02 Infinite-degree Equation with Integers as Roots

2.1 Infinite-degree Equation with Integers as Roots (Part1)

As an example, let us consider the following function $f(z)$.

$$f(z) = \prod_{r=1}^{\infty}\left(1-\frac{z}{r}\right)e^{\frac{z}{r}} = 1+c_1 z^1+c_2 z^2+c_3 z^3+\cdots$$

This function has only positive integers $z=1,2,3,\cdots$ as roots. It is because the compensation terms $\prod_{r=1}^{\infty}e^{z/r}$ does not have a zero. Now, let us make the power series equal to 0 as follows.

$$1+c_1 z^1+c_2 z^2+c_3 z^3+\cdots = 0$$

This can be called Infinite-degree equation with positive integers $z=1,2,3,\cdots$ as roots

In this section, we look for the power series which has either negative or positive integers as zeros.

Formula 2.1.1

Let $\psi_r(z)$ be the polygamma function, $B_{r,k}(f_1, f_2, \ldots)$ are Bell polynomials, γ be Euler-Mascheroni constant and a_r, b_r are the following constants.

$$a_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) \qquad r=1,2,3,\cdots$$

$$b_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{r-1}\left(\frac{1}{2}\right)\right) \qquad r=1,2,3,\cdots$$

Then, the following expressions hold.

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)e^{-\frac{z}{r}} = 1+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r \tag{1.1-}$$

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{r}\right)e^{\frac{z}{r}} = 1+\sum_{r=1}^{\infty}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r \tag{1.1+}$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r}\right)e^{-\frac{z}{2r}} = 1+\sum_{r=1}^{\infty}\frac{(-1)^r}{2^r}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r \tag{1.2-}$$

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{2r}\right)e^{\frac{z}{2r}} = 1+\sum_{r=1}^{\infty}\frac{1}{2^r}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r \tag{1.2+}$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)e^{-\frac{z}{2r-1}} = 1+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!\,(r-s)!}\right\}z^r \tag{1.3-}$$

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{2r-1}\right)e^{\frac{z}{2r-1}} = 1+\sum_{r=1}^{\infty}\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!\,(r-s)!}\right\}z^r \tag{1.3+}$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)e^{-\frac{z}{2r}} = 1+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{\gamma^r}{(2r)!!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\gamma^{r-s}}{(2s)!!(2r-2s)!!}\right\}z^r \tag{1.4-}$$

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{2r-1}\right)e^{\frac{z}{2r}} = 1+\sum_{r=1}^{\infty}\left\{\frac{\gamma^r}{(2r)!!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\gamma^{r-s}}{(2s)!!(2r-2s)!!}\right\}z^r \tag{1.4+}$$

Proof

According to Formula 11.1.1 (**11 Series Expansion of Reciprocal of Gamma Function**) , (1.1_-) is as follows.

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)e^{-\frac{z}{r}} = \frac{e^{-\gamma z}}{\Gamma(1+z)}$$

According to Formula 12.3.1 (**12 Series Expansion of Gamma Function & the Reciprocal**) , this right side is expanded to Maclaurin series as follows using Bell polynomial and the polygamma function respectively.

$$e^{-\gamma z} = 1-\frac{\gamma^1}{1!}z^1+\frac{\gamma^2}{2!}z^2-\frac{\gamma^3}{3!}z^3+-\cdots$$

$$\frac{1}{\Gamma(1+z)} = 1+\frac{a_1}{1!}z^1+\frac{a_2}{2!}z^2+\frac{a_3}{3!}z^3+\cdots$$

where,

$$a_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0(1),\psi_1(1),\cdots,\psi_{r-1}(1)\right) \qquad r=1,2,3,\cdots$$

Calculating these Cauchy products, it is as follows.

$$1-\frac{\gamma^1}{1!}z^1+\frac{\gamma^2}{2!}z^2-\frac{\gamma^3}{3!}z^3+-\cdots$$

$$\times\, 1+\frac{a_1}{1!}z^1+\frac{a_2}{2!}z^2+\frac{a_3}{3!}z^3+\cdots$$

$$= 1+\left(-\frac{\gamma^1}{1!}+\frac{a_1}{1!}\right)z^1+\left(\frac{\gamma^2}{2!}-\frac{\gamma^1}{1!}\frac{a_1}{1!}+\frac{a_2}{2!}\right)z^2+\left(-\frac{\gamma^3}{3!}+\frac{\gamma^2}{2!}\frac{a_1}{1!}-\frac{\gamma^1}{1!}\frac{a_2}{2!}+\frac{a_3}{3!}\right)z^3+\cdots$$

$$= 1+\left\{-\frac{\gamma^1}{1!}+\sum_{s=1}^{1}\frac{(-1)^{1-s}\gamma^{1-s}}{(1-s)!}\frac{a_s}{s!}\right\}z^1+\left\{\frac{\gamma^2}{2!}+\sum_{s=1}^{2}\frac{(-1)^{2-s}\gamma^{2-s}}{(2-s)!}\frac{a_s}{s!}\right\}z^2+\cdots$$

$$= 1+\sum_{r=1}^{\infty}\left\{(-1)^r\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^{r-s}\gamma^{r-s}}{(r-s)!}\frac{a_s}{s!}\right\}z^r$$

i.e.

$$\frac{e^{-\gamma z}}{\Gamma(1+z)} = 1+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!(r-s)!}\right\}z^r$$

(1.1_-) is obtained from this, and (1.1_+) is obtained by inverting the sign of z .

And, (1.2_-) and (1.2_+) are obtained by replacing z with $z/2$ in the (1.1_-) and (1.1_+) respectively.

According to Formula 11.1.1 (**11 Series Expansion of Reciprocal of Gamma Function**) , (1.3_-) is as follows.

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)e^{-\frac{z}{2r-1}} = \frac{\sqrt{\pi}}{\Gamma\{(1+z)/2\}}e^{-\left(\frac{\gamma}{2}+\log 2\right)z}$$

According to Formula 12.3.2 (**12 Series Expansion of Gamma Function & the Reciprocal**) , this right side is expanded to Maclaurin series as follows using Bell polynomial and the polygamma function respectively.

$$e^{-\left(\frac{\gamma}{2}+\log 2\right)z} = 1 - \frac{\left(\frac{\gamma}{2}+\log 2\right)^1}{1!}z^1 + \frac{\left(\frac{\gamma}{2}+\log 2\right)^2}{2!}z^2 - \frac{\left(\frac{\gamma}{2}+\log 2\right)^3}{3!}z^3 +- \cdots$$

$$\frac{\sqrt{\pi}}{\Gamma\{(1+z)/2\}} = 1 + \frac{b_1}{2!!}z^1 + \frac{b_2}{4!!}z^2 + \frac{b_3}{6!!}z^3 + \cdots$$

where,

$$b_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \cdots, \psi_{r-1}\left(\frac{1}{2}\right)\right) \quad r=1,2,3,\cdots$$

Calculating these Cauchy products,

$$\frac{\sqrt{\pi}}{\Gamma\{(1+z)/2\}}e^{-\left(\frac{\gamma}{2}+\log 2\right)z} = 1 + \sum_{r=1}^{\infty}(-1)^r\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!} + \sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!(r-s)!}\right\}z^r$$

(1.3_-) is obtained from this, and (1.3_+) is obtained by inverting the sign of z .

According to Formula 11.1.1 (**11 Series Expansion of Reciprocal of Gamma Function**) , (1.4_-) is as follows.

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)e^{-\frac{z}{2r}} = \frac{\sqrt{\pi}}{\Gamma\{(1+z)/2\}}e^{-\frac{\gamma z}{2}}$$

According to Formula 12.3.2 (**12 Series Expansion of Gamma Function & the Reciprocal**) , this right side is expanded to Maclaurin series as follows using Bell polynomial and the polygamma function respectively.

$$e^{-\frac{\gamma z}{2}} = 1 - \frac{\gamma^1}{2!!}z^1 + \frac{\gamma^2}{4!!}z^2 - \frac{\gamma^3}{6!!}z^3 +- \cdots$$

$$\frac{\sqrt{\pi}}{\Gamma\{(1+z)/2\}} = 1 + \frac{b_1}{2!!}z^1 + \frac{b_2}{4!!}z^2 + \frac{b_3}{6!!}z^3 + \cdots$$

where,

$$b_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \cdots, \psi_{r-1}\left(\frac{1}{2}\right)\right) \quad r=1,2,3,\cdots$$

(1.4_-) is obtained by calculating these Cauchy products, and (1.4_+) is obtained by inverting the sign of z .

Example1 Infinite-degree Equation with Natural Numbers as Roots

The above (1.1_+) corresponds to this. That is,

$$\begin{aligned}\prod_{r=1}^{\infty}\left(1-\frac{z}{r}\right)e^{\frac{z}{r}} = 1 &+ \left(\frac{\gamma^1}{1!} - \frac{a_1\gamma^0}{1!0!}\right)z^1 \\ &+ \left(\frac{\gamma^2}{2!} - \frac{a_1\gamma^1}{1!1!} + \frac{a_2\gamma^0}{2!0!}\right)z^2 \\ &+ \left(\frac{\gamma^3}{3!} - \frac{a_1\gamma^2}{1!2!} + \frac{a_2\gamma^1}{2!1!} - \frac{a_3\gamma^0}{3!0!}\right)z^3 + \cdots \qquad (1.1_+) \\ &\vdots\end{aligned}$$

When both sides are illustrated. it is as follows. Although 1000 terms are calculated on the left side (blue) and 20 terms are calculated on the right side (orange) , both sides are exactly overlapped and the left side (blue) is invisible. And we can see that $z=1,2,3,\cdots$ are the zeros of the right side. Here, the polynomial $B_{r,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica*** .

