{\Gamma\left(n+\sum_{k=1}^n r_k\right)}(\log x)^{n-1+\sum_{k=1}^n r_k}$$

$$+\frac{\log|\log x|}{(n-1)!}(x-1)^{n-1} - \frac{\gamma}{(n-1)!}(x-1)^{n-1}$$

$$+\frac{\gamma}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r x^{n-1-r} + \frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r x^{n-1-r}\log(r+1)$$

$$= -x^n\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^n r_k}\prod_{k=1}^n k^{r_k}\frac{\psi\left(n+\sum_{k=1}^n r_k\right)}{\Gamma\left(n+\sum_{k=1}^n r_k\right)}(\log x)^{n-1+\sum_{k=1}^n r_k}$$

$$+\frac{1}{(n-1)!}\left\{(x-1)^{n-1}\log\log x - \sum_{r=0}^{n-1}(-1)^{n-r}{}_{n-1}C_r x^r\log(n-r)\right\}$$

$$\left\{\because\ \sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r x^{n-1-r}\log(r+1) = -\sum_{r=0}^{n-1}(-1)^{n-r}{}_{n-1}C_r x^r\log(n-r)\right\}$$

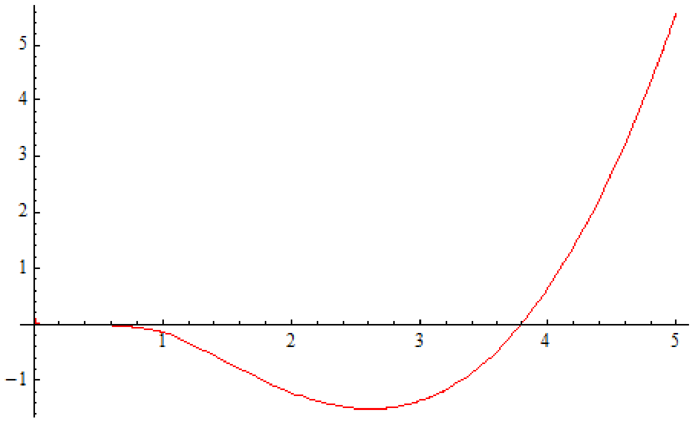
Example The 3rd order integral of $1/\log x$ (The 2nd order integral of $li(x)$)

When both sides of (1.3') are illustrated, it is as follows. Since both sides overlapp exactly, the left side (blue) is not visible.

$$m = 15;$$

$$fl[x\_] = \int_0^x \int_0^v \int_0^u \frac{1}{Log[t]} \, dt \, du \, dv;$$

$$fr[x\_] = -x^3 \sum_{r=0}^{m} \sum_{s=0}^{m} \sum_{t=0}^{m} (-1)^{r+s+t} 1^r 2^s 3^t \frac{PolyGamma[3+r+s+t]}{Gamma[3+r+s+t]} (Log[x])^{2+r+s+t}$$
$$+ \frac{1}{2!} \left((x-1)^2 Log[Abs[Log[x]]] + x^2 Log[1] - 2 x^1 Log[2] + x^0 Log[3] \right);$$



23.6 Higher Integral of log|log x|

Formula 23.6.1

When $\Gamma(z), \psi(z), \gamma$ denote the gamma function, the digamma function, Euler-Mascheroni Constant (= 0.57721566...) respectively, the following expressions hold for $x \geqq 0$.

$$\int_0^x log|log\,x|\,dx = x\sum_r^{\infty}(-1)^r 1^r \frac{log|log\,x| - \psi(2+r) - \gamma}{\Gamma(2+r)}(log\,x)^{1+r} - \gamma \tag{1.1}$$

$$\int_0^x\int_0^x log|log\,x|\,dx^2 = x^2\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}1^r 2^s \frac{log|log\,x| - \psi(3+r+s) - \gamma}{\Gamma(3+r+s)}(log\,x)^{2+r+s}$$

$$- \frac{1}{2!}\{2(\gamma + log\,1)x - (\gamma + log\,2)\} \tag{1.2}$$

$$\int_0^x\int_0^x\int_0^x log|log\,x|\,dx^3$$

$$= x^3\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}1^r 2^s 3^t \frac{log|log\,x| - \psi(4+r+s+t) - \gamma}{\Gamma(4+r+s+t)}(log\,x)^{3+r+s+t}$$

$$- \frac{1}{3!}\left\{3(\gamma+log\,1)x^2 - 3(\gamma+log\,2)x + (\gamma+log\,3)\right\} \tag{1.3}$$

