25 Series of Higher Integral with Geometric Coefficients

This chapter is a generalization of " **24 Sugioka's Theorem on the Series of Higher Integral** ". The origin of theses paper is " **e ^x** に関する公式の発見 " (Mikio Sugioka 2003) .

25.1 Series of the n-th order Integrals

In this section, we ask for the sum of series of higher integral with geometric coefficients, such as

$$c^1\int_a^x f(x)\,dx \pm c^2\int_a^x\int_a^x f(x)\,dx^2 + c^3\int_a^x\int_a^x\int_a^x f(x)\,dx^3 \pm c^4\int_a^x\cdots\int_a^x f(x)\,dx^4 +\pm \cdots$$

Theorem 25.1.1

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\sum_{r=1}^{m} c^r\int_a^x\cdots\int_a^x f(x)\,dx^r = ce^{cx}\int_a^x f(x)e^{-cx}dx - c^{m+1}e^{cx}\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}e^{-cx}dx \tag{1.1}$$

Especially, when $\lim_{m\to\infty} c^m\int_a^x\cdots\int_a^x f(x)\,dx^m = 0$,

$$\sum_{r=1}^{\infty} c^r\int_a^x\cdots\int_a^x f(x)\,dx^r = ce^{cx}\int_a^x f(x)e^{-cx}dx \tag{1.1'}$$

Proof

Let

$$f^{<r>}(x) = \int_a^x\cdots\int_a^x f(x)\,dx^r \qquad r=1,2,\cdots,m \tag{1.r}$$

Then $f^{<r>}(a)=0\ \ (r=1,2,\cdots,m)$. So,

$$\int_a^x f(x)e^{-cx}dx = \left[f^{<1>}(x)e^{-cx}\right]_a^x + c\int_a^x f^{<1>}(x)e^{-cx}dx$$

$$= f^{<1>}(x)e^{-cx} + c\int_a^x f^{<1>}(x)e^{-cx}dx$$

$$= f^{<1>}(x)e^{-cx} + c\left[f^{<2>}(x)e^{-cx}\right]_a^x + c^2\int_a^x f^{<2>}(x)e^{-cx}dx$$

$$= f^{<1>}(x)e^{-cx} + c^1 f^{<2>}(x)e^{-cx} + c^2\int_a^x f^{<2>}(x)e^{-cx}dx$$

$$\vdots$$

$$= e^{-cx}\sum_{r=1}^{m} c^{r-1}f^{<r>}(x) + c^m\int_a^x f^{<m>}e^{-cx}dx$$

Substituting (1.r) for this,

$$\int_a^x f(x)e^{-cx}dx = e^{-cx}\sum_{r=1}^{m} c^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^r + c^m\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}e^{-cx}dx$$

Multiplying by e^{cx} the both sides,

$$e^{cx}\int_a^x f(x)e^{-cx}dx = \sum_{r=1}^{m} c^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^r + c^m e^{cx}\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}e^{-cx}dx$$

Multiplying by c the both sides and transposing them, we obtain (1.1) .

Example 1 $c^1\int_a^x dx + c^2\int_a^x\int_a^x dx^2 + c^3\int_a^x\int_a^x\int_a^x dx^3 + \cdots$

$$f(x)=1\,, \qquad \int_a^x\cdots\int_a^x dx^m = \frac{(x-a)^m}{m\,!}$$

Substituting these for (1.1) ,

$$\sum_{r=1}^{m} c^r\int_a^x\cdots\int_a^x dx^r = ce^{cx}\int_a^x e^{-cx}dx - c^{m+1}e^{cx}\int_a^x \frac{(x-a)^m}{m\,!}e^{-cx}dx$$

The 1st term of the right sideis

$$ce^{cx}\int_a^x e^{-cx}dx = ce^{cx}\left[-\frac{e^{-cx}}{c}\right]_a^x = e^{c(x-a)}-1$$

The 2ndt term of the right sideis is expressed as follows using the incomplete gamma function $\Gamma(x,y)$.

$$\int_a^x \frac{(x-a)^m}{m\,!}e^{-cx}dx = \frac{e^{-ac}(x-a)^m(c(x-a))^{-m}\{\Gamma(m+1)-\Gamma(m+1,\,c(x-a))\}}{c\,m\,!}$$

$$= \frac{e^{-ac}}{c\,m\,!}\frac{(x-a)^m}{\{c(x-a)\}^m}\{\Gamma(m+1)-\Gamma(m+1,\,c(x-a))\}$$

When $c>0$ & $x-a>0$,

$$\int_a^x \frac{(x-a)^m}{m\,!}e^{-cx}dx = \frac{e^{-ac}}{c^{m+1}\,m\,!}\{\Gamma(m+1)-\Gamma(m+1,c(x-a))\}$$

Thus,

$$\sum_{r=1}^{m} c^r\int_a^x\cdots\int_a^x dx^r = e^{c(x-a)}-1 - \frac{e^{c(x-a)}}{m\,!}\{\Gamma(m+1)-\Gamma(m+1,c(x-a))\} \tag{1.2}$$

And, since $\lim_{m\to\infty}\frac{e^{c(x-a)}}{m\,!}\{\Gamma(m+1)-\Gamma(m+1,c(x-a))\} = 0$,

$$\sum_{r=1}^{\infty} c^r\int_a^x\cdots\int_a^x dx^r = e^{c(x-a)}-1 \tag{1.2'}$$

The higher order integral of the left side of (1.2) is as follows.

$$\int_a^x\cdots\int_a^x dx^r = \frac{(x-a)^r}{r\,!}$$

Then, the left side is

$$\sum_{r=1}^{m} c^r\int_a^x\cdots\int_a^x dx^r = \sum_{r=1}^{m} c^r\frac{(x-a)^r}{r\,!}$$

Using this as the left side of (1.2) and (1.2') ,

$$\sum_{r=1}^{m} c^r\frac{(x-a)^r}{r\,!} = e^{c(x-a)}-1 - \frac{e^{c(x-a)}}{m\,!}\{\Gamma(m+1)-\Gamma(m+1,c(x-a))\} \tag{1.3}$$

$$\sum_{r=1}^{\infty} c^r\frac{(x-a)^r}{r\,!} = e^{c(x-a)}-1 \tag{1.3'}$$

When $x=2,\ a=0,\ c=3.3,\ m=4$, both sides of (1.3) are calculated as follows.

```
Γ[x_] := Gamma[x]          Γ[x_, y_] := Gamma[x, y]

f[x_, a_, c_, m_] := Sum[c^r (x - a)^r / r!, {r, 1, m}]

g[x_, a_, c_, m_] := E^(c (x-a)) - 1 - E^(c (x-a))/m! (Γ[m + 1] - Γ[m + 1, c (x - a)])

N[f[2, 0, 3.3, 4]]               N[g[2, 0, 3.3, 4]]
155.357                          155.357
```

Also, it is clear from the following that (1.3') is correct.

$$\sum_{r=1}^{\infty} c^r \frac{(x-a)^r}{r!} = \sum_{r=0}^{\infty} \frac{\{c(x-a)\}^r}{r!} - \frac{\{c(x-a)\}^0}{0!} = e^{c(x-a)} - 1$$

Example 2 $c^1 \int_a^x e^x dx + c^2 \int_a^x \int_a^x e^x dx^2 + c^3 \int_a^x \int_a^x \int_a^x e^x dx^3 + \cdots$

$$f(x) = e^x \quad , \quad \int_a^x \cdots \int_a^x e^x dx^m = e^x - e^a \sum_{r=0}^{m-1} \frac{(x-a)^r}{r!}$$

Substituting these for (1.1) and using the incomplete gamma function $\Gamma(x,y)$,

$$\sum_{r=1}^{m} c^r \int_a^x \cdots \int_a^x e^x dx^r = ce^{cx} \int_a^x e^x e^{-cx} dx - c^{m+1} e^{cx} \int_a^x \left\{ \int_a^x \cdots \int_a^x e^x dx^m \right\} e^{-cx} dx$$

$$= ce^{cx} \int_a^x e^{(1-c)x} dx - c^{m+1} e^{cx} \int_a^x \left\{ e^x - e^a \sum_{r=0}^{m-1} \frac{(x-a)^r}{r!} \right\} e^{-cx} dx$$

$$= ce^{cx} \int_a^x e^{(1-c)x} dx - c^{m+1} e^{cx} \int_a^x e^{(1-c)x} dx + c^{m+1} e^{cx+a} \sum_{r=0}^{m-1} \int_a^x \frac{(x-a)^r}{r!} e^{-cx} dx$$

$$= \frac{ce^{cx}\left\{e^{(1-c)x} - e^{(1-c)a}\right\}}{1-c} - \frac{c^{m+1}e^{cx}\left\{e^{(1-c)x} - e^{(1-c)a}\right\}}{1-c}$$

$$+ c^{m+1} e^{cx+a} \sum_{r=0}^{m-1} \frac{e^{-ac}(x-a)^r (c(x-a))^{-r} (\Gamma(1+r) - \Gamma(1+r, c(x-a)))}{c\, r!}$$

$$= \frac{ce^{cx}\left\{e^{(1-c)x} - e^{(1-c)a}\right\}}{1-c} - \frac{c^{m+1}e^{cx}\left\{e^{(1-c)x} - e^{(1-c)a}\right\}}{1-c}$$

$$- c^m e^{cx+(1-c)a} \sum_{r=0}^{m-1} \frac{1}{c^r} \left\{ \frac{\Gamma(1+r, c(x-a))}{r!} - 1 \right\}$$

i.e.

$$\sum_{r=1}^{m} c^r \int_a^x \cdots \int_a^x e^x dx^r = \frac{ce^{cx}\left\{e^{(1-c)x} - e^{(1-c)a}\right\}}{1-c} - \frac{c^{m+1}e^{cx}\left\{e^{(1-c)x} - e^{(1-c)a}\right\}}{1-c}$$

$$- c^m e^{cx+(1-c)a} \sum_{r=0}^{m-1} \frac{1}{c^r} \left\{ \frac{\Gamma(1+r, c(x-a))}{r!} - 1 \right\} \tag{1.4}$$

Especially, when $a = -\infty$,

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} e^x dx^r = e^x \qquad r=1,2,3,\cdots \qquad \text{: Lineal Higher Integral}$$

$$e^{(1-c)a} = 0 \quad , \quad \frac{\Gamma(1+r, c(x-a))}{c^r r!} = 0$$

So, (1.4) becoms as follows.

$$e^x \sum_{r=1}^{m} c^r = ce^x \frac{1-c^m}{1-c}$$

And from this, the following well-known equation is obtained.

$$\sum_{r=0}^{m-1} c^r = \frac{1-c^m}{1-c}$$

When $0<c<1$, $\lim_{m\to\infty} c^m = 0$. Then,

$$\sum_{r=1}^{\infty} c^r \int_a^x \cdots \int_a^x e^x dx^r = \frac{ce^{cx}\left\{e^{(1-c)x} - e^{(1-c)a}\right\}}{1-c} \tag{1.4'}$$

Especially, when $a=-\infty$,

$$e^{cx} \sum_{r=1}^{\infty} c^r = \frac{ce^{cx}}{1-c}$$

And from this, the following well-known equation is obtained.

$$\sum_{r=0}^{\infty} c^r = \frac{1}{1-c}$$

The higher order integral of the left side of (1.4) is as follows.

$$\int_a^x \cdots \int_a^x e^x dx^r = e^x - e^a \sum_{s=0}^{r-1} \frac{(x-a)^s}{s!}$$

Then, the left side is

$$\sum_{r=1}^{m} c^r \int_a^x \cdots \int_a^x dx^r = \sum_{r=1}^{m} c^r \left\{ e^x - e^a \sum_{s=0}^{r-1} \frac{(x-a)^s}{s!} \right\}$$

Using this as the left side of (1.4) and (1.4'),

$$\sum_{r=1}^{m} c^r \left\{ e^x - e^a \sum_{s=0}^{r-1} \frac{(x-a)^s}{s!} \right\} = \frac{1-c^m}{1-c} ce^{cx}\left\{e^{(1-c)x} - e^{(1-c)a}\right\}$$
$$- c^m e^{cx+(1-c)a} \sum_{r=0}^{m-1} \frac{1}{c^r} \left\{ \frac{\Gamma(1+r, c(x-a))}{r!} - 1 \right\} \tag{1.5}$$

$$\sum_{r=1}^{\infty} c^r \left\{ e^x - e^a \sum_{s=0}^{r-1} \frac{(x-a)^s}{s!} \right\} = \frac{ce^{cx}\left\{e^{(1-c)x} - e^{(1-c)a}\right\}}{1-c} \tag{1.5'}$$

When $x=4, a=1, c=2, m=3$, both sides of (1.5) are calculated as follows. Both sides coincide exactly. So, (1.4) is confirmed to be correct.

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Γ[x_, y_] := Gamma[x, y]
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f[x_, a_, c_, m_] := Sum[c^r (e^x - e^a Sum[(x - a)^s / s!, {s, 0, r-1}]), {r, 1, m}]

g[x_, a_, c_, m_] := (1 - c^m)/(1 - c) c e^(c x) (e^((1-c) x) - e^((1-c) a))
                      - c^m e^(c x+(1-c) a) Sum[1/c^r (Γ[1 + r, c (x - a)]/r! - 1), {r, 0, m-1}]

N[f[4, 1, 2, 3]]            N[g[4, 1, 2, 3]]
530.602                     530.602
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$$\sum_{r=1}^{m} c^r \left(e^x - e^a \sum_{s=0}^{r-1} \frac{(x-a)^s}{s!} \right)$$

$$\frac{1-c^m}{1-c}\, c\, e^{cx} \left(e^{(1-c)x} - e^{(1-c)a} \right) - c^m e^{cx+(1-c)a} \sum_{r=0}^{m-1} \frac{1}{c^r} \left(\frac{\Gamma[1+r,\, c(x-a)]}{r!} - 1 \right)$$

Example 3 $c^1 \int_0^x \log x\, dx + c^2 \int_0^x \int_0^x \log x\, dx^2 + c^3 \int_0^x \int_0^x \int_0^x \log x\, dx^3 + \cdots$

Substituting $f(x) = \log x$, $a = 0$ for (1.1') ,

$$\sum_{r=1}^{\infty} c^r \int_a^x \cdots \int_a^x \log x\, dx^r = c e^{cx} \int_a^x \log x\, e^{-cx} dx$$

The integral of the right side is as follows.

$$\int_0^x \log x\, e^{-cx} dx = \left[\frac{-e^{-cx} \log|x| + Ei(-cx)}{c} \right]_0^x$$

$$= \frac{-e^{-cx} \log|x| + Ei(-cx)}{c} - \frac{\log c + \gamma}{c}$$

Where, $Ei(x) = \int_{-\infty}^x \frac{e^t}{t} dt$, $\gamma = 0.5772\cdots$ (Euler-Mascheroni Constant).

