16 Higher Integral of the Product of Two Functions

16.1 Higher Integral of f (x) g (x)

16.1.1 Higher Intagration by parts

Formula 16.1.1

Let $f^{<r>}$ be the arbitrary r th order primitive function of $f(x)$ and $g^{<r>}$ be the r th order derivative function of $g(x)$ for $r=1,2,\cdots,n$. And let $f_{a_k}^{<r>},\ g_{a_k}^{(r)}$ be the function values of $f^{<r>},\ g^{(r)}$ on a_k $k=1,\cdots,n$

Then, the following expressions hold.

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<0>}g^{(0)}dx^n = f^{<n>}g^{(0)} - \sum_{r=0}^{n-1}\sum_{s=0}^{r} {}_rC_s f_{a_{n-r}}^{<n-r+s>} g_{a_{n-r}}^{(s)} \int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

$$- \sum_{r=1}^{n} {}_nC_r \int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<r>}g^{(r)}dx^n \tag{1.1}$$

Especially, when $a_r = a \quad for \ \ r=1,2,\cdots,n$,

$$\int_{a}^{x}\cdots\int_{a}^{x} f^{<0>}g^{(0)}dx^n = f^{<n>}g^{(0)} - \sum_{r=0}^{n-1}\sum_{s=0}^{r} {}_rC_s f_{a}^{<n-r+s>} g_{a}^{(s)} \frac{(x-a)^r}{r\,!}$$

$$- \sum_{r=1}^{n} {}_nC_r \int_{a}^{x}\cdots\int_{a}^{x} f^{<r>}g^{(r)}dx^n \tag{1.2}$$

Especially, when $f^{<r>}(a)=0 \ \ (r=1,2,\cdots,n) \quad or \quad g^{(s)}(a)=0 \ \ (s=0,1,\cdots,n-1)$

$$\int_{a}^{x}\cdots\int_{a}^{x} f^{<0>}g^{(0)}dx^n = f^{<n>}g^{(0)} - \sum_{r=1}^{n} {}_nC_r \int_{a}^{x}\cdots\int_{a}^{x} f^{<r>}g^{(r)}dx^n \tag{1.3}$$

Proof

The following equations hold according to the formula of integration by parts.

$$\int_{a_1}^{x} f^{<0>}g^{(0)}dx = f^{<1>}g^{(0)} - f_{a_1}^{<1>}g_{a_1}^{(0)} - \int_{a_1}^{x} f^{<1>}g^{(1)}dx \tag{001}$$

$$\int_{a_2}^{x} f^{<1>}g^{(0)}dx = f^{<2>}g^{(0)} - f_{a_2}^{<2>}g_{a_2}^{(0)} - \int_{a_2}^{x} f^{<2>}g^{(1)}dx \tag{102}$$

$$\int_{a_1}^{x} f^{<1>}g^{(1)}dx = f^{<2>}g^{(1)} - f_{a_1}^{<2>}g_{a_1}^{(1)} - \int_{a_1}^{x} f^{<2>}g^{(2)}dx \tag{111}$$

$$\int_{a_3}^{x} f^{<2>}g^{(0)}dx = f^{<3>}g^{(0)} - f_{a_3}^{<3>}g_{a_3}^{(0)} - \int_{a_3}^{x} f^{<3>}g^{(1)}dx \tag{203}$$

$$\int_{a_2}^{x} f^{<2>}g^{(1)}dx = f^{<3>}g^{(1)} - f_{a_2}^{<3>}g_{a_2}^{(1)} - \int_{a_2}^{x} f^{<3>}g^{(2)}dx \tag{212}$$

$$\int_{a_1}^{x} f^{<2>}g^{(2)}dx = f^{<3>}g^{(2)} - f_{a_1}^{<3>}g_{a_1}^{(2)} - \int_{a_1}^{x} f^{<3>}g^{(3)}dx \tag{221}$$

$\vdots$

Integrating both sides of (001) with respect to x from a_2 to x ,

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>} g^{(0)} dx^2 = \int_{a2}^{x} f^{<1>} g^{(0)} dx - f_{a_1}^{<1>} g_{a_1}^{(0)} \int_{a_2}^{x} dx - \int_{a_2}^{x}\int_{a_1}^{x} f^{<1>} g^{(1)} dx^2$$

Substituting (102) for this,

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>} g^{(0)} dx^2 = f^{<2>} g^{(0)} - f_{a_2}^{<2>} g_{a_2}^{(0)} - \int_{a_2}^{x} f^{<2>} g^{(1)} dx$$

$$- f_{a_1}^{<1>} g_{a_1}^{(0)} \int_{a_2}^{x} dx - \int_{a_2}^{x}\int_{a_1}^{x} f^{<1>} g^{(1)} dx^2$$

Here, integrating both sides of (111) with respect to x from a_2 to x ,

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<1>} g^{(1)} dx^2 = \int_{a_2}^{x} f^{<2>} g^{(1)} dx - f_{a_1}^{<2>} g_{a_1}^{(1)} \int_{a_2}^{x} dx - \int_{a_2}^{x}\int_{a_1}^{x} f^{<2>} g^{(2)} dx^2$$

From this,

$$\int_{a_2}^{x} f^{<2>} g^{(1)} dx = f_{a_1}^{<2>} g_{a_1}^{(1)} \int_{a_2}^{x} dx + \int_{a_2}^{x}\int_{a_1}^{x} f^{<1>} g^{(1)} dx^2 + \int_{a_2}^{x}\int_{a_1}^{x} f^{<2>} g^{(2)} dx^2$$

Substituting this for the above,

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>} g^{(0)} dx^2 = f^{<2>} g^{(0)} - f_{a_2}^{<2>} g_{a_2}^{(0)} - \left(f_{a_1}^{<1>} g_{a_1}^{(0)} + f_{a_1}^{<2>} g_{a_1}^{(1)} \right) \int_{a_2}^{x} dx$$

$$- 2\int_{a_2}^{x}\int_{a_1}^{x} f^{<1>} g^{(1)} dx^2 - \int_{a_2}^{x}\int_{a_1}^{x} f^{<2>} g^{(2)} dx^2 \qquad (2)$$

Next, integrating both sides of (2) with respect to x from a_3 to x ,

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>} g^{(0)} dx^3 = \int_{a_3}^{x} f^{<2>} g^{(0)} dx$$

$$- f_{a_2}^{<2>} g_{a_2}^{(0)} \int_{a_3}^{x} dx - \left(f_{a_1}^{<1>} g_{a_1}^{(0)} + f_{a_1}^{<2>} g_{a_1}^{(1)} \right) \int_{a_3}^{x}\int_{a_2}^{x} dx$$

$$- 2\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<1>} g^{(1)} dx^3 - \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<2>} g^{(2)} dx^3$$

Substituting (203) for this,

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>} g^{(0)} dx^3 = f^{<3>} g^{(0)} - f_{a_3}^{<3>} g_{a_3}^{(0)} - \int_{a_3}^{x} f^{<3>} g^{(1)} dx$$

$$- f_{a_2}^{<2>} g_{a_2}^{(0)} \int_{a_3}^{x} dx - \left(f_{a_1}^{<1>} g_{a_1}^{(0)} + f_{a_1}^{<2>} g_{a_1}^{(1)} \right) \int_{a_3}^{x}\int_{a_2}^{x} dx$$

$$- 2\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<1>} g^{(1)} dx^3 - \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<2>} g^{(2)} dx^3$$

Here, integrating both sides of (212) with respect to x from a_3 to x ,

$$\int_{a_3}^{x} f^{<3>} g^{(1)} dx = f_{a_2}^{<3>} g_{a_2}^{(1)} \int_{a_3}^{x} dx + \int_{a_3}^{x}\int_{a_2}^{x} f^{<2>} g^{(1)} dx^2 + \int_{a_3}^{x}\int_{a_2}^{x} f^{<3>} g^{(2)} dx^2$$

Substituting this for the above,

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>}g^{(0)}dx^3 = f^{<3>}g^{(0)} - f_{a_3}^{<3>}g_{a_3}^{(0)} - \left(f_{a_2}^{<2>}g_{a_2}^{(0)} + f_{a_2}^{<3>}g_{a_2}^{(1)}\right)\int_{a_3}^{x}dx$$

$$- \left(f_{a_1}^{<1>}g_{a_1}^{(0)} + f_{a_1}^{<2>}g_{a_1}^{(1)}\right)\int_{a_3}^{x}\int_{a_2}^{x}dx$$

$$- \int_{a_3}^{x}\int_{a_2}^{x} f^{<2>}g^{(1)}dx^2 - \int_{a_3}^{x}\int_{a_2}^{x} f^{<3>}g^{(2)}dx^2$$

$$- 2\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<1>}g^{(1)}dx^3 - \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<2>}g^{(2)}dx^3$$

Here, integrating both sides of (111) , (221) with respect to x from a_2 to x respectivly,

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<1>}g^{(1)}dx^2 = \int_{a_2}^{x} f^{<2>}g^{(1)}dx - f_{a_1}^{<2>}g_{a_1}^{(1)}\int_{a_2}^{x}dx - \int_{a_2}^{x}\int_{a_1}^{x} f^{<2>}g^{(2)}dx^2$$

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<2>}g^{(2)}dx^2 = \int_{a_2}^{x} f^{<3>}g^{(2)}dx - f_{a_1}^{<3>}g_{a_1}^{(2)}\int_{a_2}^{x}dx - \int_{a_2}^{x}\int_{a_1}^{x} f^{<3>}g^{(3)}dx^2$$

Furthermore, integrating both sides of these with respect to x from a_3 to x respectivly,

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<1>}g^{(1)}dx^3 = \int_{a_3}^{x}\int_{a_2}^{x} f^{<2>}g^{(1)}dx^2 - f_{a_1}^{<2>}g_{a_1}^{(1)}\int_{a_3}^{x}\int_{a_2}^{x}dx^2 - \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<2>}g^{(2)}dx^3$$

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<2>}g^{(2)}dx^3 = \int_{a_3}^{x}\int_{a_2}^{x} f^{<3>}g^{(2)}dx^2 - f_{a_1}^{<3>}g_{a_1}^{(2)}\int_{a_3}^{x}\int_{a_2}^{x}dx^2 - \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<3>}g^{(3)}dx^3$$

i.e.

$$\int_{a_3}^{x}\int_{a_2}^{x} f^{<2>}g^{(1)}dx^2 = \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<1>}g^{(1)}dx^3 + f_{a_1}^{<2>}g_{a_1}^{(1)}\int_{a_3}^{x}\int_{a_2}^{x}dx^2 + \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<2>}g^{(2)}dx^3$$

$$\int_{a_3}^{x}\int_{a_2}^{x} f^{<3>}g^{(2)}dx^2 = \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<2>}g^{(2)}dx^3 + f_{a_1}^{<3>}g_{a_1}^{(2)}\int_{a_3}^{x}\int_{a_2}^{x}dx^2 + \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<3>}g^{(3)}dx^3$$

Substituting this for the above,

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>}g^{(0)}dx^3 = f^{<3>}g^{(0)} - f_{a_3}^{<3>}g_{a_3}^{(0)}$$

$$- \left(f_{a_2}^{<2>}g_{a_2}^{(0)} + f_{a_2}^{<3>}g_{a_2}^{(1)}\right)\int_{a_3}^{x}dx$$

$$- \left(f_{a_1}^{<1>}g_{a_1}^{(0)} + 2f_{a_1}^{<2>}g_{a_1}^{(1)} + f_{a_1}^{<3>}g_{a_1}^{(2)}\right)\int_{a_3}^{x}\int_{a_2}^{x}dx^2$$

$$-3\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<1>}g^{(1)}dx^3 - 3\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<2>}g^{(2)}dx^3 - \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<3>}g^{(3)}dx^3$$

$$= f^{<3>}g^{(0)} - f_{a_3}^{<3>}g_{a_3}^{(0)}$$

$$- \sum_{s=0}^{1} {}_1C_s f_{a_2}^{<2+s>}g_{a_2}^{(s)}\int_{a_3}^{x}\cdots\int_{a_{3-1+1}}^{x}dx^1 - \sum_{s=0}^{2} {}_2C_s f_{a_1}^{<1+s>}g_{a_1}^{(s)}\int_{a_3}^{x}\cdots\int_{a_{3-2+1}}^{x}dx^2$$

$$- \sum_{r=1}^{3} {}_3C_r\int_{a_3}^{x}\cdots\int_{a_1}^{x} f^{<r>}g^{(r)}dx^3$$

$$= f^{<3>}g^{(0)} - f_{a_3}^{<3>}g_{a_3}^{(0)} - \sum_{r=1}^{2}\sum_{s=0}^{r} {}_rC_s f_{a_{3-r}}^{<3-r+s>} g_{a_{3-r}}^{(s)} \int_{a_3}^{x}\cdots\int_{a_{3-r+1}}^{x} dx^r$$

$$- \sum_{r=1}^{3} {}_3C_r \int_{a_3}^{x}\cdots\int_{a_1}^{x} f^{<r>}g^{(r)}dx^3$$

Thus,

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>}g^{(0)}dx^3 = f^{<3>}g^{(0)} - f_{a_3}^{<3>}g_{a_3}^{(0)} - \sum_{r=1}^{2}\sum_{s=0}^{r} {}_rC_s f_{a_{3-r}}^{<3-r+s>} g_{a_{3-r}}^{(s)} \int_{a_3}^{x}\cdots\int_{a_{3-r+1}}^{x} dx^r$$

$$- \sum_{r=1}^{3} {}_3C_r \int_{a_3}^{x}\cdots\int_{a_1}^{x} f^{<r>}g^{(r)}dx^3 \qquad (3)$$

Hereafter by induction, we obtain the following expression.

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<0>}g^{(0)}dx^n = f^{<n>}g^{(0)} - f_{a_n}^{<n>}g_{a_n}^{(0)} - \sum_{r=1}^{n-1}\sum_{s=0}^{r} {}_rC_s f_{a_{n-r}}^{<n-r+s>} g_{a_{n-r}}^{(s)} \int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

$$- \sum_{r=1}^{n} {}_nC_r \int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<r>}g^{(r)}dx^n$$

And pushing $f_{a_n}^{<n>}g_{a_n}^{(0)}$ into $\sum\sum$, we obtain (1.1) .

Next, when $a_r = a \;\; for \;\; r=1,2, \cdots ,n$,

$$\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r = \int_{a}^{x}\cdots\int_{a}^{x} dx^r = \frac{(x-a)^r}{r!} \quad for \;\; r=0,1, \cdots ,n-1$$

Then, substituting this for (1.1) , we obtain (1.2) .

Last, when $f^{<r>}(a)=0 \;\; (r=1,2,\cdots ,n) \;\; or \;\; g^{(s)}(a)=0 \;\; (s=0,1,\cdots ,n-1)$, (1.3) is clear.

Note

Different from the integration by parts of the 1st order, These formulas are seldom used directly.

16.1.2 Higher Integral of the Product of Two Functions

Theorem 16.1.2

Let $f^{<r>}$ be the arbitrary r th order primitive function of $f(x)$ and $g^{<r>}$ be the r th order derivative function of $g(x)$ for $r=1,2, \cdots ,m+n-1$. Let $f_{a_k}^{<r>}, g_{a_k}^{(r)}$ be the function values of $f^{<r>}, g^{(r)}$ on a_k for $k=1,2,\cdots ,n$. And let $B(n,m)$ be the beta function. Then, the following formulas hold.

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<0>}g^{(0)}dx^n = \sum_{r=0}^{m-1}\binom{-n}{r} f^{<n+r>}g^{(r)}$$

$$- \sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s} f_{a_{n-r}}^{<n-r+s>} g_{a_{n-r}}^{(s)} \int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

$$+ (-1)^m \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1} f_{a_{n-r}}^{\langle m+n-r+s\rangle} g_{a_{n-r}}^{(m+s)} \int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<m+k>}g^{(m+k)}dx^n \qquad (2.1)$$

Especially, when $a_r = a \quad for \quad r=1,2,\cdots,n$,

$$\int_a^x \cdots \int_a^x f^{<0>} g^{(0)} dx^n = \sum_{r=0}^{m-1} \binom{-n}{r} f^{<n+r>} g^{(r)}$$

$$- \sum_{r=0}^{n-1} \sum_{s=0}^{m-1} \binom{-n+r}{s} f_a^{<n-r+s>} g_a^{(s)} \frac{(x-a)^r}{r!}$$

$$+ (-1)^m \sum_{r=1}^{n-1} \sum_{s=0}^{r-1} \sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1} f_a^{\langle m+n-r+s \rangle} g_a^{(m+s)} \frac{(x-a)^r}{r!}$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \int_a^x \cdots \int_a^x f^{<m+k>} g^{(m+k)} dx^n \tag{2.2}$$

Especially, when $f^{<r>}(a) = 0 \ (r=1,2,\cdots,m+n-1) \ \ or \ \ g^{(s)}(a) = 0 \ (s=0,1,\cdots,m+n-2)$

$$\int_a^x \cdots \int_a^x f^{<0>} g^{(0)} dx^n = \sum_{r=0}^{m-1} \binom{-n}{r} f^{<n+r>} g^{(r)}$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \int_a^x \cdots \int_a^x f^{<m+k>} g^{(m+k)} dx^n \tag{2.3}$$

Proof

According to Formula 16.1.1 , the 3rd order integral of $f^{<k>} g^{(k)} \ \ k=0,1,2,\cdots$ are as follows

$$\int_{a_3}^x \int_{a_2}^x \int_{a_1}^x f^{<0>} g^{(0)} dx^3 = f^{<3>} g^{(0)}$$

$$- f_{a_3}^{<3>} g_{a_3}^{(0)} - \left(f_{a_2}^{<2>} g_{a_2}^{(0)} + f_{a_2}^{<3>} g_{a_2}^{(1)} \right) \int_{a_3}^x dx$$

$$- \left(f_{a_1}^{<1>} g_{a_1}^{(0)} + 2 f_{a_1}^{<2>} g_{a_1}^{(1)} + f_{a_1}^{<3>} g_{a_1}^{(2)} \right) \int_{a_3}^x \int_{a_2}^x dx^2$$

$$-3 \int_{a_3}^x \int_{a_2}^x \int_{a_1}^x f^{<1>} g^{(1)} dx^3 - 3 \int_{a_3}^x \int_{a_2}^x \int_{a_1}^x f^{<2>} g^{(2)} dx^3 - \int_{a_3}^x \int_{a_2}^x \int_{a_1}^x f^{<3>} g^{(3)} dx^3$$

$$\int_{a_3}^x \int_{a_2}^x \int_{a_1}^x f^{<1>} g^{(1)} dx^3 = f^{<4>} g^{(1)}$$

$$- f_{a_3}^{<4>} g_{a_3}^{(1)} - \left(f_{a_2}^{<3>} g_{a_2}^{(1)} + f_{a_2}^{<4>} g_{a_2}^{(2)} \right) \int_{a_3}^x dx$$

$$- \left(f_{a_1}^{<2>} g_{a_1}^{(1)} + 2 f_{a_1}^{<3>} g_{a_1}^{(2)} + f_{a_1}^{<4>} g_{a_1}^{(3)} \right) \int_{a_3}^x \int_{a_2}^x dx^2$$

$$-3 \int_{a_3}^x \int_{a_2}^x \int_{a_1}^x f^{<2>} g^{(2)} dx^3 - 3 \int_{a_3}^x \int_{a_2}^x \int_{a_1}^x f^{<3>} g^{(3)} dx^3 - \int_{a_3}^x \int_{a_2}^x \int_{a_1}^x f^{<4>} g^{(4)} dx^3$$

$$\int_{a_3}^x \int_{a_2}^x \int_{a_1}^x f^{<2>} g^{(2)} dx^3 = f^{<5>} g^{(2)}$$

$$- f_{a_3}^{<5>} g_{a_3}^{(2)} - \left(f_{a_2}^{<4>} g_{a_2}^{(2)} + f_{a_2}^{<5>} g_{a_2}^{(3)} \right) \int_{a_3}^x dx$$

$$- \left(f_{a_1}^{<3>} g_{a_1}^{(2)} + 2f_{a_1}^{<4>} g_{a_1}^{(3)} + f_{a_1}^{<5>} g_{a_1}^{(4)} \right) \int_{a_3}^{x}\int_{a_2}^{x} dx^2$$

$$-3\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<3>} g^{(3)} dx^3 -3\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<4>} g^{(4)} dx^3 - \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<5>} g^{(5)} dx^3$$

⋮

Substituting the 2nd less than expressions for the 1st expression one by one,

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>} g^{(0)} dx^3 = f^{<3>} g^{(0)} -3 f^{<4>} g^{(1)}$$

$$- \left(f_{a_3}^{<3>} g_{a_3}^{(0)} - 3f_{a_3}^{<4>} g_{a_3}^{(1)} \right)$$

$$- \left(f_{a_2}^{<2>} g_{a_2}^{(0)} - 2f_{a_2}^{<3>} g_{a_2}^{(1)} - 3f_{a_2}^{<4>} g_{a_2}^{(2)} \right) \int_{a_3}^{x} dx$$

$$- \left(f_{a_1}^{<1>} g_{a_1}^{(0)} - f_{a_1}^{<2>} g_{a_1}^{(1)} - 5f_{a_1}^{<3>} g_{a_1}^{(2)} - 3f_{a_1}^{<4>} g_{a_1}^{(3)} \right) \int_{a_3}^{x}\int_{a_2}^{x} dx^2$$

$$+6\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<2>} g^{(2)} dx^3 + 8\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<3>} g^{(3)} dx^3 +3\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<4>} g^{(4)} dx^3$$

$$= f^{<3>} g^{(0)} -3 f^{<4>} g^{(1)} + 6f^{<5>} g^{(2)}$$

$$- \left(f_{a_3}^{<3>} g_{a_3}^{(0)} - 3f_{a_3}^{<4>} g_{a_3}^{(1)} + 6f_{a_3}^{<5>} g_{a_3}^{(2)} \right)$$

$$- \left(f_{a_2}^{<2>} g_{a_2}^{(0)} - 2f_{a_2}^{<3>} g_{a_2}^{(1)} + 3f_{a_2}^{<4>} g_{a_2}^{(2)} + 6f_{a_2}^{<5>} g_{a_2}^{(3)} \right) \int_{a_3}^{x} dx$$

$$- \left(f_{a_1}^{<1>} g_{a_1}^{(0)} - f_{a_1}^{<2>} g_{a_1}^{(1)} + f_{a_1}^{<3>} g_{a_1}^{(2)} + 9f_{a_1}^{<4>} g_{a_1}^{(3)} + 6f_{a_1}^{<5>} g_{a_1}^{(4)} \right) \int_{a_3}^{x}\int_{a_2}^{x} dx^2$$

$$-10\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<3>} g^{(3)} dx^3 -15\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<4>} g^{(4)} dx^3 -6\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<5>} g^{(5)} dx^3$$

⋮

This formula can be expressed as follows.

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>} g^{(0)} dx^3 = \binom{-3}{0} f^{<3>} g^{(0)} + \binom{-3}{1} f^{<4>} g^{(1)}$$

$$- \left(\binom{-3}{0} f_{a_3}^{<3>} g_{a_3}^{(0)} + \binom{-3}{1} f_{a_3}^{<4>} g_{a_3}^{(1)} \right)$$

$$- \left(\binom{-2}{0} f_{a_2}^{<2>} g_{a_2}^{(0)} + \binom{-2}{1} f_{a_2}^{<3>} g_{a_2}^{(1)} - 3f_{a_2}^{<4>} g_{a_2}^{(2)} \right) \int_{a_3}^{x} dx$$

$$- \left(\binom{-1}{0} f_{a_1}^{<1>} g_{a_1}^{(0)} + \binom{-1}{1} f_{a_1}^{<2>} g_{a_1}^{(1)} - 5f_{a_1}^{<3>} g_{a_1}^{(2)} - 3f_{a_1}^{<4>} g_{a_1}^{(3)} \right) \int_{a_3}^{x}\int_{a_2}^{x} dx^2$$

$$+(-1)^2 \sum_{k=0}^{2} \frac{{}_2C_k}{2!} \frac{2(2+1)(2+2)}{2+k} \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<2+k>} g^{(2+k)} dx^3$$

$$= \binom{-3}{0} f^{<3>} g^{(0)} + \binom{-3}{1} f^{<4>} g^{(1)} + \binom{-3}{2} f^{<5>} g^{(2)}$$

$$- \left(\binom{-3}{0} f_{a_3}^{<3>} g_{a_3}^{(0)} + \binom{-3}{1} f_{a_3}^{<4>} g_{a_3}^{(1)} + \binom{-3}{2} f_{a_3}^{<5>} g_{a_3}^{(2)} \right)$$

$$- \left(\binom{-2}{0} f_{a_2}^{<2>} g_{a_2}^{(0)} + \binom{-2}{1} f_{a_2}^{<3>} g_{a_2}^{(1)} + \binom{-2}{2} f_{a_2}^{<4>} g_{a_2}^{(2)} \right) \int_{a_3}^{x} dx$$

$$- \left(\binom{-1}{0} f_{a_1}^{<1>} g_{a_1}^{(0)} + \binom{-1}{1} f_{a_1}^{<2>} g_{a_1}^{(1)} + \binom{-1}{2} f_{a_1}^{<3>} g_{a_1}^{(2)} \right) \int_{a_3}^{x} \int_{a_2}^{x} dx^2$$

$$- \left(6 f_{a_2}^{<5>} g_{a_2}^{(3)} \right) \int_{a_3}^{x} dx$$

$$- \left(9 f_{a_1}^{<4>} g_{a_1}^{(3)} + 6 f_{a_1}^{<5>} g_{a_1}^{(4)} \right) \int_{a_3}^{x} \int_{a_2}^{x} dx^2$$

$$+(-1)^3 \sum_{k=0}^{2} \frac{{}_2C_k}{2!} \frac{3(3+1)(3+2)}{3+k} \int_{a_3}^{x} \int_{a_2}^{x} \int_{a_1}^{x} f^{<3+k>} g^{(3+k)} dx^3$$

$$\vdots$$

Here, red and magenta coefficients are the elements of the following Pascal type triangles. Bule coefficients are given, magenta coefficients are equal to c_{00}^{3m} , and a red coefficient is obtained as the sum of two coefficients of the upper row.

$$\begin{matrix} & & c_{00}^{32} & & \\ & c_{10}^{32} & & c_{11}^{32} & \\ c_{20}^{32} & & c_{21}^{32} & & c_{22}^{32} \end{matrix} = \begin{matrix} & & 3 & & \\ & 2 & & 3 & \\ 1 & & 5 & & 3 \end{matrix} = \begin{matrix} & & {}_3C_1 & & \\ & {}_2C_1 & & {}_3C_1 & \\ {}_1C_1 & & 5 & & {}_3C_1 \end{matrix}$$

$$\begin{matrix} & & c_{00}^{33} & & \\ & c_{10}^{33} & & c_{11}^{33} & \\ c_{20}^{33} & & c_{21}^{33} & & c_{22}^{33} \end{matrix} = \begin{matrix} & & 6 & & \\ & 3 & & 6 & \\ 1 & & 9 & & 6 \end{matrix} = \begin{matrix} & & {}_4C_2 & & \\ & {}_3C_2 & & {}_4C_2 & \\ {}_2C_2 & & 9 & & {}_4C_2 \end{matrix}$$

If bule coefficients are given, the other coefficients can be calculated by the following expression. (See **Lemma 16.8.1**)

$$c_{rs}^{3m} = \sum_{t=s-1}^{r-1} {}_tC_{s-1}\, c_{r-1-t\,0}^{3m} \qquad r, s \geqq 1$$

When $m=2$, $c_{00}^{32} = {}_3C_1$, $c_{10}^{32} = {}_2C_1$, $c_{20}^{32} = {}_1C_1$ i.e. $c_{t0}^{32} = {}_{3-t}C_1 \qquad t=0,1,2$

From this, $c_{r-1-t\,0}^{32} = {}_{3-(r-1-t)}C_1 = {}_{4-r+t}C_1$

$$\therefore \qquad c_{rs}^{32} = \sum_{t=s-1}^{r-1} {}_tC_{s-1}\, {}_{4-r+t}C_1 \qquad r, s \geqq 1$$

When $m=3$, $c_{00}^{33} = {}_4C_2$, $c_{10}^{33} = {}_3C_2$, $c_{20}^{33} = {}_2C_2$ i.e. $c_{t0}^{33} = {}_{4-t}C_2 \qquad t=0,1,2$

From this, $c_{r-1-t\,0}^{33} = {}_{4-(r-1-t)}C_2 = {}_{5-r+t}C_2$

$$\therefore \qquad c_{rs}^{33} = \sum_{t=s-1}^{r-1} {}_tC_{s-1\ 5-r+t}C_2 \qquad r,s \geqq 1$$

Thus, $$c_{rs}^{3m} = \sum_{t=s-1}^{r-1} {}_tC_{s-1} \cdot {}_{m+2-r+t}C_{m-1}$$

If these are used, the above formula is expressed as follows.