⋮

$$\int_0^x\cdots\int_0^x log|log\,x|\,dx^n$$

$$= x^n\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\prod_{k=1}^{n}k^{r_k}\frac{log|log\,x| - \psi\left(1+n+\sum_{k=1}^{n}r_k\right) - \gamma}{\Gamma\left(1+n+\sum_{k=1}^{n}r_k\right)}(log\,x)^{n+\sum_{k=1}^{n}r_k}$$

$$+ \frac{1}{n!}\sum_{r=1}^{n}(-1)^r {}_nC_r(\gamma+log\,r)x^{n-r} \tag{1.n}$$

Calculation

Let $g = log f$, $f = log\,x$. Then,

$$g^{\left\langle n+\sum_{k=1}^{n}r_k\right\rangle} = \frac{log f - \psi\left(1+n+\sum_{k=1}^{n}r_k\right) - \gamma}{\Gamma\left(1+n+\sum_{k=1}^{n}r_k\right)} f^{n+\sum_{k=1}^{n}r_k}$$

$$g^{\langle m_1\rangle} = \frac{log f - \psi(1+m_1) - \gamma}{\Gamma(1+m_1)} f^{m_1}$$

$$g^{\langle 1+r_1+m_2\rangle} = \frac{log f - \psi(1+1+r_1+m_2) - \gamma}{\Gamma(1+1+r_1+m_2)} f^{1+r_1+m_2}$$

$$g^{\langle 2+r_1+r_2+m_3\rangle} = \frac{log f - \psi(1+2+r_1+r_2+m_3) - \gamma}{\Gamma(1+2+r_1+r_2+m_3)} f^{2+r_1+r_2+m_3}$$

$$g^{\left\langle n-1+\sum_{k=1}^{n-1}r_k+m_n\right\rangle} = \frac{log f - \psi\left(1+n-1+\sum_{k=1}^{n-1}r_k+m_n\right) - \gamma}{\Gamma\left(1+n-1+\sum_{k=1}^{n-1}r_k+m_n\right)} f^{n-1+\sum_{k=1}^{n-1}r_k+m_n}$$

Substitute these for Formula 23.3.1 and returning f to x in the non-remainder terms, since $[-\infty, f] \to [0, x]$,

$$\int_0^x \cdots \int_0^x log\,|log\,x|\,dx^n = x^n \sum_{r_1=0}^{m_1-1} \sum_{r_2=0}^{m_2-1} \cdots \sum_{r_n=0}^{m_n-1} (-1)^{\sum_{k=1}^{n} r_k} \prod_{k=1}^{n} k^{r_k}$$

$$\frac{log\,|log\,x| - \psi\left(1+n+\sum_{k=1}^{n} r_k\right) - \gamma}{\Gamma\left(1+n+\sum_{k=1}^{n} r_k\right)} (log\,x)^{n+\sum_{k=1}^{n} r_k} + R_{m_1}^n \qquad \text{(n)}$$

$R_{m_1}^n =$

$$(-1)^{m_1} 1^{m_1} \int_{-\infty}^{f} \left(\int_{-\infty}^{f} \left(\int_{-\infty}^{f} \cdots \left(\int_{-\infty}^{f} \frac{log\,f - \psi(1+m_1) - \gamma}{\Gamma(1+m_1)} f^{m_1} e^f df \right) \cdots e^f df \right) e^f df \right) e^f df$$

$$+ \sum_{r_1=0}^{m_1-1} (-1)^{r_1+m_2} 1^{m_1} 2^{m_2}$$

$$\times \int_{-\infty}^{f} \left(\int_{-\infty}^{f} \cdots \left(\int_{-\infty}^{f} \frac{log\,f - \psi(2+r_1+m_2) - \gamma}{\Gamma(2+r_1+m_2)} f^{1+r_1+m_2} e^{2f} df \right) \cdots e^f df \right) e^f df$$

$$+ \sum_{r_1=0}^{m_1-1} \sum_{r_2=0}^{m_2-1} (-1)^{r_1+r_2+m_3} 1^{m_1} 2^{m_2} 3^{m_3}$$