Multiplying by e^{cx} the both sides,

$$c e^{cx} \int_0^x \log x\, e^{-cx} dx = -\log|x| + e^{cx} \{ Ei(-cx) - \gamma - \log c \}$$

Substituting this for the right side of the above,

$$\sum_{r=1}^{\infty} c^r \int_a^x \cdots \int_a^x \log x\, dx^r = -\log|x| + e^{cx} \{ Ei(-cx) - \gamma - \log c \} \tag{1.6}$$

The higher order integral of the left side becomes lineal higher integral as follows.

$$\int_0^x \cdots \int_0^x \log x\, dx^n = \frac{x^n}{n!} \left(\log|x| - \sum_{s=1}^{n} \frac{1}{s} \right)$$

Then, (1.6) is

$$\sum_{r=1}^{\infty} \frac{c^r x^r}{r!} \left(\log|x| - \sum_{s=1}^{r} \frac{1}{s} \right) = -\log|x| + e^{cx} \{ Ei(-cx) - \gamma - \log c \}$$

When $c = 1.7$, the first 10 terms of $\sum$ are calculated and both sides are illustrated, it is as follows. Both sides overlap exactly and blue (left) can not be seen.

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Ei[x_] := ExpIntegralEi[x]          γ := EulerGamma
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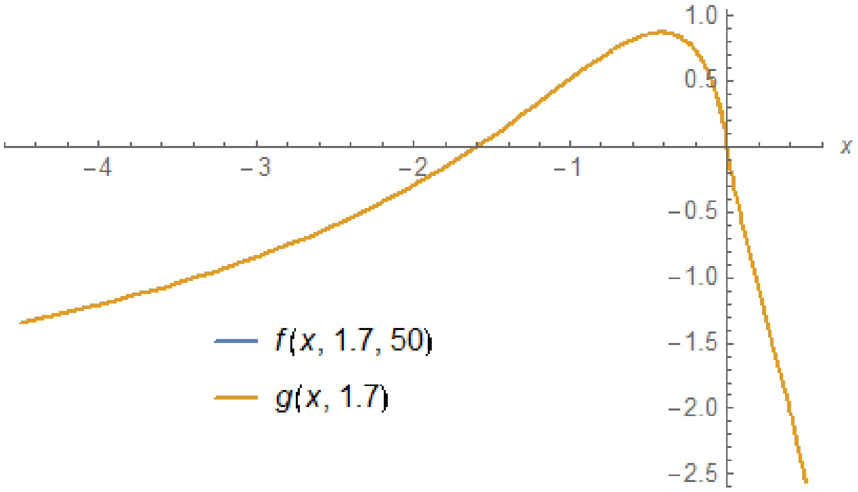
$$\text{f[x\_, c\_, m\_]} := \sum_{r=1}^{m} \frac{c^r x^r}{r!} \left(\text{Log[Abs[x]]} - \sum_{s=1}^{r} \frac{1}{s} \right)$$

$$\text{g[x\_, c\_]} := -\text{Log[Abs[x]]} + e^{c\,x} \left(\text{Ei}[-c\,x] - \gamma - \text{Log}[c] \right)$$



Theorem 25.1.2

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\sum_{r=1}^{m} (-1)^{r-1} c^r \int_a^x \cdots \int_a^x f(x)\,dx^r = c e^{-cx} \int_a^x f(x) e^{cx} dx$$
$$- (-1)^m c^{m+1} e^{-cx} \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\,dx^m \right\} e^{cx} dx \qquad (2.1)$$

Especially, when $\lim_{m\to\infty} c^m \int_a^x \cdots \int_a^x f(x)\,dx^m = 0$,

$$\sum_{r=1}^{m} (-1)^{r-1} c^r \int_a^x \cdots \int_a^x f(x)\,dx^r = c e^{-cx} \int_a^x f(x) e^{cx} dx \qquad (2.1')$$

Proof

In a similar way to the proof of Theorem 25.1.1 , we obtain the desired expressions.

Example 4

$$c^1 \int_a^x \sin x\,dx - c^2 \int_a^x \int_a^x \sin x\,dx^2 + c^3 \int_a^x \int_a^x \int_a^x \sin x\,dx^3 - c^4 \int_a^x \cdots \int_a^x \sin x\,dx^4 + - \cdots$$

Substituting $f(x) = \sin x$ for (2.1') ,

$$\sum_{r=1}^{\infty} (-1)^{r-1} c^r \int_a^x \cdots \int_a^x \sin x\,dx^r = c e^{-cx} \int_a^x \sin x\, e^{cx} dx$$

The right side is as follows.

$$\int_a^x \sin x \cdot e^{cx} dx = \frac{(c \sin x - \cos x) e^{cx} - (c \sin a - \cos a) e^{ca}}{1+c^2}$$

Multiplying by $c e^{-cx}$ the both sides,

$$ce^{-cx}\int_a^x \sin x \cdot e^{cx}\,dx = \frac{c}{1+c^2}\left\{c\sin x - \cos x - (c\sin a - \cos a)e^{c(a-x)}\right\}$$

Thus, we obtain

$$\sum_{r=1}^{\infty}(-1)^{r-1}c^r\int_a^x \cdots \int_a^x \sin x\,dx^r = \frac{c}{1+c^2}\left\{c\sin x - \cos x - (c\sin a - \cos a)e^{c(a-x)}\right\} \tag{2.2'}$$

The higher integral of the left side is as follows according to Theorem 4.1.3 (**4.1**)

$$\int_a^x \cdots \int_a^x \sin x\,dx^n = \sin\left(x - \frac{\pi n}{2}\right) - \sum_{s=0}^{n-1}\frac{(x-a)^s}{s!}\sin\left\{a - \frac{\pi(n-s)}{2}\right\}$$

Then, (2.2') is

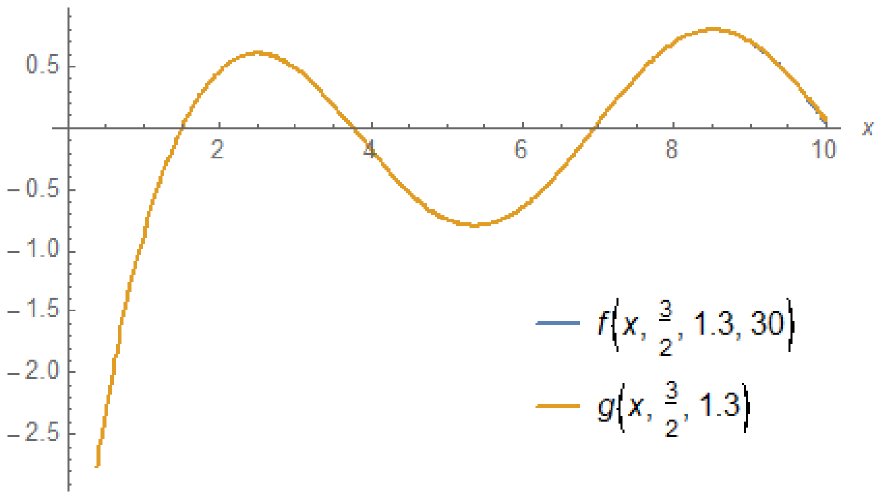
$$\sum_{r=1}^{\infty}(-1)^{r-1}c^r\left\{\sin\left(x - \frac{\pi r}{2}\right) - \sum_{s=0}^{r-1}\frac{(x-a)^s}{s!}\sin\left\{a - \frac{\pi(r-s)}{2}\right\}\right\}$$

$$= \frac{c}{1+c^2}\left\{c\sin x - \cos x - (c\sin a - \cos a)e^{c(a-x)}\right\}$$

When $a=3/2$, $c=1.3$, the first 30 terms of $\sum$ are calculated and both sides are illustrated, it is as follows. Both sides overlap exactly and blue (left) can not be seen.

$$\texttt{f[x\_, a\_, c\_, m\_] :=} \sum_{r=1}^{m}(-1)^{r-1}c^r\left(\mathrm{Sin}\left[x - \frac{\pi r}{2}\right] - \sum_{s=0}^{r-1}\frac{(x-a)^s}{s!}\mathrm{Sin}\left[a - \frac{\pi(r-s)}{2}\right]\right)$$

$$\texttt{g[x\_, a\_, c\_] :=} \frac{c}{1+c^2}\left(c\,\mathrm{Sin}[x] - \mathrm{Cos}[x] - (c\,\mathrm{Sin}[a] - \mathrm{Cos}[a])\,e^{c\,(a-x)}\right)$$



25.2 Series of the odd-th order Integrals

In this section, we ask for the sum of series of higher integral with geometric coefficients, such as

$$c^1\int_a^x f(x)\,dx \pm c^3\int_a^x\int_a^x\int_a^x f(x)\,dx^3 + c^5\int_a^x\cdots\int_a^x f(x)\,dx^5 \pm c^7\int_a^x\cdots\int_a^x f(x)\,dx^7 + \pm \cdots$$

Theorem 25.2.1

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\sum_{r=1}^{m} c^{2r-1}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \frac{c}{2}\left\{e^{cx}\int_a^x f(x)e^{-cx}dx + e^{-cx}\int_a^x f(x)e^{cx}dx\right\}$$
$$- (-1)^{2m-1}c^{2m}\cosh cx\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^{2m-1}\right\}\frac{e^{cx}+(-1)^{-2m+1}e^{-cx}}{2}dx$$
$$+ (-1)^{2m-1}c^{2m}\sinh cx\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^{2m-1}\right\}\frac{e^{cx}-(-1)^{-2m+1}e^{-cx}}{2}dx \tag{1.1}$$

Especially, when $\lim_{m\to\infty} c^{2m}\int_a^x\cdots\int_a^x f(x)\,dx^{2m-1} = 0$,

$$\sum_{r=1}^{\infty} c^{2r-1}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \frac{c}{2}\left\{e^{cx}\int_a^x f(x)e^{-cx}dx + e^{-cx}\int_a^x f(x)e^{cx}dx\right\} \tag{1.1'}$$

Proof

Formula of repeated integration by parts was as follows. (" 01 **Generalized Taylor's Theorem** " (A la Carte))

$$\int_a^x f(x)g(x)\,dx = \sum_{r=1}^{m}(-1)^{r-1}\left[f^{<r>}(x)g^{(r-1)}(x)\right]_a^x + (-1)^m\int_a^x f^{<m>}(x)g^{(m)}(x)\,dx$$

When $g(x)=\cosh cx$, $\sinh cx$,

$$(\cosh cx)^{(r-1)} = c^{r-1}\frac{e^{cx}+(-1)^{-r+1}e^{-cx}}{2}, \quad (\cosh cx)^{(m)} = c^m\frac{e^{cx}+(-1)^{-m}e^{-cx}}{2}$$

$$(\sinh cx)^{(r-1)} = c^{r-1}\frac{e^{cx}-(-1)^{-r+1}e^{-cx}}{2}, \quad (\sinh cx)^{(m)} = c^m\frac{e^{cx}-(-1)^{-m}e^{-cx}}{2}$$

Substituting these for the above,

$$\int_a^x f(x)\cosh cx\,dx = \sum_{r=1}^{m}(-1)^{r-1}\left[f^{<r>}(x)c^{r-1}\frac{e^{cx}+(-1)^{-r+1}e^{-cx}}{2}\right]_a^x,$$
$$+ (-1)^m\int_a^x f^{<m>}(x)c^m\frac{e^{cx}+(-1)^{-m}e^{-cx}}{2}dx,$$

$$\int_a^x f(x)\sinh cx\,dx = \sum_{r=1}^{m}(-1)^{r-1}\left[f^{<r>}(x)c^{r-1}\frac{e^{cx}-(-1)^{-r+1}e^{-cx}}{2}\right]_a^x,$$
$$+ (-1)^m\int_a^x f^{<m>}(x)c^m\frac{e^{cx}-(-1)^{-m}e^{-cx}}{2}dx,$$

Here, let

$$f^{<r>}(x) = \int_a^x\cdots\int_a^x f(x)\,dx^r \qquad r=1,2,\cdots,m$$

Since $f^{<r>}(a)=0 \quad (r=1,2,\cdots,m)$,

$$\int_a^x f(x)\cosh cx\,dx = \sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\frac{e^{cx}+(-1)^{-r+1}e^{-cx}}{2}\int_a^x\cdots\int_a^x f(x)\,dx^r$$
$$+(-1)^m\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}c^m\frac{e^{cx}+(-1)^{-m}e^{-cx}}{2}dx$$

$$\int_a^x f(x)\sinh x\,dx = \sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\frac{e^{cx}-(-1)^{-r+1}e^{-cx}}{2}\int_a^x\cdots\int_a^x f(x)\,dx^r$$
$$+(-1)^m\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}c^m\frac{e^{cx}-(-1)^{-m}e^{-cx}}{2}dx$$

Expanding a part of the 1st term of the right side,

$$\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\frac{e^{cx}+(-1)^{-r+1}e^{-cx}}{2} = c^0\cosh cx - c^1\sinh cx + c^2\cosh cx - c^3\sinh cx + - \cdots$$

$$\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\frac{e^{cx}-(-1)^{-r+1}e^{-cx}}{2} = c^0\sinh cx - c^1\cosh cx + c^2\sinh cx - c^3\cosh cx + - \cdots$$