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>} g^{(0)} dx^3 = \sum_{r=0}^{1}\binom{-3}{r} f^{<3+r>} g^{(r)} - \sum_{s=0}^{1}\binom{-3}{s} f_{a_3}^{<3+s>} g_{a_3}^{(s)}$$
$$- \sum_{s=0}^{1}\binom{-2}{s} f_{a_2}^{<2+s>} g_{a_2}^{(s)} \int_{a_3}^{x} dx - \sum_{s=0}^{1}\binom{-1}{0} f_{a_1}^{<1+s>} g_{a_1}^{(s)} \int_{a_3}^{x}\int_{a_2}^{x} dx^2$$
$$+ c_{11}^{32} f_{a_2}^{<4>} g_{a_2}^{(2)} \int_{a_3}^{x} dx + \left(c_{21}^{32} f_{a_1}^{<3>} g_{a_1}^{(2)} + c_{22}^{32} f_{a_1}^{<4>} g_{a_1}^{(3)} \right) \int_{a_3}^{x}\int_{a_2}^{x} dx^2$$
$$+(-1)^2 \sum_{k=0}^{2} \frac{{}_2C_k}{2!} \frac{2(2+1)(2+2)}{2+k} \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<2+k>} g^{(2+k)} dx^3$$
$$= \sum_{r=0}^{2}\binom{-3}{r} f^{<3+r>} g^{(r)} - \sum_{s=0}^{2}\binom{-3}{s} f_{a_3}^{<3+s>} g_{a_3}^{(s)}$$
$$- \sum_{s=0}^{2}\binom{-2}{s} f_{a_2}^{<2+s>} g_{a_2}^{(s)} \int_{a_3}^{x} dx - \sum_{s=0}^{2}\binom{-1}{0} f_{a_1}^{<1+s>} g_{a_1}^{(s)} \int_{a_3}^{x}\int_{a_2}^{x} dx^2$$
$$- \left(c_{11}^{33} f_{a_2}^{<5>} g_{a_2}^{(3)} \right) \int_{a_3}^{x} dx - \left(c_{21}^{33} f_{a_1}^{<4>} g_{a_1}^{(3)} + c_{22}^{33} f_{a_1}^{<5>} g_{a_1}^{(4)} \right) \int_{a_3}^{x}\int_{a_2}^{x} dx^2$$
$$+(-1)^3 \sum_{k=0}^{2} \frac{{}_2C_k}{2!} \frac{3(3+1)(3+2)}{3+k} \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<3+k>} g^{(3+k)} dx^3$$

⋮

That is,

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>} g^{(0)} dx^3 = \sum_{r=0}^{1}\binom{-3}{r} f^{<3+r>} g^{(r)}$$
$$- \sum_{r=0}^{2}\sum_{s=0}^{1}\binom{-3+r}{s} f_{a_{3-r}}^{<3-r+s>} g_{a_{3-r}}^{(s)} \int_{a_3}^{x}\cdots\int_{a_{3-r+1}}^{x} dx^r$$
$$+ (-1)^2 \sum_{r=1}^{2}\sum_{s=0}^{r-1} c_{r\,1+s}^{32} f_{a_{3-r}}^{<5-r+s>} g_{a_{3-r}}^{(2+s)} \int_{a_3}^{x}\cdots\int_{a_{3-r+1}}^{x} dx^r$$
$$+(-1)^2 \sum_{k=0}^{2} \frac{{}_2C_k}{2!} \frac{2(2+1)(2+2)}{2+k} \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<2+k>} g^{(2+k)} dx^3$$
$$= \sum_{r=0}^{2}\binom{-3}{r} f^{<3+r>} g^{(r)}$$
$$- \sum_{r=0}^{2}\sum_{s=0}^{2}\binom{-3+r}{s} f_{a_{3-r}}^{<3-r+s>} g_{a_{3-r}}^{(s)} \int_{a_3}^{x}\cdots\int_{a_{3-r+1}}^{x} dx^r$$
$$+ (-1)^3 \sum_{r=1}^{2}\sum_{s=0}^{r-1} c_{r\,1+s}^{33} f_{a_{3-r}}^{<6-r+s>} g_{a_{3-r}}^{(3+s)} \int_{a_3}^{x}\cdots\int_{a_{3-r+1}}^{x} dx^r$$

$$+(-1)^3 \sum_{k=0}^{2} \frac{{}_2C_k}{2!} \frac{3(3+1)(3+2)}{3+k} \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<3+k>} g^{(3+k)} dx^3$$

$\vdots$

Here, if $c_{r\,s+1}^{3m} = \sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+2-r+t}C_{m-1}$ is used, the expansion of $0 \sim m-1$ terms is as follows.

$$\int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>} g^{(0)} dx^3 = \sum_{r=0}^{m-1} \binom{-3}{r} f^{<3+r>} g^{(r)}$$

$$- \sum_{r=0}^{2}\sum_{s=0}^{m-1} \binom{-3+r}{s} f_{a_{3-r}}^{<3-r+s>} g_{a_{3-r}}^{(s)} \int_{a_3}^{x} \cdots \int_{a_{3-r+1}}^{x} dx^r$$

$$+ (-1)^m \sum_{r=1}^{2}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+2-r+t}C_{m-1}\, f_{a_{3-r}}^{<m+3-r+s>} g_{a_{3-r}}^{(m+s)} \int_{a_3}^{x} \cdots \int_{a_{3-r+1}}^{x} dx^r$$

$$+(-1)^m \sum_{k=0}^{2} \frac{{}_2C_k}{2!} \frac{m(m+1)(m+2)}{m+k} \int_{a_3}^{x}\int_{a_2}^{x}\int_{a_1}^{x} f^{<m+k>} g^{(m+k)} dx^3 \qquad (3)$$

In a similar way, calculating for the 4th integral,

$$\int_{a_4}^{x} \cdots \int_{a_1}^{x} f^{<0>} g^{(0)} dx^4 = f^{<4>} g^{(0)} - 4f^{<5>} g^{(1)}$$

$$- \left(f_{a_4}^{<4>} g_{a_4}^{(0)} - 4f_{a_4}^{<4>} g_{a_4}^{(1)} \right)$$

$$- \left(f_{a_3}^{<3>} g_{a_3}^{(0)} - 3f_{a_3}^{<4>} g_{a_3}^{(1)} \right) \int_{a_4}^{x} dx$$

$$- \left(f_{a_2}^{<2>} g_{a_2}^{(0)} - 2f_{a_2}^{<3>} g_{a_2}^{(1)} \right) \int_{a_4}^{x}\int_{a_3}^{x} dx^2$$

$$- \left(f_{a_1}^{<1>} g_{a_1}^{(0)} - f_{a_1}^{<2>} g_{a_1}^{(1)} \right) \int_{a_4}^{x}\int_{a_3}^{x}\int_{a_2}^{x} dx^3$$

$$+ \left(4f_{a_3}^{<5>} g_{a_3}^{(2)} \right) \int_{a_4}^{x} dx$$

$$+ \left(7f_{a_2}^{<4>} g_{a_2}^{(2)} + 4f_{a_3}^{<5>} g_{a_3}^{(3)} \right) \int_{a_4}^{x}\int_{a_3}^{x} dx^2$$

$$+ \left(9f_{a_1}^{<3>} g_{a_1}^{(2)} + 11f_{a_1}^{<4>} g_{a_1}^{(3)} + 4f_{a_1}^{<5>} g_{a_1}^{(4)} \right) \int_{a_4}^{x}\int_{a_3}^{x}\int_{a_2}^{x} dx^3$$

$$+ 10\int_{a_4}^{x} \cdots \int_{a_1}^{x} f^{<2>} g^{(2)} dx^4 + 20\int_{a_4}^{x} \cdots \int_{a_1}^{x} f^{<3>} g^{(3)} dx^4$$

$$+ 15\int_{a_4}^{x} \cdots \int_{a_1}^{x} f^{<4>} g^{(4)} dx^4 + 4\int_{a_4}^{x} \cdots \int_{a_1}^{x} f^{<5>} g^{(5)} dx^4$$

$$= f^{<4>} g^{(0)} - 4f^{<5>} g^{(1)} + 10f^{<6>} g^{(2)}$$

$$- \left(f_{a_4}^{<4>} g_{a_4}^{(0)} - 4f_{a_4}^{<5>} g_{a_4}^{(1)} + 10f_{a_4}^{<6>} g_{a_4}^{(2)} \right)$$

$$- \left(f_{a_3}^{<3>} g_{a_3}^{(0)} - 3f_{a_3}^{<4>} g_{a_3}^{(1)} + 6f_{a_3}^{<5>} g_{a_3}^{(2)} \right) \int_{a_4}^{x} dx$$

$$- \left(f_{a_2}^{\langle 2\rangle} g_{a_2}^{(0)} - 2 f_{a_2}^{\langle 3\rangle} g_{a_2}^{(1)} + 3 f_{a_2}^{\langle 4\rangle} g_{a_2}^{(2)} \right) \int_{a_4}^{x}\int_{a_3}^{x} dx^2$$

$$- \left(f_{a_1}^{\langle 1\rangle} g_{a_1}^{(0)} - f_{a_1}^{\langle 2\rangle} g_{a_1}^{(1)} + f_{a_1}^{\langle 3\rangle} g_{a_1}^{(2)} \right) \int_{a_4}^{x}\int_{a_3}^{x}\int_{a_2}^{x} dx^3$$

$$- \left(10 f_{a_3}^{\langle 6\rangle} g_{a_3}^{(3)} \right) \int_{a_4}^{x} dx$$

$$- \left(16 f_{a_2}^{\langle 5\rangle} g_{a_2}^{(3)} + 10 f_{a_2}^{\langle 6\rangle} g_{a_2}^{(4)} \right) \int_{a_4}^{x}\int_{a_3}^{x} dx^2$$

$$- \left(19 f_{a_1}^{\langle 4\rangle} g_{a_1}^{(3)} + 26 f_{a_1}^{\langle 5\rangle} g_{a_1}^{(4)} + 10 f_{a_1}^{\langle 6\rangle} g_{a_1}^{(5)} \right) \int_{a_4}^{x}\int_{a_3}^{x}\int_{a_2}^{x} dx^3$$

$$- 20 \int_{a_4}^{x}\cdots\int_{a_1}^{x} f^{\langle 3\rangle} g^{(3)} dx^4 - 45 \int_{a_4}^{x}\cdots\int_{a_1}^{x} f^{\langle 4\rangle} g^{(4)} dx^4$$

$$- 36 \int_{a_4}^{x}\cdots\int_{a_1}^{x} f^{\langle 5\rangle} g^{(5)} dx^4 - 10 \int_{a_4}^{x}\cdots\int_{a_1}^{x} f^{\langle 6\rangle} g^{(6)} dx^4$$

$\vdots$

Here, red and magenta coefficients are the elements of the following Pascal type triangles. Bule coefficients are given, magenta coefficients are equal to c_{00}^{4m} , and a red coefficient is obtained as the sum of two coefficients of the upper row.

$$\begin{array}{ccccccc} & & & c_{00}^{42} & & & \\ & & c_{10}^{42} & & c_{11}^{42} & & \\ & c_{20}^{42} & & c_{21}^{42} & & c_{22}^{42} & \\ c_{30}^{42} & & c_{31}^{42} & & c_{32}^{42} & & c_{33}^{42} \end{array} = \begin{array}{ccccccc} & & & 4 & & & \\ & & 3 & & 4 & & \\ & 2 & & 7 & & 4 & \\ 1 & & 9 & & 11 & & 4 \end{array} = \begin{array}{ccccccc} & & & {}_4C_1 & & & \\ & & {}_3C_1 & & {}_4C_1 & & \\ & {}_2C_1 & & 7 & & {}_4C_1 & \\ {}_1C_1 & & 9 & & 11 & & {}_4C_1 \end{array}$$

$$\begin{array}{ccccccc} & & & c_{00}^{43} & & & \\ & & c_{10}^{43} & & c_{11}^{43} & & \\ & c_{20}^{43} & & c_{21}^{43} & & c_{22}^{43} & \\ c_{30}^{43} & & c_{31}^{43} & & c_{32}^{43} & & c_{33}^{43} \end{array} = \begin{array}{ccccccc} & & & 10 & & & \\ & & 6 & & 10 & & \\ & 3 & & 16 & & 10 & \\ 1 & & 19 & & 26 & & 10 \end{array} = \begin{array}{ccccccc} & & & {}_5C_2 & & & \\ & & {}_4C_2 & & {}_5C_2 & & \\ & {}_3C_2 & & 16 & & {}_5C_2 & \\ {}_2C_2 & & 19 & & 26 & & {}_5C_2 \end{array}$$

As well as before, if bule coefficients are given, the other coefficients can be calculated by the following equation.

$$c_{rs}^{4m} = \sum_{t=s-1}^{r-1} {}_tC_{s-1} \cdot {}_{m+3-r+t}C_{m-1}$$

If these are used, the above formula is expressed as follows.

$$\int_{a_4}^{x}\cdots\int_{a_1}^{x} f^{\langle 0\rangle} g^{(0)} dx^4 = \sum_{r=0}^{1} \binom{-4}{r} f^{\langle 4+r\rangle} g^{(r)}$$

$$- \sum_{r=0}^{3}\sum_{s=0}^{1} \binom{-4+r}{s} f_{a_{4-r}}^{\langle 4-r+s\rangle} g_{a_{4-r}}^{(s)} \int_{a_4}^{x}\cdots\int_{a_{4-r+1}}^{x} dx^r$$

$$+ \sum_{r=1}^{3}\sum_{s=0}^{r-1} c_{r\,1+s}^{42} f_{a_{4-r}}^{\langle 6-r+s\rangle} g_{a_{4-r}}^{(2+s)} \int_{a_4}^{x}\cdots\int_{a_{4-r+1}}^{x} dx^r$$

$$+ (-1)^2 \sum_{k=0}^{3} \frac{{}_3C_k}{3!} \frac{2(2+1)(2+2)(2+3)}{2+k} \int_{a_4}^{x} \cdots \int_{a_1}^{x} f^{<2+k>} g^{(2+k)} dx^4$$

$$= \sum_{r=0}^{2} \binom{-4}{r} f^{<4+r>} g^{(r)}$$

$$- \sum_{r=0}^{3} \sum_{s=0}^{2} \binom{-4+r}{s} f_{a_{4-r}}^{<4-r+s>} g_{a_{4-r}}^{(s)} \int_{a_4}^{x} \cdots \int_{a_{4-r+1}}^{x} dx^r$$

$$- \sum_{r=1}^{3} \sum_{s=0}^{r-1} c_{r1+s}^{43} f_{a_{4-r}}^{\langle 7-r+s \rangle} g_{a_{4-r}}^{(3+s)} \int_{a_4}^{x} \cdots \int_{a_{4-r+1}}^{x} dx^r$$

$$+ (-1)^3 \sum_{k=0}^{3} \frac{{}_3C_k}{3!} \frac{3(3+1)(3+2)(3+3)}{3+k} \int_{a_4}^{x} \cdots \int_{a_1}^{x} f^{<3+k>} g^{(3+k)} dx^4$$

$\vdots$

Here, if $c_{rs+1}^{4m} = \sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+3-r+t}C_{m-1}$ is used, the expansion of $0 \sim m-1$ terms is as follows.

$$\int_{a_4}^{x} \cdots \int_{a_1}^{x} f^{<0>} g^{(0)} dx^4 = \sum_{r=0}^{m-1} \binom{-4}{r} f^{<4+r>} g^{(r)}$$

$$- \sum_{r=0}^{3} \sum_{s=0}^{m-1} \binom{-4+r}{s} f_{a_{4-r}}^{<4-r+s>} g_{a_{4-r}}^{(s)} \int_{a_4}^{x} \cdots \int_{a_{4-r+1}}^{x} dx^r$$

$$+ (-1)^m \sum_{r=1}^{3} \sum_{s=0}^{r-1} \sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+3-r+t}C_{m-1} f_{a_{4-r}}^{\langle m+4-r+s \rangle} g_{a_{4-r}}^{(m+s)} \int_{a_4}^{x} \cdots \int_{a_{4-r+1}}^{x} dx^r$$

$$+ (-1)^m \sum_{k=0}^{3} \frac{{}_3C_k}{3!} \frac{m(m+1)(m+2)(m+3)}{m+k} \int_{a_4}^{x} \cdots \int_{a_1}^{x} f^{<m+k>} g^{(m+k)} dx^4$$

Hereafter, in a similar way, the expansion of $0 \sim m-1$ terms of n th order integral is obtained as follows.

$$\int_{a_n}^{x} \cdots \int_{a_1}^{x} f^{<0>} g^{(0)} dx^n = \sum_{r=0}^{m-1} \binom{-n}{r} f^{<n+r>} g^{(r)}$$

$$- \sum_{r=0}^{n-1} \sum_{s=0}^{m-1} \binom{-n+r}{s} f_{a_{n-r}}^{<n-r+s>} g_{a_{n-r}}^{(s)} \int_{a_n}^{x} \cdots \int_{a_{n-r+1}}^{x} dx^r$$

$$+ (-1)^m \sum_{r=1}^{n-1} \sum_{s=0}^{r-1} \sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1} f_{a_{n-r}}^{\langle m+n-r+s \rangle} g_{a_{n-r}}^{(m+s)} \int_{a_n}^{x} \cdots \int_{a_{n-r+1}}^{x} dx^r$$

$$+ (-1)^m \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{(n-1)!} \frac{m(m+1)\cdots(m+n-1)}{m+k} \int_{a_n}^{x} \cdots \int_{a_1}^{x} f^{<m+k>} g^{(m+k)} dx^4$$

Here,

$$\frac{m(m+1)\cdots(m+n-1)}{(n-1)!} = \frac{(m+n-1)!}{(n-1)!(m-1)!} = \frac{\Gamma(m+n)}{\Gamma(n)\Gamma(m)} = \frac{1}{B(n,m)}$$

Then, substituting this for the above,

$$\int_{a_n}^{x} \cdots \int_{a_1}^{x} f^{<0>} g^{(0)} dx^n = \sum_{r=0}^{m-1} \binom{-n}{r} f^{<n+r>} g^{(r)}$$

$$- \sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s} f_{a_{n-r}}^{<n-r+s>} g_{a_{n-r}}^{(s)} \int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

$$+ (-1)^m \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1}\, f_{a_{n-r}}^{\langle m+n-r+s\rangle} g_{a_{n-r}}^{(m+s)} \int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<m+k>} g^{(m+k)} dx^4 \qquad (2.1)$$

Especially, when $a_r = a \quad for \ \ r=1,2, \cdots ,n$,

$$\int_a^x\cdots\int_a^x f^{<0>} g^{(0)} dx^n = \sum_{r=0}^{m-1}\binom{-n}{r} f^{<n+r>} g^{(r)}$$

$$- \sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s} f_a^{<n-r+s>} g_a^{(s)} \int_a^x\cdots\int_a^x dx^r$$

$$+ (-1)^m \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1}\, f_a^{\langle m+n-r+s\rangle} g_a^{(m+s)} \int_a^x\cdots\int_a^x dx^r$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \int_a^x\cdots\int_a^x f^{<m+k>} g^{(m+k)} dx^4$$

Here, since

$$\int_a^x\cdots\int_a^x dx^r = \frac{(x-a)^r}{r!} \qquad r = 0,1,2, \cdots ,n-1$$

substituting this for the above, we obtain (2.2) .

Last, when $f^{<r>}(a)=0 \ \ (r=1,2, \cdots ,m+n-1) \ \ or \ \ g^{(s)}(a)=0 \ \ (s=0,1, \cdots ,m+n-2)$,
(2.3) is clear. Q.E.D.

Note

As mentioned in **4.1.3**, the polynomial expression for the higher integral of 1 is difficult.

Example 1 $\int_2^x\int_1^x e^x \sin x \, dx^2$

When $n=2, m=3$, from (2.1) ,

$$\int_{a_2}^x\int_{a_1}^x f^{<0>} g^{(0)} dx^2 = \sum_{r=0}^{3-1}\binom{-2}{r} f^{<2+r>} g^{(r)}$$

$$- \sum_{r=0}^{2-1}\sum_{s=0}^{3-1}\binom{-2+r}{s} f_{a_{2-r}}^{<2-r+s>} g_{a_{2-r}}^{(s)} \int_{a_2}^x\int_{a_{2-r+1}}^x dx^r$$

$$+ (-1)^3 \sum_{r=1}^{2-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{3+2-1-r+t}C_{3-1}\, f_{a_{2-r}}^{\langle 3+2-r+s\rangle} g_{a_{2-r}}^{(3+s)} \int_{a_2}^x\cdots\int_{a_{2-r+1}}^x dx^r$$

$$+ \frac{(-1)^3}{B(2,3)} \sum_{k=0}^{2-1} \frac{{}_{2-1}C_k}{3+k} \int_{a_2}^x\int_{a_1}^x f^{<3+k>} g^{(3+k)} dx^2$$

$$= \sum_{r=0}^{2}\binom{-2}{r} f^{<2+r>} g^{(r)}$$

$$- \sum_{s=0}^{2}\binom{-2}{s} f_{a_2}^{<2+s>} g_{a_2}^{(s)} - \sum_{s=0}^{2}\binom{-1}{s} f_{a_1}^{<1+s>} g_{a_1}^{(s)} \int_{a_2}^{x} dx^1$$

$$- 3 f_{a_1}^{<4>} g_{a_1}^{(3)} \int_{a_2}^{x} dx$$

$$- 4\int_{a_2}^{x}\int_{a_1}^{x} f^{<3>} g^{(3)} dx^2 - 3\int_{a_2}^{x}\int_{a_1}^{x} f^{<4>} g^{(4)} dx^2$$

Let $f = e^x$, $g = \sin x$, $a_1 = 1$, $a_2 = 2$, then

Left: $\displaystyle \int_{a_2}^{x}\int_{a_1}^{x} f^{<0>} g^{(0)} dx^2 = \int_{2}^{x}\int_{1}^{x} e^x \sin x \, dx^2$

$$= -\frac{e^x \cos x - e^2 \cos 2}{2} - \frac{e^1 x(\sin 1 - \cos 1)}{2} + e^1(\sin 1 - \cos 1)$$

Next,

$$f^{<r>} = f^{<2+r>} = e^x \quad , \quad g^{(r)} = \sin\left(x + \frac{r\pi}{2}\right)$$

$$f_{a_2}^{<2+s>} = e^2 , \; f_{a_1}^{<1+s>} = e^1 \quad , \quad g_{a_2}^{(s)} = \sin\left(2 + \frac{s\pi}{2}\right) , \; g_{a_1}^{(s)} = \sin\left(1 + \frac{s\pi}{2}\right)$$

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<3>} g^{(3)} dx^2 = \int_{2}^{x}\int_{1}^{x} e^x \sin\left(x + \frac{3\pi}{2}\right) dx^2$$

$$= -\frac{e^x \sin x - e^2 \sin 2}{2} + \frac{e^1 x(\sin 1 + \cos 1)}{2} - e^1(\sin 1 + \cos 1)$$

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<4>} g^{(4)} dx^2 = \int_{2}^{x}\int_{1}^{x} e^x \sin\left(x + \frac{4\pi}{2}\right) dx^2$$

$$= -\frac{e^x \cos x - e^2 \cos 2}{2} - \frac{e^1 x(\sin 1 - \cos 1)}{2} + e^1(\sin 1 - \cos 1)$$

Substituting these for the right,

Right: $\displaystyle = e^x \sum_{r=0}^{2}\binom{-2}{r} \sin\left(x + \frac{r\pi}{2}\right) - e^2 \sum_{r=0}^{2}\binom{-2}{r} \sin\left(2 + \frac{r\pi}{2}\right)$

$$- e^1 \sum_{s=0}^{2}\binom{-1}{s} \sin\left(1 + \frac{s\pi}{2}\right)\frac{x-2}{1!} - 3e^1 \sin\left(1 + \frac{3\pi}{2}\right)\frac{x-2}{1!}$$

$$- 4\left\{ -\frac{e^x \sin x - e^2 \sin 2}{2} + \frac{e^1 x(\sin 1 + \cos 1)}{2} - e^1(\sin 1 + \cos 1)\right\}$$

$$- 3\left\{ -\frac{e^x \cos x - e^2 \cos 2}{2} - \frac{e^1 x(\sin 1 - \cos 1)}{2} + e^1(\sin 1 - \cos 1)\right\}$$

$$= -2e^x \cos x - 2e^x \sin x + 2e^2 \cos 2 + 2e^2 \sin 2$$

$$+ 4x e^1 \cos 1 - 8e^1 \cos 1$$

$$+\ 2e^{x}\sin x-2e^{2}\sin 2-2e^{1}x\sin 1-2e^{1}x\cos 1+4e^{1}\sin 1+4e^{1}\cos 1$$

$$+\ \frac{3}{2}e^{x}\cos x-\frac{3}{2}e^{2}\cos 2+\frac{3}{2}e^{1}x\sin 1-\frac{3}{2}e^{1}x\cos 1-3e^{1}\sin 1+3e^{1}\cos 1$$

$$=-\frac{1}{2}e^{x}\cos x+\frac{1}{2}e^{2}\cos 2-\frac{1}{2}e^{1}x\sin 1+\frac{1}{2}e^{1}x\cos 1+e^{1}\sin 1-e^{1}\cos 1$$

This is consistent with the left. And, since this result contains the constant-of-integration polynomial with degree 1, we can see that this 2nd order integral is collateral.