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Tblψ[r_ , z_] := Table[PolyGamma[k, z], {k, 0, r - 1}]
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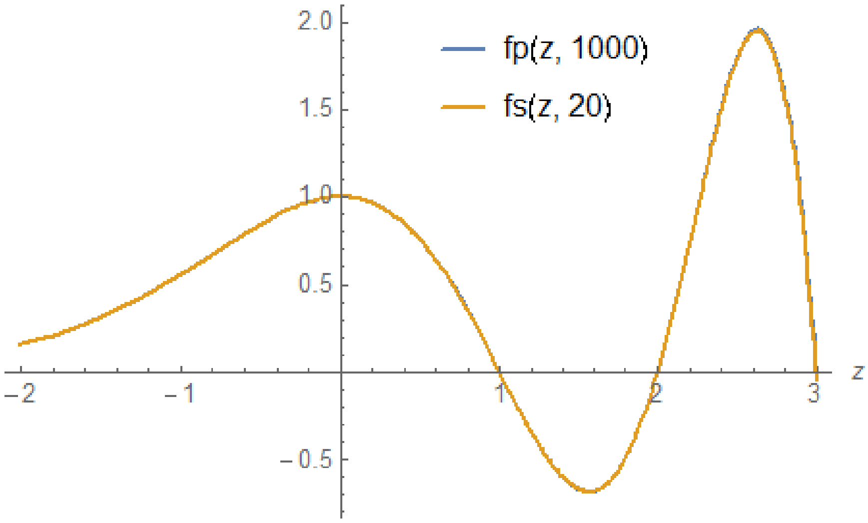
$$a[r\_] := \sum_{k=1}^{r} (-1)^k \text{BellY}[r, k, \text{Tbl}\psi[r, 1]]$$

$$fp[z\_, m\_] := \prod_{r=1}^{m} \left(1 - \frac{z}{r}\right) e^{\frac{z}{r}}$$

$$fs[z\_, m\_] := 1 + \sum_{r=1}^{m} \left(\frac{\text{EulerGamma}^r}{r!} + \sum_{s=1}^{r} \frac{(-1)^s a[s]\, \text{EulerGamma}^{r-s}}{s!\,(r-s)!} \right) z^r$$



Example2 Infinite-degree Equation with Positive Odd Numbers as Roots

The above (1.3_+) and (1.4_+) correspond to this. Here, we illustrate (1.4_+) .

$$\begin{aligned}\prod_{r=1}^{\infty}\left(1-\frac{z}{2r-1}\right)e^{\frac{z}{2r}} = 1 &+ \left(\frac{\gamma^1}{2!!} - \frac{b_1\gamma^0}{2!!0!!}\right)z^1 \\ &+ \left(\frac{\gamma^2}{4!!} - \frac{b_1\gamma^1}{2!!2!!} + \frac{b_2\gamma^0}{4!!0!!}\right)z^2 \\ &+ \left(\frac{\gamma^3}{6!!} - \frac{b_1\gamma^2}{2!!4!!} + \frac{b_2\gamma^1}{4!!2!!} - \frac{b_3\gamma^0}{6!!0!!}\right)z^3 \\ &\vdots \end{aligned} \qquad (1.4_+)$$

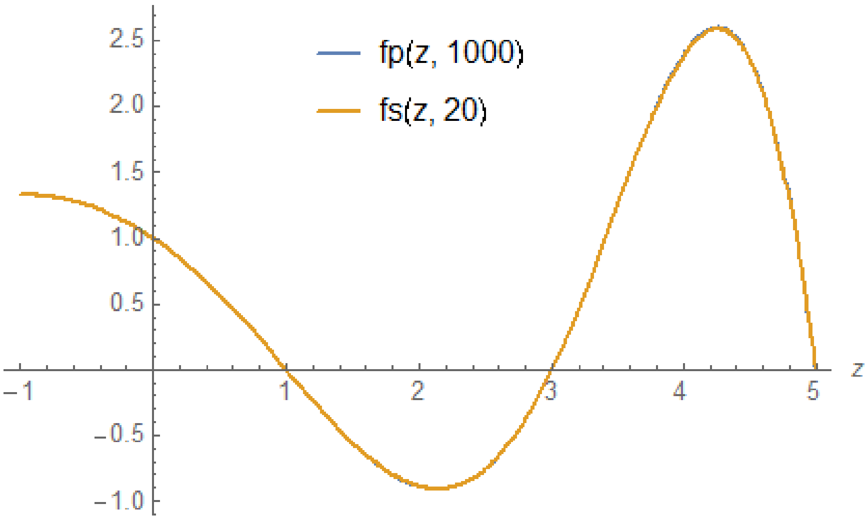
When both sides are illustrated. it is as follows. Although 1000 terms are calculated on the left side (blue) and 20 terms are calculated on the right side (orange) , both sides are exactly overlapped and the left side (blue) is invisible. And we can see that $z=1,3,5,\cdots$ are the zeros of the right side. Here, the polynomial $B_{r,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica*** .

$$\text{Tbl}\psi[r\_, z\_] := \text{Table}[\text{PolyGamma}[k, z], \{k, 0, r-1\}]$$

$$b[r\_] := \sum_{k=1}^{r} (-1)^k \,\text{BellY}\left[r, k, \text{Tbl}\psi\left[r, \frac{1}{2}\right]\right]$$

$$\text{fp}[z\_, m\_] := \prod_{r=1}^{m} \left(1 - \frac{z}{2r-1}\right) e^{\frac{z}{2r}}$$

$$\text{fs}[z\_, m\_] := 1 + \sum_{r=1}^{m} \left(\frac{\text{EulerGamma}^r}{(2r)!!} + \sum_{s=1}^{r} \frac{(-1)^s \, b[s] \, \text{EulerGamma}^{r-s}}{(2s)!! \, (2r-2s)!!} \right) z^r$$



The coefficient c_1 of z^1

The coefficient of z^1 is $c_1 = 0$ for $(1.1_-) \sim (1.3_+)$ and $c_1 = \pm log2$ for (1.4_-) & (1.4_+) .

Calculating c_1 for the example1 (1.1_+) and the example2 (1.4_+) ,

$$\frac{\gamma^1}{1!} - \frac{a_1\gamma^0}{1!0!} = \gamma + \psi_0(1) = \gamma - \gamma = 0 \qquad \left(= \sum_{r=1}^{\infty} \frac{1}{r} - \sum_{r=1}^{\infty} \frac{1}{r} \right)$$

$$\frac{\gamma^1}{2!!} - \frac{b_1\gamma^0}{2!!0!!} = \frac{1}{2}\left\{ \gamma + \psi_0\left(\frac{1}{2}\right) \right\} = \frac{\gamma}{2} - \frac{\gamma + 2log2}{2} = -log2 \qquad \left(= \sum_{r=1}^{\infty} \frac{1}{2r} - \sum_{r=1}^{\infty} \frac{1}{2r-1} \right)$$

The insides of parenthesis are the total of the coefficient of z of the compensation term and the body.

c_1 becomes a constant by this work of the compensation term. If there is no compensation term, $c_1 = -\infty$, and the series itself can not exist.

2.2 Infinite-degree Equation with Integers as Roots (Part2)

As an example, let us consider the following function $f(z)$.

$$f(z) = \prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)\left(1-\frac{z}{r}\right) = 1+c_1 z^1+c_2 z^2+c_3 z^3+\cdots$$

This function has integers $z=\pm 1,\pm 2,\pm 3,\cdots$ as roots. Now, let the power series equal to 0 .

$$1+c_1 z^1+c_2 z^2+c_3 z^3+\cdots = 0$$

This can be called Infinite-degree equation with negative and positive integers $z=\pm 1,\pm 2,\pm 3,\cdots$ as roots. In this section, we look for the power series which has both negative and positive integers as zeros.

Formula 2.2.1

Let $\psi_r(z)$ be the polygamma function, $B_{r,k}(f_1, f_2, \ldots)$ are Bell polynomials, γ be Euler-Mascheroni constant and a_r, b_r are the following constants.

$$a_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) \qquad r=1,2,3,\cdots$$

$$b_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{r-1}\left(\frac{1}{2}\right)\right) \qquad r=1,2,3,\cdots$$

Then, the following expressions hold.

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)\left(1-\frac{z}{r}\right) = \sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(2r+1)!}z^{2r} \qquad \left(=\frac{\sin \pi z}{\pi z}\right) \tag{2.1}$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r}\right)\left(1-\frac{z}{2r}\right) = \sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{2^{2r}(2r+1)!}z^{2r} \qquad \left(=\frac{\sin(\pi z/2)}{\pi z/2}\right) \tag{2.2}$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)\left(1-\frac{z}{2r-1}\right) = \sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(4r)!!}z^{2r} \qquad \left(=\cos\frac{\pi z}{2}\right) \tag{2.3}$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)\left(1-\frac{z}{2r}\right) = 1+\frac{b_1-a_1}{2!!}z^1 + \sum_{r=2}^{\infty}\left\{\frac{b_r+(-1)^r a_r}{(2r)!!}+\sum_{s=1}^{r-1}\frac{(-1)^{r-s}b_s a_{r-s}}{(2s)!!\{2(r-s)\}!!}\right\}z^r \tag{2.4}$$

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{2r-1}\right)\left(1+\frac{z}{2r}\right) = 1+\frac{a_1-b_1}{2!!}z^1 + \sum_{r=2}^{\infty}\left\{\frac{a_r+(-1)^r b_r}{(2r)!!}+\sum_{s=1}^{r-1}\frac{(-1)^{r-s}a_s b_{r-s}}{(2s)!!\{2(r-s)\}!!}\right\}z^r \tag{2.5}$$

Proof

According to Formula 11.1.1 (**11 Series Expansion of Reciprocal of Gamma Function**), (2.1) is as follows.