Multiplying both sides by $\cosh cx, \sinh cx$ respectively,

$$\cosh cx\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\frac{e^{cx}+(-1)^{-r+1}e^{-cx}}{2}$$
$$= c^0\cosh^2 cx - c^1\cosh cx\sinh cx + c^2\cosh^2 cx - c^3\cosh cx\sinh cx + - \cdots$$

$$\sinh cx\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\frac{e^{cx}-(-1)^{-r+1}e^{-cx}}{2}$$
$$= c^0\sinh^2 cx - c^1\sinh cx\cosh cx + c^2\sinh^2 cx - c^3\sinh cx\cosh cx + - \cdots$$

The 2nd term of the right side is as follows.

$$(-1)^m\cosh cx\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}c^m\frac{e^{cx}+(-1)^{-m}e^{-cx}}{2}dx$$

$$(-1)^m\sinh cx\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}c^m\frac{e^{cx}-(-1)^{-m}e^{-cx}}{2}dx$$

Since $\cosh^2 cx - \sinh^2 cx = 1$,

$$\cosh cx\int_a^x f(x)\cosh cx\,dx - \sinh cx\int_a^x f(x)\sinh cx\,dx = \sum_{r=1}^{m}c^{2r-2}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1}$$
$$+(-1)^{2m-1}\cosh cx\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^{2m-1}\right\}c^{2m-1}\frac{e^{cx}+(-1)^{-2m+1}e^{-cx}}{2}dx$$
$$-(-1)^{2m-1}\sinh cx\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^{2m-1}\right\}c^{2m-1}\frac{e^{cx}-(-1)^{-2m+1}e^{-cx}}{2}dx$$

Here, m in the 2nd term and the 3rd term on the right side has to be an odd number corresponding to the 1st term. Furthermore,

$$\cosh cx\int_a^x f(x)\cosh cx\,dx - \sinh cx\int_a^x f(x)\sinh cx\,dx$$

$$= \frac{e^{cx}+e^{-cx}}{2}\int_a^x f(x)\frac{e^{cx}+e^{-cx}}{2}dx - \frac{e^{cx}-e^{-cx}}{2}\int_a^x f(x)\frac{e^{cx}-e^{-cx}}{2}dx$$

$$= \frac{1}{2}\left\{e^{cx}\int_a^x f(x)e^{-cx}dx + e^{-cx}\int_a^x f(x)e^{cx}dx\right\}$$

Using this, we obtain

$$\sum_{r=1}^{m} c^{2r-2}\int_a^x \cdots \int_a^x f(x)dx^{2r-1} = \frac{1}{2}\left\{e^{cx}\int_a^x f(x)e^{-cx}dx + e^{-cx}\int_a^x f(x)e^{cx}dx\right\}$$

$$-(-1)^{2m-1}c^{2m-1}\cosh cx\int_a^x\left\{\int_a^x \cdots \int_a^x f(x)dx^{2m-1}\right\}\frac{e^{cx}+(-1)^{-2m+1}e^{-cx}}{2}dx$$

$$+(-1)^{2m-1}c^{2m-1}\sinh cx\int_a^x\left\{\int_a^x \cdots \int_a^x f(x)dx^{2m-1}\right\}\frac{e^{cx}-(-1)^{-2m+1}e^{-cx}}{2}dx$$

Multiplying by c the both sides, we obtain (1.1) .

Example 1 $c^1\int_a^x dx + c^3\int_a^x\int_a^x\int_a^x dx^3 + c^5\int_a^x \cdots \int_a^x dx^5 + c^7\int_a^x \cdots \int_a^x dx^7 + \cdots$

$$f(x)=1 , \qquad \int_a^x \cdots \int_a^x dx^m = \frac{(x-a)^m}{m!}$$

Substituting these for (1.1) ,

$$\sum_{r=1}^{m} c^{2r-1}\int_a^x \cdots \int_a^x dx^{2r-1} = \frac{c}{2}\left\{e^{cx}\int_a^x e^{-cx}dx + e^{-cx}\int_a^x e^{cx}dx\right\}$$

$$-(-1)^{2m-1}c^{2m}\cosh cx\int_a^x \frac{(x-a)^{2m-1}}{(2m-1)!}\frac{e^{cx}+(-1)^{-2m+1}e^{-cx}}{2}dx$$

$$+(-1)^{2m-1}c^{2m}\sinh cx\int_a^x \frac{(x-a)^{2m-1}}{(2m-1)!}\frac{e^{cx}-(-1)^{-2m+1}e^{-cx}}{2}dx$$

Here,

$$\frac{c}{2}\left\{e^{cx}\int_a^x e^{-cx}dx + e^{-cx}\int_a^x e^{cx}dx\right\} = \frac{ce^{cx}}{2}\left[-\frac{e^{-cx}}{c}\right]_a^x + \frac{ce^{cx}}{2}\left[\frac{e^{cx}}{c}\right]_a^x$$

$$= \frac{e^{c(x-a)}-e^{-c(x-a)}}{2} = \sinh\{c(x-a)\}$$

Then,

$$\sum_{r=1}^{m} c^{2r-1}\int_a^x \cdots \int_a^x dx^{2r-1} = \sinh\{c(x-a)\}$$

$$-(-1)^{2m-1}c^{2m}\cosh cx\int_a^x \frac{(x-a)^{2m-1}}{(2m-1)!}\frac{e^{cx}+(-1)^{-2m+1}e^{-cx}}{2}dx$$

$$+(-1)^{2m-1}c^{2m}\sinh cx\int_a^x \frac{(x-a)^{2m-1}}{(2m-1)!}\frac{e^{cx}-(-1)^{-2m+1}e^{-cx}}{2}dx \tag{1.2}$$

$$\sum_{r=1}^{\infty} c^{2r-1}\int_a^x \cdots \int_a^x dx^{2r-1} = \sinh\{c(x-a)\} \tag{1.2'}$$

The left side is

$$\sum_{r=1}^{m} c^{2r-1}\int_a^x \cdots \int_a^x dx^{2r-1} = \sum_{r=1}^{m} c^{2r-1}\frac{(x-a)^{2r-1}}{(2r-1)!}$$

Using this as the left side of (1.2) and (1.2') ,

$$\sum_{r=1}^{m} c^{2r-1}\frac{(x-a)^{2r-1}}{(2r-1)!} = \sinh\{c(x-a)\}$$
$$-(-1)^{2m-1}c^{2m}\cosh cx\int_a^x \frac{(x-a)^{2m-1}}{(2m-1)!}\frac{e^{cx}+(-1)^{-2m+1}e^{-cx}}{2}dx$$
$$+(-1)^{2m-1}c^{2m}\sinh cx\int_a^x \frac{(x-a)^{2m-1}}{(2m-1)!}\frac{e^{cx}-(-1)^{-2m+1}e^{-cx}}{2}dx \tag{1.3}$$

$$\sum_{r=1}^{\infty} c^{2r-1}\frac{(x-a)^{2r-1}}{(2r-1)!} = \sinh\{c(x-a)\} \tag{1.3'}$$

When $x=3,\ a=1,\ c=2.1,\ m=5$, Both sides of (1.3) are calculated as follows.

$$f[x\_,a\_,c\_,m\_] := \sum_{r=1}^{m} c^{2r-1}\frac{(x-a)^{2r-1}}{(2r-1)!}$$

$$g[x\_,a\_,c\_,m\_] := \text{Sinh}[c\,(x-a)]$$
$$-(-1)^{2m-1}c^{2m}\,\text{Cosh}[c\,x]\int_a^x \frac{(t-a)^{2m-1}}{(2m-1)!}\frac{e^{ct}+(-1)^{-2m+1}e^{-ct}}{2}dt$$
$$+(-1)^{2m-1}c^{2m}\,\text{Sinh}[c\,x]\int_a^x \frac{(t-a)^{2m-1}}{(2m-1)!}\frac{e^{ct}-(-1)^{-2m+1}e^{-ct}}{2}dt$$

```
N[f[3, 1, 2.1, 5]]        N[g[3, 1, 2.1, 5]]
33.1338                   33.1338
```

Also, (1.3') is correct. Because the left side is Taylor expansion of the right side.

Theorem 25.2.2

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\sum_{r=1}^{m}(-1)^{r-1}c^{2r-1}\int_a^x \cdots \int_a^x f(x)dx^{2r-1} = c\cos cx\int_a^x f(x)\cos cx\,dx + c\sin cx\int_a^x f(x)\sin cx\,dx$$
$$-(-1)^{2m-1}c^{2m}\cos cx\int_a^x\left\{\int_a^x \cdots \int_a^x f(x)dx^{2m-1}\right\}\cos\left\{cx+\frac{(2m-1)\pi}{2}\right\}dx$$
$$-(-1)^{2m-1}c^{2m}\sin cx\int_a^x\left\{\int_a^x \cdots \int_a^x f(x)dx^{2m-1}\right\}\sin\left\{cx+\frac{(2m-1)\pi}{2}\right\}dx \tag{2.1}$$

Especially, when $\lim_{m\to\infty} c^{2m}\int_a^x \cdots \int_a^x f(x)dx^{2m-1} = 0$,

$$\sum_{r=1}^{\infty}(-1)^{r-1}c^{2r-1}\int_a^x \cdots \int_a^x f(x)dx^{2r-1} = c\cos cx\int_a^x f(x)\cos cx\,dx + c\sin cx\int_a^x f(x)\sin cx\,dx \tag{2.1'}$$

Proof

Formula of repeated integration by parts was as follows (" **1 Generalized Taylor's Theorem** ") .

$$\int_a^x f(x)g(x)dx = \sum_{r=1}^{m}(-1)^{r-1}\left[f^{<r>}(x)g^{(r-1)}(x)\right]_a^x + (-1)^m\int_a^x f^{<m>}(x)g^{(m)}(x)dx$$

When $g(x)= sin\,cx\,,\,cos\,cx$,

$$(cos\,cx)^{(r-1)} = c^{r-1}cos\left\{cx+\frac{(r-1)\pi}{2}\right\} \quad , \quad (cos\,cx)^{(m)} = c^m cos\left(cx+\frac{m\pi}{2}\right)$$

$$(sin\,cx)^{(r-1)} = c^{r-1}sin\left\{cx+\frac{(r-1)\pi}{2}\right\} \quad , \quad (sin\,cx)^{(m)} = c^m sin\left(cx+\frac{m\pi}{2}\right)$$

Substituting these for the above,

$$\int_a^x f(x)cos\,cx\,dx = \sum_{r=1}^{m}(-1)^{r-1}\left[f^{<r>}(x)c^{r-1}cos\left\{cx+\frac{(r-1)\pi}{2}\right\}\right]_a^x$$
$$+ (-1)^m\int_a^x f^{<m>}(x)c^m cos\left(cx+\frac{m\pi}{2}\right)dx$$

$$\int_a^x f(x)sin\,cx\,dx = \sum_{r=1}^{m}(-1)^{r-1}\left[f^{<r>}(x)c^{r-1}sin\left\{cx+\frac{(r-1)\pi}{2}\right\}\right]_a^x$$
$$+ (-1)^m\int_a^x f^{<m>}(x)c^m sin\left(cx+\frac{m\pi}{2}\right)dx$$

Here, let

$$f^{<r>}(x) = \int_a^x\cdots\int_a^x f(x)dx^r \qquad r=1,2,\cdots,m$$

Since $f^{<r>}(a)=0 \ \ (r=1,2,\cdots,m)$,

$$\int_a^x f(x)cos\,cx\,dx = \sum_{r=1}^{m}(-1)^{r-1}c^{r-1}cos\left\{cx+\frac{(r-1)\pi}{2}\right\}\int_a^x\cdots\int_a^x f(x)dx^r$$
$$+ (-1)^m\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}c^m cos\left(cx+\frac{m\pi}{2}\right)dx$$

$$\int_a^x f(x)sin\,x\,dx = \sum_{r=1}^{m}(-1)^{r-1}c^{r-1}sin\left\{cx+\frac{(r-1)\pi}{2}\right\}\int_a^x\cdots\int_a^x f(x)dx^r$$
$$+ (-1)^m\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}c^m sin\left(cx+\frac{m\pi}{2}\right)dx$$

Expanding a part of the 1st term of the right side,

$$\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}cos\left\{cx+\frac{(r-1)\pi}{2}\right\} = c^0cos\,cx + c^1sin\,cx - c^2cos\,cx - c^3sin\,cx \ ++--\cdots$$

$$\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}sin\left\{cx+\frac{(r-1)\pi}{2}\right\} = c^0sin\,cx - c^1cos\,cx - c^2sin\,cx + c^3cos\,cx \ ++--\cdots$$

Multiplying both sides by $cos\,cx,\,sin\,cx$ respectively,

$$cos\,cx\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}cos\left\{cx+\frac{(r-1)\pi}{2}\right\}$$
$$= c^0cos^2cx + c^1sin\,cx\,cos\,cx - c^2cos^2cx - c^3sin\,cx\,cos\,cx \ ++--\cdots$$

$$\sin cx \sum_{r=1}^{m} (-1)^{r-1} c^{r-1} \sin\left\{ cx + \frac{(r-1)\pi}{2} \right\}$$

$$= c^0 \sin^2 cx - c^1 \sin cx \cos cx - c^2 \sin^2 cx + c^3 \sin cx \cos cx ++-- \cdots$$

The 2nd term of the right side is as follows.

$$(-1)^m \cos cx \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^m \right\} c^m \cos\left(cx + \frac{m\pi}{2} \right) dx$$

$$(-1)^m \sin cx \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^m \right\} c^m \sin\left(cx + \frac{m\pi}{2} \right) dx$$

Since $\cos^2 cx + \sin^2 cx = 1$,

$$\cos cx \int_a^x f(x) \cos cx\, dx + \sin cx \int_a^x f(x) \sin cx\, dx = \sum_{r=1}^{m} (-1)^{r-1} c^{2r-2} \int_a^x \cdots \int_a^x f(x)\, dx^{2r-1}$$

$$+ (-1)^{2m-1} c^{2m-1} \cos cx \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^{2m-1} \right\} \cos\left\{ cx + \frac{(2m-1)\pi}{2} \right\} dx$$

$$+ (-1)^{2m-1} c^{2m-1} \sin cx \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^{2m-1} \right\} \sin\left\{ cx + \frac{(2m-1)\pi}{2} \right\} dx$$

Here, m in the 2nd term and the 3rd term on the right side has to be an odd number corresponding to the 1st term. Then, multiplying by c the both sides and transposing them, we obtain (2.1) .

