List of Theorems and Formulas (A la Carte)

01 Generalized Taylor's Theorem

Formula 1.1.1 (Formula of repeated integration by parts)

When $f(x)$ is n times differentiable function on $[a, b]$ and $f(x)$ is

$$g^{\langle n\rangle}(x) = \int g^{\langle n-1\rangle}(x)dx + c_n \qquad n=1,2,3,\cdots \qquad c_n \text{ : arbitrary constants,}$$

the following expressions hold.

$$\int_a^b g(x)f(x)dx = \sum_{r=1}^{n}(-1)^{r-1}\left[g^{\langle r\rangle}(x)f^{(r-1)}(x)\right]_a^b + (-1)^n\int_a^b g^{\langle n\rangle}(x)f^{(n)}(x)dx$$

$$\int_a^b g(x)f(x)dx = \sum_{r=1}^{n}\left[g^{\langle 2r-1\rangle}(x)f^{(2r-2)}(x)\right]_a^b - \sum_{r=1}^{n}\left[g^{\langle 2r\rangle}(x)f^{(2r-1)}(x)\right]_a^b + \int_a^b g^{\langle 2n\rangle}(x)f^{(2n)}(x)dx$$

In the following, $\Gamma(z)$, $\psi(z)$, γ and H_n are the gamma function, digamma function, Euler-Mascheroni constant and harmonic number respectively.

Formula 1.1.2 (Series expansion of Exponential Integral)

When $Ei(x) = \int_{-\infty}^{x}\frac{e^x}{x}dx$, the following expressions hold.

(1) for $x>1$

$$Ei(x) = \frac{e^x}{x}\left\{1+\frac{1!}{x^1}+\frac{2!}{x^2}+\cdots+\frac{(n-1)!}{x^{n-1}}\right\} + n!\int_{-\infty}^{x}\frac{e^x}{x^{n+1}}dx$$

(2) for $x \ne 0$

$$Ei(x) = \frac{e^x}{x}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\log|x|-\psi(r)-\gamma}{\Gamma(r)}x^r + \gamma$$

$$= \log|x| + \frac{e^x}{x}\sum_{r=1}^{\infty}(-1)^r\frac{\psi(r)}{\Gamma(r)}x^r = \log|x| + \gamma - e^x\sum_{r=0}^{\infty}(-1)^r\frac{H_r}{r!}x^r$$

Formula 1.1.3 (Series expansion of Sine Integral)

When $Si(x)=\int_0^x\frac{\sin x}{x}dx = \int_{\infty}^{x}\frac{\sin x}{x}dx + \frac{\pi}{2}$, the following expressions hold.

(1) for $x>1$

$$Si(x)= \frac{\cos x}{x}\sum_{r=1}^{n}(-1)^r\frac{(2r-2)!}{x^{2r-2}} + \frac{\sin x}{x}\sum_{r=1}^{n}(-1)^r\frac{(2r-1)!}{x^{2r-1}} + \frac{\pi}{2} + R_{2n+1}$$

$$R_{2n+1} = (-1)^n(2n)!\int_{\infty}^{x}\frac{\sin x}{x^{2n+1}}dx$$

(2) for any x

$$Si(x) = \frac{\sin x}{x}\sum_{r=1}^{n}(-1)^{r-1}\frac{\log|x|-\psi(2r-1)-\gamma}{\Gamma(2r-1)}x^{2r-1} - \frac{\cos x}{x}\sum_{r=1}^{n}(-1)^{r-1}\frac{\log|x|-\psi(2r)-\gamma}{\Gamma(2r)}x^{2r} + R_{2n+1}$$

$$R_{2n+1} = (-1)^n \int_0^x \frac{\log|x| - \psi(2n) - \gamma}{\Gamma(2n)} x^{2n-1} \sin x \, dx$$

$$Si(x) = \frac{\sin x}{x} \sum_{r=1}^{\infty} (-1)^r \frac{\psi(2r-1)}{\Gamma(2r-1)} x^{2r-1} - \frac{\cos x}{x} \sum_{r=1}^{\infty} (-1)^r \frac{\psi(2r)}{\Gamma(2r)} x^{2r}$$

$$= -\sin x \sum_{r=0}^{\infty} (-1)^r \frac{H_{2r}}{(2r)!} x^{2r} + \cos x \sum_{r=0}^{\infty} (-1)^r \frac{H_{2r+1}}{(2r+1)!} x^{2r+1}$$

Formula 1.1.4 (Series expansion of Cosine Integral)

When $Ci(x) = \int_{\infty}^{x} \frac{\cos x}{x} dx = \gamma + \log x - \int_0^x \frac{1-\cos x}{x} dx$, the following expressions hold.