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)\left(1-\frac{z}{r}\right) = \frac{1}{\Gamma(1+z)\Gamma(1-z)} = \frac{\sin \pi z}{\pi z}$$

This right side is expanded to Maclaurin series as follows.

$$\frac{\sin \pi z}{\pi z} = 1-\frac{\pi^2}{3!}z^2+\frac{\pi^4}{5!}z^4-\frac{\pi^6}{7!}z^6+-\cdots = \sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(2r+1)!}z^{2r}$$

Thus, (2.1) is obtained. And (2.2) is obtained by replacing z with $z/2$

According to Corollary 11.1.1 (**11 Series Expansion of Reciprocal of Gamma Function**) , (2.3) is as follows.

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)\left(1-\frac{z}{2r-1}\right)=\frac{\pi}{\Gamma\left(\frac{1+z}{2}\right)\Gamma\left(\frac{1-z}{2}\right)}=\cos\frac{\pi z}{2}$$

This right side is expanded to Maclaurin series as follows.

$$\cos\frac{\pi z}{2}=1-\frac{\pi^2}{4!!}z^2+\frac{\pi^4}{8!!}z^4-\frac{\pi^6}{12!!}z^6+-\cdots=\sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(4r)!!}z^{2r}$$

Thus, (2.3) is obtained.

According to Corollary 11.1.1 (**11 Series Expansion of Reciprocal of Gamma Function**) , (2.4) is as follows.

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)\left(1-\frac{z}{2r}\right)=\frac{\sqrt{\pi}}{\Gamma\left(\frac{1+z}{2}\right)\Gamma\left(1-\frac{z}{2}\right)}$$

According to Formula 12.3.1 & 12.3.2 (**12 Series Expansion of Gamma Function & the Reciprocal**), the right side is expanded to Maclaurin series as follows using Bell polynomial and the polygamma function respectively.

$$\frac{1}{\Gamma\left(1-\frac{z}{2}\right)}=1-\frac{a_1}{2!!}z^1+\frac{a_2}{4!!}z^2-\frac{a_3}{6!!}z^3+\frac{a_4}{8!!}z^4-+\cdots$$

$$\frac{\sqrt{\pi}}{\Gamma\left(\frac{1+z}{2}\right)}=1+\frac{b_1}{2!!}z^1+\frac{b_2}{4!!}z^2+\frac{b_3}{6!!}z^3+\frac{b_4}{8!!}z^4+\cdots$$

where,

$$a_r=\sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0(1),\psi_1(1),\cdots,\psi_{r-1}(1)\right)\qquad r=1,2,3,\cdots$$

$$b_r=\sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right),\psi_1\left(\frac{1}{2}\right),\cdots,\psi_{r-1}\left(\frac{1}{2}\right)\right)\qquad r=1,2,3,\cdots$$

(2.4) is obtained by calculating these Cauchy products, and (2.5) is obtained by swapping a and b .

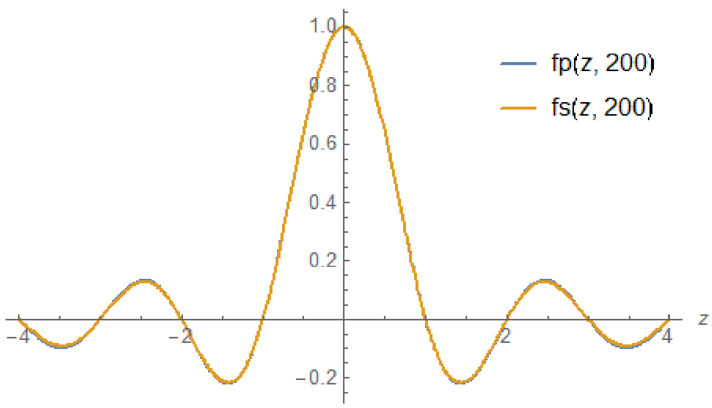
Example1 Infinite-degree Equation with All Integers as Roots

The above (2.1) corresponds to this. That is,

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)\left(1-\frac{z}{r}\right)=1-\frac{\pi^2}{3!}z^2+\frac{\pi^4}{5!}z^4-\frac{\pi^6}{7!}z^6+-\cdots \tag{2.1}$$

When both sides are illustrated, it is as follows.

$$\texttt{fp[z\_, m\_]} := \prod_{r=1}^{m}\left(1+\frac{z}{r}\right)\left(1-\frac{z}{r}\right)\qquad \texttt{fs[z\_, m\_]} := \sum_{r=0}^{m}\frac{(-1)^r\pi^{2r}}{(2r+1)!}z^{2r}$$



Both sides are calculated by 200 terms. Both are overlapping and the left side (blue) can not be seen.

And we can see that $z=\pm 1,\pm 2,\pm 3,\cdots$ are the zeros of the right side.

Example2 Negative Even Numbers & Positive Odd Numbers

The above (2.5) corresponds to this. That is,

$$\begin{aligned}\prod_{r=1}^{\infty}\left(1-\frac{z}{2r-1}\right)\left(1+\frac{z}{2r}\right) = 1 &+ \frac{a_1-b_1}{2!!}z^1 \\ &+\left(\frac{a_2+b_2}{4!!}-\frac{a_1}{2!!}\frac{b_1}{2!!}\right)z^2 \\ &+\left(\frac{a_3-b_3}{6!!}+\frac{a_1}{2!!}\frac{b_2}{4!!}-\frac{a_2}{4!!}\frac{b_1}{2!!}\right)z^3 \\ &+\left(\frac{a_4+b_4}{8!!}-\frac{a_1}{2!!}\frac{b_3}{6!!}+\frac{a_2}{4!!}\frac{b_2}{4!!}-\frac{a_3}{6!!}\frac{b_1}{2!!}\right)z^4 \\ &\vdots \end{aligned} \tag{2.5}$$

When both sides are illustrated. it is as follows. Though 1000 terms are calculated on the left side (blue) and 20 terms are calculated on the right side (orange), both sides are exactly overlapped and the left side (blue) is invisible. And we can see that $z=-2,-4,-5,\cdots$ and $z=1,3,5,\cdots$ are the zeros of the right side.

Here, the polynomial $B_{r,k}(f_1,f_2,\ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***.

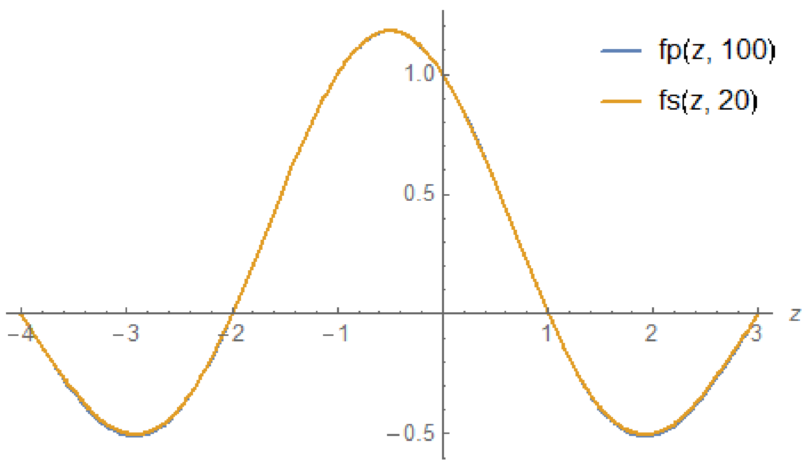
$$\text{Tbl}\psi[r\_,z\_] := \text{Table}[\text{PolyGamma}[k,z],\{k,0,r-1\}]$$

$$a[r\_] := \sum_{k=1}^{r}(-1)^k\,\text{BellY}[r,k,\text{Tbl}\psi[r,1]]$$

$$b[r\_] := \sum_{k=1}^{r}(-1)^k\,\text{BellY}\left[r,k,\text{Tbl}\psi\left[r,\frac{1}{2}\right]\right]$$

$$fp[z\_,m\_] := \prod_{r=1}^{m}\left(1-\frac{z}{2r-1}\right)\left(1+\frac{z}{2r}\right)$$

$$fs[z\_,m\_] := 1+\frac{a[1]-b[1]}{2!!}z + \sum_{r=2}^{m}\left(\frac{a[r]+(-1)^r b[r]}{(2r)!!}+\sum_{s=1}^{r-1}\frac{(-1)^{r-s}a[s]\,b[r-s]}{(2s)!!\,(2(r-s))!!}\right)z^r$$



The coefficient c_1 of z^1

$c_1 = 0$ for (2.1), (2.2) and (2.3) . Because they are even functions, they do not have odd terms in the first place.