Example 2 $c^1 \int_a^x e^x dx - c^3 \int_a^x \int_a^x \int_a^x e^x dx^3 + c^5 \int_a^x \cdots \int_a^x e^x dx^5 - c^7 \int_a^x \cdots \int_a^x e^x dx^7 +- \cdots$

Substituting $f(x) = e^x$ for (2.1') ,

$$\sum_{r=1}^{\infty} (-1)^{r-1} c^{2r-1} \int_a^x \cdots \int_a^x e^x dx^{2r-1} = c \cos cx \int_a^x e^x \cos cx\, dx + c \sin cx \int_a^x e^x \sin cx\, dx$$

The right side is as follow.

$$\int_a^x e^x \cos cx\, dx = \frac{1}{1+c^2} \left\{ e^x (c \sin cx + \cos cx) - e^a (c \sin ca + \cos ca) \right\}$$

$$\int_a^x e^x \sin cx\, dx = \frac{1}{1+c^2} \left\{ e^x (\sin cx - c \cos cx) - e^a (\sin ca - c \cos ca) \right\}$$

From these,

$$\cos x \int_a^x e^x \cos x\, dx + \sin x \int_a^x e^x \sin x\, dx = \frac{e^x}{1+c^2} - \frac{e^a}{1+c^2} [\cos\{c(x-a)\} - c \sin\{c(x-a)\}]$$

Therefore,

$$\sum_{r=1}^{\infty} (-1)^{r-1} c^{2r-1} \int_a^x \cdots \int_a^x e^x dx^{2r-1} = \frac{ce^x}{1+c^2} - \frac{ce^a}{1+c^2} [\cos\{c(x-a)\} - c \sin\{c(x-a)\}] \tag{2.2'}$$

The higher integral of the left side is as follows according to Theorem 4.1.3 (**4.1**)

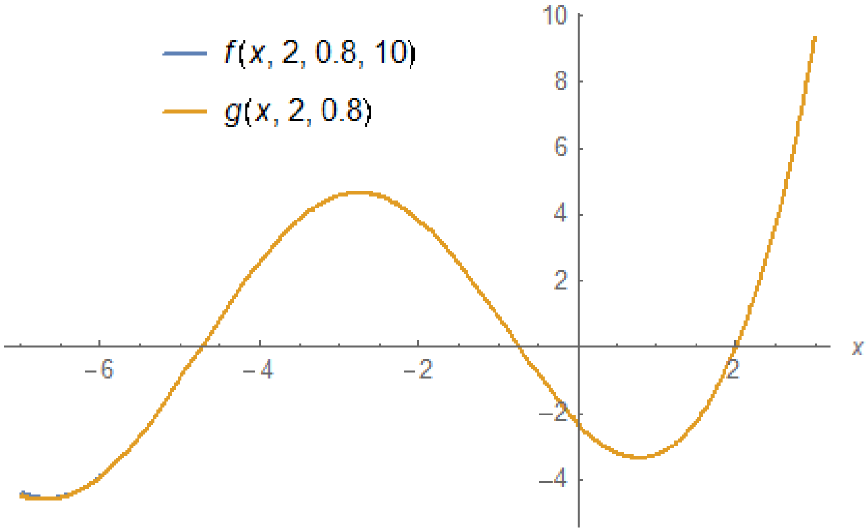
$$\int_a^x \cdots \int_a^x e^x dx^n = e^x - \sum_{s=0}^{n-1} e^a \frac{(x-a)^s}{s!}$$

Then, (2.2') becomes

$$\sum_{r=1}^{\infty}(-1)^{r-1}c^{2r-1}\left\{e^{x}-e^{a}\sum_{s=0}^{2r-2}\frac{(x-a)^{s}}{s!}\right\}=\frac{ce^{x}}{1+c^{2}}-\frac{ce^{a}}{1+c^{2}}[\cos\{c(x-a)\}-c\sin\{c(x-a)\}]$$

(2.3')

When $a=2$, $c=0.8$, the first 10 terms of $\sum$ are calculated and both sides are illustrated, it is as follows. Both sides overlap exactly and blue (left) can not be seen.



25.3 Series of the even-th order Integrals

In this section, we ask for the sum of series of higher integral with geometric coefficients, such as

$$c^2\int_a^x\int_a^x f(x)dx^2 \pm c^4\int_a^x\cdots\int_a^x f(x)dx^4 + c^6\int_a^x\cdots\int_a^x f(x)dx^6 \pm c^8\int_a^x\cdots\int_a^x f(x)dx^8 +\pm \cdots$$

Theorem 25.3.1

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\begin{aligned}\sum_{r=1}^{m} c^{2r}\int_a^x\cdots\int_a^x f(x)dx^{2r} = \frac{c}{2}\left\{e^{cx}\int_a^x f(x)e^{-cx}dx - e^{-cx}\int_a^x f(x)e^{cx}dx\right\}\\ - c^{2m+1}\sinh cx\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^{2m}\right\}\cosh cx\,dx\\ + c^{2m+1}\cosh cx\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^{2m}\right\}\sinh cx\,dx \qquad (1.1)\end{aligned}$$

Especially, when $\lim_{m\to\infty} c^{2m+1}\int_a^x\cdots\int_a^x f(x)dx^{2m} = 0$,

$$\sum_{r=1}^{\infty} c^{2r}\int_a^x\cdots\int_a^x f(x)dx^{2r} = \frac{c}{2}\left\{e^{cx}\int_a^x f(x)e^{-cx}dx - e^{-cx}\int_a^x f(x)e^{cx}dx\right\} \qquad (1.1')$$

Proof

The following expressions were obtained during the proof of Theorem 25.2.1 .

$$\begin{aligned}\int_a^x f(x)\cosh cx\,dx = \sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\frac{e^{cx}+(-1)^{-r+1}e^{-cx}}{2}\int_a^x\cdots\int_a^x f(x)dx^r\\ + (-1)^m\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}c^m\frac{e^{cx}+(-1)^{-m}e^{-cx}}{2}dx\end{aligned}$$

$$\begin{aligned}\int_a^x f(x)\sinh cx\,dx = \sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\frac{e^{cx}-(-1)^{-r+1}e^{-cx}}{2}\int_a^x\cdots\int_a^x f(x)dx^r\\ + (-1)^m\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}c^m\frac{e^{cx}-(-1)^{-m}e^{-cx}}{2}dx\end{aligned}$$

Expanding a part of the 1st term of the right side,

$$\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\frac{e^{cx}+(-1)^{-r+1}e^{-cx}}{2} = c^0\cosh cx - c^1\sinh cx + c^2\cosh cx - c^3\sinh cx +-\cdots$$

$$\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\frac{e^{cx}-(-1)^{-r+1}e^{-cx}}{2} = c^0\sinh cx - c^1\cosh cx + c^2\sinh cx - c^3\cosh cx +-\cdots$$

Multiplying both sides by $\sinh cx$, $\cosh cx$ respectively,

$$\begin{aligned}\sinh cx\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\frac{e^{cx}+(-1)^{-r+1}e^{-cx}}{2}\\ = c^0\sinh cx\cosh cx - c^1\sinh^2 cx + c^2\sinh cx\cosh cx - c^3\sinh^2 cx +-\cdots\end{aligned}$$

$$\begin{aligned}\cosh cx\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\frac{e^{cx}-(-1)^{-r+1}e^{-cx}}{2}\\ = c^0\cosh cx\sinh cx - c^1\cosh^2 cx + c^2\cosh cx\sinh cx - c^3\cosh^2 cx +-\cdots\end{aligned}$$

The 2nd term of the right side is as follows.

$$(-1)^m \sinh cx \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^m \right\} c^m \frac{e^{cx} + (-1)^{-m} e^{-cx}}{2} dx$$

$$(-1)^m \cosh cx \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^m \right\} c^m \frac{e^{cx} - (-1)^{-m} e^{-cx}}{2} dx$$

Since $\cosh^2 cx - \sinh^2 cx = 1$

$$\sinh cx \int_a^x f(x) \cosh cx\, dx - \cosh cx \int_a^x f(x) \sinh cx\, dx = \sum_{r=1}^{m} c^{2r-1} \int_a^x \cdots \int_a^x f(x)\, dx^{2r}$$

$$+ (-1)^{2m} \sinh cx \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^{2m} \right\} c^{2m} \frac{e^{cx} + (-1)^{-2m} e^{-cx}}{2} dx$$

$$- (-1)^{2m} \cosh cx \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^{2m} \right\} c^{2m} \frac{e^{cx} - (-1)^{-2m} e^{-cx}}{2} dx$$

Here, m in the 2nd term and the 3rd term on the right side has to be an even number corresponding to the 1st term. Furthermore,

$$\sinh cx \int_a^x f(x) \cosh cx\, dx - \cosh cx \int_a^x f(x) \sinh cx\, dx$$

$$= \frac{e^{cx} - e^{-cx}}{2} \int_a^x f(x) \frac{e^{cx} + e^{-cx}}{2} dx - \frac{e^{cx} + e^{-cx}}{2} \int_a^x f(x) \frac{e^{cx} - e^{-cx}}{2} dx$$

$$= \frac{1}{2} \left\{ e^{cx} \int_a^x f(x) e^{-cx} dx - e^{-cx} \int_a^x f(x) e^{cx} dx \right\}$$

Using this, we obtain

$$\sum_{r=1}^{m} c^{2r-1} \int_a^x \cdots \int_a^x f(x)\, dx^{2r} = \frac{1}{2} \left\{ e^{cx} \int_a^x f(x) e^{-cx} dx - e^{-cx} \int_a^x f(x) e^{cx} dx \right\}$$

$$- c^{2m} \sinh cx \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^{2m} \right\} \cosh cx\, dx$$

$$+ c^{2m} \cosh cx \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^{2m} \right\} \sinh cx\, dx$$

Multiplying by c the both sides, we obtain (1.1) .

Example 1 $c^2 \int_a^x \int_a^x dx^2 + c^4 \int_a^x \cdots \int_a^x dx^4 + c^6 \int_a^x \cdots \int_a^x dx^6 + c^8 \int_a^x \cdots \int_a^x dx^8 + \cdots$

$$f(x) = 1 , \quad \int_a^x \cdots \int_a^x dx^m = \frac{(x-a)^m}{m!}$$

Substituting these for (1.1) ,

$$\sum_{r=1}^{m} c^{2r} \int_a^x \cdots \int_a^x dx^{2r} = \frac{c}{2} \left\{ e^{cx} \int_a^x e^{-cx} dx - e^{-cx} \int_a^x e^{cx} dx \right\}$$

$$- c^{2m+1} \sinh cx \int_a^x \frac{(x-a)^{2m}}{(2m)!} \cosh cx\, dx$$

$$+ c^{2m+1} \cosh cx \int_a^x \frac{(x-a)^{2m}}{(2m)!} \sinh cx\, dx$$

Here,

$$\frac{c}{2}\left\{e^{cx}\int_a^x e^{-cx}dx - e^{-cx}\int_a^x e^{cx}dx\right\} = \frac{ce^{cx}}{2}\left[-\frac{e^{-cx}}{c}\right]_a^x - \frac{ce^{cx}}{2}\left[\frac{e^{cx}}{c}\right]_a^x$$

$$= \frac{e^{c(x-a)}-1}{2} - \frac{-e^{-c(x-a)}+1}{2}$$

$$= cosh\{c(x-a)\} - 1$$

Then,

$$\sum_{r=1}^{m} c^{2r}\int_a^x \cdots \int_a^x dx^{2r} = cosh\{c(x-a)\} - 1 - c^{2m+1} sinh\, cx \int_a^x \frac{(x-a)^{2m}}{(2m)!} cosh\, cx\, dx$$

$$+ c^{2m+1} cosh\, cx \int_a^x \frac{(x-a)^{2m}}{(2m)!} sinh\, cx\, dx \tag{1.2}$$

$$\sum_{r=1}^{\infty} c^{2r}\int_a^x \cdots \int_a^x dx^{2r} = cosh\{c(x-a)\} - 1 \tag{1.2'}$$

The left side is

$$\sum_{r=1}^{m} c^{2r}\int_a^x \cdots \int_a^x dx^{2r} = \sum_{r=1}^{m} c^{2r}\frac{(x-a)^{2r}}{(2r)!}$$

Using this as the left side of (1.2) and (1.2') ,

$$\sum_{r=1}^{m} c^{2r}\frac{(x-a)^{2r}}{(2r)!} = cosh\{c(x-a)\} - 1 - c^{2m+1} sinh\, cx \int_a^x \frac{(x-a)^{2m}}{(2m)!} cosh\, cx\, dx$$

$$+ c^{2m+1} cosh\, cx \int_a^x \frac{(x-a)^{2m}}{(2m)!} sinh\, cx\, dx \tag{1.3}$$

$$\sum_{r=1}^{\infty} c^{2r}\frac{(x-a)^{2r}}{(2r)!} = cosh\{c(x-a)\} - 1 \tag{1.3'}$$

When $x=3,\ a=1,\ c=2.3,\ m=4$, Both sides of (1.3) are calculated as follows.

$$f[x\_, a\_, c\_, m\_] := \sum_{r=1}^{m} c^{2r}\frac{(x-a)^{2r}}{(2r)!}$$

$$g[x\_, a\_, c\_, m\_] := Cosh[c\,(x-a)] - 1$$
$$- c^{2m+1} Sinh[c\,x] \int_a^x \frac{(t-a)^{2m}}{(2m)!} Cosh[c\,t]\, dt$$
$$+ c^{2m+1} Cosh[c\,x] \int_a^x \frac{(t-a)^{2m}}{(2m)!} Sinh[c\,t]\, dt$$

```
N[f[3, 1, 2.3, 4]]          N[g[3, 1, 2.3, 4]]
47.3669                     47.3669
```

Also, it is clear that (1.3') is correct if $cosh\{c(x-a)\}$ is expanded to Taylor series.