Example 2 $\displaystyle\int_{\frac{2\pi}{4}}^{x}\int_{\frac{1\pi}{4}}^{x} e^{x}\sin x\,dx^{2}$

When $n=2,\ m=3$, from (2.1) ,

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>}g^{(0)}dx^{2}=\sum_{r=0}^{2}\binom{-2}{r}f^{<2+r>}g^{(r)}$$

$$-\sum_{s=0}^{2}\binom{-2}{s}f_{a_2}^{<2+s>}g_{a_2}^{(s)}-\sum_{s=0}^{2}\binom{-1}{s}f_{a_1}^{<1+s>}g_{a_1}^{(s)}\int_{a_2}^{x}dx$$

$$-\ 3f_{a_1}^{<4>}g_{a_1}^{(3)}\int_{a_2}^{x}dx$$

$$-\ 4\int_{a_2}^{x}\int_{a_1}^{x} f^{<3>}g^{(3)}dx^{2}-3\int_{a_2}^{x}\int_{a_1}^{x} f^{<4>}g^{(4)}dx^{2}$$

Let $f=e^{x}$, $g=\sin x$, $a_1=1\pi/4$, $a_2=2\pi/4$, then

Left: $\displaystyle\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>}g^{(0)}dx^{2}=\int_{\frac{2\pi}{4}}^{x}\int_{\frac{1\pi}{4}}^{x} e^{x}\sin x\,dx^{2}=-\frac{e^{x}\cos x}{2}$

Next,

$$f^{<r>}=f^{<2+r>}=e^{x}\qquad ,\qquad g^{(r)}=\sin\left(x+\frac{r\pi}{2}\right)$$

$$f_{a_2}^{<2+s>}=e^{\frac{2\pi}{4}},\ f_{a_1}^{<1+s>}=e^{\frac{1\pi}{4}},\ g_{a_2}^{(s)}=\sin\left(\frac{2\pi}{4}+\frac{s\pi}{2}\right),\ g_{a_1}^{(s)}=\sin\left(\frac{1\pi}{4}+\frac{s\pi}{2}\right)$$

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<3>}g^{(3)}dx^{2}=\int_{\frac{2\pi}{4}}^{x}\int_{\frac{1\pi}{4}}^{x} e^{x}\sin\left(x+\frac{3\pi}{2}\right)dx^{2}$$

$$=-\frac{1}{2}\left(e^{x}\sin x-e^{\frac{2\pi}{4}}\sin\frac{2\pi}{4}\right)+\frac{e^{\frac{1\pi}{4}}x\sqrt{2}}{2}-\frac{e^{\frac{1\pi}{4}}\pi\sqrt{2}}{4}$$

$$\int_{a_2}^{x}\int_{a_1}^{x} f^{<4>}g^{(4)}dx^{2}=\int_{\frac{2\pi}{4}}^{x}\int_{\frac{1\pi}{4}}^{x} e^{x}\sin\left(x+\frac{4\pi}{2}\right)dx^{2}=-\frac{e^{x}\cos x}{2}$$

Substituting these for the right side,

Right: $\displaystyle e^{x}\sum_{s=0}^{2}\binom{-2}{s}\sin\left(x+\frac{s\pi}{2}\right)-e^{\frac{2\pi}{4}}\sum_{s=0}^{2}\binom{-2}{s}\sin\left(\frac{2\pi}{4}+\frac{s\pi}{2}\right)$

$$- e^{\frac{1\pi}{4}} \sum_{s=0}^{2} \binom{-1}{s} \sin\left(\frac{1\pi}{4}+\frac{s\pi}{2}\right)\left(x-\frac{2\pi}{4}\right)$$

$$- 3 e^{\frac{1\pi}{4}} \sin\left(\frac{1\pi}{4}+\frac{3\pi}{2}\right)\left(x-\frac{2\pi}{4}\right)$$

$$- 4\left\{-\frac{1}{2}\left(e^x \sin x - e^{\frac{2\pi}{4}} \sin\frac{2\pi}{4}\right) + \frac{e^{\frac{1\pi}{4}} x\sqrt{2}}{2} - \frac{e^{\frac{1\pi}{4}} \pi\sqrt{2}}{4}\right\} + \frac{3e^x \cos x}{2}$$

$$= -2e^x \cos x - 2e^x \sin x$$

$$+ 2e^{\frac{2\pi}{4}} + 2e^{\frac{1\pi}{4}} x\sqrt{2}$$

$$- e^{\frac{1\pi}{4}} \pi\sqrt{2}$$

$$+ 2e^x \sin x - 2e^{\frac{2\pi}{4}} - 2e^{\frac{1\pi}{4}} x\sqrt{2} + e^{\frac{1\pi}{4}} \pi\sqrt{2} + \frac{3e^x \cos x}{2}$$

$$= -\frac{e^x \cos x}{2}$$

This is consistent with the left. And, since this result does not contains the constant-of-integration polynomial, we can see that this 2nd order integral is lineal.

Example 3 $\int_0^x \int_0^x e^x \sin x \, dx^2$

When $n=2, m=3$, from (2.2) ,

$$\int_a^x \int_a^x f^{<0>} g^{(0)} dx^2 = \sum_{r=0}^{2} \binom{-2}{r} f^{<2+r>} g^{(r)}$$

$$- \sum_{s=0}^{2} \binom{-2}{s} f_a^{<2+s>} g_a^{(s)} \frac{(x-a)^0}{0!} - \sum_{s=0}^{2} \binom{-1}{s} f_a^{<1+s>} g_a^{(s)} \frac{(x-a)^1}{1!}$$

$$- 3 f_a^{<4>} g_a^{(3)} \frac{(x-a)^1}{1!}$$

$$- 4\int_a^x \int_a^x f^{<3>} g^{(3)} dx^2 - 3\int_a^x \int_a^x f^{<4>} g^{(4)} dx^2$$

Let $f = e^x$, $g = \sin x$, $a = 0$, then

Left: $\int_a^x \int_a^x f^{<0>} g^{(0)} dx^2 = \int_0^x \int_0^x e^x \sin x \, dx^2 = -\frac{e^x \cos x}{2} + \frac{x+1}{2}$

Next

$$f^{<r>} = f^{<2+r>} = e^x \quad , \quad g^{(r)} = \sin\left(x+\frac{r\pi}{2}\right)$$

$$f_a^{<1+s>} = f_a^{<4>} = e^0 \quad , \quad g_a^{(s)} = \sin\frac{s\pi}{2}$$

$$\int_a^x \int_a^x f^{<3>} g^{(3)} dx^2 = \int_0^x \int_0^x e^x \sin\left(x+\frac{3\pi}{2}\right) dx^2 = -\frac{e^x \sin x}{2} + \frac{x}{2}$$

$$\int_a^x \int_a^x f^{<4>} g^{(4)} dx^2 = \int_0^x \int_0^x e^x \sin\left(x+\frac{4\pi}{2}\right) dx^2 = -\frac{e^x \cos x}{2} + \frac{x+1}{2}$$

Substituting these for the right side,

Right:
$$= e^x \sum_{r=0}^{2} \binom{-2}{r} \sin\left(x+\frac{r\pi}{2}\right)$$
$$- e^0 \sum_{r=0}^{1}\sum_{s=0}^{2} \binom{-2+r}{s} \sin\frac{s\pi}{2}\frac{x^r}{r!} - 3e^0 \sin\frac{3\pi}{2}\frac{x^1}{1!}$$
$$- 4\left(-\frac{e^x \sin x}{2}+\frac{x}{2}\right) - 3\left(-\frac{e^x \cos x}{2}+\frac{x+1}{2}\right)$$
$$= -2e^x \cos x - 2e^x \sin x + 2 + x + 3x$$
$$+ 2e^x \sin x - 2x + \frac{3e^x \cos x}{2} - \frac{3x+3}{2}$$
$$= -\frac{e^x \cos x}{2} + \frac{x}{2} + \frac{1}{2}$$

This is consistent with the left. And, since this result contains the constant-of-integration polynomial with degree 1, we can see that this 2nd order integral is collateral.

Example 4 $\int_{-\infty}^{x}\int_{-\infty}^{x} e^x \sin x \, dx^2$

When $n=2, m=3$, from (2.3) ,

$$\int_a^x\int_a^x f^{<0>}g^{(0)}dx^2 = \sum_{r=0}^{2}\binom{-2}{r} f^{<n+r>}g^{(r)} + \frac{(-1)^3}{B(2,3)}\sum_{k=0}^{1}\frac{{}_1C_k}{3+k}\int_a^x\int_a^x f^{<3+k>}g^{(3+k)}dx^2$$
$$= \sum_{s=0}^{2}\binom{-2}{s} f^{<n+s>}g^{(s)} - 4\int_a^x\int_a^x f^{<3>}g^{(3)}dx^2 - 3\int_a^x\int_a^x f^{<3>}g^{(3)}dx^2$$

Let $f = e^x$, $g = \sin x$, $a = -\infty$, then

Left: $\int_a^x\int_a^x f^{<0>}g^{(0)}dx^2 = \int_{-\infty}^{x}\int_{-\infty}^{x} e^x \sin x \, dx^2 = -\frac{e^x \cos x}{2}$

Next,

$$f^{<r>} = f^{<2+r>} = e^x \quad , \quad g^{(r)} = \sin\left(x+\frac{r\pi}{2}\right)$$
$$\int_a^x\int_a^x f^{<3>}g^{(3)}dx^2 = \int_{-\infty}^{x}\int_{-\infty}^{x} e^x \sin\left(x+\frac{3\pi}{2}\right)dx^2 = -\frac{e^x \sin x}{2}$$
$$\int_a^x\int_a^x f^{<4>}g^{(4)}dx^2 = \int_{-\infty}^{x}\int_{-\infty}^{x} e^x \sin\left(x+\frac{4\pi}{2}\right)dx^2 = -\frac{e^x \cos x}{2}$$

Substituting these for the right side,

Right:
$$= e^x \sum_{r=0}^{2}\binom{-2}{r}\sin\left(x+\frac{r\pi}{2}\right) - 4\left(-\frac{e^x \sin x}{2}\right) - 3\left(-\frac{e^x \cos x}{2}\right)$$
$$= e^x(\sin x - 2\cos x - 3\sin x) + 2e^x \sin x + \frac{3e^x \cos x}{2}$$
$$= -\frac{e^x \cos x}{2}$$

This is consistent with the left. And, since this result does not contains the constant-of-integration polynomial, we can see that this 2nd order integral is lineal.

Remark 1

The following thing can be seen from Example 2 and Example 4 which are lineal together.

$$\left(e^x \sin x\right)^{<2>} = -\frac{e^x \cos x}{2} = \int_{\frac{2\pi}{4}}^{x}\int_{\frac{1\pi}{4}}^{x} e^x \sin x\, dx^2 = \int_{-\infty}^{x}\int_{-\infty}^{x} e^x \sin x\, dx^2$$

That is, **the higher integral which gives the higher primitive function of the product of two functions is not necessarily unique.**

Remark 2

As understood from the above proof process and Example 1,2 , (2.1) holds unconditionally and is perfect. However, it is too complicated and the practical use is difficult. (2.2) is not simplified considering the conditions.

(2.3) is the most practicable compared with these. The condition looks severe apparently. But, let

$$f^{<r>} = \int_a^x \cdots \int_a^x f(x)\, dx^r \quad (r=1,2,\cdots,n)$$

Then $f^{<r>}(a)=0 \quad (r=1,2,\cdots,n)$ holds without trouble.

Therefore, in Example 3, if not $f^{<r>} = e^x$ but $f^{<r>} = e^x - \sum_{k=0}^{r-1}\frac{x^k}{k!}$ was adopted, not (2.2) but (2.3) was applicable. But if it does so, since $f^{<r>}$ will become a collateral higher primitive function, the calculation may be complicated. Like Example4, In the case that the zero of the lineal higher primitive function is consistent with the lower limit a of the integral of fg, the calculation is the easiest. Therefore, below, we adopt this case as much as possible, and will mainly use (2.3).

16.1.3 Riemann-Liouville Integral Expression

According to Formula 4.2.1 in **4.2** , Higher Integral is expressed by Riemann-Liouville Integral as follows.

$$\int_a^x \cdots \int_a^x f(x)\, dx^n = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)\, dt$$

If (2.2), (2.3) in Theorem 16.1.2 are rewritten using this, it is as follows.

Theorem 16.1.3

Let $f^{<r>}$ be the arbitrary r th order primitive function of $f(x)$ and $g^{<r>}$ be the r th order derivative function of $g(x)$ for $r=1,2,\cdots,m+n-1$. Let $f_{a_k}^{<r>}, g_{a_k}^{(r)}$ be the function values of $f^{<r>}, g^{(r)}$ on a_k *for* $k=1,2,\cdots,n$. And let $\Gamma(n)$ be the gamma function and $B(n,m)$ be the beta function. Then, the following formulas hold.

$$\begin{aligned}\frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} fg\, dt &= \sum_{r=0}^{m-1}\binom{-n}{r} f^{<n+r>} g^{(r)} \\ &- \sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s} f_a^{<n-r+s>} g_a^{(s)} \frac{(x-a)^r}{r!} \\ &+ (-1)^m \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1}\, f_a^{\langle m+s+n-r\rangle} g_a^{(m+s)} \frac{(x-a)^r}{r!}\end{aligned}$$

$$+\frac{(-1)^m}{B(n,m)\Gamma(n)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x (x-t)^{n-1} f^{<m+k>} g^{(m+k)} dt \tag{3.1}$$

Especially, when $f^{<r>}(a)=0\ (r=1,2,\cdots,m+n-1)$ *or* $g^{(s)}(a)=0\ (s=0,1,\cdots,m+n-2)$

$$\frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f g\, dt = \sum_{r=0}^{m-1}\binom{-n}{r} f^{<n+r>} g^{(r)}$$

$$+\frac{(-1)^m}{B(n,m)\Gamma(n)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x (x-t)^{n-1} f^{<m+k>} g^{(m+k)} dt \tag{3.2}$$

16.2 Higher Integral of x^a f (x) (general)

There are two features in the higher integral of $x^{\alpha}f(x)$. That is

1 When α is a positive integer, the remainder and a part of the constant-of-integration polynomial disappear.

2 When the commonn lower limit of Higher Integral is 0, the constant-of-integration polynomial disappear.

Formula 16.2.0

Let $\Gamma(z)$ be the gamma function, $B(n,m)$ be the beta function, $f^{<r>}$ be the arbitrary r th primitive function of $f(x)$ and $f_{a_k}^{<r>}$ be the function values of $f^{<r>}$ on a_k . Then the following expressions hold for a natural number n.

(1)

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<0>}x^{\alpha}dx^n = \sum_{r=0}^{m-1}\binom{-n}{r} f^{<n+r>}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}$$

$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s} f_{a_{n-r}}^{<n-r+s>}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-s)}a_{n-r}^{\alpha-s}\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x}dx^r$$

$$+(-1)^m\sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1}{}_tC_s\cdot{}_{m+n-1-r+t}C_{m-1}$$

$$\times f_{a_{n-r}}^{\langle m+n-r+s\rangle}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m-s)}a_{n-r}^{\alpha-m-s}\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x}dx^r$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<m+k>}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m-k)}x^{\alpha-m-k}dx^n \qquad (0.1)$$

Especially, when $\alpha=m=0,1,2,\cdots$

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<0>}x^{m}dx^n = \sum_{r=0}^{m}\binom{-n}{r} f^{<n+r>}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}$$

$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m}\binom{-n+r}{s} f_{a_{n-r}}^{<n-r+s>}\frac{\Gamma(1+m)}{\Gamma(1+m-s)}a_{n-r}^{m-s}\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x}dx^r \qquad (0.2)$$

Especially, when $f^{<r>}(a)=0 \ \ (r=1,2,\cdots,m+n-1) \ \ or \ \ a=0$

$$\int_{a}^{x}\cdots\int_{a}^{x} f^{<0>}x^{\alpha}dx^n = \sum_{r=0}^{m-1}\binom{-n}{r} f^{<n+r>}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_{a}^{x}\cdots\int_{a}^{x} f^{<m+k>}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m-k)}x^{\alpha-m-k}dx^n \qquad (0.3)$$

Where, if $\alpha=-1,-2,-3,\cdots$, it shall read as follows.

$$\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} \longrightarrow (-1)^{-r}\frac{\Gamma(-\alpha+r)}{\Gamma(-\alpha)} \qquad r=r,\ s,\ m+s\ ,\ m+k$$

(2) When $\alpha\neq-1,-2,-3,\cdots$ & $\alpha+n\neq-1,-2,-3,\cdots$

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} x^{\alpha}f^{(0)}dx^n = \sum_{r=0}^{m-1}\binom{-n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)}x^{\alpha+n+r}f^{(r)}$$

$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n-r+s)}a_{n-r}^{\alpha+n-r+s} f_{a_{n-r}}^{(s)}\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x}dx^r$$

$$+ (-1)^m \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1}$$

$$\times \quad \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+n-r+s)} a_{n-r}^{\alpha+m+n-r+s} f_{a_{n-r}}^{(m+s)} \int_{a_n}^{x} \cdots \int_{a_{n-r+1}}^{x} dx^r$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \int_{a_n}^{x} \cdots \int_{a_1}^{x} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} x^{\alpha+m+k} f^{(m+k)} dx^n \qquad (0.4)$$

Especially, when $a=0 \quad or \quad f^{(s)}(a)=0 \ \ (s=0,1,\cdots,m+n-2)$

$$\int_a^x \cdots \int_a^x x^{\alpha} f^{(0)} dx^n = \sum_{r=0}^{m-1} \binom{-n}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)} x^{\alpha+n+r} f^{(r)}$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \int_a^x \cdots \int_a^x \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} x^{\alpha+m+k} f^{(m+k)} dx^n \qquad (0.5)$$

Proof (1)

When $\alpha \ne -1,-2,-3,\cdots$

$$\left(x^{\alpha}\right)^{(r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} x^{\alpha-r}, \qquad \left(x^{\alpha}\right)^{(m+k)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m-k)} x^{\alpha-m-k}$$

Then, substituting these for (2.1) in Theorem 16.1.2,

$$\int_{a_n}^x \cdots \int_{a_1}^x f^{<0>} x^{\alpha} dx^n = \sum_{r=0}^{m-1} \binom{-n}{r} f^{<n+r>} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} x^{\alpha-r}$$

$$- \sum_{r=0}^{n-1}\sum_{s=0}^{m-1} \binom{-n+r}{s} f_{a_{n-r}}^{<n-r+s>} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-s)} a_{n-r}^{\alpha-s} \int_{a_n}^{x} \cdots \int_{a_{n-r+1}}^{x} dx^r$$

$$\times f_{a_{n-r}}^{\langle m+s+n-r\rangle} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m-s)} a_{n-r}^{\alpha-m-s} \int_{a_n}^{x} \cdots \int_{a_{n-r+1}}^{x} dx^r$$

$$+ (-1)^m \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1}$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \int_{a_n}^{x} \cdots \int_{a_1}^{x} f^{<m+k>} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m-k)} x^{\alpha-m-k} dx^n \qquad (0.1)$$

When $\alpha=m-1 \quad m=0,1,2,\cdots$

$$\Gamma(1+\alpha-m-s) = \pm\infty \quad s=0,1,2,3,\cdots$$

$$\Gamma(1+\alpha-m-k) = \pm\infty \quad k=0,1,2,3,\cdots$$

Hence, ΣΣΣ and the remainder term in (0.1) disappear, and is as follows.

$$\int_{a_n}^x \cdots \int_{a_1}^x f^{<0>} x^{m-1} dx^n = \sum_{r=0}^{m-1} \binom{-n}{r} f^{<n+r>} \frac{\Gamma(m)}{\Gamma(m-r)} x^{m-1-r}$$

$$- \sum_{r=0}^{n-1}\sum_{s=0}^{m-1} \binom{-n+r}{s} f_{a_{n-r}}^{<n-r+s>} \frac{\Gamma(m)}{\Gamma(m-s)} a_{n-r}^{m-1-s} \int_{a_n}^{x} \cdots \int_{a_{n-r+1}}^{x} dx^r$$

Then, replacing m-1 with m, we obtain (0.2) .

Next, when $f^{<r>}(a)=0 \ \ (r=1,2,\cdots,m+n-1) \ \ or \ \ a=0$, the condition of (2.3) in Thorem 16.1.2 is satisfied. Then we obtain (0.3) .

When $\alpha = -1, -2, -3, \cdots$, from (5.5) in **1.1.5**

$$\frac{\Gamma(-z)}{\Gamma(-z-n)} = (-1)^{-n}\frac{\Gamma(1+z+n)}{\Gamma(1+z)} \qquad \text{(n is a nonnegative integer)}$$

Then replaceng $-z = 1+\alpha$, $n = r$, we obtain the proviso.

Proof (2)

When $\alpha \neq -1, -2, -3, \cdots$ & $\alpha+n \neq -1, -2, -3, \cdots$

$$f^{<n+r>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)}x^{\alpha+n+r} \quad , \quad f^{<m+k>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)}x^{\alpha+m+k}$$

$$f_{a_{n-r}}^{\langle n-r+s\rangle} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n-r+s)}a_{n-r}^{\alpha+n-r+s}$$

$$f_{a_{n-r}}^{\langle m+n-r+s\rangle} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+n-r+s)}a_{n-r}^{\alpha+m+n-r+s}$$

Then, substituting these for (2.1) in Theorem 16.1.2 ,

$$\begin{aligned}\int_{a_n}^{x}\cdots\int_{a_1}^{x} x^{\alpha} g^{(0)} dx^n = &\sum_{r=0}^{m-1}\binom{-n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)}x^{\alpha+n+r}g^{(r)} \\ &- \sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n-r+s)}a_{n-r}^{\alpha+n-r+s}\, g_{a_{n-r}}^{(s)}\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r \\ &+ (-1)^m\sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1} \\ &\qquad \times \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+n-r+s)}a_{n-r}^{\alpha+m+n-r+s}\, g_{a_{n-r}}^{(m+s)}\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r \\ &+ \frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_{a_n}^{x}\cdots\int_{a_1}^{x}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)}x^{\alpha+m+k}g^{(m+k)}dx^n\end{aligned}$$

Here, replacing g with f , we obtain (0.4) .

Last, when $a = 0$ *or* $f^{(s)}(a) = 0$ $(s=0, 1, \cdots, m+n-2)$, since the condition of the Theorem 16.1.2 (2.3) is satisfied, we obtain (0.5) .