$c_1 = \pm log 2$ for (2.4) and (2.5) .

Calculating c_1 for the example2 (2.5) ,

$$\frac{a_1 - b_1}{2!!} = \frac{1}{2}\left\{ \gamma + \psi_0\left(\frac{1}{2} \right) \right\} = \frac{\gamma}{2} - \frac{\gamma + 2log 2}{2} = -log 2 \qquad \left(= \sum_{r=1}^{\infty} \frac{1}{2r} - \sum_{r=1}^{\infty} \frac{1}{2r-1} \right)$$

The insides of parenthesis are the total of the coefficient of z of the left side. We may interpret that the sum of even reciprocals is 2 less than the sum of odd reciprocals

2.3 Infinite-degree Equation with Imaginary Integers as Roots (Part1)

By replacing z with iz , Formula 2.1.1 can be easily converted to a power series with imaginary integers as zeros. Now, let us make the power series equal to 0 as follows.

$$1+c_1 z^1+c_2 z^2+c_3 z^3+\cdots = 0$$

Then, this becomes an ifinite-degree equation with imaginary negative or positive Integers as roots.

Formula 2.3.1

Let $\psi_r(z)$ be the polygamma function, $B_{r,k}(f_1, f_2, \ldots)$ are Bell polynomials, γ be Euler-Mascheroni constant and a_r, b_r are the following constants.

$$a_r = \sum_{k=1}^{r} (-1)^k B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) \qquad r=1,2,3,\cdots$$

$$b_r = \sum_{k=1}^{r} (-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{r-1}\left(\frac{1}{2}\right)\right) \qquad r=1,2,3,\cdots$$

Then, the following expressions hold.

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{ir}\right)e^{-\frac{z}{ir}} = 1+\sum_{r=1}^{\infty} i^r\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r \tag{3.1-}$$

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{ir}\right)e^{\frac{z}{ir}} = 1+\sum_{r=1}^{\infty} (-i)^r\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r \tag{3.1+}$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2ir}\right)e^{-\frac{z}{2ir}} = 1+\sum_{r=1}^{\infty} \frac{i^r}{2^r}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r \tag{3.2-}$$

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{2ir}\right)e^{\frac{z}{2ir}} = 1+\sum_{r=1}^{\infty} \frac{(-i)^r}{2^r}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r \tag{3.2+}$$

$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{i(2r-1)}\right\}e^{-\frac{z}{i(2r-1)}} = 1+\sum_{r=1}^{\infty} i^r\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!\,(r-s)!}\right\}z^r \tag{3.3-}$$

$$\prod_{r=1}^{\infty}\left\{1-\frac{z}{i(2r-1)}\right\}e^{\frac{z}{i(2r-1)}} = 1+\sum_{r=1}^{\infty} (-i)^r\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!\,(r-s)!}\right\}z^r \tag{3.3+}$$

$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{i(2r-1)}\right\}e^{-\frac{z}{2ir}} = 1+\sum_{r=1}^{\infty} i^r\left\{\frac{\gamma^r}{(2r)!!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\gamma^{r-s}}{(2s)!!\,(2r-2s)!!}\right\}z^r \tag{3.4-}$$

$$\prod_{r=1}^{\infty}\left\{1-\frac{z}{i(2r-1)}\right\}e^{\frac{z}{2ir}} = 1+\sum_{r=1}^{\infty} (-i)^r\left\{\frac{\gamma^r}{(2r)!!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\gamma^{r-s}}{(2s)!!\,(2r-2s)!!}\right\}z^r \tag{3.4+}$$

Proof

From Formula 2.1.1 ,

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{r}\right)e^{\frac{z}{r}} = 1+\sum_{r=1}^{\infty}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r \tag{1.1+}$$

Replacing z with iz ,

$$\prod_{r=1}^{\infty}\left(1-\frac{iz}{r}\right)e^{\frac{iz}{r}}=\prod_{r=1}^{\infty}\left(1+\frac{z}{ir}\right)e^{-\frac{z}{ir}}=1+\sum_{r=1}^{\infty}i^{r}\left\{\frac{\gamma^{r}}{r\,!}+\sum_{s=1}^{r}\frac{(-1)^{s}a_{s}\gamma^{r-s}}{s\,!\,(r-s)!}\right\}z^{r} \tag{3.1$_-$}$$

Next, from Formula 2.1.1 ,

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)e^{-\frac{z}{r}}=1+\sum_{r=1}^{\infty}(-1)^{r}\left\{\frac{\gamma^{r}}{r\,!}+\sum_{s=1}^{r}\frac{(-1)^{s}a_{s}\gamma^{r-s}}{s\,!\,(r-s)!}\right\}z^{r} \tag{1.1$_-$}$$

Replacing z with iz ,

$$\prod_{r=1}^{\infty}\left(1+\frac{iz}{r}\right)e^{-\frac{iz}{r}}=\prod_{r=1}^{\infty}\left(1-\frac{z}{ir}\right)e^{\frac{z}{ir}}=1+\sum_{r=1}^{\infty}(-i)^{r}\left\{\frac{\gamma^{r}}{r\,!}+\sum_{s=1}^{r}\frac{(-1)^{s}a_{s}\gamma^{r-s}}{s\,!\,(r-s)!}\right\}z^{r} \tag{3.1$_+$}$$

Hereafter in a similer way, we obtain the desired expressions.

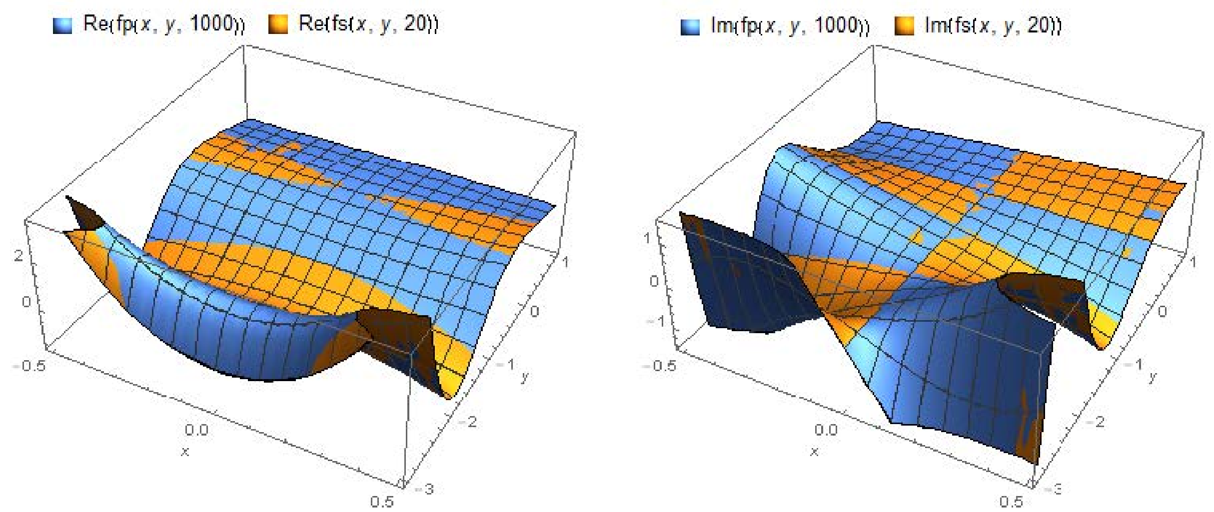
Example1 Infinite-degree Equation with Negative Imaginary Integers as Roots

The above (3.1$_-$) corresponds to this. That is,

$$\begin{aligned}\prod_{r=1}^{\infty}\left(1+\frac{z}{ir}\right)e^{\frac{iz}{r}}&=1+i^{1}\left(\frac{\gamma^{1}}{1!}-\frac{a_{1}\gamma^{0}}{1!0!}\right)z^{1}\\&+i^{2}\left(\frac{\gamma^{2}}{2!}-\frac{a_{1}\gamma^{1}}{1!1!}+\frac{a_{2}\gamma^{0}}{2!0!}\right)z^{2}\\&+i^{3}\left(\frac{\gamma^{3}}{3!}-\frac{a_{1}\gamma^{2}}{1!2!}+\frac{a_{2}\gamma^{1}}{2!1!}-\frac{a_{3}\gamma^{0}}{3!0!}\right)z^{3}+\cdots\\&\vdots\end{aligned} \tag{3.1$_-$}$$

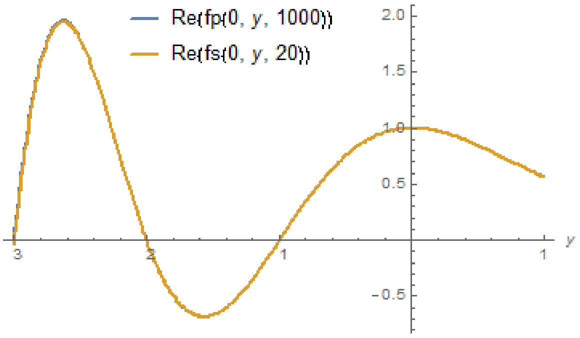
If z is replaced with $x+iy$ and both sides are illustrated in 3D figures, it is as follws.