Theorem 25.3.2

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\sum_{r=1}^{m}(-1)^{r-1}c^{2r}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = c\sin cx\int_a^x f(x)\cos cx\,dx - c\cos cx\int_a^x f(x)\sin cx\,dx$$
$$- c^{2m+1}\sin cx\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^{2m}\right\}\cos(cx+m\pi)\,dx$$
$$+ c^{2m+1}\cos cx\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^{2m}\right\}\sin(cx+m\pi)\,dx \tag{2.1}$$

Especially, when $\lim_{m\to\infty} c^{2m+1}\int_a^x\cdots\int_a^x f(x)\,dx^{2m} = 0$,

$$\sum_{r=1}^{\infty}(-1)^{r-1}c^{2r}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = c\sin cx\int_a^x f(x)\cos cx\,dx - c\cos cx\int_a^x f(x)\sin cx\,dx \tag{2.1'}$$

Proof

The following expressions were obtained during the proof of Theorem 25.2.2.

$$\int_a^x f(x)\cos cx\,dx = \sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\cos\left\{cx+\frac{(r-1)\pi}{2}\right\}\int_a^x\cdots\int_a^x f(x)\,dx^r$$
$$+ (-1)^m\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}c^m\cos\left(cx+\frac{m\pi}{2}\right)dx$$

$$\int_a^x f(x)\sin cx\,dx = \sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\sin\left\{cx+\frac{(r-1)\pi}{2}\right\}\int_a^x\cdots\int_a^x f(x)\,dx^r$$
$$+ (-1)^m\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}c^m\sin\left(cx+\frac{m\pi}{2}\right)dx$$

Expanding a part of the 1st term of the right side,

$$\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\cos\left\{cx+\frac{(r-1)\pi}{2}\right\} = c^0\cos cx + c^1\sin cx - c^2\cos cx - c^3\sin cx ++--\cdots$$
$$\sum_{r=1}^{m}(-1)^{r-1}c^{r-1}\sin\left\{cx+\frac{(r-1)\pi}{2}\right\} = c^0\sin cx - c^1\cos cx - c^2\sin cx + c^3\cos cx ++--\cdots$$

Multiplying both sides by $\sin cx$, $\cos cx$ respectively,

$$\sin cx\sum_{r=1}^{m}(-1)^{r-1}\cos\left\{x+\frac{(r-1)\pi}{2}\right\}$$
$$= c^0\sin cx\cos cx + c^1\sin^2 cx - c^2\sin cx\cos cx - c^3\sin^2 cx ++--\cdots$$

$$\cos cx\sum_{r=1}^{m}(-1)^{r-1}\sin\left\{x+\frac{(r-1)\pi}{2}\right\}$$
$$= c^0\cos cx\sin cx - c^1\cos^2 cx - c^2\cos cx\sin cx + c^3\cos^2 cx ++--\cdots$$

The 2nd term of the right side is as follows.

$$(-1)^m\sin cx\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}c^m\cos\left(cx+\frac{m\pi}{2}\right)dx$$

$$(-1)^m\cos cx\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}c^m\sin\left(cx+\frac{m\pi}{2}\right)dx$$

Since $\cos^2 cx + \sin^2 cx = 1$,

$$\sin cx \int_a^x f(x)\cos cx\,dx - \cos cx \int_a^x f(x)\sin cx\,dx = \sum_{r=1}^{m} (-1)^{r-1} c^{2r-1} \int_a^x \cdots \int_a^x f(x)\,dx^{2r}$$
$$+ c^{2m}\sin cx \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\,dx^{2m} \right\} \cos(cx+m\pi)\,dx$$
$$- c^{2m}\cos cx \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\,dx^{2m} \right\} \sin(cx+m\pi)\,dx$$

Here, m in the 2nd term and the 3rd term on the right side has to be an even number corresponding to the 1st term. Then, multiplying by c the both sides and transposing them, we obtain (2.1) .

Example 2 $c^2\int_0^x\int_0^x x^2 dx^2 - c^4\int_0^x \cdots \int_0^x x^2 dx^4 + c^6\int_0^x \cdots \int_0^x x^2 dx^6 - c^8\int_0^x \cdots \int_0^x x^2 dx^8 +- \cdots$

Substituting $f(x) = x^2$ for (2.1') ,

$$c^2\int_0^x\int_0^x x^2 dx^2 - c^4\int_0^x \cdots \int_0^x x^2 dx^4 + c^6\int_0^x \cdots \int_0^x x^2 dx^6 - c^8\int_0^x \cdots \int_0^x x^2 dx^8 +- \cdots$$

The right side is,

$$\int_0^x x^2 \cos cx\,dx = \frac{2cx\cos cx + (c^2x^2-2)\sin cx}{c^3}$$
$$\int_0^x x^2 \sin cx\,dx = \frac{2cx\sin cx - (c^2x^2-2)\cos cx - 2}{c^3}$$

So,

$$c\sin cx \int_0^x x^2\cos cx\,dx - c\cos cx\int_0^x x^2 \sin cx\,dx = \frac{c^2x^2-2+2\cos cx}{c^2}$$

Therefore,

$$\sum_{r=1}^{\infty} (-1)^{r-1} c^{2r} \int_0^x \cdots \int_0^x x^2 dx^{2r} = \frac{c^2x^2-2+2\cos cx}{c^2} \qquad (2.2')$$

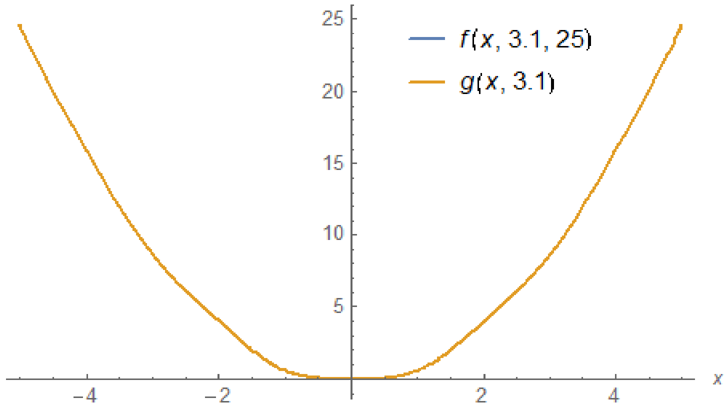
The higher integral of the left side is

$$\int_0^x \cdots \int_0^x x^2 dx^n = \frac{2!}{(2+n)!}x^{2+n}$$

Then, (2.2') becomes

$$\sum_{r=1}^{\infty} (-1)^{r-1} c^{2r} \frac{2!}{(2+2r)!} x^{2+2r} = \frac{c^2x^2-2+2\cos cx}{c^2} \qquad (2.3')$$

When $c=3.1$, the first 25 terms of $\sum$ are calculated and both sides are illustrated, it is as follows. Both sides overlap exactly and blue (left) can not be seen.



25.4 Series of Higher Integrals with coefficients

In this section, we ask for the sum of series of higher integral with arithmetic coefficients and geometric coefficients, such as

$$1c^1\int_a^x f(x)dx \pm 2c^2\int_a^x\int_a^x f(x)dx^2 + 3c^3\int_a^x\int_a^x\int_a^x f(x)dx^3 \pm 4c^4\int_a^x\cdots\int_a^x f(x)dx^4 +\pm \cdots$$

Theorem 25.4.1

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\sum_{r=1}^{m} rc^r\int_a^x\cdots\int_a^x f(x)dx^r = ce^{cx}\int_a^x f(x)e^{-cx}dx + c^2e^{cx}\int_a^x\int_a^x f(x)e^{-cx}dx^2$$

$$-(m+1)c^{m+1}e^{cx}\int_a^x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^{m-1}\right\}e^{-cx}dx^2$$

$$+mc^{m+2}e^{cx}\int_a^x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}e^{-cx}dx^2 \qquad (1.1)$$

Especially, when $\lim_{m\to\infty} mc^{m+2}\int_a^x\cdots\int_a^x f(x)dx^m = 0$,

$$\sum_{r=1}^{\infty} rc^r\int_a^x\cdots\int_a^x f(x)dx^r = ce^{cx}\int_a^x f(x)e^{-cx}dx + c^2e^{cx}\int_a^x\int_a^x f(x)e^{-cx}dx^2 \qquad (1.1')$$

Proof

When $f^{<r>}(a)=0\ \ (r=1,2,\cdots,m+n-1)$, Theorem 16.1.2 (**16.1**) was as follows.

$$\int_a^x\cdots\int_a^x f^{<0>}g^{(0)}dx^n = \sum_{r=0}^{m-1}\binom{-n}{r}f^{<n+r>}g^{(r)}$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x\cdots\int_a^x f^{<m+k>}g^{(m+k)}dx^n$$

Here, let $g=e^{-cx}$, $n=2$. Then, since $g^{(r)}=(-1)^r c^r e^{-cx}$,

$$\int_a^x\int_a^x f^{<0>}e^{-cx}dx^2 = \sum_{r=0}^{m-1}\binom{-2}{r}f^{<2+r>}(-1)^r c^r e^{-cx}$$

$$+\frac{(-1)^m}{B(2,m)}\frac{{}_1C_0}{m+0}\int_a^x\int_a^x f^{<m+0>}(-1)^{m+0}c^{m+0}e^{-cx}dx^2$$

$$+\frac{(-1)^m}{B(2,m)}\frac{{}_1C_1}{m+1}\int_a^x\int_a^x f^{<m+1>}(-1)^{m+1}c^{m+1}e^{-cx}dx^2$$

i.e.

$$\int_a^x\int_a^x f^{<0>}e^{-cx}dx^2 = \sum_{r=0}^{m-1}(1+r)f^{<2+r>}c^r e^{-cx}$$

$$+(m+1)c^m\int_a^x\int_a^x f^{<m>}e^{-cx}dx^2 - mc^{m+1}\int_a^x\int_a^x f^{<m+1>}e^{-cx}dx^2$$

$$\left\{\because\ \ (-1)^r\binom{-2}{r} = 1+r\ ,\ \ \frac{1}{B(2,m)} = m(m+1)\right\}$$

Here, subtracting 1 from the index $<r>$ of the integration operator of the function f,

$$\int_a^x\int_a^x f^{<-1>}e^{-cx}dx^2 = \sum_{r=0}^{m-1}(1+r)f^{<1+r>}c^r e^{-cx}$$

$$+(m+1)c^m\int_a^x\int_a^x f^{<m-1>}e^{-cx}dx^2-mc^{m+1}\int_a^x\int_a^x f^{<m>}e^{-cx}dx^2$$

Left side becomes as follows.

$$\int_a^x\int_a^x f^{<-1>}e^{-cx}dx^2=\int_a^x\left\{\left[f^{<0>}e^{-cx}\right]_a^x+c\int_a^x f^{<0>}e^{-cx}dx\right\}dx$$

$$=\int_a^x f^{<0>}e^{-cx}dx+c\int_a^x\int_a^x f^{<0>}e^{-cx}dx^2$$

Substituting this for the above,

$$\int_a^x f^{<0>}e^{-cx}dx+c\int_a^x\int_a^x f^{<0>}e^{-cx}dx^2=\sum_{r=0}^{m-1}(1+r)f^{<1+r>}c^r e^{-cx}$$

$$+(m+1)c^m\int_a^x\int_a^x f^{<m-1>}e^{-cx}dx^2-mc^{m+1}\int_a^x\int_a^x f^{<m>}e^{-cx}dx^2$$

From this,

$$e^{-cx}\sum_{r=1}^{m}rc^{r-1}f^{<r>}(x)=\int_a^x f^{<0>}e^{-cx}dx+c\int_a^x\int_a^x f^{<0>}e^{-cx}dx^2$$

$$-(m+1)c^m\int_a^x\int_a^x f^{<m-1>}e^{-cx}dx^2+mc^{m+1}\int_a^x\int_a^x f^{<m>}e^{-cx}dx^2$$

Here, let

$$f^{<r>}=\int_a^x\cdots\int_a^x f(x)dx^r\ ,\ \ f^{<m-1>}=\int_a^x\cdots\int_a^x f(x)dx^{m-1}\ ,\ \ f^{<m>}=\int_a^x\cdots\int_a^x f(x)dx^m$$

Since these satisfy the condition $f^{<r>}(a)=0\ (r=1,2,\cdots,m+1)$, substituting these for the above and multiplying by ce^{cx} the both sides,

$$\sum_{r=1}^{m}rc^r\int_a^x\cdots\int_a^x f(x)dx^r=ce^{cx}\int_a^x f(x)e^{-cx}dx+c^2e^{cx}\int_a^x\int_a^x f(x)e^{-cx}dx^2$$

$$-(m+1)c^{m+1}e^{cx}\int_a^x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^{m-1}\right\}e^{-cx}dx^2$$

$$+mc^{m+2}e^{cx}\int_a^x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}e^{-cx}dx^2 \qquad (1.1)$$

Example 1 $1c^1\int_a^x dx+2c^2\int_a^x\int_a^x dx^2+3c^3\int_a^x\int_a^x\int_a^x dx^3+\cdots$

$$f(x)=1\ \ ,\ \ \int_a^x\cdots\int_a^x dx^{m-1}=\frac{(x-a)^{m-1}}{(m-1)!}\ \ ,\ \ \int_a^x\cdots\int_a^x dx^m=\frac{(x-a)^m}{m!}$$