Example 1 $\int_{2}^{x}\int_{1}^{x}\sqrt{x}\,\log x\, dx^2$

Let $n=2, m=3$, $f = \log x$, $\alpha=1/2$, $a_1=1$, $a_2=2$. And integrating the left directly

Left: $\int_{a_2}^{x}\int_{a_1}^{x} f^{<0>}x^{\alpha}dx^2 = \int_{2}^{x}\int_{1}^{x}\sqrt{x}\,\log x\, dx^2$

$$= \left(\frac{4\log x}{15} - \frac{64}{225}\right)x^{\frac{5}{2}} - \left(\frac{4\log 2}{15} - \frac{64}{225}\right)2^{\frac{5}{2}} + \frac{4}{9}(x-2)$$

Next ,

$$f^{<r>} = (\log x)^{<r>} = \frac{\log x - \psi(1+r) - \gamma}{\Gamma(1+r)}x^r \ , \qquad \int_{a_2}^{x} dx^1 = \int_{2}^{x} dx = x - 2$$

$$f^{<2+r>} = \frac{\log x - \psi(1+2+r) - \gamma}{\Gamma(1+2+r)}x^{2+r} \quad , \quad f_{a_1}^{<4>} = \frac{\log 1 - \psi(1+4) - \gamma}{\Gamma(1+4)}1^4$$

$$f_{a_2}^{\langle 2+s\rangle} = \frac{\log 2 - \psi(1+2+s) - \gamma}{\Gamma(1+2+s)} 2^{2+s} \quad , \quad f_{a_1}^{\langle 1+s\rangle} = \frac{\log 1 - \psi(1+1+s) - \gamma}{\Gamma(1+1+s)} 1^{1+s}$$

Substituting these for the right side of (0.1) ,

Right:

$$= \sum_{r=0}^{2} \binom{-2}{r} \frac{\log x - \psi(1+2+r) - \gamma}{\Gamma(1+2+r)} x^{2+r} \frac{\Gamma(1+1/2)}{\Gamma(1+1/2-r)} x^{\frac{1}{2}-r}$$

$$- \sum_{s=0}^{2} \binom{-2}{s} \frac{\log 2 - \psi(1+2+s) - \gamma}{\Gamma(1+2+s)} 2^{2+s} \frac{\Gamma(1+1/2)}{\Gamma(1+1/2-s)} 2^{\frac{1}{2}-s}$$

$$- \sum_{s=0}^{2} \binom{-1}{s} \frac{\log 1 - \psi(1+1+s) - \gamma}{\Gamma(1+1+s)} 1^{1+s} \frac{\Gamma(1+1/2)}{\Gamma(1+1/2-s)} 1^{\frac{1}{2}-s} (x-2)$$

$$- 3 \frac{\log 1 - \psi(1+4) - \gamma}{\Gamma(1+4)} 1^{4} \frac{\Gamma(1+1/2)}{\Gamma(1+1/2-3)} 1^{\frac{1}{2}-3} (x-2)$$

$$- 4 \int_2^x \int_1^x \frac{\log x - \psi(1+3) - \gamma}{\Gamma(1+3)} x^{3} \frac{\Gamma(1+1/2)}{\Gamma(1+1/2-3)} x^{\frac{1}{2}-3} dx^2$$

$$- 3 \int_2^x \int_1^x \frac{\log x - \psi(1+4) - \gamma}{\Gamma(1+4)} x^{3} \frac{\Gamma(1+1/2)}{\Gamma(1+1/2-4)} x^{\frac{1}{2}-4} dx^2$$

$$= \left\{ \frac{1}{2}\left(\log x - \frac{3}{2}\right) - \frac{1}{6}\left(\log x - \frac{11}{6}\right) - \frac{1}{32}\left(\log x - \frac{25}{12}\right) \right\} x^{\frac{5}{2}}$$

$$- \left\{ \frac{1}{2}\left(\log 2 - \frac{3}{2}\right) - \frac{1}{6}\left(\log 2 - \frac{11}{6}\right) - \frac{1}{32}\left(\log 2 - \frac{25}{12}\right) \right\} 2^{\frac{5}{2}}$$

$$+ \frac{79}{144}(x-2) + \frac{25}{256}(x-2)$$

$$- \frac{1}{15}\left(\log x - \frac{29}{10}\right) x^{\frac{5}{2}} + \frac{1}{15}\left(\log 2 - \frac{29}{10}\right) 2^{\frac{5}{2}} - \frac{5}{12}(x-2)$$

$$+ \frac{1}{32}\left(\log x - \frac{63}{20}\right) x^{\frac{5}{2}} - \frac{1}{32}\left(\log 2 - \frac{63}{20}\right) 2^{\frac{5}{2}} + \frac{55}{256}(x-2)$$

$$= \left(\frac{4 \log x}{15} - \frac{64}{225} \right) x^{\frac{5}{2}} - \left(\frac{4 \log 2}{15} - \frac{64}{225} \right) 2^{\frac{5}{2}} + \frac{4}{9}(x-2)$$

This is consistent with the left side.

Example 2 $\displaystyle \int_2^x \int_1^x x^3 \log x \, dx^2$

Let $n=2, m=3$, $f = \log x$, $\alpha = 1/2$, $a_1 = 1$, $a_2 = 2$. And integrating the left directly

Left: $\displaystyle \int_{a_2}^x \int_{a_1}^x f^{\langle 0\rangle} x^{\alpha} dx^2 = \int_2^x \int_1^x x^3 \log x \, dx^2$

$$= \left(\frac{\log x}{20} - \frac{9}{400} \right) x^5 - \left(\frac{\log 2}{20} - \frac{9}{400} \right) 2^5 + \frac{1}{16}(x-2)$$

Next ,

$$f^{<r>} = (\log x)^{<r>} = \frac{\log x - \psi(1+r) - \gamma}{\Gamma(1+r)} x^r \ , \quad \int_{a_2}^{x} dx^1 = \int_2^x dx = x-2$$

$$f^{<2+r>} = \frac{\log x - \psi(1+2+r) - \gamma}{\Gamma(1+2+r)} x^{2+r}$$

$$f_{a_2}^{<2+s>} = \frac{\log 2 - \psi(1+2+s) - \gamma}{\Gamma(1+2+s)} 2^{2+s} \quad , \quad f_{a_1}^{<1+s>} = \frac{\log 1 - \psi(1+1+s) - \gamma}{\Gamma(1+1+s)} 1^{1+s}$$

Substituting these for the right side of (0.2) ,

Right:

$$= \sum_{r=0}^{3} \binom{-2}{r} \frac{\log x - \psi(1+2+r) - \gamma}{\Gamma(1+2+r)} x^{2+r} \frac{\Gamma(1+3)}{\Gamma(1+3-r)} x^{3-r}$$

$$- \sum_{s=0}^{3} \binom{-2}{s} \frac{\log 2 - \psi(1+2+s) - \gamma}{\Gamma(1+2+s)} 2^{2+s} \frac{\Gamma(1+3)}{\Gamma(1+3-s)} 2^{3-s}$$

$$- \sum_{s=0}^{3} \binom{-1}{s} \frac{\log 1 - \psi(1+1+s) - \gamma}{\Gamma(1+1+s)} 1^{1+s} \frac{\Gamma(1+3)}{\Gamma(1+3-s)} 1^{3-s} (x-2)$$

$$= \left\{ \frac{1}{2} \left(\log x - \frac{3}{2} \right) - \frac{1}{1} \left(\log x - \frac{11}{6} \right) + \frac{3}{4} \left(\log x - \frac{25}{12} \right) - \frac{1}{5} \left(\log x - \frac{137}{60} \right) \right\} x^5$$

$$- \left(\frac{\log 2}{20} - \frac{9}{400} \right) 2^5 + \frac{1}{16} (x-2)$$

$$= \left(\frac{\log x}{20} - \frac{9}{400} \right) x^5 - \left(\frac{\log 2}{20} - \frac{9}{400} \right) 2^5 + \frac{1}{16} (x-2)$$

Example 3 $\int_0^x \int_0^x \sqrt{x} \, \log x \, dx^2$

Let $n=2, m=3$, $f = \log x$, $\alpha = 1/2$. And integrating the left side directly

Left: $\int_0^x \int_0^x f^{<0>} x^{\alpha} dx^2 = \int_0^x \int_0^x \sqrt{x} \, \log x \, dx^2 = \left(\frac{4 \log x}{15} - \frac{64}{225} \right) x^{\frac{5}{2}}$

Next,

$$f^{<r>} = (\log x)^{<r>} = \frac{\log x - \psi(1+r) - \gamma}{\Gamma(1+r)} x^r$$

Substituting these for the right side of (0.3) ,

Right:

$$= \sum_{r=0}^{2} \binom{-2}{r} \frac{\log x - \psi(1+2+r) - \gamma}{\Gamma(1+2+r)} x^{2+r} \frac{\Gamma(1+1/2)}{\Gamma(1+1/2-r)} x^{\frac{1}{2}-r}$$

$$- 4 \int_0^x \int_0^x \frac{\log x - \psi(1+3) - \gamma}{\Gamma(1+3)} x^3 \frac{\Gamma(1+1/2)}{\Gamma(1+1/2-3)} x^{\frac{1}{2}-3} dx^2$$

$$- 3 \int_0^x \int_0^x \frac{\log x - \psi(1+4) - \gamma}{\Gamma(1+4)} x^3 \frac{\Gamma(1+1/2)}{\Gamma(1+1/2-4)} x^{\frac{1}{2}-4} dx^2$$

$$= \left\{ \frac{1}{2} \left(\log x - \frac{3}{2} \right) - \frac{1}{6} \left(\log x - \frac{11}{6} \right) - \frac{1}{32} \left(\log x - \frac{25}{12} \right) \right\} x^{\frac{5}{2}}$$

$$- \frac{1}{15} \left(\log x - \frac{29}{10} \right) x^{\frac{5}{2}} + \frac{1}{32} \left(\log x - \frac{63}{20} \right) x^{\frac{5}{2}}$$

$$= \left(\frac{4 \log x}{15} - \frac{64}{225} \right) x^{\frac{5}{2}}$$

Example 4 $\int_2^x \int_1^x \sqrt{x} \sin x \, dx^2$

Let $f = \sin x$, then

$$f^{(r)} = \sin\left(x + \frac{r\pi}{2} \right) \quad , \quad f^{(m+k)} = \sin\left\{ x + \frac{(m+k)\pi}{2} \right\}$$

$$f^{(s)}_{a_{n-r}} = \sin\left(a_{n-r} + \frac{r\pi}{2} \right) \quad , \quad f^{(m+s)}_{a_{n-r}} = \sin\left\{ a_{n-r} + \frac{(m+s)\pi}{2} \right\}$$

Substitute these for (0.4) ,

$$\int_{a_n}^x \cdots \int_{a_1}^x x^\alpha \sin x \, dx^n = \sum_{r=0}^{m-1} \binom{-n}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)} x^{\alpha+n+r} \sin\left(x + \frac{r\pi}{2} \right)$$

$$- \sum_{r=0}^{n-1} \sum_{s=0}^{m-1} \binom{-n+r}{s} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n-r+s)} a_{n-r}^{\alpha+n-r+s} \sin\left(a_{n-r} + \frac{r\pi}{2} \right) \int_{a_n}^x \cdots \int_{a_{n-r+1}}^x dx^r$$

$$+ (-1)^m \sum_{r=1}^{n-1} \sum_{s=0}^{r-1} \sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1}$$

$$\times \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+n-r+s)} a_{n-r}^{\alpha+m+n-r+s} \sin\left\{ a_{n-r} + \frac{(m+s)\pi}{2} \right\} \int_{a_n}^x \cdots \int_{a_{n-r+1}}^x dx^r$$

$$+ \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \int_{a_n}^x \cdots \int_{a_1}^x \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} x^{\alpha+m+k} \sin\left\{ x + \frac{(m+k)\pi}{2} \right\} dx^n$$

This right side is very complicated. However, fortunately, in the case of this function $f = \sin x$, if $m \to \infty$ ΣΣΣ and the remainder term converge to 0. And the following expression holds.

$$\int_{a_n}^x \cdots \int_{a_1}^x x^\alpha \sin x \, dx^n = \sum_{r=0}^{\infty} \binom{-n}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)} x^{\alpha+n+r} \sin\left(x + \frac{r\pi}{2} \right)$$

$$- \sum_{r=0}^{n-1} \sum_{s=0}^{\infty} \binom{-n+r}{s} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n-r+s)} a_{n-r}^{\alpha+n-r+s} \sin\left(a_{n-r} + \frac{r\pi}{2} \right) \int_{a_n}^x \cdots \int_{a_{n-r+1}}^x dx^r$$

Substituting $n=2, \alpha=1/2, a_1=1, a_2=2$ for this,

$$\int_2^x \int_1^x \sqrt{x} \sin x \, dx^2 = \sum_{r=0}^{\infty} \binom{-2}{r} \frac{\Gamma(1+1/2)}{\Gamma(1+1/2+2+r)} x^{\frac{1}{2}+2+r} \sin\left(x + \frac{r\pi}{2} \right)$$

$$- \sum_{s=0}^{\infty} \binom{-2}{s} \frac{\Gamma(1+1/2)}{\Gamma(1+1/2+2+s)} 2^{\frac{1}{2}+2+s} \sin\left(2 + \frac{0\pi}{2} \right)$$

$$- \sum_{s=0}^{\infty} \binom{-1}{s} \frac{\Gamma(1+1/2)}{\Gamma(1+1/2+1+s)} 1^{\frac{1}{2}+2+s} \sin\left(1 + \frac{1\pi}{2} \right) (x-2)$$

When the both sides are illustrated by mathematical software, it is as follows. Both overlap exactly and blue (left) can not be seen.

Left:

```
• a:=1/2:
• Fl := int(int(t^a*sin(t),t=1..u),u=2..x)
```

$$\int_2^x \int_1^u \sqrt{t} \cdot \sin(t)\ \mathrm{d}t\ \mathrm{d}u$$

Right:

```
• fx := r-> sin(x+r*PI/2)
```

$$r \to \sin\left(x + \frac{\pi \cdot r}{2}\right)$$

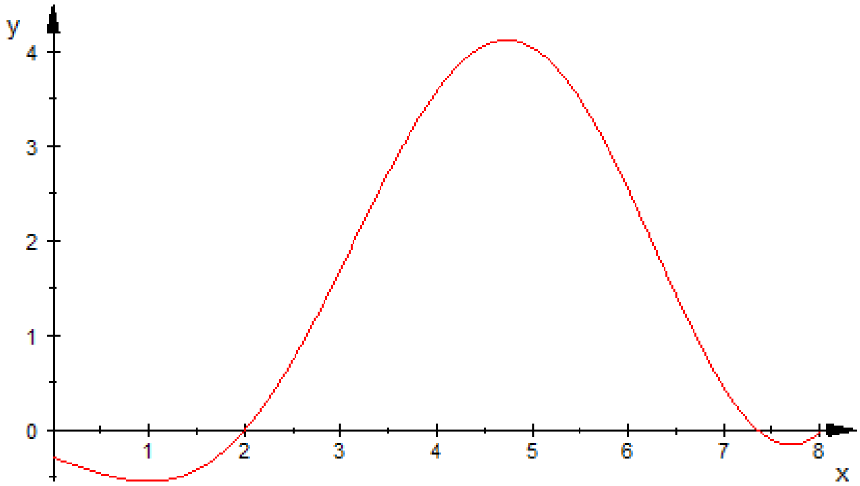
```
• gx := (r,s)-> gamma(1+a)/gamma(1+a+2-r+s)*x^(a+2-r+s)
```

$$(r, s) \to \frac{\Gamma(1+a)}{\Gamma(1+a+2-r+s)} \cdot x^{a+2-r+s}$$

```
• m:=30:
• f1 := sum(binomial(-2,s)*gx(0,s)*fx(s), s=0..m):
• f2 := -sum(binomial(-2,s)*subs(gx(0,s)*fx(s),x=2), s=0..m):
• f3 := -sum(binomial(-1,s)*subs(gx(1,s)*fx(s),x=1), s=0..m)*(x-2):
• Fr := f1+f2+f3:
```

- **Blue: Left , Red: Right**

```
• plotfunc2d(Fl,Fr, x=0..8)
```



16.3 Higher Integral of x^a f (x) (particulars)

In this section, substituting various functions f for the Formula 16.2.0 in previous section, we obtain a various formula. There are (1) and (2) in Formula 16.2.0, and we may also choose whichever. However, what we want is the expression or approximation of higher integral of $x^{\alpha}f(x)$ by the series. So, in the selection (1) or (2), we choose the way where such a well-behaved series is obtained.

Moreover, also in which formula, if $\alpha = -1, -2, -3, \cdots$, it shall read as follows.

$$\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} \longrightarrow (-1)^{-r}\frac{\Gamma(-\alpha+r)}{\Gamma(-\alpha)} \qquad r = r, s, m+s, m+k$$

16.3.1 Higher Integral of $(ax+b)^p(cx+d)^q$

Formula 16.3.1

The following expressions hold for p > 0 and a natural number n.

$$\int_{-\frac{b}{a}}^{x}\cdots\int_{-\frac{b}{a}}^{x}(ax+b)^p(cx+d)^q dx^n$$

$$= \sum_{r=0}^{m-1}\binom{-n}{r}\frac{(1/a)^{n+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(1+p+n+r)\Gamma(1+q-r)}\frac{(ax+b)^{p+n+r}}{(cx+d)^{r-q}} + R_m^n \qquad (1.1)$$

$$R_m^n = \frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\left(\frac{c}{a}\right)^{m+k}\frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(1+p+m+k)\Gamma(1+q-m-k)}$$

$$\times\int_{-\frac{b}{a}}^{x}\cdots\int_{-\frac{b}{a}}^{x}(ax+b)^{p+m+k}(cx+d)^{q-m-k}dx^n \qquad (1.1r)$$

$$\lim_{m\to\infty}R_m^n = 0$$

Especially, when $m = 0, 1, 2, \cdots$

$$\int_{-\frac{b}{a}}^{x}\cdots\int_{-\frac{b}{a}}^{x}(ax+b)^p\,(cx+d)^m dx^n$$

$$= \sum_{r=0}^{m}\binom{-n}{r}\frac{(1/a)^{n+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+m)}{\Gamma(1+p+n+r)\Gamma(1+m-r)}\frac{(ax+b)^{p+n+r}}{(cx+d)^{r-m}} \qquad (1.1')$$

Example 1st order integral of $\sqrt{x-2}\,\sqrt[3]{3x+4}$

The zeros of this primitive function are $x=-4/3$, 2 . If $-4/3$ is adopted, since $\sqrt{x-2}$ is a complex number, this higher integral becomes a complex function. It is inconvenient. Then if we adopt $x=2$ as the lower limit of the integral, since $a=1, b=-2, p=1/2, c=3, d=4, q=1/3, n=1$, substituting these for (1.1), (1.1r) , we obtain

$$\int_2^x\sqrt{x-2}\,\sqrt[3]{3x+4}\,dx$$

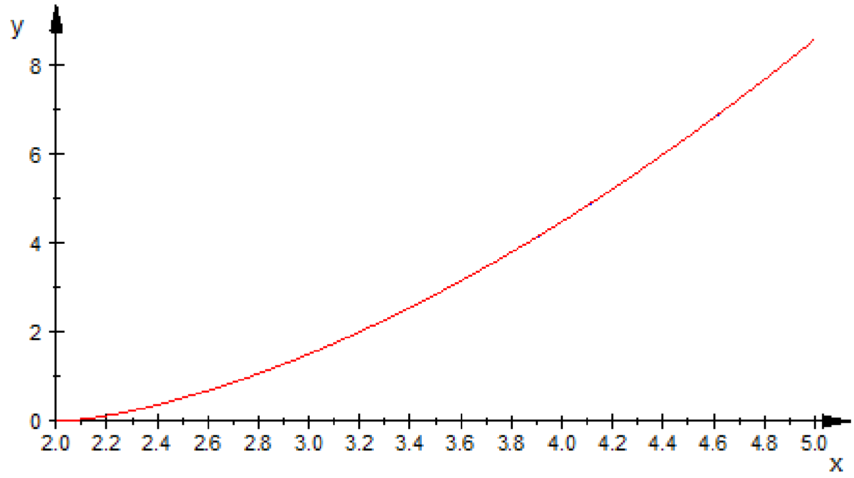
$$= \sum_{r=0}^{m-1}\binom{-1}{r}3^r\frac{\Gamma(3/2)\Gamma(4/3)}{\Gamma(5/2+r)\Gamma(4/3-r)}(x-2)^{\frac{3}{2}+r}(3x+4)^{\frac{1}{3}-r} + R_m^1$$

$$R_m^1 = (-1)^m 3^m \frac{\Gamma(3/2)\Gamma(4/3)}{\Gamma(3/2+m)\Gamma(4/3-m)} \int_2^x (x-2)^{\frac{1}{2}+m}(3x+4)^{\frac{1}{3}-m} dx$$

This remainder becomes $\lim_{m\to\infty} R_m^1 = 0$. And the convergence is quick. When $m=5$, if both sides are illustrated, it is as follows. Both overlap exactly and blue (left) can not be seen. In addition, this integral is not an elementary function.

Blue: Left , Red: Right

```
• plotfunc2d(g,f, x=2..5)
```



16.3.2 Higher Integral of $x^\alpha \log x$

Lineal higher primitive functions of $x^\alpha \log x$ are as follows.

$$(x^\alpha \log x)^{<1>} = \frac{x^{\alpha+1}\{(\alpha+1)\log x - 1\}}{(\alpha+1)^2}$$

$$(x^\alpha \log x)^{<2>} = \frac{x^{\alpha+2}\{(\alpha+1)(\alpha+2)\log x - 2\alpha - 3\}}{(\alpha+1)^2(\alpha+2)^2}$$

$$(x^\alpha \log x)^{<3>} = \frac{x^{\alpha+3}\{(\alpha+1)(\alpha+2)(\alpha+3)\log x - 3\alpha^2 - 12\alpha - 11\}}{(\alpha+1)^2(\alpha+2)^2(\alpha+3)^2}$$

$\vdots$

And when $\alpha > -n$, the zeros of these are all $x=0$.

Formula 16.3.2

When $\Gamma(z), \psi(z)$ are the gamma function and the psi function respectively, the following expressions hold for α, n such that $\alpha + n > 0$.

$$\int_0^x \cdots \int_0^x x^\alpha \log x \, dx^n = \sum_{r=0}^{m-1} \binom{-n}{r} \frac{\log x - \psi(1+n+r) - \gamma}{\Gamma(1+n+r)} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} x^{\alpha+n} + R_m^n \qquad (2.1)$$

$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \int_0^x \cdots \int_0^x \frac{\log x - \psi(1+m+k) - \gamma}{\Gamma(1+m+k)} \frac{\Gamma(1+\alpha)x^\alpha}{\Gamma(1+\alpha-m-k)} dx^n \qquad (2.1r)$$

$$\lim_{m\to\infty} R_m^n = 0$$

Especially, when $m = 0, 1, 2, \cdots$

$$\int_0^x \cdots \int_0^x x^m \log x\, dx^n = \sum_{r=0}^{m} \binom{-n}{r} \frac{\log x - \psi(1+n+r) - \gamma}{\Gamma(1+n+r)} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m+n} \tag{2.1'}$$

Complete Automorphism

Observing well the Formula 16.3.2, we notice that the integral of the completely same type as the left side is included in the remainder. In such a case, we can take out the integral of the purpose by transposition.

Formula 16.3.2'

The following expression holds for α , n such that $\alpha + n > 0$.

$$\int_0^x \cdots \int_0^x x^\alpha \log x\, dx^n$$

$$= \frac{\dfrac{\log x - \psi(1+n) - \gamma}{\Gamma(1+n)} + n \dfrac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n)} \displaystyle\sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{(1+k)^2} \frac{\psi(2+k)+\gamma}{B(1+k, \alpha-k)}}{1 + n \displaystyle\sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{(1+k)^2} \frac{1}{B(1+k, \alpha-k)}} x^{\alpha+n} \tag{2.2}$$

Calculation

Let $m=1$ in (2.1), (2.1r), then

$$\int_0^x \cdots \int_0^x x^\alpha \log x\, dx^n = \sum_{r=0}^{1-1} \binom{-n}{0} \frac{\log x - \psi(1+n+0) - \gamma}{\Gamma(1+n+0)} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-0)} x^{\alpha+n} + R_1^n$$

$$R_1^n = \frac{(-1)^1}{B(n,1)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{1+k} \int_0^x \cdots \int_0^x \frac{\log x - \psi(1+1+k) - \gamma}{\Gamma(1+1+k)} \frac{\Gamma(1+\alpha) x^\alpha}{\Gamma(1+\alpha-1-k)} dx^n$$

$$= -n \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{1+k} \frac{\Gamma(1+\alpha)}{\Gamma(2+k)\Gamma(\alpha-k)} \int_0^x \cdots \int_0^x x^\alpha \log x dx^n$$

$$+ n \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{1+k} \frac{\Gamma(1+\alpha)\{\psi(2+k)+\gamma\}}{\Gamma(2+k)\Gamma(\alpha-k)} \int_0^x \cdots \int_0^x x^\alpha dx^n$$

Here,

$$\Gamma(2+k) = (1+k)\Gamma(1+k) \quad , \quad \int_0^x \cdots \int_0^x x^\alpha dx^n = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n)} x^{\alpha+n}$$

Substituting these for the remainder,

$$R_m^n = -n \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{(1+k)^2} \frac{1}{B(1+k, \alpha-k)} \int_0^x \cdots \int_0^x x^\alpha \log x dx^n$$

$$+ n \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{(1+k)^2} \frac{\psi(2+k)+\gamma}{B(1+k, \alpha-k)} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n)} x^{\alpha+n}$$

Thus

$$\int_0^x \cdots \int_0^x x^\alpha \log x\, dx^n = \frac{\log x - \psi(1+n) - \gamma}{\Gamma(1+n)} x^{\alpha+n}$$

$$- n \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{(1+k)^2} \frac{1}{B(1+k, \alpha-k)} \int_0^x \cdots \int_0^x x^\alpha \log x dx^n$$

$$+ n \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{(1+k)^2} \frac{\psi(2+k)+\gamma}{B(1+k, \alpha-k)} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n)} x^{\alpha+n}$$

From this, we obtain (2.2).

Example When $n=1$,

$$\int_0^x x^\alpha \log x\,dx = \frac{\dfrac{\log x-\psi(2)-\gamma}{\Gamma(2)}+\dfrac{\Gamma(1+\alpha)}{\Gamma(2+\alpha)}\dfrac{{}_0C_0}{1^2}\dfrac{\psi(2)+\gamma}{B(1,\alpha)}}{1+\dfrac{{}_0C_0}{1^2}\dfrac{1}{B(1,\alpha)}}x^{\alpha+1}$$

$$=\frac{\log x-1+\dfrac{\alpha}{1+\alpha}}{1+\alpha}x^{\alpha+1}=\frac{x^{\alpha+1}}{\alpha+1}\left(\log x-\frac{1}{\alpha+1}\right)$$

But in this case, direct calculation is far easier !

16.3.3 Higher Integral of $x^\alpha \sin x$, $x^\alpha \cos x$

Lineal higher primitive functions of $x^3 \sin x$ are as follows.

$$(x^3\sin x)^{<1>}=(3x^2-6)\sin x-(x^3-6x)\cos x$$

$$(x^3\sin x)^{<2>}=-(6x^2-24)\cos x-(x^3-18x)\sin x$$

$$(x^3\sin x)^{<3>}=-(9x^2-60)\sin x+(x^3-36x)\cos x$$

$$(x^3\sin x)^{<4>}=(12x^2-120)\cos x+(x^3-60x)\sin x$$

$\vdots$

And the zeros of these are $x=0$, $4.9762\cdots$, 0 , $3.1224\cdots$, 0 , $\cdots$ respectively. Then the lower limit of the higher integral of $x^3\sin x$ is variable. Similarly, the lower limit of the lineal higher integral of $x^\alpha \sin x$ for any α is variable.