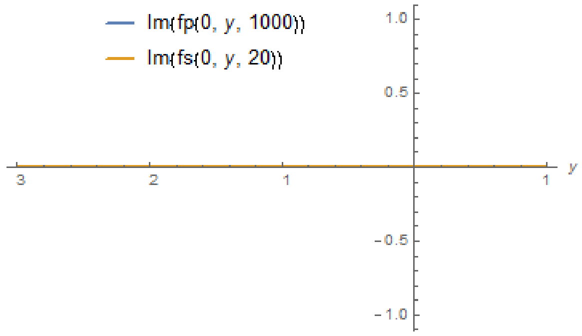
```
Tblψ[r_, z_] := Table[PolyGamma[k, z], {k, 0, r - 1}]

a[r_] := Sum[(-1)^k BellY[r, k, Tblψ[r, 1]], {k, 1, r}]

fp[x_, y_, m_] := Product[(1 + (x + i y)/(i r)) e^(-(x + i y)/(i r)), {r, 1, m}]

fs[x_, y_, m_] := 1 + Sum[i^r (EulerGamma^r/r! + Sum[(-1)^s a[s] EulerGamma^(r-s)/(s! (r - s)!), {s, 1, r}]) (x + i y)^r, {r, 1, m}]
```



The left figure is the real part and the right figure is the imaginary part. although 1000 terms are calculated on the left side (blue) and 20 terms are calculated on the right side (orange), both sides are drawn on the checkered pattern. This is because the convergence of the left side (infinite product) is slow. Here, the polynomial $B_{r,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***.

Next, the 2D figures of both sides at $x=0$ are as follows. The left figure is the real part and the right figure is the imaginary part. Although 1000 terms are calculated on the left side (blue) and 20 terms are calculated on the right side (orange) , both sides are exactly overlapped and the left side (blue) is invisible. And we can see that $z=-1,-2,-3,\cdots$ are the zeros of the right side.





Example2 Infinite-degree Equation with Negative Imaginary Odd Numbers as Roots

The above (3.3.) and (3.4.) correspond to this. Here, we illustrate (3.3.) .

$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{i(2r-1)}\right\}e^{-\frac{z}{i(2r-1)}}=1+\sum_{r=1}^{\infty}i^r\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!(r-s)!}\right\}z^r \tag{3.3.}$$

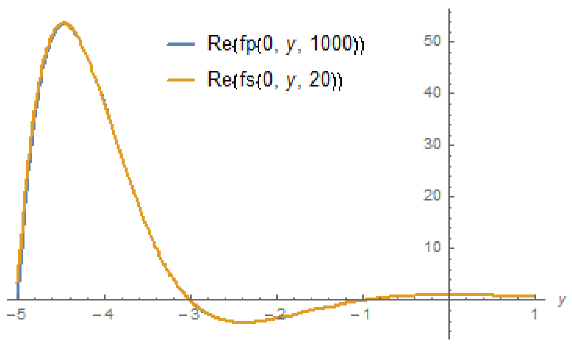
After z is replaced with $0+iy$, 2D figure of both sids are as follows. The left figure is the real part and the right figure is the imaginary part. Although 1000 terms are calculated on the left side (blue) and 20 terms are calculated on the right side (orange) , both sides are exactly overlapped and the left side (blue) is invisible. And we can see that $z=-1,-3,-5,\cdots$ are the zeros of the right side.

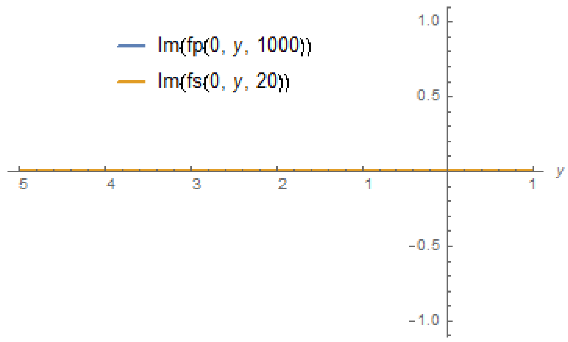
$$\mathrm{Tbl}\psi[r\_,\ z\_] := \mathrm{Table}[\mathrm{PolyGamma}[k,\ z],\ \{k,\ 0,\ r-1\}]$$

$$b[r\_] := \sum_{k=1}^{r}(-1)^k\,\mathrm{BellY}\left[r,\ k,\ \mathrm{Tbl}\psi\left[r,\ \frac{1}{2}\right]\right] \qquad \gamma := \mathrm{EulerGamma}$$

$$\mathrm{fp}[x\_,\ y\_,\ m\_] := \prod_{r=1}^{m}\left(1+\frac{x+i\,y}{i\,(2r-1)}\right)e^{-\frac{x+i\,y}{i\,(2r-1)}}$$

$$\mathrm{fs}[x\_,\ y\_,\ m\_] := 1+\sum_{r=1}^{m}i^r\left(\frac{\left(\frac{\gamma}{2}+\mathrm{Log}[2]\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s\,b[s]\left(\frac{\gamma}{2}+\mathrm{Log}[2]\right)^{r-s}}{(2s)!!\,(r-s)!}\right)(x+i\,y)^r$$





2.4 Infinite-degree Equation with Imaginary Integers as Roots (Part2)

By replacing z with iz , Formula 2.2.1 can be easily converted to a power series with imaginary integers as zeros. Now, let us make the power series equal to 0 as follows.

$$1+c_1 z^1+c_2 z^2+c_3 z^3+\cdots = 0$$

Then, this becomes an ifinite-degree equation with imaginary negative and positive Integers as roots.

Formula 2.4.1

Let $\psi_r(z)$ be the polygamma function, $B_{r,k}(f_1, f_2, \ldots)$ are Bell polynomials, γ be Euler-Mascheroni constant and a_r, b_r are the following constants.

$$a_r = \sum_{k=1}^{r} (-1)^k B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) \qquad r=1,2,3,\cdots$$

$$b_r = \sum_{k=1}^{r} (-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{r-1}\left(\frac{1}{2}\right)\right) \qquad r=1,2,3,\cdots$$

Then, the following expressions hold.

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{ir}\right)\left(1-\frac{z}{ir}\right) = \sum_{r=0}^{\infty}\frac{\pi^{2r}}{(2r+1)!}z^{2r} \qquad \left(=\frac{\sinh \pi z}{\pi z}\right) \tag{4.1}$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2ir}\right)\left(1-\frac{z}{2ir}\right) = \sum_{r=0}^{\infty}\frac{\pi^{2r}}{2^{2r}(2r+1)!}z^{2r} \qquad \left(=\frac{\sinh(\pi z/2)}{\pi z/2}\right) \tag{4.2}$$

$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{i(2r-1)}\right\}\left\{1-\frac{z}{i(2r-1)}\right\} = \sum_{r=0}^{\infty}\frac{\pi^{2r}}{(4r)!!}z^{2r} \qquad \left(=\cosh\frac{\pi z}{2}\right) \tag{4.3}$$

$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{i(2r-1)}\right\}\left(1-\frac{z}{2ir}\right) = 1 + i\frac{a_1-b_1}{2!!}z$$
$$+\sum_{r=2}^{\infty} i^r\left\{\frac{a_r+(-1)^r b_r}{(2r)!!}+\sum_{s=1}^{r-1}\frac{(-1)^{r-s}a_s b_{r-s}}{(2s)!!\{2(r-s)\}!!}\right\}z^r \tag{4.4}$$

$$\prod_{r=1}^{\infty}\left\{1-\frac{z}{i(2r-1)}\right\}\left(1+\frac{z}{2ir}\right) = 1 + i\frac{b_1-a_1}{2!!}z$$
$$+\sum_{r=2}^{\infty} i^r\left\{\frac{b_r+(-1)^r a_r}{(2r)!!}+\sum_{s=1}^{r-1}\frac{(-1)^{r-s}b_s a_{r-s}}{(2s)!!\{2(r-s)\}!!}\right\}z^r \tag{4.5}$$

Proof

From Formula 2.2.1 ,

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)\left(1-\frac{z}{r}\right) = \sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(2r+1)!}z^{2r} \tag{2.1}$$

Replacing z with iz ,

Left: $$\prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)\left(1-\frac{z}{r}\right) = \prod_{r=1}^{\infty}\left(1+\frac{iz}{r}\right)\left(1-\frac{iz}{r}\right) = \prod_{r=1}^{\infty}\left(1-\frac{z}{ir}\right)\left(1+\frac{z}{ir}\right)$$

Right: $$\sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(2r+1)!}(iz)^{2r} = \sum_{r=0}^{\infty}\frac{\left(i^2\right)^r(-1)^r\pi^{2r}}{(2r+1)!}z^{2r} = \sum_{r=0}^{\infty}\frac{(-1)^{2r}\pi^{2r}}{(2r+1)!}z^{2r}$$

Thus,

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{ir}\right)\left(1-\frac{z}{ir}\right)=\sum_{r=0}^{\infty}\frac{\pi^{2r}}{(2r+1)!}z^{2r}$$

(4.2) and (4.3) are obtained in a similer way.

(4.4) is obtained by replacing z with iz in Formula 2.2.1 (2.5) , (4.5) is obtained by swapping a and b .