Substituting these for (1.1) ,

$$\sum_{r=1}^{m}rc^r\int_a^x\cdots\int_a^x dx^r=ce^{cx}\int_a^x e^{-cx}dx+c^2e^{cx}\int_a^x\int_a^x e^{-cx}dx^2$$

$$-(m+1)c^{m+1}e^{cx}\int_a^x\int_a^x\frac{(x-a)^{m-1}}{(m-1)!}e^{-cx}dx^2$$

$$+mc^{m+2}e^{cx}\int_a^x\int_a^x\frac{(x-a)^m}{m!}e^{-cx}dx^2$$

The 1st term and the 2nd term of the right side are as follows.

$$ce^{cx}\int_a^x e^{-cx}dx + ce^{cx}\int_a^x\int_a^x e^{-cx}dx^2 = ce^{cx}\frac{e^{ca}-e^{cx}}{c} + ce^{cx}\int_a^x \frac{e^{ca}-e^{cx}}{c}dx$$
$$= ce^{c(x-a)}(x-a)$$

After the long calculation, the 3rd term and the 4th term on the right side become as follows.

$$-(m+1)c^{m+1}e^{cx}\int_a^x\int_a^x \frac{(x-a)^{m-1}}{(m-1)!}e^{-cx}dx^2 + mc^{m+2}e^{cx}\int_a^x\int_a^x \frac{(x-a)^m}{m!}e^{-cx}dx^2$$
$$= -ce^{c(x-a)}(x-a)$$
$$+\frac{m+1}{(m-1)!}e^{c(x-a)}[c(x-a)\Gamma\{m,c(x-a)\}-\Gamma\{1+m,c(x-a)\}]$$
$$-\frac{1}{(m-1)!}e^{c(x-a)}[c(x-a)\Gamma\{1+m,c(x-a)\}-\Gamma\{2+m,c(x-a)\}]$$

Substituting these for the above,

$$\sum_{r=1}^{m} rc^r\int_a^x\cdots\int_a^x dx^r = ce^{c(x-a)}(x-a) - ce^{c(x-a)}(x-a)$$
$$+\frac{m+1}{(m-1)!}e^{c(x-a)}[c(x-a)\Gamma\{m,c(x-a)\}-\Gamma\{1+m,c(x-a)\}]$$
$$-\frac{1}{(m-1)!}e^{c(x-a)}[c(x-a)\Gamma\{1+m,c(x-a)\}-\Gamma\{2+m,c(x-a)\}] \tag{1.2}$$

$$\sum_{r=1}^{\infty} rc^r\int_a^x\cdots\int_a^x dx^r = ce^{c(x-a)}(x-a) \tag{1.2'}$$

The left side is

$$\sum_{r=1}^{m} rc^r\int_a^x\cdots\int_a^x dx^r = \sum_{r=1}^{m}\frac{rc^r(x-a)^r}{r!}$$

Using this as the left side of (1.2) and (1.2') ,

$$\sum_{r=1}^{m}\frac{rc^r(x-a)^r}{r!} = ce^{c(x-a)}(x-a) - ce^{c(x-a)}(x-a)$$
$$+\frac{m+1}{(m-1)!}e^{c(x-a)}[c(x-a)\Gamma\{m,c(x-a)\}-\Gamma\{1+m,c(x-a)\}]$$
$$-\frac{1}{(m-1)!}e^{c(x-a)}[c(x-a)\Gamma\{1+m,c(x-a)\}-\Gamma\{2+m,c(x-a)\}] \tag{1.3}$$

$$\sum_{r=1}^{\infty}\frac{rc^r(x-a)^r}{r!} = ce^{c(x-a)}(x-a) \tag{1.3'}$$

When $x=4, a=3, c=2, m=7$, both sides of (1.3) are calculated as follows.

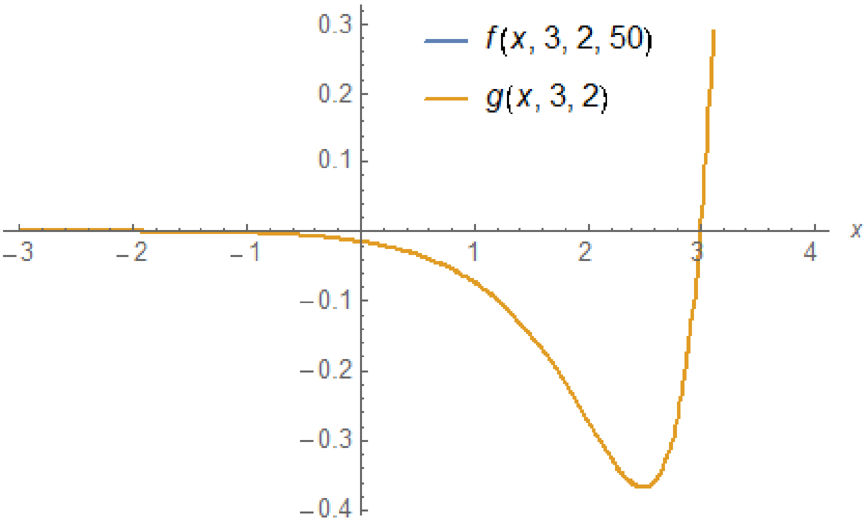
$$\texttt{f[x\_, a\_, c\_, m\_] :=} \sum_{r=1}^{m}\frac{r\,c^r\,(x-a)^r}{r!} \qquad \texttt{Γ[x\_, y\_] := Gamma[x, y]}$$

$$\texttt{g[x\_, a\_, c\_, m\_] :=} c\,e^{c\,(x-a)}\,(x-a) - c\,e^{c\,(x-a)}\,(x-a)$$
$$+\frac{m+1}{(m-1)!}e^{c\,(x-a)}\,(c\,(x-a)\,\Gamma[m,\,c\,(x-a)] - \Gamma[1+m,\,c\,(x-a)])$$

$$-\frac{1}{(m-1)!}e^{c(x-a)}\left(c\,(x-a)\,\Gamma[1+m,\ c\,(x-a)]-\Gamma[2+m,\ c\,(x-a)]\right)$$

```
N[f[4, 3, 2, 7]]          N[g[4, 3, 2, 7]]
14.7111                   14.7111
```

Further, when $a=3, c=2$, the first 50 terms of $\sum$ are calculated and both sides of (1.3') are illustrated, it is as follows. Both sides overlap exactly and blue (left) can not be seen.



Theorem 25.4.2

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\sum_{r=1}^{m}(-1)^{r-1}rc^r\int_a^x\cdots\int_a^x f(x)dx^r = ce^{-cx}\int_a^x f(x)e^{cx}dx - c^2e^{-cx}\int_a^x\int_a^x f(x)e^{cx}dx^2$$

$$-(-1)^m(m+1)c^{m+1}e^{-cx}\int_a^x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^{m-1}\right\}e^{cx}dx^2$$

$$+(-1)^{m+1}mc^{m+2}e^{-cx}\int_a^x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}e^{cx}dx^2 \tag{2.1}$$

Especially, when $\lim_{m\to\infty} mc^{m+2}\int_a^x\cdots\int_a^x f(x)dx^m = 0$,

$$\sum_{r=1}^{\infty}(-1)^{r-1}rc^r\int_a^x\cdots\int_a^x f(x)dx^r = ce^{-cx}\int_a^x f(x)e^{cx}dx - c^2e^{-cx}\int_a^x\int_a^x f(x)e^{cx}dx^2 \tag{2.1'}$$

Proof

In a way similar to Theorem 25.4.1, we obtain the desired expressioons

Example 2

$$1c\int_0^x \log x\,dx - 2c^2\int_0^x\int_0^x \log x\,dx^2 + 3c^3\int_0^x\int_0^x\int_0^x \log x\,dx^3 - 4c^4\int_0^x\cdots\int_0^x \log x\,dx^4 + - \cdots$$

Substituting $f(x)=\log x$, $a=0$ for (2.1) ,

$$\sum_{r=1}^{m}(-1)^{r-1}rc^r\int_0^x\cdots\int_0^x \log x\,dx^r = ce^{-cx}\int_0^x \log x\,e^{cx}dx - c^2e^{-cx}\int_0^x\int_0^x \log x\,e^{cx}dx^2$$
$$-(-1)^m(m+1)c^{m+1}e^{-cx}\int_0^x\int_0^x\left\{\int_0^x\cdots\int_0^x \log x\,dx^{m-1}\right\}e^{cx}dx^2$$
$$+(-1)^{m+1}mc^{m+2}e^{-cx}\int_0^x\int_0^x\left\{\int_0^x\cdots\int_0^x \log x\,dx^{m}\right\}e^{cx}dx^2$$

The 1st term and the 2nd term of the right side are as follows.

$$\int_0^x \log x e^{cx}dx = \frac{1}{c}\left[e^{cx}\log|x| - Ei(cx)\right]_0^x = \frac{1}{c}\left\{e^{cx}\log|x| - Ei(cx)+\gamma+\log c\right\}$$
$$\int_0^x\int_0^x \log x e^{cx}dx = \frac{1}{c}\int_0^x\left\{e^{cx}\log|x| - Ei(cx)+\gamma+\log c\right\}dx$$
$$= \frac{1}{c^2}\left[e^{cx}(\log|x|+1)-(cx+1)Ei(cx)+(\gamma+\log c)cx\right]_0^x$$
$$= \frac{1}{c^2}\left\{e^{cx}(\log|x|+1)-(cx+1)Ei(cx)+(\gamma+\log c)cx\right\}-\frac{1}{c^2}(1-\gamma-\log c)$$
$$= \frac{1}{c^2}\left\{e^{cx}(\log|x|+1)-(cx+1)Ei(cx)+cx(\gamma+\log c)+\gamma+\log c-1\right\}$$

Where, $Ei(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt,\quad \gamma=0.5772\cdots$ (Euler-Mascheroni Constant).

From this,

$$ce^{-cx}\int_0^x \log x\,e^{cx}dx - c^2e^{-cx}\int_0^x\int_0^x \log x\,e^{cx}dx^2 = \frac{ce^{-cx}}{c}\left\{e^{cx}\log|x| - Ei(cx)+\gamma+\log c\right\}$$
$$-\frac{c^2e^{-cx}}{c^2}\left\{e^{cx}(\log|x|+1)-(cx+1)Ei(cx)+cx(\gamma+\log c)+\gamma+\log c-1\right\}$$
$$= -1+e^{-cx}-\gamma cxe^{-cx}+cxe^{-cx}Ei(cx)-cxe^{-cx}\log c$$
$$= cxe^{-cx}\{Ei(cx)-\log c-\gamma\}+e^{-cx}-1$$

i.e.

$$ce^{-cx}\int_0^x \log x\,e^{cx}dx - c^2e^{-cx}\int_0^x\int_0^x \log x\,e^{cx}dx^2 = cxe^{-cx}\{Ei(cx)-\log c-\gamma\}+e^{-cx}-1$$

Next,

$$\int_0^x\cdots\int_0^x \log x\,dx^m = \frac{x^m}{m!}\left(\log|x| - \sum_{s=1}^{m}\frac{1}{s}\right)$$

Using thes for the 3rd term and the 4th term on the right side, the right side becomes

$$cxe^{-cx}\{Ei(cx)-\log c-\gamma\}+e^{-cx}-1$$
$$-(-1)^m(m+1)c^{m+1}e^{-cx}\int_0^x\int_0^x\frac{x^{m-1}}{(m-1)!}\left(\log|x| - \sum_{s=1}^{m-1}\frac{1}{s}\right)e^{cx}dx^2$$
$$+(-1)^{m+1}mc^{m+2}e^{-cx}\int_0^x\int_0^x\frac{x^{m}}{m!}\left(\log|x| - \sum_{s=1}^{m}\frac{1}{s}\right)e^{cx}dx^2$$

i.e.

$$cxe^{-cx}\{Ei(cx)-\log c-\gamma\}+e^{-cx}-1$$
$$-(-1)^m\frac{m+1}{(m-1)!}c^{m+1}e^{-cx}\int_0^x\int_0^x x^{m-1}\left(\log|x|-\sum_{s=1}^{m-1}\frac{1}{s}\right)e^{cx}dx^2$$
$$+(-1)^{m+1}\frac{1}{(m-1)!}c^{m+2}e^{-cx}\int_0^x\int_0^x x^m\left(\log|x|-\sum_{s=1}^{m}\frac{1}{s}\right)e^{cx}dx^2$$

Thus, we obtain

$$\sum_{r=1}^{m}(-1)^{r-1}rc^r\int_0^x\cdots\int_0^x\log x\,dx^r=cxe^{-cx}\{Ei(cx)-\log c-\gamma\}+e^{-cx}-1$$
$$-(-1)^m\frac{m+1}{(m-1)!}c^{m+1}e^{-cx}\int_0^x\int_0^x x^{m-1}\left(\log|x|-\sum_{s=1}^{m-1}\frac{1}{s}\right)e^{cx}dx^2$$
$$+(-1)^{m+1}\frac{1}{(m-1)!}c^{m+2}e^{-cx}\int_0^x\int_0^x x^m\left(\log|x|-\sum_{s=1}^{m}\frac{1}{s}\right)e^{cx}dx^2 \quad (2.2)$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}rc^r\int_0^x\cdots\int_0^x\log x\,dx^r=cxe^{-cx}\{Ei(cx)-\log c-\gamma\}+e^{-cx}-1 \quad (2.2')$$

The left side is

$$\int_0^x\cdots\int_0^x\log x\,dx^n=\frac{x^n}{n!}\left(\log|x|-\sum_{s=1}^{n}\frac{1}{s}\right)$$

Using this,

$$\sum_{r=1}^{m}(-1)^{r-1}\frac{rc^rx^r}{r!}\left(\log|x|-\sum_{s=1}^{r}\frac{1}{s}\right)=cxe^{-cx}\{Ei(cx)-\log c-\gamma\}+e^{-cx}-1$$
$$-(-1)^m\frac{m+1}{(m-1)!}c^{m+1}e^{-cx}\int_0^x\int_0^x x^{m-1}\left(\log|x|-\sum_{s=1}^{m-1}\frac{1}{s}\right)e^{cx}dx^2$$
$$+(-1)^{m+1}\frac{1}{(m-1)!}c^{m+2}e^{-cx}\int_0^x\int_0^x x^m\left(\log|x|-\sum_{s=1}^{m}\frac{1}{s}\right)e^{cx}dx^2 \quad (2.3)$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{rx^r}{r!}\left(\log|x|-\sum_{s=1}^{r}\frac{1}{s}\right)=e^{-x}x\{Ei(x)-\gamma\}+e^{-x}-1 \quad (2.3')$$

When $x=-0.2$, $c=1.7$, $m=3$, Both sides of (2.3) are calculated as follows.