Formula 16.3.3

(1) When $m=0,1,2,\cdots$.

$$\int_{a_n}^x\cdots\int_{a_1}^x x^m \sin x\,dx^n=\sum_{r=0}^{m}\binom{-n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\sin\left\{x-\frac{(n+r)\pi}{2}\right\}$$

$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m}\binom{-n+r}{s}\frac{\Gamma(1+m)}{\Gamma(1+m-s)}a_{n-r}^{m-s}\sin\left\{a_{n-r}-\frac{(n-r+s)\pi}{2}\right\}\int_{a_n}^x\cdots\int_{a_{n-r+1}}^x dx^r \tag{3.1}$$

Especially, when a_1 , a_2 , $\cdots$, a_n are the zeros of the lineal heigher primitive of $x^m\sin x$

$$\int_{a_n}^x\cdots\int_{a_1}^x x^m \sin x\,dx^n=\sum_{r=0}^{m}\binom{-n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\sin\left\{x-\frac{(n+r)\pi}{2}\right\} \tag{3.1'}$$

(2) When $\alpha \ne -1,-2,-3,\cdots$ & $\alpha+n \ne -1,-2,-3,\cdots$.

$$\int_{a_n}^x\cdots\int_{a_1}^x x^\alpha \sin x\,dx^n=\sum_{r=0}^{m-1}\binom{-n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)}x^{\alpha+n+r}\sin\left(x+\frac{r\pi}{2}\right)$$

$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n-r+s)}a_{n-r}^{\alpha+n-r+s}\sin\left(a_{n-r}+\frac{s\pi}{2}\right)\int_{a_n}^x\cdots\int_{a_{n-r+1}}^x dx^r$$

$$+R_m^n \tag{3.2}$$

$$R_m^n = (-1)^m \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1}$$

$$\times \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+n-r+s)} a_{n-r}^{\alpha+m+n-r+s} \sin\left\{a_{n-r}+\frac{(m+s)\pi}{2}\right\} \int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_{a_n}^{x}\cdots\int_{a_1}^{x}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)}x^{\alpha+m+k}\sin\left\{x+\frac{(m+k)\pi}{2}\right\}dx^n$$

$$\lim_{m\to\infty} R_m^n = 0$$

In addition, the formula of $x^{\alpha}\cos x$ is obtained only by replacing $\sin x$ by $\cos x$.

Example 1

$$\int_2^x\int_1^x x^3 \sin x \, dx^2 = \sum_{r=0}^{3}\binom{-2}{r}\sin\left\{x-\frac{(2+r)\pi}{2}\right\}\frac{\Gamma(1+3)}{\Gamma(1+3-r)}x^{3-r}$$

$$-\sum_{s=0}^{3}\binom{-2}{s}\sin\left\{2-\frac{(2+s)\pi}{2}\right\}\frac{\Gamma(1+3)}{\Gamma(1+3-s)}2^{3-s}$$

$$-\sum_{s=0}^{3}\binom{-1}{s}\sin\left\{1-\frac{(1+s)\pi}{2}\right\}\frac{\Gamma(1+3)}{\Gamma(1+3-s)}1^{3-s}(x-2)$$

$$= -(x^3-18x)\sin x - (6x^2-24)\cos x$$

$$+ (3x-6)\sin 1 - (5x-10)\cos 1 - 28\sin 2$$

Especially, when $a_1=0$, $a_2=4.9762\cdots$,

$$\int_{a_2}^x\int_{a_1}^x x^3 \sin x \, dx^2 = \sum_{r=0}^{3}\binom{-2}{r}\sin\left\{x-\frac{(2+r)\pi}{2}\right\}\frac{\Gamma(1+3)}{\Gamma(1+3-r)}x^{3-r}$$

$$= -(x^3-18x)\sin x - (6x^2-24)\cos x$$

Example 2

See Example 4 in **16.2** .

16.3.4 Higher Integrals of $x^{\alpha}\sinh x$, $x^{\alpha}\cosh x$

Lineal higher primitive functions of $x^3\sinh x$ are as follows.

$$\left(x^3\sinh x\right)^{<1>} = e^{-x}\left(\frac{x^3}{2}+\frac{3x^2}{2}+3x+3\right) + e^{x}\left(\frac{x^3}{2}-\frac{3x^2}{2}+3x-3\right)$$

$$\left(x^3\sinh x\right)^{<2>} = -e^{-x}\left(\frac{x^3}{2}+3x^2+9x+12\right) - e^{x}\left(-\frac{x^3}{2}+3x^2-9x+12\right)$$

$$\vdots$$

And the zeros of these are $x=0$, $2.7085\cdots$, 0 , $3.1224\,i\cdots$, 0 , $\cdots$ respectively. Then the lower limit of the higher integral of $x^3\sinh x$ is variable. Similarly, the lower limit of the lineal higher integral of $x^{\alpha}\sinh x$ for any α is variable.

Formula 16.3.4

(1) When $m=0,1,2,\cdots$.

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} x^m \sinh x\, dx^n = \sum_{r=0}^{m}\binom{-n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\frac{e^x-(-1)^{n+r}e^{-x}}{2}$$

$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m}\binom{-n+r}{s}\frac{\Gamma(1+m)}{\Gamma(1+m-s)}a_{n-r}^{m-s}\frac{e^{a_{n-r}}-(-1)^{n-r+s}e^{-a_{n-r}}}{2}\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r \tag{4.1}$$

Especially, when $a_1, a_2, \cdots, a_n$ are the zeros of the lineal heigher primitive of $x^m \sinh x$

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} x^m \sinh x\, dx^n = \sum_{r=0}^{m}\binom{-n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\frac{e^x-(-1)^{n+r}e^{-x}}{2} \tag{4.1'}$$

(2) When $\alpha \neq -1,-2,-3,\cdots$ & $\alpha+n \neq -1,-2,-3,\cdots$.

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} x^\alpha \sinh x\, dx^n = \sum_{r=0}^{m-1}\binom{-n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)}x^{\alpha+n+r}\frac{e^x-(-1)^{-r}e^{-x}}{2}$$

$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n-r+s)}a_{n-r}^{\alpha+n-r+s}\frac{e^{a_{n-r}}-(-1)^{-s}e^{-a_{n-r}}}{2}\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

$$+R_m^n \tag{4.2}$$

$$R_m^n = (-1)^m\sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1}{}_tC_s\cdot{}_{m+n-1-r+t}C_{m-1}$$

$$\times\quad\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+n-r+s)}a_{n-r}^{\alpha+m+n-r+s}\frac{e^{a_{n-r}}-(-1)^{-m-s}e^{-a_{n-r}}}{2}\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_{a_n}^{x}\cdots\int_{a_1}^{x}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)}x^{\alpha+m+k}\frac{e^x-(-1)^{-m-k}e^{-x}}{2}dx^n$$

$$\lim_{m\to\infty}R_m^n = 0$$

In addition, the formula of $x^\alpha \cosh x$ is obtained by only replacing $-(-1)$ by $+(-1)$.

Example $\int_2^x\int_1^x \sqrt{x}\sinh x\, dx^2$

From (4.2) ,

$$\int_2^x\int_1^x \sqrt{x}\sinh x\, dx^2 = \sum_{r=0}^{\infty}\binom{-2}{r}\frac{\Gamma(1+1/2)}{\Gamma(1+1/2+2+r)}x^{\frac{1}{2}+2+r}\frac{e^x-(-1)^{-r}e^{-x}}{2}$$

$$-\sum_{s=0}^{\infty}\binom{-2}{s}\frac{\Gamma(1+1/2)}{\Gamma(1+1/2+2+s)}2^{\frac{1}{2}+2+s}\frac{e^2-(-1)^{-s}e^{-2}}{2}$$

$$-\sum_{s=0}^{\infty}\binom{-1}{s}\frac{\Gamma(1+1/2)}{\Gamma(1+1/2+1+s)}1^{\frac{1}{2}+2+s}\frac{e^1-(-1)^{-s}e^{-1}}{2}(x-2)$$

When the both sides are illustrated by mathematical software, it is as follows. Both overlap exactly and blue (left) can not be seen.

Left:

- `a:=1/2:`
- `Fl := int(int(t^a*sinh(t),t=1..u),u=2..x)`

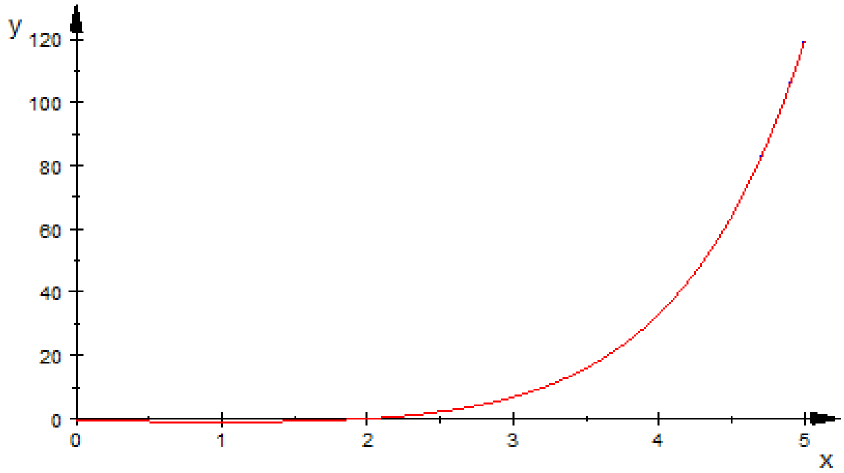
$$\int_2^x \int_1^u \sqrt{t} \cdot \sinh(t)\, \mathrm{d}\,t\, \mathrm{d}\,u$$

Right:

- `fx := r-> (E^x-(-1)^-r*E^-x)/2:`
- `gx := (r,s)-> gamma(1+a)/gamma(1+a+2-r+s)*x^(a+2-r+s):`
- `m:=15:`
- `f1 := sum(binomial(-2,s)*gx(0,s)*fx(s), s=0..m):`
- `f2 := -sum(binomial(-2,s)*subs(gx(0,s)*fx(s),x=2), s=0..m):`
- `f3 := -sum(binomial(-1,s)*subs(gx(1,s)*fx(s),x=1), s=0..m)*(x-2):`
- `Fr := f1+f2+f3:`

Blue: Left, Red: Right

- `plotfunc2d(Fl,Fr, x=0..5)`



16.4 Higher Integral of log x f (x)

16.4.1 Higher Integral of $(\log x)^2$

Lineal higher primitive functions of $(\log x)^2$ are as follows.

$$\left\{(\log x)^2\right\}^{<1>} = x(\log x(\log x - 2) + 2)$$

$$\left\{(\log x)^2\right\}^{<2>} = \frac{1}{4}x^2(2\log x(\log x - 3) + 7)$$

$$\left\{(\log x)^2\right\}^{<3>} = \frac{1}{108}x^3(6\log x(3\log x - 11) + 85)$$

$$\vdots$$

And the zeros of these are all $x=0$.

Formula 16.4.1

When $H_n = \sum_{j=1}^{n}\frac{1}{j} = \psi(1+n)+\gamma$ is a harmonic number, the following expression holds.

$$\int_0^x \cdots \int_0^x \log^2 x\, dx^n = \log x(\log x - H_n)\frac{x^n}{n!}$$

$$+ \sum_{r=1}^{m-1}(-1)^{r-1}\binom{-n}{r}\frac{\Gamma(r)}{\Gamma(1+n+r)}x^n(\log x - H_{n+r}) + R_m^n \qquad (1.1)$$

$$R_m^n = \frac{1}{B(n,m)}\frac{x^n}{n!}\sum_{k=0}^{n-1}(-1)^{k-1}\frac{{}_{n-1}C_k}{(m+k)^2}(\log x - H_n - H_{m+k}) \qquad (1.1r)$$

$$\lim_{m\to\infty} R_m^1 = 0$$

Calculation

Since $f(x) = g(x) = \log x$, $f^{<r>}(0) = 0 \quad (r=1,2,\cdots,m+n-1)$. Then (2.3) in Theorem 16.1.2 is applicable.

$$(\log x)^{<n+r>} = \frac{\log x - \psi(1+n+r) - \gamma}{\Gamma(1+n+r)}x^{n+r} \quad , \quad (\log x)^{(r)} = (-1)^{r-1}\frac{\Gamma(r)}{x^r}$$

Substituting these for (2.3) in Theorem 16.1.2 ,

$$\int_0^x \cdots \int_0^x \log^2 x\, dx^n = \frac{\{\log x - \psi(1+n) - \gamma\}\log x}{\Gamma(1+n)}x^n$$

$$+ \sum_{r=1}^{m-1}(-1)^{r-1}\binom{-n}{r}\frac{\{\log x - \psi(1+n+r) - \gamma\}\Gamma(r)}{\Gamma(1+n+r)}x^n + R_m^n$$

$$= \log x(\log x - H_n)\frac{x^n}{n!} + \sum_{r=1}^{m-1}(-1)^{r-1}\binom{-n}{r}\frac{(\log x - H_{n+r})\Gamma(r)}{\Gamma(1+n+r)}x^n + R_m^n$$

$$R_m^n = \frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_0^x \cdots \int_0^x \frac{\log x - \psi(1+m+k) - \gamma}{\Gamma(1+m+k)}x^{m+k}(-1)^{m+k-1}\frac{\Gamma(m+k)}{x^{m+k}}dx^n$$

$$= \frac{1}{B(n,m)}\sum_{k=0}^{n-1}(-1)^{k-1}\frac{{}_{n-1}C_k}{(m+k)^2}\int_0^x \cdots \int_0^x \{\log x - \psi(1+m+k) - \gamma\}\, dx^n$$

$$= \frac{1}{B(n,m)} \sum_{k=0}^{n-1} (-1)^{k-1} \frac{{}_{n-1}C_k}{(m+k)^2} \left\{ \frac{\log x - \psi(1+n) - \gamma}{\Gamma(1+n)} - \frac{\psi(1+m+k)+\gamma}{\Gamma(1+n)} \right\} x^n$$

$$= \frac{1}{B(n,m)} \frac{x^n}{n!} \sum_{k=0}^{n-1} (-1)^{k-1} \frac{{}_{n-1}C_k}{(m+k)^2} \left(\log x - H_n - H_{m+k} \right)$$

Example 3rd order integral of $\log^2 x$

Let $n=3, m=1000$ in (1.1). And if both sides are illustrated, it is s follows. Since the convergence of logarithm family is usually slow, both sides are not overlapping completely.

Left:

```
• g := int(int(int(ln(x)^2,x),x),x)
```

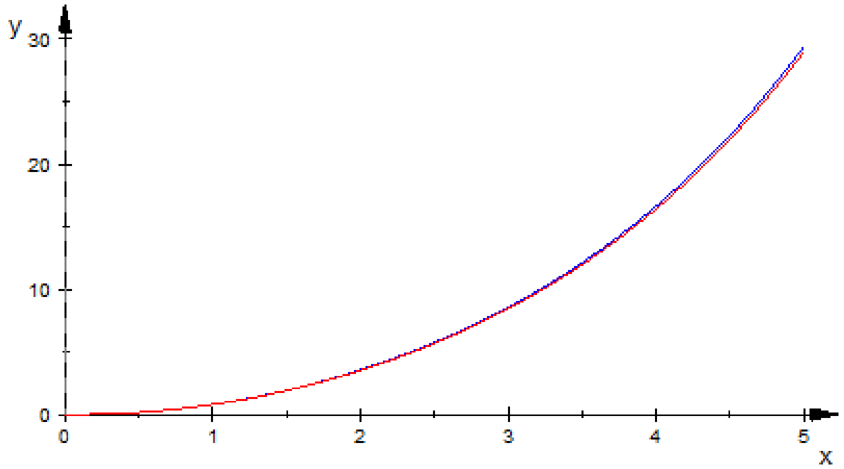
$$\frac{x^3 \cdot \ln(x)^2}{6} - \frac{11 \cdot x^3 \cdot \ln(x)}{18} + \frac{85 \cdot x^3}{108}$$

Right: Series

```
• n:=3: m:=1000: MAXDEPTH:=1000:
• f := ln(x)*(ln(x)-psi(1+n)-EULER)*x^n/gamma(1+n) +
        sum((-1)^(r-1)*binomial(-n,r)*(ln(x)-psi(1+n+r)-EULER)
                          *gamma(r)/gamma(1+n+r)*x^n, r=1..m-1):
```

Blue:Left, Red: Right

```
• plotfunc2d(g,f, x=0..5)
```



16.5 Higher Integral of e^x f (x)

16.5.1 Higher Integral of $e^x x^\alpha$

When $\Gamma(a,x)$ denotes the incomplete gamma function, the lineal higher primitive functions of $e^x x^\alpha$ can be expressed as follows.

$$\left(e^x x^\alpha\right)^{<1>} = \frac{x^\alpha}{(-x)^\alpha}\frac{\Gamma(1+\alpha,-x)x^0}{0!} \qquad \left\{= -x^{\alpha+1}E_{-\alpha}(-x)\right\}$$

$$\left(e^x x^\alpha\right)^{<2>} = \frac{x^\alpha}{(-x)^\alpha}\frac{\Gamma(2+\alpha,-x)x^0+\Gamma(1+\alpha,-x)x^1}{1!}$$

$$\left(e^x x^\alpha\right)^{<3>} = \frac{x^\alpha}{(-x)^\alpha}\frac{\Gamma(3+\alpha,-x)x^0+2\Gamma(2+\alpha,-x)x^1+\Gamma(1+\alpha,-x)x^2}{2!}$$

$$\vdots$$

$$\left(e^x x^\alpha\right)^{<n>} = \frac{x^\alpha}{(-x)^\alpha}\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha,\ -x)x^r}{(n-1)!}$$

Since these zeros are all $-\infty$ and these do not include constant-of-integration polynomial, surely these are the lineal higher primitive functions of $e^x x^\alpha$. And these are complex functions generally so that clearly from $(-x)^\alpha$.

Although it seems that the lineal higher primitive functions of $e^x x^\alpha$ can be immediately obtained from these, it does not go so. When these are calculated numerically, the sighn of the imaginary number part is contrary to the result of Riemann-Liouville integration at the time of $x>0$. That is, these are right at the time of $x \leqq 0$, and are not right at the time of $x>0$. So, I found the following formula that holds also at the time of $x>0$.

Formula 16.5.1

When $\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt$ denots the incomplete gamma function, the following expressions hold for $\alpha \ne -1,-2,-3,\cdots$.

(1) When $x \leqq 0$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} x^\alpha e^x\,dx^n = \frac{x^\alpha}{(-x)^\alpha}\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha,\ -x)x^r}{(n-1)!} \tag{1.n$^-$}$$

(2) When $x>0$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} x^\alpha e^x\,dx^n = \frac{x^\alpha}{(-x)^\alpha}\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha,\ -x)x^r}{(n-1)!}$$
$$+\ 2i\sin\alpha\pi\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha)x^r}{(n-1)!} \tag{1.n$^+$}$$

Proof

As mentioned above, the lineal higher primitive function of $e^x x^\alpha$ was as follows.

$$\left(e^x x^\alpha\right)^{<n>} = \frac{x^\alpha}{(-x)^\alpha}\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha,\ -x)x^r}{(n-1)!}$$

From this, when $x<0$, (1.n$^-$) holds immediately.

And when $\alpha \neq -1, -2, -3, \cdots$,

$$\frac{x^{\alpha}}{(-x)^{\alpha}} = \begin{cases} (-1)^{\alpha} & x \leqq 0 \\ (-1)^{-\alpha} & x > 0 \end{cases}$$

Then, (1.n$^-$) holds as the limit of $x \to -0$ also at the time of $x = 0$.

When $x > 0$, we calculate separately for $[-\infty, 0]$ and $[0, x]$. If the lineal higher integral is displayed separately by Riemann-Liouville Integral, it is as follows..

$$\frac{1}{\Gamma(n)} \int_{-\infty}^{x} (x-t)^{n-1} t^{\alpha} e^{t} dt$$

$$= \frac{1}{\Gamma(n)} \int_{-\infty}^{0} (x-t)^{n-1} t^{\alpha} e^{t} dt + \frac{1}{\Gamma(n)} \int_{0}^{x} (x-t)^{n-1} t^{\alpha} e^{t} dt \qquad \text{(w)}$$

Calculation of the 1st term

Since $x \leqq 0$, then $x^{\alpha}/(-x)^{\alpha} = (-1)^{\alpha}$. So, the 1st,

$$\int_{-\infty}^{0} (x-t)^{n-1} t^{\alpha} e^{t} dt$$

$$= \left[(x-t)^{n-1} \left(t^{\alpha} e^{t}\right)^{<1>} \right]_{-\infty}^{0} + (n-1) \int_{-\infty}^{0} \left\{ (x-t)^{n-2} \left(t^{\alpha} e^{t}\right)^{<1>} \right\} dt$$

$$= \left[(x-t)^{n-1} \frac{t^{\alpha}}{(-t)^{\alpha}} \frac{t^{0}\, \Gamma(1+\alpha, -t)}{0!} \right]_{-\infty}^{0} + (n-1) \int_{-\infty}^{0} \left\{ (x-t)^{n-2} \left(t^{\alpha} e^{t}\right)^{<1>} \right\} dt$$

$$= (-1)^{\alpha} x^{n-1} \frac{\Gamma(1+\alpha)}{0!} + (n-1) \int_{-\infty}^{0} \left\{ (x-t)^{n-2} \left(t^{\alpha} e^{t}\right)^{<1>} \right\} dt$$

the 2nd,

$$\int_{-\infty}^{0} \left\{ (x-t)^{n-2} \left(t^{\alpha} e^{t}\right)^{<1>} \right\} dt$$

$$= \left[(x-t)^{n-2} \left(t^{\alpha} e^{t}\right)^{<2>} \right]_{-\infty}^{0} + (n-2) \int_{-\infty}^{0} \left\{ (x-t)^{n-3} \left(t^{\alpha} e^{t}\right)^{<2>} \right\} dt$$

$$= (-1)^{\alpha} x^{n-2} \frac{\Gamma(2+\alpha)}{1!} + (n-2) \int_{-\infty}^{0} \left\{ (x-t)^{n-3} \left(t^{\alpha} e^{t}\right)^{<2>} \right\} dt$$

the 3rd,,

$$\int_{-\infty}^{0} \left\{ (x-t)^{n-3} \left(t^{\alpha} e^{t}\right)^{<2>} \right\} dt$$

$$= (-1)^{\alpha} x^{n-3} \frac{\Gamma(3+\alpha)}{2!} + (n-3) \int_{-\infty}^{0} \left\{ (x-t)^{n-4} \left(t^{\alpha} e^{t}\right)^{<3>} \right\} dt$$

$$\vdots$$

the last (n-1)th,

$$\int_{-\infty}^{0} \left\{ (x-t)^{1} \left(t^{\alpha} e^{t}\right)^{<n-1>} \right\} dt$$

$$= (-1)^{\alpha} x^{1} \frac{\Gamma(n-1+\alpha)}{(n-2)!} + 1 \cdot \int_{-\infty}^{0} \left\{ (x-t)^{0} \left(t^{\alpha} e^{t}\right)^{<n-1>} \right\} dt$$

Substituting these for the former expression one by one, we obtain the following expression.

$$\int_{-\infty}^{0}(x-t)^{n-1}t^{\alpha}e^{t}\,dt=(-1)^{\alpha}\sum_{r=1}^{n-1}\frac{(n-1)!}{r!(n-1-r)!}\Gamma(n-r+\alpha)x^{r}$$

$$+(n-1)!\int_{-\infty}^{0}\left\{(x-t)^{0}\left(t^{\alpha}e^{t}\right)^{<n-1>}\right\}dt$$

Thus,

$$\frac{1}{\Gamma(n)}\int_{-\infty}^{0}(x-t)^{n-1}t^{\alpha}e^{t}\,dt=(-1)^{\alpha}\sum_{r=1}^{n-1}{}_{n-1}C_{r}\,\frac{\Gamma(n-r+\alpha)x^{r}}{(n-1)!}$$

$$+\int_{-\infty}^{0}\left(t^{\alpha}e^{t}\right)^{<n-1>}dt \qquad \text{(w1)}$$

Calculation of the 2nd term

Since $x>0$, then $x^{\alpha}/(-x)^{\alpha}=1/(-1)^{\alpha}$. So, the 1st,

$$\int_{0}^{x}(x-t)^{n-1}t^{\alpha}e^{t}\,dt$$

$$=\left[(x-t)^{n-1}\left(t^{\alpha}e^{t}\right)^{<1>}\right]_{0}^{x}+(n-1)\int_{0}^{x}\left\{(x-t)^{n-2}\left(t^{\alpha}e^{t}\right)^{<1>}\right\}dt$$

$$=\left[(x-t)^{n-1}\frac{t^{\alpha}}{(-t)^{\alpha}}\frac{t^{0}\,\Gamma(1+\alpha,-t)}{0!}\right]_{0}^{x}+(n-1)\int_{0}^{x}\left\{(x-t)^{n-2}\left(t^{\alpha}e^{t}\right)^{<1>}\right\}dt$$

$$=-\frac{x^{n-1}}{(-1)^{\alpha}}\frac{\Gamma(1+\alpha)}{0!}+(n-1)\int_{0}^{x}\left\{(x-t)^{n-2}\left(t^{\alpha}e^{t}\right)^{<1>}\right\}dt$$

2nd,

$$\int_{0}^{x}\left\{(x-t)^{n-2}\left(t^{\alpha}e^{t}\right)^{<1>}\right\}dt$$

$$=\left[(x-t)^{n-2}\left(t^{\alpha}e^{t}\right)^{<2>}\right]_{0}^{x}+(n-2)\int_{0}^{x}\left\{(x-t)^{n-3}\left(t^{\alpha}e^{t}\right)^{<2>}\right\}dt$$

$$=-\frac{x^{n-2}}{(-1)^{\alpha}}\frac{\Gamma(2+\alpha)}{1!}+(n-2)\int_{0}^{x}\left\{(x-t)^{n-3}\left(t^{\alpha}e^{t}\right)^{<2>}\right\}dt$$

⋮

the last (n-1)th,

$$\int_{0}^{x}\left\{(x-t)^{1}\left(t^{\alpha}e^{t}\right)^{<n-1>}\right\}dt$$

$$=-\frac{x^{1}}{(-1)^{\alpha}}\frac{\Gamma(n-1+\alpha)}{(n-2)!}+1\cdot\int_{0}^{x}\left\{(x-t)^{0}\left(t^{\alpha}e^{t}\right)^{<n-1>}\right\}dt$$

Substituting these for the former expression one by one, we obtain the following expression.

$$\int_{0}^{x}(x-t)^{n-1}t^{\alpha}e^{t}\,dt=-\frac{1}{(-1)^{\alpha}}\sum_{r=1}^{n-1}\frac{(n-1)!}{r!(n-1-r)!}\Gamma(n-r+\alpha)x^{r}$$

$$+(n-1)!\int_{0}^{x}\left\{(x-t)^{0}\left(t^{\alpha}e^{t}\right)^{<n-1>}\right\}dt$$

Thus,

$$\frac{1}{\Gamma(n)}\int_0^x (x-t)^{n-1}t^{\alpha}e^t dt = -\frac{1}{(-1)^{\alpha}}\sum_{r=1}^{n-1}{}_{n-1}C_r\frac{\Gamma(n-r+\alpha)x^r}{(n-1)!}$$

$$+\int_0^x \left(t^{\alpha}e^t\right)^{<n-1>}dt \qquad \text{(w2)}$$

Total

Substituting (w1) , (w2) for (w) ,

$$\frac{1}{\Gamma(n)}\int_{-\infty}^x (x-t)^{n-1}t^{\alpha}e^t dt = \left\{(-1)^{\alpha}-\frac{1}{(-1)^{\alpha}}\right\}\sum_{r=1}^{n-1}{}_{n-1}C_r\frac{\Gamma(n-r+\alpha)x^r}{(n-1)!}$$

$$+\int_{-\infty}^0 \left(t^{\alpha}e^t\right)^{<n-1>}dt + \int_0^x \left(t^{\alpha}e^t\right)^{<n-1>}dt$$

Here,

$$(-1)^{\alpha}-\frac{1}{(-1)^{\alpha}} = 2i\sin\alpha\pi \qquad \left\{=\frac{2\pi i}{\Gamma(\alpha)\Gamma(1-\alpha)}\right\}$$

$$\int_{-\infty}^0 \left(t^{\alpha}e^t\right)^{<n-1>}dt + \int_0^x \left(t^{\alpha}e^t\right)^{<n-1>}dt = \int_{-\infty}^x \left(t^{\alpha}e^t\right)^{<n-1>}dt = \left(x^{\alpha}e^x\right)^{<n>}$$

$$= \frac{x^{\alpha}}{(-x)^{\alpha}}\sum_{r=0}^{n-1}\frac{{}_{n-1}C_r\,\Gamma(n-r+\alpha,\,-x)x^r}{(n-1)!}$$

Substituting these for the above, we obtain $(1.n^+)$.