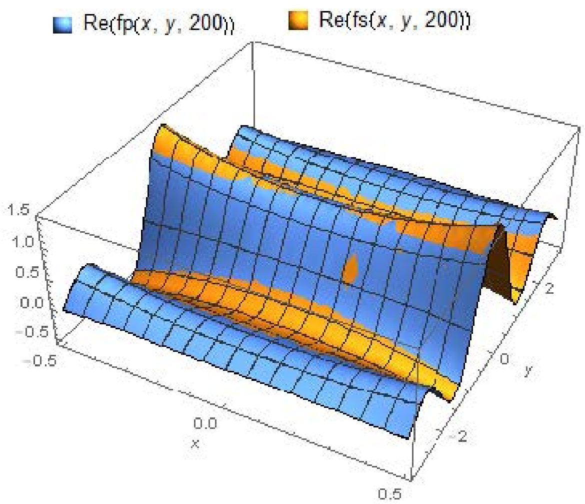
Example1 Infinite-degree Equation with All Imaginary Integers as Roots

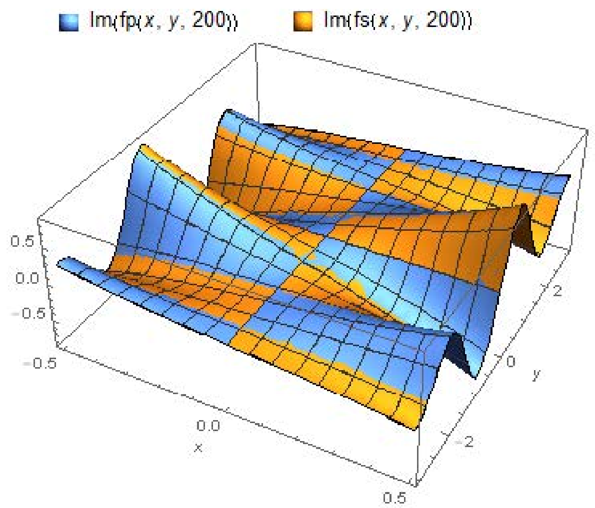
The above (4.1) corresponds to this. That is,

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{ir}\right)\left(1-\frac{z}{ir}\right)=1+\frac{\pi^2}{3!}z^2+\frac{\pi^4}{5!}z^4+\frac{\pi^6}{7!}z^6+-\cdots \tag{4.1}$$

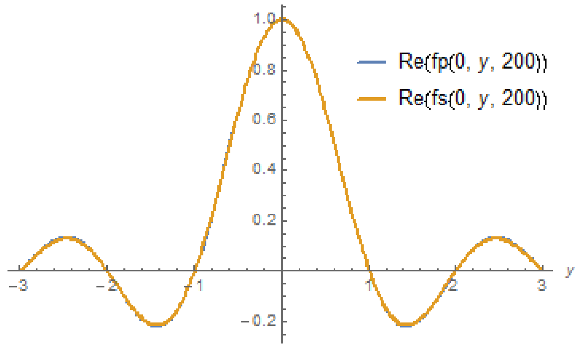
If z is replaced with $x+iy$ and both sides are illustrated in 3D figures, it is as follws. The left figure is the real part and the right figure is the imaginary part. Though both sides are calculated by 200 terms, both sides are drawn on the checkered pattern. This is because the convergence of the left side (infinite product) is slow.

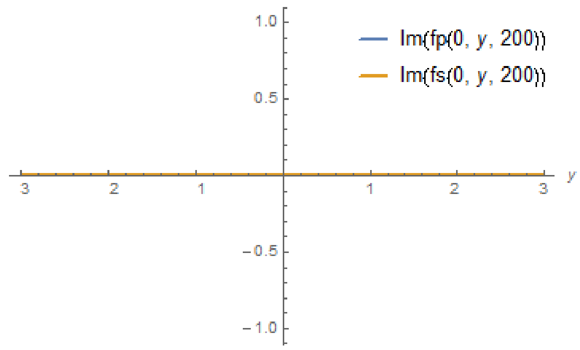
$$\mathrm{fp}[x\_,y\_,m\_]:=\prod_{r=1}^{m}\left(1+\frac{x+i\,y}{r\,i}\right)\left(1-\frac{x+i\,y}{r\,i}\right)\qquad \mathrm{fs}[x\_,y\_,m\_]:=\sum_{r=0}^{m}\frac{\pi^{2r}}{(2r+1)!}(x+i\,y)^{2r}$$





Next, the 2D figures of both sides at $x=0$ are as follows. The left figure is the real part and the right figure is the imaginary part. Although both sides are calculated by 200 terms, both sides are exactly overlapped and the left side (blue) is invisible. And we can see that $z=\pm1,\pm2,\pm3,\cdots$ are the zeros of the right side.





Example2 Negative Imaginary Odd Numbers & Positive Imaginary Even Numbers

The above (4.5) corresponds to this. That is,

$$\prod_{r=1}^{\infty}\left\{1-\frac{z}{i(2r-1)}\right\}\left(1+\frac{z}{2ir}\right)=1+i\frac{b_1-a_1}{2!!}z$$
$$+i^2\left(\frac{b_2+a_2}{4!!}-\frac{b_1}{2!!}\frac{a_1}{2!!}\right)z^2$$
$$+i^3\left(\frac{b_3-a_3}{6!!}+\frac{b_1}{2!!}\frac{a_2}{4!!}-\frac{b_2}{4!!}\frac{a_1}{2!!}\right)z^3$$
$$+i^4\left(\frac{b_4+a_4}{8!!}-\frac{b_1}{2!!}\frac{a_3}{6!!}+\frac{b_2}{4!!}\frac{a_2}{4!!}-\frac{b_3}{6!!}\frac{a_1}{2!!}\right)z^4$$
$$\vdots \tag{4.5}$$

After z is replaced with $0+iy$, 2D figure of both sids are as follows. The left figure is the real part and the right figure is the imaginary part. Although 1000 terms are calculated on the left side (blue) and 20 terms are calculated on the right side (orange), both sides are exactly overlapped and the left side (blue) is invisible. And we can see that $z=-2,-4,\cdots$ & $z=1,3,\cdots$ are the zeros of the right side. Here, the polynomial $B_{r,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***.

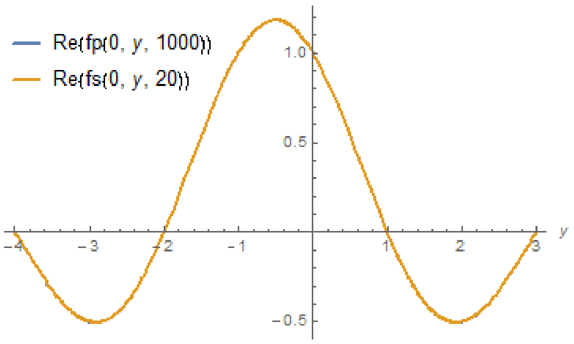
```
Tblψ[r_, z_] := Table[PolyGamma[k, z], {k, 0, r - 1}]
```

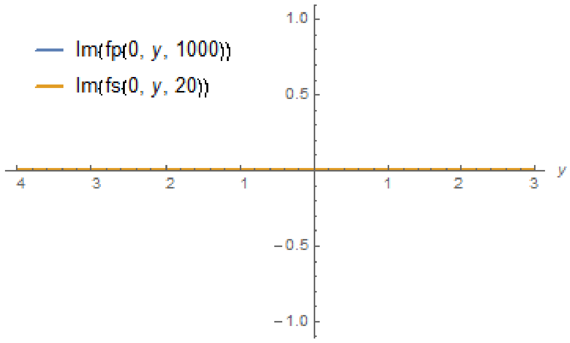
$$\text{a}[r\_] := \sum_{k=1}^{r}(-1)^k\,\text{BellY}[r, k, \text{Tbl}\psi[r, 1]]$$

$$\text{b}[r\_] := \sum_{k=1}^{r}(-1)^k\,\text{BellY}\left[r, k, \text{Tbl}\psi\left[r, \frac{1}{2}\right]\right]$$

$$\text{fp}[x\_, y\_, m\_] := \prod_{r=1}^{m}\left(1-\frac{x+i\,y}{i\,(2\,r-1)}\right)\left(1+\frac{x+i\,y}{2\,i\,r}\right)$$

$$\text{fs}[x\_, y\_, m\_] := 1+i\,\frac{\text{b}[1]-\text{a}[1]}{2\,!!}\,(x+i\,y)$$
$$+\sum_{r=2}^{m}i^r\left(\frac{\text{b}[r]+(-1)^r\,\text{a}[r]}{(2\,r)\,!!}+\sum_{s=1}^{r-1}\frac{(-1)^{r-s}\,\text{b}[s]\,\text{a}[r-s]}{(2\,s)\,!!\,(2\,(r-s))\,!!}\right)(x+i\,y)^r$$





2.5 Infinite-degree Equation with Square Roots of Integers

By replacing z with z^2, Formula 2.1.1 can be easily converted to a power series with square roots of integers. Now, let us make the power series equal to 0 as follows.

$$1+c_1 z^1+c_2 z^2+c_3 z^3+\cdots = 0$$

Then, this becomes an ifinite-degree equation with square roots of Integers.

Formula 2.5.1 (Infinite-degree Equation with Square Roots of Positive Integers)

Let $\psi_r(z)$ be the polygamma function, $B_{r,k}(f_1, f_2, \ldots)$ are Bell polynomials, γ be Euler-Mascheroni constant and a_r, b_r are the following constants.

$$a_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) \qquad r=1,2,3,\cdots$$

$$b_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{r-1}\left(\frac{1}{2}\right)\right) \qquad r=1,2,3,\cdots$$

Then, the following expressions hold.