$$\times \int_{-\infty}^{f} \cdots \left(\int_{-\infty}^{f} \frac{log\,f - \psi(3+r_1+r_2+m_3) - \gamma}{\Gamma(3+r_1+r_2+m_3)} f^{2+r_1+r_2+m_3} e^{3f} df \right) \cdots e^f df$$

$$\vdots$$

$$+ \sum_{r_1=0}^{m_1-1} \sum_{r_2=0}^{m_2-1} \cdots \sum_{r_{n-1}=0}^{m_{n-1}-1} (-1)^{\sum_{k=1}^{n-1} r_k + m_n} \prod_{k=1}^{n} k^{m_k}$$

$$\times \int_{-\infty}^{f} \frac{log\,f - \psi\left(n+\sum_{k=1}^{n-1} r_k + m_n\right) - \gamma}{\Gamma\left(n+\sum_{k=1}^{n-1} r_k + m_n\right)} f^{n-1+\sum_{k=1}^{n-1} r_k + m_n} e^{nf} df \qquad \text{(nr)}$$

When the 1st order,

$$\int_0^x log\,|log\,x|\,dx = x \sum_{r}^{m_1-1} (-1)^r 1^r \frac{log\,|log\,x| - \psi(2+r) - \gamma}{\Gamma(2+r)} (log\,x)^{1+r} + R_{m_1}^1$$

$$R_{m_1}^1 = (-1)^{m_1} 1^{m_1} \int_{-\infty}^{f} \frac{log\,|f| - \psi(1+m_1) - \gamma}{\Gamma(1+m_1)} f^{m_1} e^f df$$

Here, surprisingly, when $m_1 \to \infty$,

$$\lim_{m_1 \to \infty} R_{m_1}^1 = \lim_{m_1 \to \infty} (-1)^{m_1} 1^{m_1} \int_{-\infty}^{f} \frac{log\,|f| - \psi(1+m_1) - \gamma}{\Gamma(1+m_1)} f^{m_1} e^f df = -\gamma \qquad \text{(1r)}$$

Therefore,

$$\int_0^x log\,|log\,x|\,dx = x \sum_{r}^{\infty} (-1)^r 1^r \frac{log\,|log\,x| - \psi(2+r) - \gamma}{\Gamma(2+r)} (log\,x)^{1+r} - \gamma \qquad \text{(1.1)}$$

When the 2nd order,

$$\int_0^x\int_0^x \log|\log x|\,dx^2$$

$$= x^2\sum_{r=0}^{m_1-1}\sum_{s=0}^{m_2-1}(-1)^{r+s}1^r 2^s\frac{\log|\log x|-\psi(3+r+s)-\gamma}{\Gamma(3+r+s)}(\log x)^{2+r+s}+R_{m_1}^2$$

$$R_{m_1}^2=\sum_{r=0}^{m_1-1}(-1)^{r+m_2}1^{m_1}2^{m_2}\int_{-\infty}^f\frac{\log|f|-\psi(2+r+m_2)-\gamma}{\Gamma(2+r+m_2)}f^{1+r+m_2}e^{2f}df$$

$$+(-1)^{m_1}1^{m_1}\int_{-\infty}^f\left(\int_{-\infty}^f\frac{\log|f|-\psi(1+m_1)-\gamma}{\Gamma(1+m_1)}f^{m_1}e^f df\right)e^f df$$

Here, surprisingly again, when $m_1, m_2 \to \infty$,

$$\lim_{m_i\to\infty}R_{m_1}^2 = -\frac{1}{2!}\{2(\gamma+\log 1)x-(\gamma+\log 2)\} \qquad (i=1,2) \tag{2r}$$