Example 2nd order integral of $e^x\sqrt{x}$

Let us substitute $\alpha=1/2$, $n=2$ for $(1.n^-)$, $(1.n^+)$ and calculate the value on the arbitrary two points $x=\pm 1.7$. Then, it is as follows.

Left: Riemann-Liouville integral

```
• a:=1/2: n:=2:
• G := x-> 1/gamma(n)*int((x-t)^(n-1)*t^a*E^t, t=-infinity..x)
```

$$x \to \frac{1}{\Gamma(n)}\cdot\int_{-\infty}^{x}(x-t)^{n-1}\cdot t^a\cdot E^t\,\mathrm{d}t$$

```
• float(G(-1.7)); float(G(1.7))
```

$0.3457294964\cdot \mathrm{i}$

$2.255551158 + 2.835926162\cdot \mathrm{i}$

Right: Higher int by incompleate gamma

x <= 0

```
• Fm := x-> x^a/(-x)^a*(1/(n-1)!)*sum(binomial(n-1,r)
                          *igamma(n-r+a,-x)*x^r, r=0..n-1)
```

$$x \to \frac{x^a}{(-x)^a}\cdot\frac{1}{(n-1)!}\cdot\left(\sum_{r=0}^{n-1}\binom{n-1}{r}\cdot\Gamma(n-r+a,\,-x)\cdot x^r\right)$$

```
• float(Fm(-1.7))
```

$0.3457294964\cdot \mathrm{i}$

x > 0

```
• Fp := x-> Fm(x)+2*I*sin(a*PI)/(n-1)!
              *sum(binomial(n-1,r)*gamma(n-r+a)*x^r, r=0..n-1)
```

$$x \to \mathrm{Fm}(x) + \frac{2 \cdot \mathrm{i} \cdot \sin(\pi \cdot a)}{(n-1)!} \cdot \left(\sum_{r=0}^{n-1} \binom{n-1}{r} \cdot \Gamma(n-r+a) \cdot x^r \right)$$

```
• float(Fp(1.7))
```

$$2.255551158 + 2.835926161 \cdot \mathrm{i}$$

When α is a natural number, we can differentiate x^{α} completly. And the following formula holds.

Formula 16.5.1'

When $m=0,1,2,\cdots$,

$$\int_{-\infty}^{x} \cdots \int_{-\infty}^{x} e^x x^m dx^n = e^x \sum_{r=0}^{m} \binom{-n}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \tag{1.n'}$$

Proof

Let $f(x) = e^x$ in the Theorem16.2.0 (0.2). Then, since $a_1 = a_2 = \cdots = a_n = -\infty$,

$$\int_{-\infty}^{x} \cdots \int_{-\infty}^{x} e^x x^m dx^n = e^x \sum_{r=0}^{m} \binom{-n}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r}$$

$$- e^{-\infty} \sum_{r=0}^{n-1} \sum_{s=0}^{m} \binom{-n+r}{s} \frac{\Gamma(1+m)}{\Gamma(1+m-s)} (-\infty)^{m-s} \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} dx^r$$

Since this 2nd line is 0 clearly, we obtain (1.n') immediately.

Example 5th order integral of $e^x x^4$

Substituting $m=4$, $n=5$ for (1.n'),

$$\int_{-\infty}^{x} \cdots \int_{-\infty}^{x} e^x x^4 dx^5 = e^x \sum_{r=0}^{4} \binom{-5}{r} \frac{\Gamma(5)}{\Gamma(5-r)} x^{4-r} = e^x \sum_{r=0}^{4} (-1)^r \binom{4+r}{4} \frac{4!}{(4-r)!} x^{4-r}$$

When the both sides are calculated by mathematical software, it is as follows.

Left: Riemann-Liouville integral

```
• m:=4: n:=5:
• g := 1/gamma(n)*int((x-t)^(n-1)*t^m*E^t, t=-infinity..x)
```

$$\mathrm{e}^x \cdot \left(x^4 - 20 \cdot x^3 + 180 \cdot x^2 - 840 \cdot x + 1680\right)$$

Right: Formula

```
• f := E^x*sum(binomial(-n,r)*gamma(1+m)/gamma(1+m-r)
                                         *x^(m-r), r=0..m)
```

$$\mathrm{e}^x \cdot \left(x^4 - 20 \cdot x^3 + 180 \cdot x^2 - 840 \cdot x + 1680\right)$$

Collateral higher integral of $e^x x^{\alpha}$

About the collateral higher integral whose zero is 0, the following formula holds from the Formula 16.2.0 (0.5) .

Formula 16.5.1"

The following expressions hold for α, n such that $\alpha \neq -1, -2, -3, \cdots$ & $\alpha+n>0$.

$$\int_0^x \cdots \int_0^x e^x x^\alpha dx^n = e^x \sum_{r=0}^{m-1} \binom{-n}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)} x^{\alpha+n+r} + R_m^n \tag{1.n''}$$

$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} \int_0^x \cdots \int_0^x e^x x^{\alpha+m+k} dx^n$$

$$\lim_{m \to \infty} R_m^n = 0$$

Example Collateral the 2nd order integral of $e^x \sqrt[3]{x}$

This function can be illustrated in a positive domain. Blue (Riemann-Liouville integral) hidden from red (Series) cannot be seen. Although this turns into complex function in a negative domain, of course, both sides are corresponding.

Left: Riemann-Liouville integral

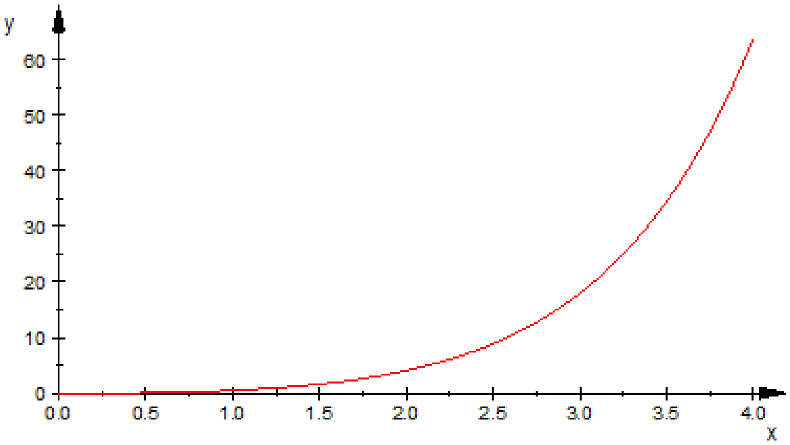
```
• a:=1/3: n:=2:
• g := x-> 1/gamma(n)*int((x-t)^(n-1)*t^a*E^t, t=0..x):
```

Right: Series

```
• m:=20:
• f := x-> E^x*sum(binomial(-n,r)*gamma(1+a)/gamma(1+a+n+r)
                                   *x^(a+n+r), r=0..m-1):
```

Blue: Left, Red: Right

```
• plotfunc2d(g(x),f(x), x=0..4, ViewingBoxYMin=0)
```



16.5.2 Higher Integral of $e^x \log x$

Formula 16.5.2

When $Ei(x) = \int_{-\infty}^x \frac{e^t}{t} dt$ is the Exponential Integral, the following expressions hold.

$$\int_{-\infty}^x e^x \log x \, dx = e^x \log x - Ei(x)$$

$$\int_{-\infty}^x \int_{-\infty}^x e^x \log x \, dx^2 = e^x \left\{ \log x + \frac{0!}{1!} x^0 \right\} - \left(\frac{x^0}{0!} + \frac{x^1}{1!} \right) Ei(x)$$

$$\int_{-\infty}^{x}\int_{-\infty}^{x}\int_{-\infty}^{x} e^x \log x\, dx^3 = e^x\left\{\log x + \frac{0!}{2!}x^1 + \left(\frac{0!}{1!}+\frac{1!}{2!}\right)x^0\right\} - \left(\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}\right)Ei(x)$$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \log x\, dx^4 = e^x\left\{\log x + \frac{0!}{3!}x^2 + \left(\frac{0!}{2!}+\frac{1!}{3!}\right)x^1 + \left(\frac{0!}{1!}+\frac{1!}{2!}+\frac{2!}{3!}\right)x^0\right\} - \left(\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)Ei(x)$$

$\vdots$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \log x\, dx^n = e^x\left\{\log x + \sum_{r=0}^{n-2}\sum_{s=0}^{n-2-r}\frac{s!\,x^r}{(r+s+1)!}\right\} - Ei(x)\sum_{r=0}^{n-1}\frac{x^r}{r!} \tag{2.n}$$

Proof

When $Ei(x) = \int_{-\infty}^{x}\frac{e^t}{t}dt$, lineal higher primitive functions of $e^x \log x$ are as follows.

$$\left(e^x \log x\right)^{<1>} = e^x \log x - Ei(x)$$

$$\left(e^x \log x\right)^{<2>} = e^x(\log x + 1) - (x+1)Ei(x)$$

$$\left(e^x \log x\right)^{<3>} = e^x\left(\log x + \frac{x+3}{2}\right) - \frac{x^2+2x+2}{2}Ei(x)$$

$$\left(e^x \log x\right)^{<4>} = e^x\left(\log x + \frac{x^2+4x+11}{3!}\right) - \frac{x^3+3x^2+6x+6}{6}Ei(x)$$

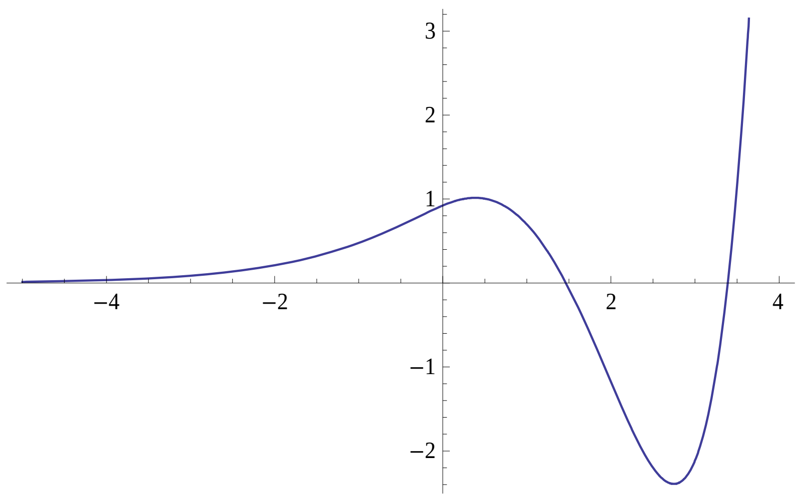
$\vdots$

And the zeros of these are all $x = -\infty$. Therefore, 1st ~ 4th order integrals can be written as mentioned above. And hereafter, by induction, we obtain (2.n) .

Example 3rd order integral of $e^x \log x$

$$F[n\_] := e^x\left(\text{Log}[\text{Abs}[x]] + \sum_{r=0}^{n-2}\sum_{s=0}^{n-2-r}\frac{s!\,x^r}{(r+s+1)!}\right) - \text{ExpIntegralEi}[x]\sum_{r=0}^{n-1}\frac{x^r}{r!}$$

```
Plot [F[3], {x, -5, 4}]
```



Note

All polynomials obtained by applying Theorem 16.1.2 to $e^x \log x$ become the asymptotic expansions, and they are hardly useful.

16.5.3 Higher Integrals of $e^x \sin x$, $e^x \cos x$

Formula 16.5.3

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \sin x\, dx^n = \left(\sin\frac{\pi}{4}\right)^n e^x \sin\left(x-\frac{n\pi}{4}\right) \tag{3.0s}$$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \cos x\, dx^n = \left(\sin\frac{\pi}{4}\right)^n e^x \cos\left(x-\frac{n\pi}{4}\right) \tag{3.0c}$$

Proof

About the higher derivative of $e^x \sin x$, the following formula is known.(See 共立 数学公式 p187).

$$\left(e^x \sin x\right)^{(n)} = \left(\sin\frac{\pi}{4}\right)^{-n} e^x \sin\left(x+\frac{n\pi}{4}\right)$$

Replacing n with $-n$,

$$\left(e^x \sin x\right)^{(-n)} = \left(\sin\frac{\pi}{4}\right)^{n} e^x \sin\left(x-\frac{n\pi}{4}\right)$$

Since a differentiation operator $(-n)$ is equal to an integration operator $<n>$,

$$\left(e^x \sin x\right)^{<n>} = \left(\sin\frac{\pi}{4}\right)^{n} e^x \sin\left(x-\frac{n\pi}{4}\right)$$

And since $x=-\infty$ is zero of this clearly, rewriting the left side, we obtain (3.0s).

The following formula holds also about $e^x \cos x$.

$$\left(e^x \cos x\right)^{(n)} = \left(\sin\frac{\pi}{4}\right)^{-n} e^x \cos\left(x+\frac{n\pi}{4}\right)$$

From this, we obtain (3.0c) in a similar way.

Example

$$\int_{-\infty}^{x} e^x \sin x\, dx = \left(\sin\frac{\pi}{4}\right) e^x \sin\left(x-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} e^x \frac{\sin x - \cos x}{\sqrt{2}}$$

$$= \frac{1}{2} e^x (\sin x - \cos x)$$

$$\int_{-\infty}^{x}\int_{-\infty}^{x} e^x \sin x\, dx^2 = \left(\sin\frac{\pi}{4}\right)^2 e^x \sin\left(x-\frac{2\pi}{4}\right) = -\frac{1}{2} e^x \cos x$$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \cos x\, dx^3 = \left(\sin\frac{\pi}{4}\right)^3 e^x \cos\left(x-\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{4} e^x \sin\left(x-\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{4} e^x \frac{\sin x - \cos x}{\sqrt{2}} = \frac{1}{4} e^x (\sin x - \cos x)$$

Higher Integral of $e^x \sin x$, $e^x \cos x$ ends now. There is no necessity for Theorem 16.1.2. However, daring use Theorem 16.1.2, we obtain an interesting result.

Formula 16.5.3'

(1)

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \sin x\, dx^n = e^x \sum_{r=0}^{m-1}\binom{-n}{r}\sin\left(x+\frac{r\pi}{2}\right) + R_m^n \tag{3.1}$$

$$R_m^n = \frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \sin\left\{x+\frac{(m+k)\pi}{2}\right\}dx^n \tag{3.1r}$$

(2)

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \sin x\, dx^n = e^x \sum_{r=0}^{m-1}\binom{-n}{r}\sin\left\{x-\frac{(n+r)\pi}{2}\right\} + R_m^n \tag{3.2}$$

$$R_m^n = \frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \sin\left\{x-\frac{(m+k)\pi}{2}\right\}dx^n \tag{3.2r}$$

(3)

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \cos x\, dx^n = e^x \sum_{r=0}^{m-1}\binom{-n}{r}\cos\left(x+\frac{r\pi}{2}\right) + R_m^n \tag{3.3}$$

$$R_m^n = \frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \cos\left\{x+\frac{(m+k)\pi}{2}\right\}dx^n \tag{3.3r}$$

(4)

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \cos x\, dx^n = e^x \sum_{r=0}^{m-1}\binom{-n}{r}\cos\left\{x-\frac{(n+r)\pi}{2}\right\} + R_m^n \tag{3.4}$$

$$R_m^n = \frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \cos\left\{x-\frac{(m+k)\pi}{2}\right\}dx^n \tag{3.4r}$$

Calculation

Let $f(x)=e^x,\ g(x)=\sin x$. Then

$$(\sin x)^{(r)} = \sin\left(x+\frac{r\pi}{2}\right)$$

Substituting this for (2.3) in Theorem 16.1.2 , we obtain (3.1), (3.1r).

Next, let $f(x)=\sin x$, $g(x)=e^x$. Then

$$(\sin x)^{<n+r>} = \sin\left\{x-\frac{(n+r)\pi}{2}\right\}$$

Substituting this for (2.3) in Theorem 16.1.2, we obtain (3.2), (3.2r).

Also about $e^x\cos x$, in a similar way, we obtain (3.3) ~ (3.4r).

Example 2nd order integral of $e^x\sin x$

Let $n=2, m=10$ in (3.1), (3.1r) . And calculating the function value on the arbitrary point $x=3$ by mathematical software, it is as follows.

Left: Formula 16.5.3'

```
• n:=2:
• g := (sin(PI/4))^n*E^x*sin(x-n*PI/4):
• float(subs(g, x=3))
      9.942265422
```

Series

```
• m:=100:
• f := E^x*sum(binomial(-n,r)*sin(x+r*PI/2), r=0..m-1):
• float(subs(f, x=3))
```

-1135.950099

Remainder

```
• r0 := -1/(m+0)*binomial(n-1,0)*E^x/2*cos(x+(m+0)*PI/2):
• r1 := -1/(m+1)*binomial(n-1,1)*E^x/2*cos(x+(m+1)*PI/2):
• R := (-1)^m/beta(n,m)*(r0+r1):
• float(subs(R, x=3))
```

1145.892364

Series + Remainder

```
• float(subs(f+R, x=3))
```

9.942265422

Complete Automorphism

As understood from this example, although (3.1), (3.1r) hold as an equation, it is not helpful at all. The same is said for (3.2), (3.2r). However, if these are used in combination, the complete automorphism of the higher integral of $e^x \sin x$ is drawn as follows. This is the same for $e^x \cos x$

Formula 16.5.3"

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \sin x\, dx^n = \frac{e^x \sin\left(x-\frac{n\pi}{4}\right)\cos\frac{n\pi}{4}}{1+n\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{1+k}\cos\frac{(1+k)\pi}{2}} \tag{3.5}$$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \cos x\, dx^n = \frac{e^x \cos\left(x-\frac{n\pi}{4}\right)\sin\frac{n\pi}{4}}{n\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{1+k}\sin\frac{(1+k)\pi}{2}} \tag{3.6}$$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \cos x\, dx^n = \frac{e^x \cos\left(x-\frac{n\pi}{4}\right)\cos\frac{n\pi}{4}}{1+n\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{1+k}\cos\frac{(1+k)\pi}{2}} \tag{3.7}$$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \sin x\, dx^n = \frac{e^x \sin\left(x-\frac{n\pi}{4}\right)\sin\frac{n\pi}{4}}{n\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{1+k}\sin\frac{(1+k)\pi}{2}} \tag{3.8}$$

Calculation

Let $m=1$ in (3.1) ~ (3.2r). Since $B(n,1)=1/n$,

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \sin x\, dx^n = e^x \sin x - n\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{1+k}\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x \sin\left\{x+\frac{(1+k)\pi}{2}\right\}dx^n$$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin x\,dx^{n}=e^{x}\sin\left(x-\frac{n\pi}{2}\right)\;-n\sum_{k=0}^{n-1}\frac{{}_{n-1}C_{k}}{1+k}\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin\left\{x-\frac{(1+k)\pi}{2}\right\}dx^{n}$$

Here, let $C_k=\dfrac{{}_{n-1}C_k}{1+k}$, $B_k=\dfrac{(1+k)\pi}{2}$. Then

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin x\,dx^{n}=e^{x}\sin x\;-n\sum_{k=0}^{n-1}C_k\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin(x+B_k)dx^{n}$$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin x\,dx^{n}=e^{x}\sin\left(x-\frac{n\pi}{2}\right)\;-n\sum_{k=0}^{n-1}C_k\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin(x-B_k)dx^{n}$$

Using the sum and difference formulas,

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin x\,dx^{n}=e^{x}\sin x\;-n\sum_{k=0}^{n-1}C_k\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}(\sin x\cos B_k+\cos x\sin B_k)dx^{n} \qquad \text{(a)}$$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin x\,dx^{n}=e^{x}\sin\left(x-\frac{n\pi}{2}\right)-n\sum_{k=0}^{n-1}C_k\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}(\sin x\cos B_k-\cos x\sin B_k)dx^{n} \qquad \text{(b)}$$

Adding (a) to (b),

$$2\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin x\,dx^{n}=e^{x}\left\{\sin x+\sin\left(x-\frac{n\pi}{2}\right)\right\}-2n\sum_{k=0}^{n-1}C_k\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin x\cos B_k dx^{n}$$

$$=2e^{x}\sin\left(x-\frac{n\pi}{4}\right)\cos\frac{n\pi}{4}\;-\;2n\sum_{k=0}^{n-1}C_k\cos B_k\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin x\,dx^{n}$$

From this,

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin x\,dx^{n}=\frac{e^{x}\sin\left(x-\frac{n\pi}{4}\right)\cos\frac{n\pi}{4}}{1+n\sum_{k=0}^{n-1}C_k\cos B_k}$$

Returning C_k , B_k before,

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin x\,dx^{n}=\frac{e^{x}\sin\left(x-\frac{n\pi}{4}\right)\cos\frac{n\pi}{4}}{1+n\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{1+k}\cos\frac{(1+k)\pi}{2}} \qquad (3.5)$$

Next, subtract (b) from (a) . Then

$$0=e^{x}\left\{\sin x dx^{n}-\sin\left(x-\frac{n\pi}{2}\right)\right\}\;-2n\sum_{k=0}^{n-1}C_k\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\cos x\sin B_k\,dx^{n}$$

$$=2\cos\left(x-\frac{n\pi}{4}\right)\sin\frac{n\pi}{4}\;-2n\sum_{k=0}^{n-1}C_k\sin B_k\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\cos x\,dx^{n}$$

From this,

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\cos x\,dx^{n}=\frac{e^{x}\cos\left(x-\frac{n\pi}{4}\right)\sin\frac{n\pi}{4}}{n\sum_{k=0}^{n-1}C_k\sin B_k}$$

Returning C_k , B_k before,

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\cos x\,dx^{n}=\frac{e^{x}\cos\left(x-\frac{n\pi}{4}\right)\sin\frac{n\pi}{4}}{n\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{1+k}\sin\frac{(1+k)\pi}{2}} \qquad (3.6)$$

Also about $e^x cos x$, in a similar way, we obtain (3.7), (3.8).

Example When n=2

$$\int_{-\infty}^{x}\int_{-\infty}^{x} e^x sin x dx^2 = \frac{e^x sin\left(x-\frac{\pi}{2}\right) cos\frac{\pi}{2}}{1+2\sum_{k=0}^{1}\frac{{}_1C_k}{k+1}cos\frac{(1+k)\pi}{2}} = \frac{e^x sin\left(x-\frac{\pi}{2}\right) cos\frac{\pi}{2}}{1+2\left\{\frac{1}{1}cos\frac{\pi}{2}+\frac{1}{2}cos\frac{2\pi}{2}\right\}}$$

$$= \frac{e^x sin\left(x-\frac{\pi}{2}\right) cos\frac{\pi}{2}}{1+2\left\{\frac{1}{1}cos\frac{\pi}{2}-\frac{1}{2}\right\}} = \frac{e^x sin\left(x-\frac{\pi}{2}\right)}{2} = -\frac{e^x cos x}{2}$$

$$\int_{-\infty}^{x}\int_{-\infty}^{x} e^x cos x\, dx^2 = \frac{e^x cos\left(x-\frac{\pi}{2}\right) sin\frac{\pi}{2}}{2\sum_{k=0}^{1}\frac{{}_1C_k}{1+k}sin\frac{(1+k)\pi}{2}} = \frac{e^x cos\left(x-\frac{\pi}{2}\right) sin\frac{\pi}{2}}{2\left\{\frac{1}{1}sin\frac{\pi}{2}+\frac{1}{2}sin\frac{2\pi}{2}\right\}}$$

$$= \frac{e^x cos\left(x-\frac{\pi}{2}\right) sin\frac{\pi}{2}}{2sin\frac{\pi}{2}} = \frac{e^x cos\left(x-\frac{\pi}{2}\right)}{2} = \frac{e^x sin x}{2}$$

16.5.4 Trigonometric Polynomial and Binomial Polynomial

(1) Trigonometric Polynomial

Although Formla 16.5.3" is helpful, it is fearfully complicated and cannot be compared to Formula 16.5.3. However, if we dare compare this with Formula16.5.3 , the following Trigonometric Polynomial is obtained.