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{\sqrt{r}}\right)\left(1-\frac{z}{\sqrt{r}}\right)e^{\frac{z^2}{r}} = 1+\sum_{r=1}^{\infty}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s \gamma^{r-s}}{s!(r-s)!}\right\}z^{2r} \qquad (5.1_+)$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{\sqrt{2r}}\right)\left(1-\frac{z}{\sqrt{2r}}\right)e^{\frac{z^2}{2r}} = 1+\sum_{r=1}^{\infty}\frac{1}{2^r}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s \gamma^{r-s}}{s!(r-s)!}\right\}z^{2r} \qquad (5.2_+)$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{\sqrt{2r-1}}\right)\left(1-\frac{z}{\sqrt{2r-1}}\right)e^{\frac{z^2}{2r-1}}$$

$$= 1+\sum_{r=1}^{\infty}\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!(r-s)!}\right\}z^{2r} \qquad (5.3_+)$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{\sqrt{2r-1}}\right)\left(1-\frac{z}{\sqrt{2r-1}}\right)e^{\frac{z^2}{2r}}$$

$$= 1+\sum_{r=1}^{\infty}\left\{\frac{\gamma^r}{(2r)!!}+\sum_{s=1}^{r}\frac{(-1)^s b_s \gamma^{r-s}}{(2s)!!(2r-2s)!!}\right\}z^{2r} \qquad (5.4_+)$$

Proof

Replacing z with z^2 in (1.1_+), (1.2_+), (1.3_+) and (1.4_+), we obtain the desired expressions.

Example Infinite-degree Equation with Square Roots of Natural Numbers

The above (5.1_+) corresponds to this. That is,

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{\sqrt{r}}\right)\left(1-\frac{z}{\sqrt{r}}\right)e^{\frac{z^2}{r}} = 1+\left(\frac{\gamma^1}{1!}-\frac{a_1\gamma^0}{1!0!}\right)z^2$$

$$+\left(\frac{\gamma^2}{2!}-\frac{a_1\gamma^1}{1!1!}+\frac{a_2\gamma^0}{2!0!}\right)z^4$$

$$+\left(\frac{\gamma^3}{3!}-\frac{a_1\gamma^2}{1!2!}+\frac{a_2\gamma^1}{2!1!}-\frac{a_3\gamma^0}{3!0!}\right)z^6$$

$$\vdots \qquad (5.1_+)$$

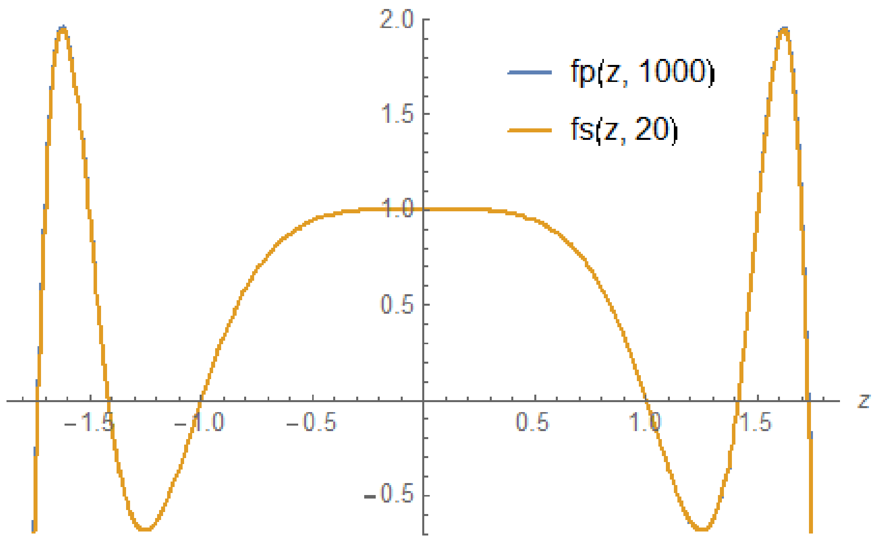
When both sides are illustrated. it is as follows. Although 1000 terms are calculated on the left side (blue) and 20 terms are calculated on the right side (orange) , both sides are exactly overlapped and the left side (blue) is invisible. And we can see that $z=\pm\sqrt{1},\pm\sqrt{2},\pm\sqrt{3},\cdots$ are the zeros of the right side. Here, the polynomial $B_{r,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica*** .

$$\mathtt{Tbl\psi[r\_,\, z\_] := Table[PolyGamma[k,\, z],\, \{k,\, 0,\, r-1\}]}$$

$$\mathtt{a[r\_] :=} \sum_{\mathtt{k}=1}^{\mathtt{r}} \mathtt{(-1)^{k}\, BellY[r,\, k,\, Tbl\psi[r,\, 1]]}$$

$$\mathtt{fp[z\_,\, m\_] :=} \prod_{\mathtt{r}=1}^{\mathtt{m}} \left(1+\frac{\mathtt{z}}{\sqrt{\mathtt{r}}}\right)\left(1-\frac{\mathtt{z}}{\sqrt{\mathtt{r}}}\right) e^{\frac{\mathtt{z}^2}{\mathtt{r}}}$$

$$\mathtt{fs[z\_,\, m\_] := 1 +} \sum_{\mathtt{r}=1}^{\mathtt{m}} \left(\frac{\mathtt{EulerGamma^{r}}}{\mathtt{r\,!}} + \sum_{\mathtt{s}=1}^{\mathtt{r}} \frac{\mathtt{(-1)^{s}\, a[s]\, EulerGamma^{r-s}}}{\mathtt{s\,!\,(r-s)\,!}}\right) \mathtt{z^{2\,r}}$$



Formula 2.5.2 (Infinite-degree Equation with Square Roots of Negative Integers)

Let $\psi_r(z)$ be the polygamma function, $B_{r,k}(f_1, f_2, \ldots)$ are Bell polynomials, γ be Euler-Mascheroni constant and a_r, b_r are the following constants.

$$a_r = \sum_{k=1}^{r} (-1)^k B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) \qquad r=1,2,3,\cdots$$

$$b_r = \sum_{k=1}^{r} (-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{r-1}\left(\frac{1}{2}\right)\right) \qquad r=1,2,3,\cdots$$

Then, the following expressions hold.

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{i\sqrt{r}}\right)\left(1-\frac{z}{i\sqrt{r}}\right)e^{-\frac{z^2}{r}} = 1+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{\gamma^r}{r\,!}+\sum_{s=1}^{r}\frac{(-1)^s a_s \gamma^{r-s}}{s\,!\,(r-s)!}\right\} z^{2r} \qquad (5.1_{-})$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{i\sqrt{2r}}\right)\left(1-\frac{z}{i\sqrt{2r}}\right)e^{-\frac{z^2}{2r}} = 1+\sum_{r=1}^{\infty}\frac{(-1)^r}{2^r}\left\{\frac{\gamma^r}{r\,!}+\sum_{s=1}^{r}\frac{(-1)^s a_s \gamma^{r-s}}{s\,!\,(r-s)!}\right\} z^{2r} \qquad (5.2_{-})$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{i\sqrt{2r-1}}\right)\left(1-\frac{z}{i\sqrt{2r-1}}\right)e^{-\frac{z^2}{2r-1}}$$

$$=1+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r\,!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!\,(r-s)\,!}\right\}z^{2r} \quad (5.3.)$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{i\sqrt{2r-1}}\right)\left(1-\frac{z}{i\sqrt{2r-1}}\right)e^{-\frac{z^2}{2r}}$$

$$=1+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{\gamma^r}{(2r)!!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\gamma^{r-s}}{(2s)!!\,(2r-2s)!!}\right\}z^{2r} \quad (5.4.)$$

Proof

Replacing z with z^2 in (1.1.) , (1.2.) , (1.3.) and (1.4.) , we obtain the desired expressions.

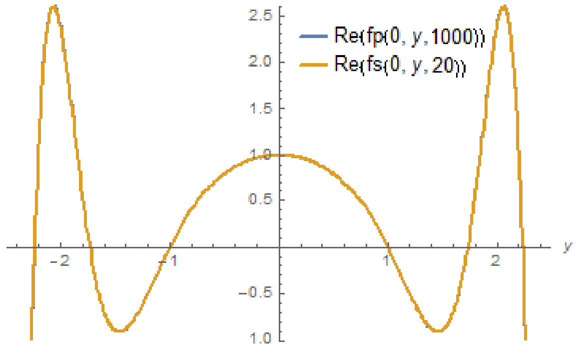
Example Infinite-degree Equation with Square Roots of Negative Odd Numbers

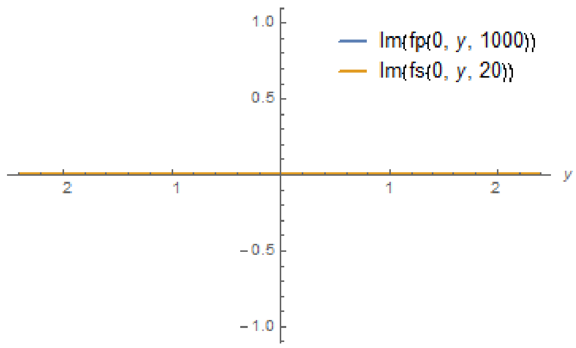
The above (5.3.) and (5.4.) correspond to this. Here, we illustrate (5.4.) .