Therefore,

$$\int_0^x\int_0^x \log|\log x|\,dx^2 = x^2\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}1^r 2^s\frac{\log|\log x|-\psi(3+r+s)-\gamma}{\Gamma(3+r+s)}(\log x)^{2+r+s}$$

$$-\frac{1}{2!}\{2(\gamma+\log 1)x-(\gamma+\log 2)\} \tag{1.2}$$

Also, when the 3rd order,

$$\int_0^x\int_0^x\int_0^x \log|\log x|\,dx^3$$

$$= x^3\sum_{r=0}^{m_1-1}\sum_{s=0}^{m_2-1}\sum_{t=0}^{m_3-1}(-1)^{r+s+t}1^r 2^s 3^t\frac{\log|\log x|-\psi(4+r+s+t)-\gamma}{\Gamma(4+r+s+t)}(\log x)^{3+r+s+t}$$

$$+R_{m_1}^3$$

$$R_{m_1}^3=\sum_{r=0}^{m_1-1}\sum_{s=0}^{m_2-1}(-1)^{r+s+m_3}1^{m_1}2^{m_2}3^{m_3}\int_{-\infty}^f\frac{\log|f|-\psi(3+r+s+m_3)-\gamma}{\Gamma(3+r+s+m_3)}f^{2+r+s+m_3}e^{3f}df$$

$$+\sum_{r=0}^{m_1-1}(-1)^{r+m_2}1^{m_1}2^{m_2}\int_{-\infty}^f\left(\int_{-\infty}^f\frac{\log|f|-\psi(2+r+m_2)-\gamma}{\Gamma(2+r+m_2)}f^{1+r+m_2}e^{2f}df\right)e^f df$$

$$+(-1)^{m_1}1^{m_1}\int_{-\infty}^f\left(\int_{-\infty}^f\left(\int_{-\infty}^f\frac{\log|f|-\psi(1+m_1)-\gamma}{\Gamma(1+m_1)}f^{m_1}e^f df\right)e^f df\right)e^f df$$

And

$$\lim_{m_i\to\infty}R_{m_1}^3 = -\frac{1}{3!}\left\{3(\gamma+\log 1)x^2-3(\gamma+\log 2)x+(\gamma+\log 3)\right\} \quad (i=1,2,3) \tag{3r}$$

Therefore,

$$\int_0^x\int_0^x\int_0^x \log|\log x|\,dx^3$$

$$= x^3\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}1^r 2^s 3^t\frac{\log|\log x|-\psi(4+r+s+t)-\gamma}{\Gamma(4+r+s+t)}(\log x)^{3+r+s+t}$$

$$-\frac{1}{3!}\left\{3(\gamma+\log 1)x^2-3(\gamma+\log 2)x+(\gamma+\log 3)\right\} \tag{1.3}$$

Hereafter, by induction, we obtain the general expression.

Although the problems are (1r), (2r), (3r), ⋯ , these proofs are difficult. However, surely they hold on the numerical computation.

Example The 2nd integral of $log\,|log\,x\,|$

When the one arbitrary point $x=4.7$ is given in (1.2) , the values of the both sides are as follows.

```
m = 15;
f1[x_] := ∫_0^x ∫_0^u Log[Abs[Log[t]]] dt du
fr[x_] := x^2 Sum_{r=0}^{m} Sum_{s=0}^{m} (-1)^(r+s) 1^r 2^s (Log[Abs[Log[x]]] - PolyGamma[3 + r + s] - EulerGamma)/Gamma[3 + r + s] (Log[x])^(2+r+s)
          - 1/2! (2 (EulerGamma + Log[1]) x - (EulerGamma + Log[2]))
N[f1[4.7]]        N[fr[4.7]]
-6.07051          -6.07051
```

Note

Formula14.5.1 (**14.5**) using $Ei(x)$ is easier as the formula of the higher integral of $log\,|log(x)|$. In contrast, Formula 23.6.1 looks truly complicated and not useful. However, both have a decisive difference. For example, if (1.2) is rewritten using a harmonic number H_n and a factorial ! , it is as follows.

$$\int_0^x\!\!\int_0^x log\,|log\,x\,|\,dx^2 = x^2\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}1^r2^s\frac{log\,|log\,x\,| - H_{2+r+s}}{(2+r+s)\,!}(log\,x)^{2+r+s} - \frac{1}{2\,!}\{2(\gamma + log\,1)x - (\gamma + log\,2)\} \tag{1.2}$$

That is, (1.2) is expressed by the elementary functions while Formula 14.5.1 is expressed by non-elementary functions. Here is the meaning of Formula 23.6.1.

Now, Formula 23.6.1 can be simplified a little more. For the purpose, the following two lemmas are necessary.