Formula 16.5.4

$$\sum_{k=1}^{n}\frac{{}_{n-1}C_{k-1}}{k}sin\frac{k\pi}{2} = \frac{1}{n}\left(sin\frac{\pi}{4}\right)^{-n} sin\frac{n\pi}{4} \tag{4.1}$$

$$\sum_{k=1}^{n}\frac{{}_{n-1}C_{k-1}}{k}cos\frac{k\pi}{2} = \frac{1}{n}\left(sin\frac{\pi}{4}\right)^{-n} cos\frac{n\pi}{4} - \frac{1}{n} \tag{4.2}$$

Proof

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x cos x dx^n = \left(sin\frac{\pi}{4}\right)^n e^x cos\left(x-\frac{n\pi}{4}\right) \tag{3.0c}$$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x cos x\, dx^n = \frac{e^x cos\left(x-\frac{n\pi}{4}\right) sin\frac{n\pi}{4}}{n\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{1+k}sin\frac{(1+k)\pi}{2}} \tag{3.6}$$

From these,

$$\left(sin\frac{\pi}{4}\right)^n = sin\frac{n\pi}{4} \Big/ \left\{n\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{1+k}sin\frac{(1+k)\pi}{2}\right\}$$

Replacing k with $k-1$, we obtain (4.1).

Next,

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin x\,dx^{n} = \left(\sin\frac{\pi}{4}\right)^{n} e^{x}\sin\left(x-\frac{n\pi}{4}\right) \tag{3.0s}$$

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^{x}\sin x\,dx^{n} = \frac{e^{x}\sin\left(x-\frac{n\pi}{4}\right)\cos\frac{n\pi}{4}}{1+n\sum_{k=0}^{n-1}\frac{{}_{n-1}C_{k}}{1+k}\cos\frac{(1+k)\pi}{2}} \tag{3.5}$$

From these,

$$\left(\sin\frac{\pi}{4}\right)^{n} = \cos\frac{n\pi}{4} \Bigg/ \left\{1+n\sum_{k=0}^{n-1}\frac{{}_{n-1}C_{k}}{1+k}\cos\frac{k\pi}{2}\right\}$$

Replacing k with $k-1$, we obtain (4.2).

(2) Binomial Polynomial

However, Formula 16.5.4 is tedious for a natural number n and is not interesting. Then removing $\sin\frac{k\pi}{2}, \cos\frac{k\pi}{2}$ from the formula, we obtain the following interesting polynomials.

Formula 16.5.4'

When $\downarrow$, $\uparrow$ denote a floor function and a ceiling function respectivly,

$$\sum_{k=0}^{n/2\downarrow}\frac{(-1)^{k}}{2k+1}{}_{n}C_{2k} = \frac{1}{n+1}\left(\sin\frac{\pi}{4}\right)^{-n-1}\sin\frac{(n+1)\pi}{4} \tag{4.3}$$

$$\sum_{k=1}^{n/2\uparrow}\frac{(-1)^{k}}{2k}{}_{n}C_{2k-1} = \frac{1}{n+1}\left\{\left(\sin\frac{\pi}{4}\right)^{-n-1}\cos\frac{(n+1)\pi}{4} - 1\right\} \tag{4.4}$$

Proof

Replacing n,k with $n+1,k+1$ in (4.1) ,

$$\sum_{k=0}^{n}\frac{{}_{n}C_{k}}{k+1}\sin\frac{(k+1)\pi}{2} = \frac{1}{n+1}\left(\sin\frac{\pi}{4}\right)^{-n-1}\sin\frac{(n+1)\pi}{4}$$

Since the even-numbered terms of the left side are all 0,

$$\sum_{k=0}^{n}\frac{{}_{n}C_{k}}{k+1}\sin\frac{(k+1)\pi}{2} = \frac{{}_{n}C_{0}}{1}-\frac{{}_{n}C_{2}}{3}+\frac{{}_{n}C_{4}}{5}-+\cdots+\frac{(-1)^{k}}{2k+1}{}_{n}C_{2k} \qquad 2k\leqq n$$

$$= \sum_{k=0}^{n/2\downarrow}\frac{(-1)^{k}}{2k+1}{}_{n}C_{2k}$$

$$\therefore \quad \sum_{k=0}^{n/2\downarrow}\frac{(-1)^{k}}{2k+1}{}_{n}C_{2k} = \frac{1}{n+1}\left(\sin\frac{\pi}{4}\right)^{-n-1}\sin\frac{(n+1)\pi}{4} \tag{4.3}$$

Replacing n,k with $n+1,k+1$ in (4.2) ,

$$\sum_{k=0}^{n}\frac{{}_{n}C_{k}}{k+1}\cos\frac{(k+1)\pi}{2} = \frac{1}{n+1}\left\{\left(\sin\frac{\pi}{4}\right)^{-n-1}\cos\frac{(n+1)\pi}{4} - 1\right\}$$

Since the odd-numbered terms of the left side are all 0,

$$\sum_{k=0}^{n}\frac{{}_{n}C_{k}}{k+1}\cos\frac{(k+1)\pi}{2} = -\frac{{}_{n}C_{1}}{2}+\frac{{}_{n}C_{3}}{4}-\frac{{}_{n}C_{5}}{6}+-\cdots+\frac{(-1)^{k}}{2k}{}_{n}C_{2k-1} \quad 2k-1\leqq n$$

$$= \sum_{k=1}^{n/2\uparrow} \frac{(-1)^k}{2k} {}_nC_{2k-1}$$

$$\therefore \quad \sum_{k=1}^{n/2\uparrow} \frac{(-1)^k}{2k} {}_nC_{2k-1} = \frac{1}{n+1} \left\{ \left(sin\frac{\pi}{4} \right)^{-n-1} cos\frac{(n+1)\pi}{4} - 1 \right\} \tag{4.4}$$

Example

$$\frac{{}_9C_0}{1} - \frac{{}_9C_2}{3} + \frac{{}_9C_4}{5} - \frac{{}_9C_6}{7} + \frac{{}_9C_8}{9} = \frac{1}{9+1} \left(sin\frac{\pi}{4} \right)^{-9-1} sin\frac{(9+1)\pi}{4} = \frac{16}{5}$$

$$\frac{{}_8C_0}{1} - \frac{{}_8C_2}{3} + \frac{{}_8C_4}{5} - \frac{{}_8C_6}{7} + \frac{{}_8C_8}{9} = \frac{1}{8+1} \left(sin\frac{\pi}{4} \right)^{-8-1} sin\frac{(8+1)\pi}{4} = \frac{16}{9}$$

$$-\frac{{}_7C_1}{2} + \frac{{}_7C_3}{4} - \frac{{}_7C_5}{6} + \frac{{}_7C_7}{8} = \frac{1}{7+1} \left\{ \left(sin\frac{\pi}{4} \right)^{-7-1} cos\frac{(7+1)\pi}{4} - 1 \right\} = \frac{15}{8}$$

$$-\frac{{}_8C_1}{2} + \frac{{}_8C_3}{4} - \frac{{}_8C_5}{6} + \frac{{}_8C_7}{8} = \frac{1}{8+1} \left\{ \left(sin\frac{\pi}{4} \right)^{-8-1} cos\frac{(8+1)\pi}{4} - 1 \right\} = \frac{5}{3}$$

16.6 Higher Integral of f (x) / e^x

16.6.1 Higher Integral of $e^{-x}x^{\alpha}$

Formula 16.6.1

When $\Gamma(a,x)=\int_x^{\infty} t^{a-1}e^{-t}dt$ is the incomplete gamma function,

$$\int_{\infty}^{x} e^{-x}x^{\alpha}dx = -\frac{1}{0!}\Gamma(1+\alpha,x)$$

$$\int_{\infty}^{x}\int_{\infty}^{x} e^{-x}x^{\alpha}dx^2 = \frac{1}{1!}\{\Gamma(2+\alpha,x)-\Gamma(1+\alpha,x)x\}$$

$$\int_{\infty}^{x}\int_{\infty}^{x}\int_{\infty}^{x} e^{-x}x^{\alpha}dx^3 = -\frac{1}{2!}\{\Gamma(3+\alpha,x)-2\Gamma(2+\alpha,x)x+\Gamma(1+\alpha,x)x^2\}$$

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} e^{-x}x^{\alpha}dx^4 = \frac{1}{3!}\{\Gamma(4+\alpha,x)-3\Gamma(3+\alpha,x)x+3\Gamma(2+\alpha,x)x^2-\Gamma(1+\alpha,x)x^3\}$$

$$\vdots$$

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} e^{-x}x^{\alpha}dx^n = \frac{(-1)^n}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\,\Gamma(n-r+\alpha,x)x^r \qquad (1.n)$$

Proof

If the integration is obediently repeated, they are led naturally. And these zeros are all $x=\infty$. Then these are lineal higher integrals.

Example 2nd order integral of $e^{-x}\sqrt{x}$

Let substitute $\alpha=1/2$, $n=2$ for (1.n), and calculate the value on thearbitrary point $x=-2.3$. Then, it is as follows. Both sides are corresponding exactry.

Left: Riemann-Liouville integral

```
• a := 1/2: n:=2:
• g := x-> 1/gamma(n)*int((x-t)^(n-1)*t^a/E^t,t=infinity..x):
• float(g(-2.3))
```

3.367662317 + 6.637309509 · i

Right: Integral by incomplete gamma function

```
• f := x-> (-1)^n/(n-1)! *
      sum((-1)^r*binomial(n-1,r)*igamma(n-r+a,x)*x^r, r=0..n-1)
```

$$x \rightarrow \frac{(-1)^n}{(n-1)!}\cdot\left(\sum_{r=0}^{n-1}(-1)^r\cdot\binom{n-1}{r}\cdot\Gamma(n-r+a,\,x)\cdot x^r\right)$$

```
• float(f(-2.3))
```

3.367662317 + 6.637309509 · i

When α is a natural number, we can differentiate x^{α} completly. And the following formula holds.

Formula 16.6.1'

When $m=0,1,2,\cdots$,

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} e^{-x}x^m dx^n = \frac{(-1)^n}{e^x}\sum_{r=0}^{m}(-1)^r\binom{-n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r} \qquad (1.n')$$

Example 5th order integral of $e^{-x}x^4$

Substituting $m=4$, $n=5$ for (1.n') ,

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} e^{-x}x^4 dx^5 = \frac{(-1)^5}{e^x}\sum_{r=0}^{4}(-1)^r\binom{-5}{r}\frac{\Gamma(1+4)}{\Gamma(1+4-r)}x^{4-r}$$

When the both sides are calculated by mathematical software, it is as follows. Naturally, both sides are corresponding.

Left: Riemann-Liouville integral

```
• m:=4: n:=5:
• g := 1/gamma(n)*int((x-t)^(n-1)*t^m/E^t,t=infinity..x):
• expand(g)
```

$$-\frac{840\cdot x}{e^x}-\frac{180\cdot x^2}{e^x}-\frac{20\cdot x^3}{e^x}-\frac{x^4}{e^x}-\frac{1680}{e^x}$$

Right: Formula

```
• f := (-1)^n/E^x*sum((-1)^r*binomial(-n,r)*gamma(1+m)
                                /gamma(1+m-r)*x^(m-r), r=0..m):
• expand(f)
```

$$-\frac{180\cdot x^2}{e^x}-\frac{20\cdot x^3}{e^x}-\frac{x^4}{e^x}-\frac{1680}{e^x}-\frac{840\cdot x}{e^x}$$

Collateral higher integral of $e^{-x}x^{\alpha}$

The following formula holds about the collateral higher integral whose zero is 0.

Formula 16.6.1''

The following expressions hold for α, n such that $\alpha \neq -1,-2,-3,\cdots$ & $\alpha+n>0$.

$$\int_{0}^{x}\cdots\int_{0}^{x} e^{-x}x^{\alpha} dx^n = \frac{1}{e^x}\sum_{r=0}^{m-1}(-1)^{-r}\binom{-n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)}x^{\alpha+n+r} + R_m^n$$

$$R_m^n = \frac{1}{B(n,m)}\sum_{k=0}^{n-1}(-1)^{-k}\frac{{}_{n-1}C_k}{m+k}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)}\int_{0}^{x}\cdots\int_{0}^{x}\frac{x^{\alpha+m+k}}{e^x}dx^n$$

$$\lim_{m\to\infty} R_m^n = 0$$

Example Collateral the 2nd order integral of $e^{-x}\sqrt{x}$

This function can be illustrated in a positive domain. Blue (Riemann-Liouville integral) hidden from red (Series) cannot be seen. Although this turns into complex function in a negative domain, of course, both sides are corresponding.

Left: Riemann-Liouville integral

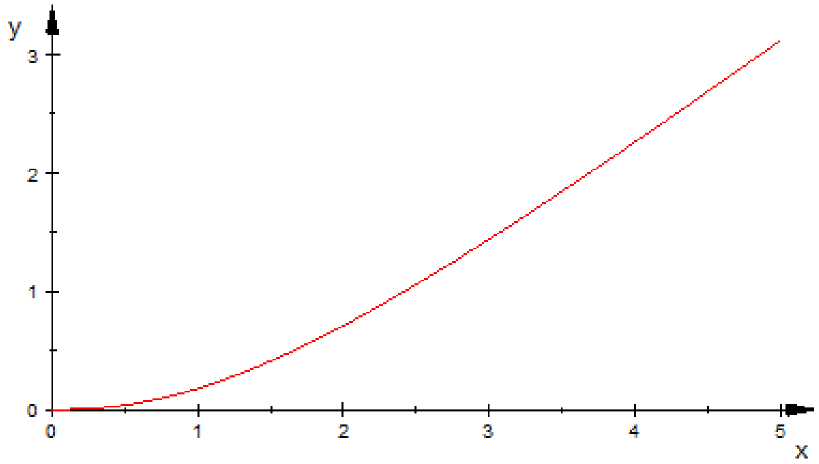
```
• a := 1/2: n:=2:
• g := x->1/gamma(n)*int((x-t)^(n-1)*t^a/E^t,t=0..x):
```

Right: Series

```
• m:=20:
• f := x->sum((-1)^-r*binomial(-n,r)*gamma(1+a)/gamma(1+a+n+r)
                                       *x^(a+n+r)/E^x, r=0..m-1):
```

Blue: Left , Red: Right

```
• plotfunc2d(g(x),f(x), x=0..5)
```



16.6.2 Higher Integral of $e^{-x}\log x$

Formula 16.6.2

When $Ei(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt$ is the Exponential Integral, the following expressions hold.

$$\int_{\infty}^{x} e^{-x}\log x\, dx = -e^{-x}\log x + Ei(-x)$$

$$\int_{\infty}^{x}\int_{\infty}^{x} e^{-x}\log x\, dx^2 = e^{-x}\left\{\log x + \frac{0!}{1!}x^0\right\} - \left(\frac{x^0}{0!}-\frac{x^1}{1!}\right)Ei(-x)$$

$$\int_{\infty}^{x}\int_{\infty}^{x}\int_{\infty}^{x} e^{-x}\log x\, dx^3 = -e^{-x}\left\{\log x - \frac{0!}{2!}x^1 + \left(\frac{0!}{1!}+\frac{1!}{2!}\right)x^0\right\}$$
$$+\left(\frac{x^0}{0!}-\frac{x^1}{1!}+\frac{x^2}{2!}\right)Ei(-x)$$

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} e^{-x}\log x\, dx^4 = e^{-x}\left\{\log x + \frac{0!}{3!}x^2 - \left(\frac{0!}{2!}+\frac{1!}{3!}\right)x^1 + \left(\frac{0!}{1!}+\frac{1!}{2!}+\frac{2!}{3!}\right)x^0\right\}$$
$$-\left(\frac{x^0}{0!}-\frac{x^1}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}\right)Ei(-x)$$

$$\vdots$$

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} e^{-x}\log x\, dx^n = (-1)^n e^{-x}\left\{\log x + \sum_{r=0}^{n-2}(-x)^r\sum_{s=0}^{n-2-r}\frac{s!}{(r+s+1)!}\right\}$$

$$-(-1)^n Ei(-x)\sum_{r=0}^{n-1}\frac{(-x)^r}{r!} \tag{2.n}$$

Proof

Lineal higher primitive functions of $e^{-x}\log x$ are as follows.

$$\left(e^{-x}\log x\right)^{<1>} = -e^{-x}\log x + Ei(-x)$$

$$\left(e^{-x}\log x\right)^{<2>} = e^{-x}(\log x+1) + (x-1)Ei(-x)$$

$$\left(e^{-x}\log x\right)^{<3>} = -e^{-x}\left(\log x-\frac{x}{2}+\frac{3}{2}\right) + \frac{x^2-2x+2}{2!}Ei(-x)$$

$$\left(e^{-x}\log x\right)^{<4>} = e^{-x}\left(\log x+\frac{x^2-4x}{3!}+\frac{11}{6}\right) + \frac{x^3-3x^2+6x-6}{3!}Ei(-x)$$

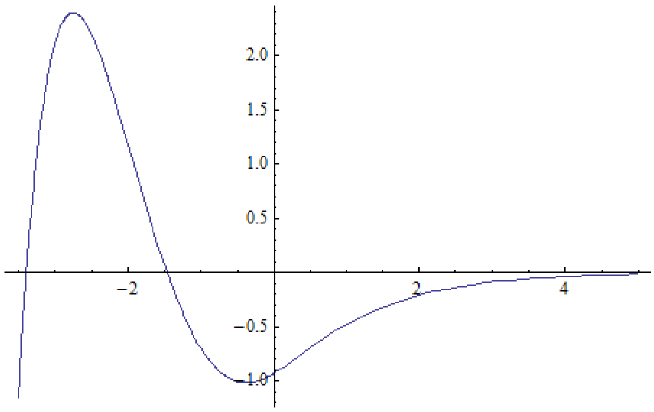
$$\vdots$$

And the zeros of these are all $x=\infty$. Therefore, 1st ~ 4th order integrals can be written as mentioned above. And hereafter, by induction, we obtain (2.n).

Example 3rd order integral of $e^{-x}\log x$

```
F[n_] := (-1)^n e^-x (Log[Abs[x]] + Sum[Sum[s! (-x)^r/(r + s + 1)!, {s, 0, n-2-r}], {r, 0, n-2}]) - (-1)^n ExpIntegralEi[-x] Sum[(-x)^r/r!, {r, 0, n-1}]

Plot[F[3], {x, -3.5, 5}]
```



Note

All polynomials obtained by applying Theorem 16.1.2 to $e^{-x}\log x$ become the asymptotic expansions, and they are hardly useful.

16.6.3 Higher Integrals of $e^{-x}\sin x$, $e^{-x}\cos x$

Formula 16.6.3

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} e^{-x}\sin x\,dx^n = (-1)^n\left(\sin\frac{\pi}{4}\right)^n e^{-x}\sin\left(x+\frac{n\pi}{4}\right) \tag{3.0s}$$

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} e^{-x}\cos x\,dx^n = (-1)^n\left(\sin\frac{\pi}{4}\right)^n e^{-x}\cos\left(x+\frac{n\pi}{4}\right) \tag{3.0c}$$

Proof

About the higher derivative of $e^x\sin x$, the following formula is known.(See 共立 数学公式 p187).

$$\left(e^x\sin x\right)^{(n)} = \left(\sin\frac{\pi}{4}\right)^{-n} e^x\sin\left(x+\frac{n\pi}{4}\right)$$

Replacing x with $-x$,

$$(-1)^n\left\{e^{-x}\sin(-x)\right\}^{(n)} = \left(\sin\frac{\pi}{4}\right)^{-n} e^{-x}\sin\left(-x+\frac{n\pi}{4}\right)$$

Using $\sin(-x) = -\sin x$

$$\left(e^{-x}\sin x\right)^{(n)} = (-1)^n\left(\sin\frac{\pi}{4}\right)^{-n} e^{-x}\sin\left(x-\frac{n\pi}{4}\right)$$

Replacing n with $-n$,

$$\left(e^{-x}\sin x\right)^{(-n)} = (-1)^n\left(\sin\frac{\pi}{4}\right)^{n} e^{-x}\sin\left(x+\frac{n\pi}{4}\right)$$

Replacing $(-n)$ with $<n>$,

$$\left(e^{-x}\sin x\right)^{<n>} = (-1)^n\left(\sin\frac{\pi}{4}\right)^{n} e^{-x}\sin\left(x+\frac{n\pi}{4}\right)$$

And since $x=\infty$ is zero of this clearly, rewriting the left side, we obtain (3.0s).

The following formula holds also about $e^x\cos x$.

$$\left(e^x\cos x\right)^{(n)} = \left(\sin\frac{\pi}{4}\right)^{-n} e^x\cos\left(x+\frac{n\pi}{4}\right)$$

From this, we obtain (3.0c) in a similar way.

Example

$$\int_{\infty}^{x} e^{-x}\sin x\,dx = (-1)^1\left(\sin\frac{\pi}{4}\right) e^{-x}\sin\left(x+\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}e^{-x}\frac{\sin x+\cos x}{\sqrt{2}}$$

$$= -\frac{1}{2}e^{-x}(\sin x+\cos x)$$

$$\int_{\infty}^{x}\int_{\infty}^{x} e^{-x}\sin x\,dx^2 = (-1)^2\left(\sin\frac{\pi}{4}\right)^2 e^{-x}\sin\left(x+\frac{2\pi}{4}\right) = \frac{1}{2}e^{-x}\cos x$$

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} e^{x}\cos x\,dx^3 = (-1)^3\left(\sin\frac{\pi}{4}\right)^3 e^{-x}\cos\left(x+\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{4}e^{-x}\sin\left(x+\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{4}e^{x}\frac{\cos x+\sin x}{\sqrt{2}} = \frac{1}{4}e^{-x}(\sin x+\cos x)$$

16.7 Series Expansion of Higher Integral

By adopting easy function such as 1 or e^x as one side of the functions f and g in Thorem 16.1.2, various function series is obtained.

16.7.1 Series Expansion of Higher Integral with fixed lower limit

First, adopting 1 as the function f in Theorem 16.1.2, we obtain the following theorem.

Theorem 16.7.1

Let m,n be natural numbers, $f^{(r)}$ $r=0,1,\cdots,m+n$ be the r th derivative of f, a be an arbitrary constant on the domain of f, $B(x,y)$ be the beta function. Then the following expressions hold.

$$\int_a^x\cdots\int_a^x f(x)dx^n=\sum_{r=0}^{m-1}\binom{-n}{r}\frac{(x-a)^{n+r}}{(n+r)!}f^{(r)}(x)+R_m^n \tag{1.1}$$

$$R_m^n=\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x\int_a^t\cdots\int_a^t\frac{(t-a)^{m+k}}{(m+k)!}f^{(m+k)}(t)dt^n \tag{1.1r}$$

$$\int_a^x\cdots\int_a^x f(x)dx^n=\sum_{r=0}^{m-1}\frac{(x-a)^{n+r}}{(n+r)!}f^{(r)}(a)+R_m^n \tag{1.2}$$

$$R_m^n=\sum_{r=1}^{n-1}\sum_{s=0}^{r-1}(-1)^{n-r+s}\sum_{t=s}^{r-1}{}_tC_s\cdot{}_{m+n-1-r+t}C_{m-1}\cdot{}_{m+n+s}C_r\frac{(x-a)^{m+n+s}}{(m+n+s)!}f^{(m+s)}(a)$$
$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x\int_a^t\cdots\int_a^t\frac{(t-x)^{m+k}}{(m+k)!}f^{(m+k)}(t)dt^n \tag{1.2r}$$

Where, there shall be no term of $\Sigma\Sigma$ at the time of $n<2$.

Proof

Exchanging f and g in Theorem 16.1.2 ,

$$\int_a^x\cdots\int_a^x g^{\langle 0\rangle}f^{(0)}dx^n=\sum_{r=0}^{m-1}\binom{-n}{r}g^{\langle n+r\rangle}f^{(r)}$$
$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s}g_a^{\langle n-r+s\rangle}f_a^{(s)}\frac{(x-a)^r}{r!}$$
$$+(-1)^m\sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1}{}_tC_s\cdot{}_{m+n-1-r+t}C_{m-1}\,g_a^{\langle m+n-r+s\rangle}f_a^{(m+s)}\frac{(x-a)^r}{r!}$$
$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x\cdots\int_a^x g^{\langle m+k\rangle}f^{(m+k)}dx^n$$

Substituting $g^{\langle r\rangle}=\dfrac{(x-c)^r}{r!}$ $(a\leqq c)$ for this,

$$\int_a^x\cdots\int_a^x 1\cdot f^{(0)}dx^n=\sum_{r=0}^{m-1}\binom{-n}{r}\frac{(x-c)^{n+r}}{(n+r)!}f^{(r)}$$
$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s}\frac{(a-c)^{n-r+s}}{(n-r+s)!}\frac{(x-a)^r}{r!}f_a^{(s)}$$

$$+(-1)^m \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1} \frac{(a-c)^{m+n-r+s}}{(m+n-r+s)!}\frac{(x-a)^r}{r!} f_a^{(m+s)}$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x \cdots \int_a^x \frac{(x-c)^{m+k}}{(m+k)!} f^{(m+k)} dx^n$$

Change this to the definite higher integral from a to $b\ (\geqq c)$, and rewrite the variable in the remainder term from x to t. Then,

$$\int_a^b\int_a^x \cdots \int_a^x f^{(0)} dx^n = \sum_{r=0}^{m-1}\binom{-n}{r}\frac{(b-c)^{n+r}}{(n+r)!} f_b^{(r)}$$

$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s}\frac{(a-c)^{n-r+s}}{(n-r+s)!}\frac{(b-a)^r}{r!} f_a^{(s)}$$

$$+(-1)^m \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1} \frac{(a-c)^{m+n-r+s}}{(m+n-r+s)!}\frac{(b-a)^r}{r!} f_a^{(m+s)}$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^b\int_a^t \cdots \int_a^t \frac{(t-c)^{m+k}}{(m+k)!} f^{(m+k)} dt^n \qquad \textbf{(1.0)}$$

Here, put $c=a$, then

$$\int_a^b\int_a^x \cdots \int_a^x f^{(0)} dx^n = \sum_{r=0}^{m-1}\binom{-n}{r}\frac{(b-a)^{n+r}}{(n+r)!} f_b^{(r)}$$

$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s}\frac{(a-a)^{n-r+s}}{(n-r+s)!}\frac{(b-a)^r}{r!} f_a^{(s)}$$

$$+(-1)^m \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1} \frac{(a-a)^{m+n-r+s}}{(m+n-r+s)!}\frac{(b-a)^r}{r!} f_a^{(m+s)}$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^b\int_a^t \cdots \int_a^t \frac{(t-a)^{m+k}}{(m+k)!} f^{(m+k)} dt^n$$

$$= \sum_{r=0}^{m-1}\binom{-n}{r}\frac{(b-a)^{n+r}}{(n+r)!} f_b^{(r)}$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^b\int_a^t \cdots \int_a^t \frac{(t-a)^{m+k}}{(m+k)!} f^{(m+k)} dt^n$$

Returning b to x, we obtain (1.1), (1.1r) .