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{i\sqrt{2r-1}}\right)\left(1-\frac{z}{i\sqrt{2r-1}}\right)e^{-\frac{z^2}{2r}}=1-\left(\frac{\gamma^1}{2!!}-\frac{b_1\gamma^0}{2!!0!!}\right)z^2$$
$$+\left(\frac{\gamma^2}{4!!}-\frac{b_1\gamma^1}{2!!2!!}+\frac{b_2\gamma^0}{4!!0!!}\right)z^4$$
$$-\left(\frac{\gamma^3}{6!!}-\frac{b_1\gamma^2}{2!!4!!}+\frac{b_2\gamma^1}{4!!2!!}-\frac{b_3\gamma^0}{6!!0!!}\right)z^6$$
$$\vdots \quad (5.4.)$$

After z is replaced with $0+iy$, 2D figure of both sids are as follows. The left figure is the real part and the right figure is the imaginary part. Although 1000 terms are calculated on the left side (blue) and 20 terms are calculated on the right side (orange) , both sides are exactly overlapped and the left side (blue) is invisible. And we can see that $y=\pm\sqrt{1},\pm\sqrt{3},\pm\sqrt{5},\cdots$ are the zeros of the right side. Here, the polynomial $B_{r,k}(f_1,f_2,\ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***.

```
γ := EulerGamma

Tblψ[r_, z_] := Table[PolyGamma[k, z], {k, 0, r - 1}]

b[r_] := Sum[(-1)^k BellY[r, k, Tblψ[r, 1/2]], {k, 1, r}]

fp[x_, y_, m_] := Product[(1 + (x + i y)/(i Sqrt[2 r - 1])) (1 - (x + i y)/(i Sqrt[2 r - 1])) e^(-(x+i y)^2/(2 r)), {r, 1, m}]

fs[x_, y_, m_] := 1 + Sum[(-1)^r (γ^r/(2 r)!! + Sum[((-1)^s b[s] γ^(r-s))/((2 s)!! (2 r - 2 s)!!), {s, 1, r}]) (x + i y)^(2 r), {r, 1, m}]
```


Re(fp(0, y, 1000))
Re(fs(0, y, 20))
y


Im(fp(0, y, 1000))
Im(fs(0, y, 20))
y

2.6 Infinite-degree Equation with Square Numbers as Roots

By replacing z with $\sqrt{z}$, a part of Formula 2.2.1 can be easily converted to a power series with square numbers as roots. Now, let us make the power series equal to 0 as follows.

$$1+c_1 z^1+c_2 z^2+c_3 z^3+\cdots = 0$$

Then, this becomes an ifinite-degree equation with square numbers as roots.

Formula 2.6.1 (Infinite-degree Equation with Square Numbers as Roots)

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{r^2}\right) = \sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(2r+1)!}z^r \qquad \left(=\frac{\sin\left(\pi\sqrt{z}\right)}{\pi\sqrt{z}}\right) \tag{6.1+}$$

$$\prod_{r=1}^{\infty}\left\{1-\frac{z}{(2r)^2}\right\} = \sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{2^{2r}(2r+1)!}z^r \qquad \left(=\frac{\sin\left(\pi\sqrt{z/4}\right)}{\pi\sqrt{z/4}}\right) \tag{6.2+}$$

$$\prod_{r=1}^{\infty}\left\{1-\frac{z}{(2r-1)^2}\right\} = \sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(4r)!!}z^r \qquad \left(=\cos\frac{\pi\sqrt{z}}{2}\right) \tag{6.3+}$$

Proof

Replacing z with $\sqrt{z}$ in (2.1) , (2.2) and (2.3) , we obtain the desired expressions.

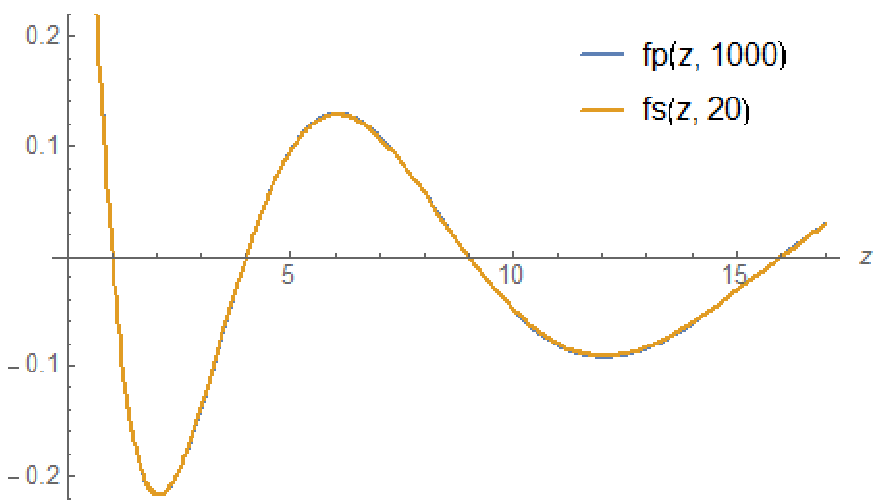
Example Infinite-degree Equation with Square of Natural Numbers as Roots

The above (6.1+) corresponds to this. That is,

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{r^2}\right) = 1 - \frac{\pi^2}{3!}z^1+\frac{\pi^4}{5!}z^2-\frac{\pi^6}{7!}z^3+-\cdots \tag{6.1+}$$

When both sides are illustrated. it is as follows. Although 1000 terms are calculated on the left side (blue) and 20 terms are calculated on the right side (orange) , both sides are exactly overlapped and the left side (blue) is invisible. And we can see that $z=1^2,2^2,3^2,4^2,\cdots$ are the zeros of the right side.

$$\mathrm{fp}[z\_, m\_] := \prod_{r=1}^{m}\left(1-\frac{z}{r^2}\right) \qquad \mathrm{fs}[z\_, m\_] := \sum_{r=0}^{m}\frac{(-1)^r\pi^{2r}}{(2r+1)!}z^r$$



Formula 2.6.2 (Infinite-degree Equation with Negative Square Numbers as Roots)

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{r^2}\right)=\sum_{r=0}^{\infty}\frac{\pi^{2r}}{(2r+1)\,!}z^r \qquad \left(=\frac{\sinh\left(\pi\sqrt{z}\right)}{\pi\sqrt{z}}\right) \qquad (6.1_{-})$$

$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{(2r)^2}\right\}=\sum_{r=0}^{\infty}\frac{\pi^{2r}}{2^{2r}(2r+1)\,!}z^r \qquad \left(=\frac{\sinh\left(\pi\sqrt{z/4}\right)}{\pi\sqrt{z/4}}\right) \qquad (6.2_{-})$$

$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{(2r-1)^2}\right\}=\sum_{r=0}^{\infty}\frac{\pi^{2r}}{(4r)!!}z^r \qquad \left(=\cosh\frac{\pi\sqrt{z}}{2}\right) \qquad (6.3_{-})$$

Proof

Replacing z with $-z$ in Formula 2.6.1 , we obtain the desired expressions.

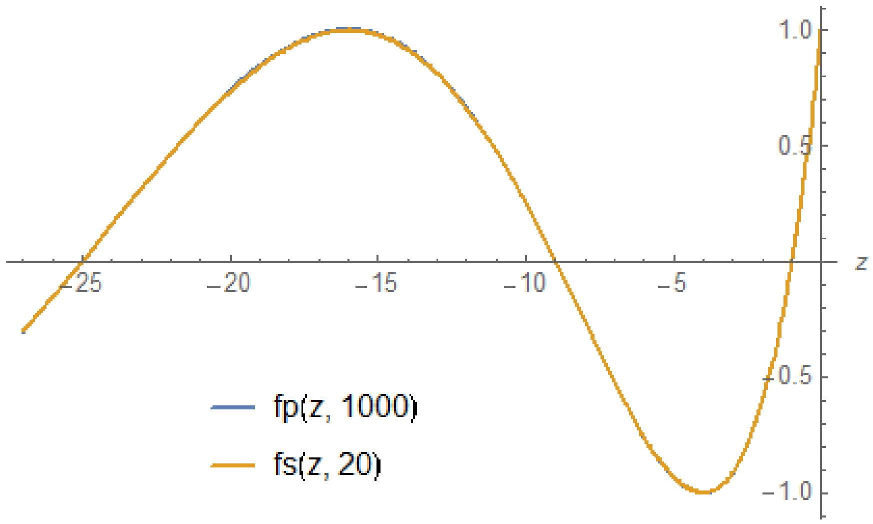
Example Infinite-degree Equation with -(Square of Odd Numbers) as Roots

The above (6.3_{+}) corresponds to this. That is,

$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{(2r-1)^2}\right\}=1+\frac{\pi^2}{4!!}z^1+\frac{\pi^4}{8!!}z^2+\frac{\pi^6}{12!!}z^3+\cdots \qquad (6.3_{-})$$

When both sides are illustrated. it is as follows. Although 1000 terms are calculated on the left side (blue) and 20 terms are calculated on the right side (orange) , both sides are exactly overlapped and the left side (blue) is invisible. And we can see that $z=-1^2,-3^2,-5^2,\cdots$ are the zeros of the right side.

$$\texttt{fp}[z\_,m\_]:=\prod_{r=1}^{m}\left(1+\frac{z}{(2r-1)^2}\right) \qquad \texttt{fs}[z\_,m\_]:=\sum_{r=0}^{m}\frac{\pi^{2r}}{(4r)!!}z^r$$



2017.08.31

2017.10.20 Added Sec.6

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Alien's Mathematics