Lemma4

$$\sum_{r=0}^{\infty}(-1)^r\frac{1^r}{\Gamma(2+r)}x^{1+r} = \frac{1}{1\,!}\left(1-\frac{1}{e^x}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{1^r2^s}{\Gamma(3+r+s)}x^{2+r+s} = \frac{1}{2\,!}\left(1-\frac{2}{e^x}+\frac{1}{e^{2x}}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r2^s3^t}{\Gamma(4+r+s+t)}x^{3+r+s+t} = \frac{1}{3\,!}\left(1-\frac{3}{e^x}+\frac{3}{e^{2x}}-\frac{1}{e^{3x}}\right)$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{\prod_{k=1}^{n}k^{r_k}}{\Gamma\left(1+n+\sum_{k=1}^{n}r_k\right)}x^{n+\sum_{k=1}^{n}r_k} = \frac{1}{n\,!}\sum_{r=0}^{n}(-1)^r\frac{{}_nC_r}{e^r}$$

Proof

By induction. Details are discussed in one chapter of "**À la carte**". In addition, this can be drawn also from Lemma3.

Replacing x with $log\,x$ in Lemma4 , we obtain the following Lemma immediately.

Lemma4'

$$\sum_{r=0}^{\infty}(-1)^r\frac{1^r}{\Gamma(2+r)}(\log x)^{1+r}=\frac{1}{1!}\left(1-\frac{1}{x}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{1^r2^s}{\Gamma(3+r+s)}(\log x)^{2+r+s}=\frac{1}{2!}\left(1-\frac{1}{x}\right)^2$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r2^s3^t}{\Gamma(4+r+s+t)}(\log x)^{3+r+s+t}=\frac{1}{3!}\left(1-\frac{1}{x}\right)^3$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{\prod_{k=1}^{n}k^{r_k}}{\Gamma\left(1+n+\sum_{k=1}^{n}r_k\right)}(\log x)^{n+\sum_{k=1}^{n}r_k}=\frac{1}{n!}\left(1-\frac{1}{x}\right)^n \tag{4'.n}$$

Using this Lemma4' , Formula 23.6.1 can be simplified as follows.

Formula 23.6.1'

When $\Gamma(z),\psi(z),\gamma$ denote the gamma function, the digamma function, Euler-Mascheroni-Constant (= 0.57721566...) respectively, the following expressions hold for $x\geqq 0$.

$$\int_0^x \log|\log x|\,dx=-x\sum_{r}^{\infty}(-1)^r1^r\frac{\psi(2+r)}{\Gamma(2+r)}(\log x)^{1+r}$$
$$+\frac{1}{1!}\{(\log|\log x|-\gamma)(x-1)^1-(\gamma+\log 1)x^0\} \tag{1.1'}$$

$$\int_0^x\int_0^x \log|\log x|\,dx^2=-x^2\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}1^r2^s\frac{\psi(3+r+s)}{\Gamma(3+r+s)}(\log x)^{2+r+s}$$
$$+\frac{1}{2!}\left\{(\log|\log x|-\gamma)(x-1)^2+\sum_{r=1}^{2}(-1)^r{}_2C_r(\gamma+\log r)x^{2-r}\right\} \tag{1.2'}$$

$$\int_0^x\int_0^x\int_0^x \log|\log x|\,dx^3=-x^3\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}1^r2^s3^t\frac{\psi(4+r+s+t)}{\Gamma(4+r+s+t)}(\log x)^{3+r+s+t}$$
$$+\frac{1}{3!}\left\{(\log|\log x|-\gamma)(x-1)^3+\sum_{r=1}^{3}(-1)^r{}_3C_r(\gamma+\log r)x^{3-r}\right\} \tag{1.3'}$$

$$\vdots$$

$$\int_0^x\!\!\cdots\!\int_0^x \log|\log x|\,dx^n=-x^n\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\prod_{k=1}^{n}k^{r_k}\frac{\psi\left(1+n+\sum_{k=1}^{n}r_k\right)}{\Gamma\left(1+n+\sum_{k=1}^{n}r_k\right)}(\log x)^{n+\sum_{k=1}^{n}r_k}$$
$$+\frac{1}{n!}\left\{(\log|\log x|-\gamma)(x-1)^n+\sum_{r=1}^{n}(-1)^r{}_nC_r(\gamma+\log r)x^{n-r}\right\} \tag{1.n'}$$