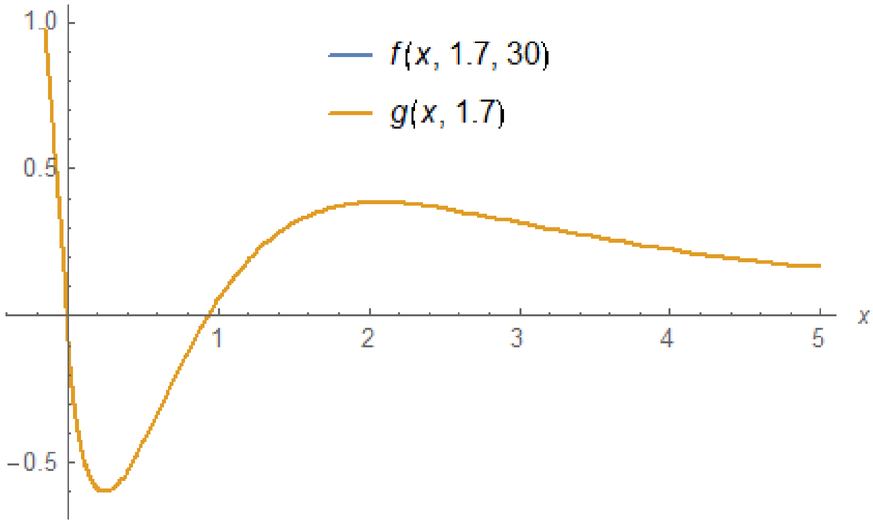
```
Ei[x_] := ExpIntegralEi[x]          γ := EulerGamma

f[x_, c_, m_] := Sum[(-1)^(r-1) r c^r x^r / r! (Log[x] - Sum[1/s, {s,1,r}]), {r,1,m}]

g[x_, c_, m_] := c x E^(-c x) (Ei[c x] - Log[c] - γ) + E^(-c x) - 1
  - (-1)^m (m+1)/(m-1)! c^(m+1) E^(-c x) ∫_0^x (∫_0^u t^(m-1) (Log[Abs[t]] - Sum[1/s, {s,1,m-1}]) E^(c t) dt) du
  + (-1)^(m+1) 1/(m-1)! c^(m+2) E^(-c x) ∫_0^x (∫_0^u t^m (Log[Abs[t]] - Sum[1/s, {s,1,m}]) E^(c t) dt) du

N[f[-0.2, 1.7, 3]]          N[g[-0.2, 1.7, 3]]
1.31432                     1.31432
```

Further, when $c=1.7$, the first 30 terms of $\sum$ are calculated and both sides of (2.3') are illustrated, it is as follows. Both sides overlap exactly and blue (left) can not be seen.



25.5 Calculation by Double Series

The sum of series of higher integral with geometric coefficients can also be calculated by a double series. That way is almost the same as **24.4 ~ 24.6** .

Theorem 25.5.1

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\sum_{r=0}^{\infty} c^r \int_a^x \cdots \int_a^x f(x)dx^r = \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=0}^{\infty} \frac{c^s(x-a)^{s+r}}{(s+r)!} \tag{1.1}$$

$$= \sum_{r=0}^{\infty} f^{(r)}(a) \int_a^x \cdots \int_a^x e^{c(x-a)}dx^r \tag{1.1'}$$

$$\sum_{r=0}^{\infty} (-1)^r c^r \int_a^x \cdots \int_a^x f(x)dx^r = \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=0}^{\infty} (-1)^s \frac{c^s(x-a)^{s+r}}{(s+r)!} \tag{1.2}$$

$$= \sum_{r=0}^{\infty} f^{(r)}(a) \int_a^x \cdots \int_a^x e^{-c(x-a)}dx^r \tag{1.2'}$$

Proof

Let $f^{(r)}(a) = f_a^r$ and expand $f(x)$ into Taylor series around a . Then

$$f(x) = f_a^0 \cdot \frac{(x-a)^0}{0!} + f_a^1 \cdot \frac{(x-a)^1}{1!} + f_a^2 \cdot \frac{(x-a)^2}{2!} + f_a^3 \cdot \frac{(x-a)^3}{3!} + \cdots \tag{0}$$

Integrating both sides of this with respect to x from a to x one by one,

$$\int_a^x f(x)dx = f_a^0 \cdot \frac{(x-a)^1}{1!} + f_a^1 \cdot \frac{(x-a)^2}{2!} + f_a^2 \cdot \frac{(x-a)^3}{3!} + f_a^3 \cdot \frac{(x-a)^4}{4!} + \cdots$$

$$\int_a^x \int_a^x f(x)dx^2 = f_a^0 \cdot \frac{(x-a)^2}{2!} + f_a^1 \cdot \frac{(x-a)^3}{3!} + f_a^2 \cdot \frac{(x-a)^4}{4!} + f_a^3 \cdot \frac{(x-a)^5}{5!} + \cdots$$

$$\int_a^x \int_a^x \int_a^x f(x)dx^3 = f_a^0 \cdot \frac{(x-a)^3}{3!} + f_a^1 \cdot \frac{(x-a)^4}{4!} + f_a^2 \cdot \frac{(x-a)^5}{5!} + f_a^3 \cdot \frac{(x-a)^6}{6!} + \cdots$$

$$\vdots$$

Multiplying these also including (0) by $c^0, c^1, c^2, \cdots$, respectively,

$$c^0 f(x) = f_a^0 \cdot \frac{c^0(x-a)^0}{0!} + f_a^1 \cdot \frac{c^0(x-a)^1}{1!} + f_a^2 \cdot \frac{c^0(x-a)^2}{2!} + f_a^3 \cdot \frac{c^0(x-a)^3}{3!} + \cdots$$

$$c^1 \int_a^x f(x)dx = f_a^0 \cdot \frac{c^1(x-a)^1}{1!} + f_a^1 \cdot \frac{c^1(x-a)^2}{2!} + f_a^2 \cdot \frac{c^1(x-a)^3}{3!} + f_a^3 \cdot \frac{c^1(x-a)^4}{4!} + \cdots$$

$$c^2 \int_a^x \int_a^x f(x)dx^2 = f_a^0 \cdot \frac{c^2(x-a)^2}{2!} + f_a^1 \cdot \frac{c^2(x-a)^3}{3!} + f_a^2 \cdot \frac{c^2(x-a)^4}{4!} + f_a^3 \cdot \frac{c^2(x-a)^5}{5!} + \cdots$$

$$c^3 \int_a^x \int_a^x \int_a^x f(x)dx^3 = f_a^0 \cdot \frac{c^3(x-a)^3}{3!} + f_a^1 \cdot \frac{c^3(x-a)^4}{4!} + f_a^2 \cdot \frac{c^3(x-a)^5}{5!} + f_a^3 \cdot \frac{c^3(x-a)^6}{6!} + \cdots$$

$$\vdots$$

Adding these perpendicularly.

$$c^0 f(x)+c^1\int_a^x f(x)dx+c^2\int_a^x\int_a^x f(x)dx^2+c^3\int_a^x\int_a^x\int_a^x f(x)dx^3\ \cdots$$

$$= f_a^0\left\{\frac{c^0(x-a)^0}{0!}+\frac{c^1(x-a)^1}{1!}+\frac{c^2(x-a)^2}{2!}+\frac{c^3(x-a)^3}{3!}+\cdots\right\}$$

$$+f_a^1\left\{\frac{c^0(x-a)^1}{1!}+\frac{c^1(x-a)^2}{2!}+\frac{c^2(x-a)^3}{3!}+\frac{c^3(x-a)^4}{4!}+\cdots\right\}$$

$$+f_a^2\left\{\frac{c^0(x-a)^2}{2!}+\frac{c^1(x-a)^3}{3!}+\frac{c^2(x-a)^4}{4!}+\frac{c^3(x-a)^5}{5!}+\cdots\right\}$$

$$\vdots$$

$$=\sum_{r=0}^{\infty} f_a^r\sum_{s=0}^{\infty}\frac{c^s(x-a)^{s+r}}{(s+r)!}$$

i.e.

$$\sum_{r=0}^{\infty} c^r\int_a^x\cdots\int_a^x f(x)dx^r=\sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}\frac{c^s(x-a)^{s+r}}{(s+r)!} \tag{1.1}$$

Furthermore,

$$\sum_{s=0}^{\infty}\frac{c^s(x-a)^{s+r}}{(s+r)!}=\int_a^x\cdots\int_a^x e^{c(x-a)}dx^r$$

Then

$$\sum_{r=0}^{\infty} c^r\int_a^x\cdots\int_a^x f(x)dx^r=\sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x\cdots\int_a^x e^{c(x-a)}dx^r \tag{1.1'}$$

(1.2) and (1.2') are also proved in a similar way.

Note

If (1.1') and (1.2') are written down, they are as follows.

$$c^0 f(x)\pm c^1\int_a^x f(x)dx+c^2\int_a^x\int_a^x f(x)dx^2\pm c^3\int_a^x\int_a^x\int_a^x f(x)dx^3\pm+\ \cdots$$

$$=f^{(0)}(a)\pm f^{(1)}(a)\int_a^x e^{\pm c(x-a)}dx+f^{(2)}(a)\int_a^x\int_a^x e^{\pm c(x-a)}dx^2\pm f^{(3)}(a)\int_a^x\int_a^x\int_a^x e^{\pm c(x-a)}dx^3+\pm\ \cdots$$

These are similar to the Taylor series and are beautiful, so these are very good for viewing. However, these are not so useful.

Example 1 $\log x+c^1\int_1^x \log x\,dx+c^2\int_1^x\int_1^x \log x\,dx^2+c^3\int_1^x\int_1^x\int_1^x \log x\,dx^3+\cdots$

$$f(x)=\log x,\quad (\log x)^{(n)}=(-1)^{n-1}(n-1)!\ x^{-n}\qquad n=1,2,3,\cdots$$

Substituting these for (1.1), (1.1') ,

$$\sum_{r=0}^{\infty} c^r\int_1^x\cdots\int_1^x \log x\,dx^r=\sum_{r=1}^{\infty}(-1)^{r-1}(r-1)!\sum_{s=0}^{\infty}\frac{c^s(x-1)^{s+r}}{(s+r)!}$$

$$=\sum_{r=1}^{\infty}(-1)^{r-1}(r-1)!\int_1^x\cdots\int_1^x e^{c(x-1)}dx^r$$

When higher integrals are replaced with Riemann-Liouville Integral and $x=0.8, c=1.7, m=10$ are given, each is calculated as follows.

```
f[x_, c_, m_] := Log[x] + Sum[c^r/Gamma[r] Integrate[(x - t)^(r-1) Log[t], {t, 1, x}], {r, 1, m}]
g[x_, c_, m_] := Sum[(-1)^(r-1) (r - 1)! Sum[c^s (x - 1)^(s+r)/(s + r)!, {s, 0, ∞}], {r, 1, m}]
h[x_, c_, m_] := Sum[(-1)^(r-1) (r - 1)! 1/Gamma[r] Integrate[(x - t)^(r-1) e^(c (t-1)), {t, 1, x}], {r, 1, m}]
```

$$f[x\_, c\_, m\_] := \mathrm{Log}[x] + \sum_{r=1}^{m} \frac{c^r}{\mathrm{Gamma}[r]} \int_1^x (x-t)^{r-1} \mathrm{Log}[t]\, dt$$

$$g[x\_, c\_, m\_] := \sum_{r=1}^{m} (-1)^{r-1} (r-1)! \sum_{s=0}^{\infty} \frac{c^s (x-1)^{s+r}}{(s+r)!}$$

$$h[x\_, c\_, m\_] := \sum_{r=1}^{m} (-1)^{r-1} (r-1)! \frac{1}{\mathrm{Gamma}[r]} \int_1^x (x-t)^{r-1} e^{c(t-1)}\, dt$$

N[f[0.8, 1.7, 10]]
-0.190362

N[g[0.8, 1.7, 10]]
-0.190362

N[h[0.8, 1.7, 10]]
-0.190362

Example 2 $\sqrt{x} - c^1\int_1^x \sqrt{x}\,dx + c^2\int_1^x\int_1^x \sqrt{x}\,dx^2 - c^3\int_1^x\int_1^x\int_1^x \sqrt{x}\,dx^3 + - \cdots$

$$f(x) = \sqrt{x}$$

$$f^{(n)}(1) = \frac{\Gamma(1+1/2)}{\Gamma(1+1/2-n)} 1^{\frac{1}{2}-n} = (-1)^{n-1} \frac{(2n-3)!!}{2^n}$$

Substituting these for (1.2),

$$\sum_{r=0}^{\infty} (-1)^r c^r \int_a^x \cdots \int_a^x \sqrt{x}\,dx^r = e^{-c(x-1)} + \sum_{r=0}^{\infty} (-1)^{r-1} \frac{(2r-3)!!}{2^r} \sum_{s=0}^{\infty} (-1)^s \frac{c^s (x-a)^{s+r}}{(s+r)!}$$

When higher integrals are replaced with Riemann-Liouville Integral and $x=0.9$, $c=1.3$, $m=15$ are given, both sides are calculated as follows.

$$f[x\_, c\_, m\_] := \sqrt{x} + \sum_{r=1}^{m} \frac{(-1)^r c^r}{\mathrm{Gamma}[r]} \int_1^x (x-t)^{r-1} \sqrt{t}\, dt$$

$$g[x\_, c\_, m\_] := e^{-c(x-1)} + \sum_{r=1}^{m} (-1)^{r-1} \frac{(2r-3)!!}{2^r} \sum_{s=0}^{\infty} (-1)^s \frac{c^s (x-1)^{s+r}}{(s+r)!}$$

N[f[0.9, 1.3, 15]]
1.08406

N[g[0.9, 1.3, 15]]
1.08406

During the proof of the previous theorem , if the calculation is done without including (0) , we obtain the following theorem.