Next, let $c=b$ in **(1.0)** . Then,

$$\int_a^b\int_a^x \cdots \int_a^x f^{(0)} dx^n = \sum_{r=0}^{m-1}\binom{-n}{r}\frac{(b-b)^{n+r}}{(n+r)!} f_b^{(r)}$$

$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s}\frac{(a-b)^{n-r+s}}{(n-r+s)!}\frac{(b-a)^r}{r!} f_a^{(s)}$$

$$+(-1)^m \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1} \frac{(a-b)^{m+n-r+s}}{(m+n-r+s)!}\frac{(b-a)^r}{r!} f_a^{(m+s)}$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^b\int_a^t \cdots \int_a^t \frac{(t-b)^{m+k}}{(m+k)!} f^{(m+k)} dt^n$$

$$= -\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}(-1)^{n-r+s}\binom{-n+r}{s}\frac{(b-a)^{n+s}}{r!\,(n-r+s)!}f_a^{(s)}$$

$$+\sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1}(-1)^{n-r+s}{}_tC_s\cdot{}_{m+n-1-r+t}C_{m-1}\frac{(b-a)^{m+n+s}}{r!\,(m+n-r+s)!}f_a^{(m+s)}$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^b\int_a^t\cdots\int_a^t\frac{(t-b)^{m+k}}{(m+k)!}f^{(m+k)}dt^n$$

Here,

$$(-1)^s\binom{-n+r}{s}={}_{n-r-1+s}C_s\quad,\quad\frac{1}{r!\,(n-r+s)!}=\frac{{}_{n+s}C_r}{(n+s)!}$$

Substituting these for the above,

$$\int_a^b\int_a^x\cdots\int_a^x f^{(0)}dx^n=-\sum_{s=0}^{m-1}\left(\sum_{r=0}^{n-1}(-1)^{n-r}{}_{n-1-r+s}C_s\cdot{}_{n+s}C_r\right)\frac{(b-a)^{n+s}}{(n+s)!}f_a^{(s)}$$

$$+\sum_{r=1}^{n-1}\sum_{s=0}^{r-1}(-1)^{n-r+s}\sum_{t=s}^{r-1}{}_tC_s\cdot{}_{m+n-1-r+t}C_{m-1}\cdot{}_{m+n+s}C_r\frac{(b-a)^{m+n+s}}{(m+n+s)!}f_a^{(m+s)}$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^b\int_a^t\cdots\int_a^t\frac{(t-b)^{m+k}}{(m+k)!}f^{(m+k)}dt^n$$

Here, the following equation holds for a natural number n and a non-negative integer s .

$$\sum_{r=0}^{n-1}(-1)^{n-r}{}_{n-1-r+s}C_s\cdot{}_{n+s}C_r=-1$$

Because,

$$\sum_{r=0}^{n-1}(-1)^{n-r}{}_{n-1-r+s}C_s\cdot{}_{n+s}C_r=\sum_{r=0}^{n-1}(-1)^{n-r}\frac{(n-1-r+s)!}{(n-1-r)!\,s!}\frac{(n+s)!}{(n-r+s)!\,r!}$$

$$=\frac{(n+s)!}{(n-1)!\,s!}\sum_{r=0}^{n-1}(-1)^{n-r}\frac{(n-1)!}{(n-1-r)!\,r!}\frac{1}{n-r+s}$$

$$=-\frac{(n+s)!}{(n-1)!\,s!}\sum_{r=0}^{n-1}(-1)^{n-1-r}\frac{{}_{n-1}C_{n-1-r}}{s+1+n-1-r}$$

$$=-\frac{(n+s)!}{(n-1)!\,s!}\sum_{r=0}^{n-1}\frac{(-1)^r}{s+1+r}{}_{n-1}C_r$$

According to 岩波数学公式Ⅱ p12,

$$\sum_{r=0}^{m}\frac{(-1)^r}{a+r}{}_mC_r=\frac{m!\,\Gamma(a)}{\Gamma(m+a+1)}\quad a\neq 0,-1,-2,\cdots$$

Then,

$$\sum_{r=0}^{n-1}\frac{(-1)^r}{s+1+r}{}_{n-1}C_r=\frac{(n-1)!\,\Gamma(s+1)}{\Gamma(n-1+s+1+1)}=\frac{(n-1)!\,s!}{(n+s)!}$$

Therefore

$$\int_a^b\int_a^x\cdots\int_a^x f^{(0)}dx^n=\sum_{s=0}^{m-1}\frac{(b-a)^{n+s}}{(n+s)!}f_a^{(s)}$$

$$+\sum_{r=1}^{n-1}\sum_{s=0}^{r-1}(-1)^{n-r+s}\sum_{t=s}^{r-1}{}_tC_s\cdot{}_{m+n-1-r+t}C_{m-1}\cdot{}_{m+n+s}C_r\frac{(b-a)^{m+n+s}}{(m+n+s)!}f_a^{(m+s)}$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^b\int_a^t\cdots\int_a^t\frac{(t-b)^{m+k}}{(m+k)!}f^{(m+k)}dt^n$$

ΣΣ of the 2nd line cannot be simplified any more. Moreover, since this is a disturbance term, we combine this with the 3rd line and make it a remainder. Needless to say, when this 2nd line doesn't exist when n is smaller than 2. Thus, rewriting s to r in the 1st line and returning b to x, we obtain (1.2), (1.2r) .

Series Expansion of Riemann-Liouville Integral

Higher Integral with fixed lower limit is Riemann-Liouville Integral itself. Therefore, the above thorem is also a series expansion of Riemann-Liouville Integral. If the above thorem is rewritten by Riemann-Liouville Integral, it is as follows.

Thorem 16.7.1'

The following formulas hold for a natural number n and an arbitrary number a on the domain of f.

$$\frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)dt = \sum_{r=0}^{m-1}\binom{-n}{r}\frac{(x-a)^{n+r}}{(n+r)!}f^{(r)}(x) + R_m^n \tag{1.1'}$$

$$R_m^n = \frac{(-1)^m}{B(n,m)\Gamma(n)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x \frac{(x-t)^{n-1}(t-a)^{m+k}}{(m+k)!}f^{(m+k)}(t)dt^n \tag{1.1'r}$$

$$\frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)dt = \sum_{r=0}^{m-1}\frac{(x-a)^{n+r}}{(n+r)!}f^{(r)}(a) + R_m^n \tag{1.2'}$$

$$R_m^n = \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}(-1)^{n-r+s}\sum_{t=s}^{r-1}{}_tC_s\cdot{}_{m+n-1-r+t}C_{m-1}\cdot{}_{m+n+s}C_r\frac{(x-a)^{m+n+s}}{(m+n+s)!}f^{(m+s)}(a)$$
$$-\frac{(-1)^{m+n}}{B(n,m)\Gamma(n)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x \frac{(t-x)^{m+n+k}}{(m+k)!}f^{(m+k)}(t)dt^n \tag{1.2'r}$$

Where, there shall be no term of ΣΣ at the time of $n<2$.

Example 1 Series expansion of the 2nd order collateral integral of e^x

Let $f(x)=e^x$, then

$$f(x)=e^x \quad , \quad f^{(r)}(x)=e^x , \quad f^{(r)}(0)=1 \qquad r=1,2,3,\cdots$$

Substituting these for Theorem 16.7.1 ,

$$\int_a^x\cdots\int_a^x e^x dx^n = e^x\sum_{r=0}^{m-1}\binom{-n}{r}\frac{(x-a)^{n+r}}{(n+r)!} + R_m^n$$

$$R_m^n = \frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x\int_a^t\cdots\int_a^t \frac{(t-a)^{m+k}}{(m+k)!}e^t dt^n$$

$$\int_a^x\cdots\int_a^x e^x dx^n = \sum_{r=0}^{m-1}\frac{(x-a)^{n+r}}{(n+r)!}e^a + R_m^n$$

$$R_m^n = \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}(-1)^{n-r+s}\sum_{t=s}^{r-1}{}_tC_s\cdot{}_{m+n-1-r+t}C_{m-1}\cdot{}_{m+n+s}C_r\frac{(x-a)^{m+n+s}}{(m+n+s)!}e^a$$
$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x\int_a^t\cdots\int_a^t \frac{(t-x)^{m+k}}{(m+k)!}e^t dt^n$$

Although the zero of the lineal higher integral of e^x is $a=-\infty$, these expressons can not have this zero. That is, by the above formulas, the lineal higher integral of e^x can not be expanded to the series. Then, we put $a=0$. and we expand the collateral higher integral of e^x to the series. When $n=2$,

$$\int_0^x\int_0^x e^x dx^2 = e^x \sum_{r=0}^{m-1}\binom{-2}{r}\frac{(x-0)^{2+r}}{(2+r)!} + R_m^2$$

$$R_m^2 = \frac{(-1)^m}{B(2,m)}\sum_{k=0}^{2-1}\frac{{}_{2-1}C_k}{m+k}\int_0^x\int_0^x \frac{t(t-0)^{m+k}}{(m+k)!}e^t dt^2$$

$$\int_0^x\int_0^x e^x dx^2 = \sum_{r=0}^{m-1}\frac{(x-0)^{2+r}}{(2+r)!}e^0 + R_m^2$$

$$R_m^2 = -{}_mC_{m-1}\cdot{}_{m+2}C_1\frac{(x-0)^{m+2}}{(m+2)!}e^0$$
$$+\frac{(-1)^m}{B(2,m)}\sum_{k=0}^{2-1}\frac{{}_{2-1}C_k}{m+k}\int_0^x\int_0^x \frac{t(t-x)^{m+k}}{(m+k)!}e^t dt^2$$

Example 2 Series expansion of the 3rd order integral of $\log x$

Let $f(x)=\log x$, then

$$f^{(0)} = \log x \quad , \quad f^{(r)} = (-1)^{r-1}\frac{(r-1)!}{x^r} \qquad r=1,2,3,\cdots$$

Substituting these for Theorem 16.7.1 ,

$$\int_a^x\cdots\int_a^x \log x\, dx^n = \frac{(x-a)^n}{n!}\log x - \sum_{r=1}^{m-1}(-1)^r\binom{-n}{r}\frac{(x-a)^{n+r}}{(n+r)!}\frac{(r-1)!}{x^r} + R_m^n$$

$$\int_a^x\cdots\int_a^x \log x\, dx^n = \frac{(x-a)^n}{n!}\log a - \sum_{r=1}^{m-1}(-1)^r\frac{(x-a)^{n+r}}{(n+r)!}\frac{(r-1)!}{a^r} + R_m^n$$

If $a=0$ then the first formula is lineal higher integral, else it is collateral higher integral. On the oter hand, the second formula can not be lineal higher integral.

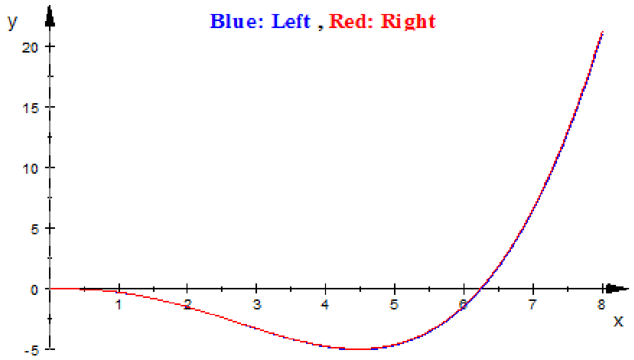
By the first formula, we calculate the lineal 3rd order integral and illustrate it as follows. Although the convergence of this series was slow, as a result of taking m very greatly, both sids overlapped somehow.

Left: lineal higher integral

```
• g := n-> x^n/n!*(ln(x) - sum(1/k,k=1..n)):
```

Right: Series

```
• m:=1000: a:=0:
• f := n-> (x-a)^n/n!*ln(x)-sum((-1)^r*binomial(-n,r)
                                *((x-a)^(n+r)/(n+r)!)
                                *((r-1)!/x^r),r=1..m-1):
```



16.7.2 Series Expansion of a Function

Sliding the index of the integration operator in Theorem 16.7.1, we obtain the series expansion of a function. Of course in these, Taylor expansion is also included.

Theorem 16.7.2

Let m,n are natural numbers, $f^{(r)}$ $r=0,1,\cdots,m+n$ be the r th derivative of f , a be the arbitrary constant on the domain of f , $B(x,y)$ be the beta function. Then the following expressions hold.

$$f(x) = \sum_{r=0}^{n-1}\frac{(x-a)^r}{r!}f^{(r)}(a) + \sum_{r=0}^{m-1}\binom{-n}{r}\frac{(x-a)^{n+r}}{(n+r)!}f^{(n+r)}(x) + R_m^n \tag{2.1}$$

$$R_m^n = \frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x\int_a^t\cdots\int_a^t\frac{(t-a)^{m+k}}{(m+k)!}f^{(n+m+k)}(t)dt^n \tag{2.1r}$$

$$f(x) = \sum_{r=0}^{m+n-1}\frac{(x-a)^r}{r!}f^{(r)}(a) + R_m^n \tag{2.2}$$

$$R_m^n = \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}(-1)^{n-r+s}\sum_{t=s}^{r-1}{}_tC_s\cdot{}_{m+n-1-r+t}C_{m-1}\cdot{}_{m+n+s}C_r\frac{(x-a)^{m+n+s}}{(m+n+s)!}f^{(m+n+s)}(a)$$
$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x\int_a^t\cdots\int_a^t\frac{(t-x)^{m+k}}{(m+k)!}f^{(n+m+k)}(t)dt^n \tag{2.2r}$$

Where, there shall be no term of $\Sigma\Sigma$ at the time of $n<2$.

Proof

Adding n to the index of the differentiation operator of the function $f^{(0)}$ in Theorem16.7.1 ,

$$\int_a^x\cdots\int_a^x f^{(n)}(x)dx^n = \sum_{r=0}^{m-1}\binom{-n}{r}\frac{(x-a)^{n+r}}{(n+r)!}f^{(n+r)}(x) + R_m^n$$

$$R_m^n = \frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x\int_a^t\cdots\int_a^t\frac{(t-a)^{m+k}}{(m+k)!}f^{(n+m+k)}(t)dt^n$$

$$\int_a^x\cdots\int_a^x f^{(n)}(x)dx^n = \sum_{r=0}^{m-1}\frac{(x-a)^{n+r}}{(n+r)!}f^{(n+r)}(a) + R_m^n$$

$$R_m^n = \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}(-1)^{n-r+s}\sum_{t=s}^{r-1}{}_tC_s\cdot{}_{m+n-1-r+t}C_{m-1}\cdot{}_{m+n+s}C_r\frac{(x-a)^{m+n+s}}{(m+n+s)!}f^{(n+m+s)}(a)$$
$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x\int_a^t\cdots\int_a^t\frac{(t-x)^{m+k}}{(m+k)!}f^{(n+m+k)}(t)dt^n$$

Here,

$$\int_a^x\cdots\int_a^x f^{(n)}dx^n = \int_a^x\cdots\int_a^x\left\{f^{(n-1)} - \frac{(x-a)^0}{0!}f_a^{(n-1)}\right\}dx^{n-1}$$
$$=\int_a^x\cdots\int_a^x\left\{f^{(n-2)} - \frac{(x-a)^0}{0!}f_a^{(n-2)} - \frac{(x-a)^1}{1!}f_a^{(n-1)}\right\}dx^{n-2}$$
$$=\int_a^x\cdots\int_a^x\left\{f^{(n-3)} - \frac{(x-a)^0}{0!}f_a^{(n-3)} - \frac{(x-a)^1}{1!}f_a^{(n-2)} - \frac{(x-a)^2}{2!}f_a^{(n-1)}\right\}dx^{n-3}$$

⋮

$$= f^{(0)} - \sum_{r=0}^{n-1} \frac{(x-a)^r}{r!} f_a^{(r)}$$

And,

$$\sum_{r=0}^{n-1} \frac{(x-a)^r}{r!} f^{(r)}(a) + \sum_{r=0}^{m-1} \frac{(x-a)^{n+r}}{(n+r)!} f^{(n+r)}(a) = \sum_{r=0}^{m+n-1} \frac{(x-a)^r}{r!} f^{(r)}(a)$$

Substituting these for the above, we obtain the desired expressions.

Remark

When $n=1$ in (2.2), (2.2r) ,

$$f(x) = \sum_{r=0}^{m} \frac{(x-a)^r}{r!} f^{(r)}(a) + (-1)^m \int_a^x \frac{(t-x)^m}{m!} f^{(1+m)}(t)\,dt$$

Needless to say, this is the usual Taylor expansion. However, in fact, (2.2), (2.2r) is alredy the usual Taylor expansion. Because, from Theorem 4.1.3 (1.2), this fearful remainder must be as follows.

$$R_m^n = \int_a^x \cdots \int_a^x f^{(m+n)}(x)\,dx^{m+n} = \frac{1}{\Gamma(m+n)} \int_a^x (x-t)^{m+n-1} f(t)\,dt$$

(2.2), (2.2r) is a trivial expression.

After all, the clarification from this theorem is the following two.

(1) A series of functions of a function $f(x)$ about a point a exists innumerably.

(2) The Taylor series of a function $f(x)$ about a point a is unique.

Example 1 Series Expansion of e^x about 0

$$f(x) = e^x \quad , \quad f^{(r)}(x) = e^x \, , \quad f^{(r)}(0) = 1 \qquad r=1,2,3,\cdots$$

Substituting these for (2.1) in the theorem,

$$e^x = \sum_{r=0}^{n-1} \frac{x^r}{r!} + e^x \sum_{r=0}^{m-1} \binom{-n}{r} \frac{x^{n+r}}{(n+r)!} + R_m^n$$

However, this is an automorphism and is not interesting. Then, using $\lim_{m\to\infty} R_m^n = 0$,

$$e^x \left\{ 1 - \sum_{r=0}^{\infty} \binom{-n}{r} \frac{x^{n+r}}{(n+r)!} \right\} = \sum_{r=0}^{n-1} \frac{x^r}{r!}$$

From this,

$$e^x = \left\{ 1 - \sum_{r=0}^{\infty} \binom{-n}{r} \frac{(-x)^{n+r}}{(n+r)!} \right\} \Big/ \sum_{r=0}^{n-1} \frac{(-x)^r}{r!}$$

And giving $n=1,2,3,\cdots$ to this, we obtain the following series.

$$e^x = \left\{ 1 - \sum_{r=0}^{\infty} \binom{-1}{r} \frac{(-x)^{1+r}}{(1+r)!} \right\} \Big/ 1 \qquad = \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

$$= \left\{ 1 - \sum_{r=0}^{\infty} \binom{-2}{r} \frac{(-x)^{2+r}}{(2+r)!} \right\} \Big/ \left(1 - \frac{x^1}{1!} \right)$$

$$= \left\{ 1 - \sum_{r=0}^{\infty} \binom{-3}{r} \frac{(-x)^{3+r}}{(3+r)!} \right\} \Big/ \left(1 - \frac{x^1}{1!} + \frac{x^2}{2!} \right)$$

⋮

Example 2 Series Expansion of $log\,x$ about 1

$$f(x)=log\,x\ , \quad f(1)=0\ , \qquad f^{(r)}(x)=(-1)^{r-1}\frac{(r-1)!}{x^r} \qquad r=1,2,3,\cdots$$

Substituting these for (2.1) in the theorem,

$$log\,x = -\sum_{r=1}^{n-1}\frac{(1-x)^r}{r} - \sum_{r=0}^{m-1}\frac{1}{n+r}\binom{-n}{r}\left(\frac{1}{x}-1\right)^{n+r} + R_m^n \qquad \frac{1}{2}<x$$

And giving $n=1,2,3,\cdots$ to this, we obtain the following series.

$$log\,x = -\sum_{r=0}^{\infty}\frac{1}{1+r}\binom{-1}{r}\left(\frac{1}{x}-1\right)^{1+r} \qquad \frac{1}{2}<x$$

$$= -\sum_{r=0}^{\infty}\frac{1}{2+r}\binom{-2}{r}\left(\frac{1}{x}-1\right)^{2+r} - \frac{(1-x)^1}{1}$$

$$= -\sum_{r=0}^{\infty}\frac{1}{3+r}\binom{-3}{r}\left(\frac{1}{x}-1\right)^{3+r} - \frac{(1-x)^1}{1} - \frac{(1-x)^2}{2}$$

⋮

16.8 Pascal Type Triangle

When there is a number triangle, such that the numbers on two sides are given and the other numbers are given by sum of the two numbers on the upper line, we will call it "Pascal Type Triangle".

Lemma 16.8.1

In the following number triangle, c_{r0} $(r=0,1,2,\cdots)$ are given, and let $c_{rr}=c_{00}$,

$c_{rs}=c_{r-1\,s}+c_{r-1\,s-1}$ $r,s \geqq 1$.

$$
\begin{array}{ccccccccc}
 & & & & c_{00} & & & & \\
 & & & c_{10} & & c_{11} & & & \\
 & & c_{20} & & c_{21} & & c_{22} & & \\
 & c_{30} & & c_{31} & & c_{32} & & c_{33} & \\
c_{40} & & c_{41} & & c_{42} & & c_{43} & & c_{44} \\
 & & & & \vdots & & & &
\end{array}
$$

Then, the following expressions hold.

$$c_{rs} = \sum_{t=0}^{r-s} {}_{r-1-t}C_{s-1}\, c_{t0} = \sum_{t=s-1}^{r-1} {}_{t}C_{s-1}\, c_{r-1-t\,0}$$

Proof

If we calculate one byone according to the condition $c_{rs}=c_{r-1\,s}+c_{r-1\,s-1}$, it is as follows.

1	c_{10}	c_{11}			
2	c_{20}	$c_{10}+c_{11}$	c_{22}		
3	c_{30}	$c_{10}+c_{11}+c_{20}$	$c_{10}+c_{11}+c_{22}$	c_{33}	
4	c_{40}	$c_{10}+c_{11}+c_{20}+c_{30}$	$2c_{10}+2c_{11}+c_{20}+c_{22}$	$c_{10}+c_{11}+c_{22}+c_{33}$	c_{44}
	$\vdots$				

When $c_{rr}=c_{00}$ $r=1,2,3,\cdots$

0	c_{00}	1	2	3	4
1	c_{10}	c_{00}			
2	c_{20}	$c_{00}+c_{10}$	$1c_{00}$		
3	c_{30}	$c_{00}+c_{10}+c_{20}$	$2c_{00}+1c_{10}$	$1c_{00}$	
4	c_{40}	$c_{00}+c_{10}+c_{20}+c_{30}$	$3c_{00}+2c_{10}+1c_{20}$	$3c_{00}+c_{10}$	$1c_{00}$
	$\vdots$				

Using binomial coefficient ${}_rC_s$,

0	c_{00}	1	2	3
1	c_{10}	${}_0C_0\, c_{00}$		
2	c_{20}	${}_1C_0\, c_{00}+{}_0C_0\, c_{10}$	${}_1C_1\, c_{00}$	
3	c_{30}	${}_2C_0\, c_{00}+{}_1C_0\, c_{10}+{}_0C_0\, c_{20}$	${}_2C_1\, c_{00}+{}_1C_1\, c_{10}$	${}_2C_2\, c_{00}$
	$\vdots$			

That is,

0	c_{00}	1	2	3	4
1	c_{10}	$\sum_{t=0}^{0} {}_{0-t}C_0\, c_{t0}$			
2	c_{20}	$\sum_{t=0}^{1} {}_{1-t}C_0\, c_{t0}$	$\sum_{t=0}^{0} {}_{1-t}C_1\, c_{t0}$		
3	c_{30}	$\sum_{t=0}^{2} {}_{2-t}C_0\, c_{t0}$	$\sum_{t=0}^{1} {}_{2-t}C_1\, c_{t0}$	$\sum_{t=0}^{0} {}_{2-t}C_2\, c_{t0}$	
4	c_{40}	$\sum_{t=0}^{3} {}_{3-t}C_0\, c_{t0}$	$\sum_{t=0}^{2} {}_{3-t}C_1\, c_{t0}$	$\sum_{t=0}^{1} {}_{3-t}C_2\, c_{t0}$	$\sum_{t=0}^{0} {}_{3-t}C_3\, c_{t0}$
	$\vdots$				

Thus, we obtain the following expression.

$$c_{rs} = \sum_{t=0}^{r-s} {}_{r-1-t}C_{s-1}\, c_{t0} \qquad r, s \geqq 1$$

And,

$$\begin{aligned} c_{rs} &= \sum_{t=0}^{r-s} {}_{r-1-t}C_{s-1}\, c_{t0} \\ &= {}_{r-1-(r-s)}C_{s-1}\, c_{r-s\,0} + {}_{r-1-(r-s-1)}C_{s-1}\, c_{r-s-1\,0} + \cdots + {}_{r-1-1}C_{s-1}\, c_{10} + {}_{r-1-0}C_{s-1}\, c_{00} \\ &= {}_{s-1}C_{s-1}\, c_{r-s\,0} + {}_{s}C_{s-1}\, c_{r-s-1\,0} + \cdots + {}_{r-2}C_{s-1}\, c_{10} + {}_{r-1}C_{s-1}\, c_{00} \\ &= \sum_{t=s-1}^{r-1} {}_{t}C_{s-1}\, c_{r-1-t\;0} \end{aligned}$$

Example When $c_{00} = 4$, $c_{10} = 3$, $c_{20} = 2$, $c_{30} = 1$

Since $c_{00} = {}_4C_1$, $c_{10} = {}_3C_1$, $c_{20} = {}_2C_1$, $c_{30} = {}_1C_1$, $c_{t0} = {}_{4-t}C_1$.

Then,

```
• C2 := (r,s)-> sum(binomial(r-1-t,s-1)*binomial(4-t,1), t=0..r-s)
```

$$(r, s) \rightarrow \sum_{t=0}^{r-s} \binom{r-1-t}{s-1} \cdot \binom{4-t}{1}$$

```
• C2(3,0);C2(3,1);C2(3,2);C2(3,3)
```

1 9 11 4

4

3 4

2 7 4

1 9 11 4

Note

When $c_{t0} = 1$ $t=0,1,2,\cdots$, this number triangle reduces to Pascal's Triangle.

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Alien's Mathematics