Proof

Substituting Lemma4' (4'.n) for Formula 23.6.1 (1.n) ,

$$\int_0^x\!\!\cdots\!\int_0^x \log|\log x|\,dx^n$$

$$= -x^n \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\prod_{k=1}^{n} k^{r_k}\frac{\psi\left(1+n+\sum_{k=1}^{n} r_k\right)}{\Gamma\left(1+n+\sum_{k=1}^{n} r_k\right)}(\log x)^{n+\sum_{k=1}^{n} r_k}$$

$$+ (\log|\log x| - \gamma)\, x^n \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\frac{\prod_{k=1}^{n} k^{r_k}}{\Gamma\left(1+n+\sum_{k=1}^{n} r_k\right)}(\log x)^{n+\sum_{k=1}^{n} r_k}$$

$$+ \frac{1}{n!}\sum_{r=1}^{n}(-1)^r {}_nC_r(\gamma+\log r)x^{n-r}$$

$$= -x^n \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\prod_{k=1}^{n} k^{r_k}\frac{\psi\left(1+n+\sum_{k=1}^{n} r_k\right)}{\Gamma\left(1+n+\sum_{k=1}^{n} r_k\right)}(\log x)^{n+\sum_{k=1}^{n} r_k}$$

$$+ (\log|\log x| - \gamma)\,\frac{x^n}{n!}\left(1-\frac{1}{x}\right)^n + \frac{1}{n!}\sum_{r=1}^{n}(-1)^r {}_nC_r(\gamma+\log r)x^{n-r}$$

$$= -x^n \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\prod_{k=1}^{n} k^{r_k}\frac{\psi\left(1+n+\sum_{k=1}^{n} r_k\right)}{\Gamma\left(1+n+\sum_{k=1}^{n} r_k\right)}(\log x)^{n+\sum_{k=1}^{n} r_k}$$

$$+ \frac{1}{n!}\left\{(\log|\log x| - \gamma)\,(x-1)^n + \sum_{r=1}^{n}(-1)^r {}_nC_r(\gamma+\log r)x^{n-r}\right\}$$

Example The 3rd order integral of $\log|\log x|$

When the one arbitrary point $x=4.9$ is given in (1.3') , the values of the both sides are as follows.

```
m = 17;

fl[x_] := ∫_0^x ∫_0^v ∫_0^u Log[Abs[Log[t]]] dt du dv

fr[x_] := -x^3 Σ_{r=0}^{m} Σ_{s=0}^{m} Σ_{t=0}^{m} (-1)^(r+s+t) 1^r 2^s 3^t PolyGamma[4 + r + s + t] / Gamma[4 + r + s + t] (Log[x])^(3+r+s+t)
     + 1/3! ((Log[Abs[Log[x]]] - EulerGamma) (x - 1)^3
             + Σ_{r=1}^{3} (-1)^r Binomial[3, r] (EulerGamma + Log[r]) x^(3-r))

N[fl[4.9]]        N[fr[4.9]]
-11.999           -11.999
```

23.7 Possibility to super integral of composition

As understood from Formula23.1.1 , the n th order integral of composision $g\{f(x)\}$ becomes $n(n+1)/2$ -fold series in general. Therefore, for example, the1.5 th order integral of composition $g\{f(x)\}$ will be expressed by 1.875-fold series. However, such a multiple series does not exist. $i.e.$ it is impossible to make Formula23.1.1 Super Integral.

However, when the core function $f(x)$ is the 1st degree, according to Formula 23.1.2 ,

$$\int_a^x \cdots \int_a^x \{g(f(x))\} dx^n = \left(\frac{1}{c}\right)^n \int_{f_a}^f \cdots \int_{f_a}^f g(f) df^n \qquad f(x) = cx + d$$

So, there is no obstacle in extending this domain from the natural number n to real number p .
Thus, the following expression holds for a real number $p > 0$.

$$\int_a^x \sim \int_a^x \{g(f(x))\} dx^p = \left(\frac{1}{c}\right)^p \int_{f_a}^f \sim \int_{f_a}^f g(f) df^p \qquad f(x) = cx + d$$

(1.3p)

This is the grounds for which we have used "Linear form" since " **07 Super Integral** " as a *fait accompli* .

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Alien's Mathematics