Theorem 25.5.1'

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\sum_{r=1}^{\infty} c^r \int_a^x \cdots \int_a^x f(x)\,dx^r = \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=1}^{\infty} \frac{c^s (x-a)^{s+r}}{(s+r)!} \tag{1.3}$$

$$\sum_{r=1}^{\infty} (-1)^{r-1} c^r \int_a^x \cdots \int_a^x f(x)\,dx^r = \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=1}^{\infty} (-1)^{s-1} \frac{c^s (x-a)^{s+r}}{(s+r)!} \tag{1.3'}$$

Combining this theorem and **25.1** , we obtain the following theorem which gives the collateral integral of the product of an exponential function and arbitrary functions.

Theorem 25.5.1"

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\int_a^x e^{cx} f(x)dx = e^{cx}\sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=1}^{\infty}(-1)^{s-1}\frac{c^{s-1}(x-a)^{s+r}}{(s+r)!} \tag{1.4}$$

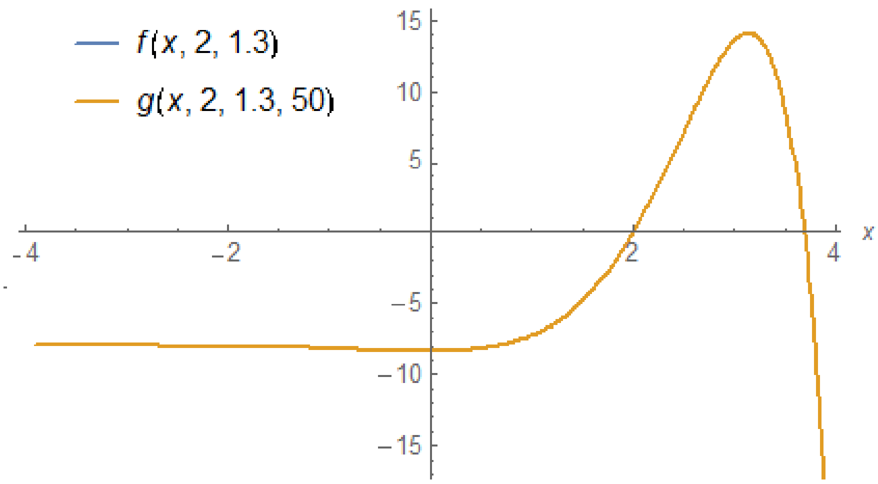
$$\int_a^x e^{-cx} f(x)dx = e^{-cx}\sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=1}^{\infty}\frac{c^{s-1}(x-a)^{s+r}}{(s+r)!} \tag{1.4'}$$

Example 2" $\int_a^x e^{cx}\sin x\,dx$

Since $(\sin x)^{(n)} = \sin(x+n\pi/2)\quad n=1,2,3,\cdots$, from (1.4),

$$\int_a^x e^{cx}\sin x\,dx = e^{cx}\sum_{r=0}^{\infty}\sin\left(a+\frac{r\pi}{2}\right)\sum_{s=1}^{\infty}(-1)^{s-1}\frac{c^{s-1}(x-a)^{s+r}}{(s+r)!}$$

When $a=2, c=1.3$ and the first 50 terms of Σ are calculated and both sides are illustrated, it is as follows. Both sides overlap exactly and blue (left) can not be seen.



Theorem 25.5.2

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\sum_{r=1}^{\infty} c^{2r-1}\int_a^x\cdots\int_a^x f(x)dx^{2r-1} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}\frac{c^{2s+1}(x-a)^{2s+1+r}}{(2s+1+r)!} \tag{2.1}$$

$$= \sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x\cdots\int_a^x \sinh\{c(x-a)\}dx^r \tag{2.1'}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}c^{2r-1}\int_a^x\cdots\int_a^x f(x)dx^{2r-1} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}(-1)^s\frac{c^{2s+1}(x-a)^{2s+1+r}}{(2s+1+r)!} \tag{2.2}$$

$$= \sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x\cdots\int_a^x \sin\{c(x-a)\}dx^r \tag{2.2'}$$

Proof

In a way similar to Theorem 25.5.1, we obtain the desired expressioons

Example 3 $c^1\int_0^x \tan x\,dx + c^3\int_0^x\int_0^x\int_0^x \tan x\,dx^3 + c^5\int_0^x\cdots\int_0^x \tan x\,dx^5 + c^7\int_0^x\cdots\int_0^x \tan x\,dx^7 + \cdots$

The higher differential quotient of $\tan x$ on $x=0$ is as follows according to Theorem 9.2.6 (**9.2**)

$$(\tan x)^{(2n-1)}\Big|_{x=0} = T_{2n-1} = \frac{2^{2n}\left(2^{2n}-1\right)|B_{2n}|}{2n} \quad , \quad (\tan x)^{(2n)}\Big|_{x=0} = 0$$

Where, T_{2n-1} is the tangent number and B_{2n} is the Bernoulli number. Thus, from (2.1) ,

$$\sum_{r=1}^{\infty} c^{2r-1}\int_0^x\cdots\int_0^x \tan x\,dx^{2r-1} = \sum_{r=1}^{\infty} T_{2r-1}\sum_{s=0}^{\infty}\frac{c^{2s+1}x^{2s+2r}}{(2s+2r)!}$$

When higher integrals are replaced with Riemann-Liouville Integral and $x=0.9, c=1.2, m=12$ are given, both sides are calculated as follows.

```
f[x_, c_, m_] := Sum[c^(2r-1)/Gamma[2r-1] Integrate[(x-t)^(2r-2) Tan[t], {t, 0, x}], {r, 1, m}]

T[r_] := 2^(2r) (2^(2r) - 1) Abs[BernoulliB[2r]] / (2r)

g[x_, c_, m_] := Sum[T[r] Sum[c^(2s+1) x^(2s+2r)/(2s+2r)!, {s, 0, ∞}], {r, 1, m}]

SetPrecision[f[0.9,1.2,12],6]
0.622605 + 0.×10^-17 i

SetPrecision[g[0.9,1.2,12],6]
0.622605
```

Example 4 $c^1\int_0^x \sec x\,dx - c^3\int_0^x\int_0^x\int_0^x \sec x\,dx^3 + c^5\int_0^x\cdots\int_0^x \sec x\,dx^5 - c^7\int_0^x\cdots\int_0^x \sec x\,dx^7 +- \cdots$

The higher differential quotient of $\sec x$ on $x=0$ is as follows according to Theorem 9.2.8 (**9.2**)

$$(\sec x)^{(2n)}\Big|_{x=0} = |E_{2n}| \quad , \quad (\sec x)^{(2n+1)}\Big|_{x=0} = 0$$

Here, E_{2n} is an Euler number. Thus, from (2.2) ,

$$\sum_{r=1}^{\infty}(-1)^{r-1}c^{2r-1}\int_0^x\cdots\int_0^x \sec x\,dx^{2r-1} = \sum_{r=0}^{\infty}|E_{2r}|\sum_{s=0}^{\infty}(-1)^s\frac{c^{2s+1}x^{2s+1+2r}}{(2s+1+2r)!}$$

When higher integrals are replaced with Riemann-Liouville Integral and $x=0.3, c=1.1, m=12$ are given, both sides are calculated as follows.

```
f[x_, c_, m_] := Sum[(-1)^(r-1) c^(2r-1)/Gamma[2r-1] Integrate[(x-t)^(2r-2) Sec[t], {t, 0, x}], {r, 1, m}]

g[x_, c_, m_] := Sum[Abs[EulerE[2r]] Sum[(-1)^s c^(2s+1) x^(2s+1+2r)/(2s+1+2r)!, {s, 0, ∞}], {r, 0, m}]

SetPrecision[f[0.3, 1.1, 12],6]
0.329080 + 0.×10^-15 i

SetPrecision[g[0.3, 1.1, 12],6]
0.329080
```

Theorem 25.5.3

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\sum_{r=0}^{\infty} c^{2r}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}\frac{c^{2s}(x-a)^{2s+r}}{(2s+r)\,!} \tag{3.1}$$

$$= \sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x\cdots\int_a^x \cosh\{c(x-a)\}\,dx^{2r} \tag{3.1'}$$

$$\sum_{r=0}^{\infty}(-1)^r c^{2r}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}(-1)^s\frac{c^{2s}(x-a)^{2s+r}}{(2s+r)\,!} \tag{3.2}$$

$$= \sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x\cdots\int_a^x \cos\{c(x-a)\}\,dx^r \tag{3.2'}$$

Proof

In a way similar to Theorem 25.5.1, we obtain the desired expressioons

Example 5 $\tan^{-1}x + c^2\int_1^x\int_1^x \tan^{-1}x\,dx^2 + c^4\int_1^x\cdots\int_1^x \tan^{-1}x\,dx^4 + c^6\int_1^x\cdots\int_1^x \tan^{-1}x\,dx^6 + \cdots$

According to " 岩波数学公式 I " p39, the following expression holds for a natural number n .

$$\left(\tan^{-1}x\right)^{(n)} = (n-1)!\cos^n\left(\tan^{-1}x\right)\sin\left(n\left(\tan^{-1}x+\frac{\pi}{2}\right)\right)$$

From this,

$$\left(\tan^{-1}x\right)^{(n)}\Big|_{x=1} = \frac{(n-1)!}{2^{n/2}}\sin\left(\frac{3n\pi}{4}\right)$$

Substituting this for (3.1) ,

$$\sum_{r=0}^{\infty} c^{2r}\int_1^x\cdots\int_1^x \tan^{-1}x\,dx^{2r} = \frac{\pi}{4}\sum_{s=0}^{\infty}\frac{c^{2s}(x-1)^{2s}}{(2s)\,!} + \sum_{r=1}^{\infty}\frac{(r-1)!}{2^{r/2}}\sin\left(\frac{3r\pi}{4}\right)\sum_{s=0}^{\infty}\frac{c^{2s}(x-1)^{2s+r}}{(2s+r)\,!}$$

When higher integrals are replaced with Riemann-Liouville Integral and $x=1.8, c=2.1, m=17$ are given, both sides are calculated as follows.

$$f[x\_, c\_, m\_] := \text{ArcTan}[x] + \sum_{r=1}^{m}\frac{c^{2r}}{\text{Gamma}[2r]}\int_1^x (x-t)^{2r-1}\,\text{ArcTan}[t]\,dt$$

$$g[x\_, c\_, m\_] := \frac{\pi}{4}\sum_{s=0}^{\infty}\frac{c^{2s}(x-1)^{2s}}{(2s)\,!} + \sum_{r=1}^{m}\frac{(r-1)\,!}{2^{r/2}}\,\text{Sin}\left[\frac{3r\pi}{4}\right]\sum_{s=0}^{\infty}\frac{c^{2s}(x-1)^{2s+r}}{(2s+r)\,!}$$

```
N[f[1.8, 2.1, 17]]
2.63984
```

```
N[g[1.8, 2.1, 17]]
2.63984
```

During the proof of the previous theorem , if the calculation is done without including (0) , we obtain the following theorem.

Theorem 25.5.3'

When c is a positive number and a is a real number on the domain of analytic function $f(x)$, the following expressions hold

$$\sum_{r=1}^{\infty} c^{2r} \int_a^x \cdots \int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=1}^{\infty} \frac{c^{2s}(x-a)^{2s+r}}{(2s+r)!} \tag{4.1}$$

$$\sum_{r=1}^{\infty} (-1)^{r-1} c^{2r} \int_a^x \cdots \int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=1}^{\infty} (-1)^{s-1} \frac{c^{2s}(x-a)^{2s+r}}{(2s+r)!} \tag{4.2}$$

Example 6

$$c^2 \int_0^x \int_0^x \sin^{-1}x\,dx^2 - c^4 \int_0^x \cdots \int_0^x \sin^{-1}x\,dx^4 + c^6 \int_0^x \cdots \int_0^x \sin^{-1}x\,dx^6 - c^8 \int_0^x \cdots \int_0^x \sin^{-1}x\,dx^8 +- \cdots$$

The higher differential quotient of $\sin^{-1}x$ on $x=0$ is as follows according to Theorem 9.3.2 (**9.3**)

$$\left(\sin^{-1}x\right)^{(2n+1)}\Big|_{x=0} = {}_{2n}C_0 (2n-1)!!\,(2n-1)!!\,0^0 = (2n-1)!!^2$$

Then, from (4.2) ,

$$\sum_{r=1}^{\infty} (-1)^{r-1} c^{2r} \int_0^x \cdots \int_0^x \sin^{-1}x\,dx^{2r} = \sum_{r=0}^{\infty} (2r-1)!!^2 \sum_{s=1}^{\infty} (-1)^{s-1} \frac{c^{2s} x^{2s+2r+1}}{(2s+2r+1)!}$$

When higher integrals are replaced with Riemann-Liouville Integral and $x=0.6$, $c=2.3$, $m=10$ are given, both sides are calculated as follows.

```
f[x_, c_, m_] := Sum[(-1)^(r-1) c^(2r)/Gamma[2r] Integrate[(x - t)^(2r-1) ArcSin[t], {t, 0, x}], {r, 1, m}]
g[x_, c_, m_] := Sum[((2r - 1)!!)^2 Sum[(-1)^(s-1) c^(2s) x^(2s+2r+1)/(2s + 2r + 1)!, {s, 1, ∞}], {r, 0, m}]

N[f[0.6, 2.3, 10]]        N[g[0.6, 2.3, 10]]
0.17668                   0.17668
```

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Alien's Mathematics