(1) for $x > 1$

$$Ci(x) = \frac{\sin x}{x} \sum_{r=1}^{n} (-1)^{r-1} \frac{(2r-2)!}{x^{2r-2}} - \frac{\cos x}{x} \sum_{r=1}^{n} (-1)^{r-1} \frac{(2r-1)!}{x^{2r-1}} + R_{2n+1}$$

$$R_{2n+1} = (-1)^n (2n)! \int_{\infty}^{x} \frac{\cos x}{x^{2n+1}} dx$$

(2) for $x > 0$

$$Ci(x) = \gamma + \frac{\cos x}{x} \sum_{r=1}^{n} (-1)^{r-1} \frac{\log x - \psi(2r-1) - \gamma}{\Gamma(2r-1)} x^{2r-1}$$

$$+ \frac{\sin x}{x} \sum_{r=1}^{n} (-1)^{r-1} \frac{\log x - \psi(2r) - \gamma}{\Gamma(2r)} x^{2r} + R_{2n+1}$$

$$R_{2n+1} = (-1)^n \int_0^x \frac{\log x - \psi(2n) - \gamma}{\Gamma(2n)} x^{2n-1} \cos x \, dx$$

$$Ci(x) = \log x + \frac{\cos x}{x} \sum_{r=1}^{\infty} (-1)^r \frac{\psi(2r-1)}{\Gamma(2r-1)} x^{2r-1} + \frac{\sin x}{x} \sum_{r=1}^{\infty} (-1)^r \frac{\psi(2r)}{\Gamma(2r)} x^{2r}$$

$$= \log x + \gamma - \cos x \sum_{r=0}^{\infty} (-1)^r \frac{H_{2r}}{(2r)!} x^{2r} - \sin x \sum_{r=0}^{\infty} (-1)^r \frac{H_{2r+1}}{(2r+1)!} x^{2r+1}$$

Lemma 1.2.1 (Lagrange form of the remainder)

When $f(x)$ is n times differentiable function on $[a, b]$ and $g^{<n>}(x)$ is

$$g^{\langle n \rangle}(x) = \int g^{\langle n-1 \rangle}(x) dx + c_n \qquad n = 1, 2, 3, \cdots \qquad c_n \text{ : arbitrary constants}$$

the following expressions hold.

$$f(b) - f(a) = \sum_{r=1}^{n} (-1)^{r-1} \left[g^{\langle r \rangle}(x) f^{(r)}(x) \right]_a^b + R_{n+1}$$

$$R_{n+1} = (-1)^n \int_a^b g^{\langle n \rangle}(x) f^{(n+1)}(x) dx = (-1)^n \left[g^{\langle n+1 \rangle}(x) \right]_a^b f^{(n+1)}(\xi) \quad (a < \xi < b)$$

Theorem 1.2.2 (Generalized Taylor's Theorem)

When $f(x)$ is $n+1$ times differentiable function on $[a, b]$, ξ exists such that $a < \xi < b$, and the followings hold.

$$f(b) - f(a) = \sum_{r=1}^{n} \frac{(b-a)^r}{r!} f^{(r)}(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

$$f(b) - f(a) = \sum_{r=1}^{n} (-1)^{r-1} \frac{(b-a)^r}{r!} f^{(r)}(b) + (-1)^n \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

Formula 1.2.3 (Taylor Expansion of Logarithmic Function)

$$\log x = \sum_{r=1}^{n} \frac{(-1)^{r-1}}{r}(x-1)^r + \frac{(-1)^n}{n+1}\left(\frac{x-1}{\xi}\right)^{n+1} \qquad 1<\xi<x$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r}(x-1)^r \qquad 0<x\leqq 2$$

$$\log x = \sum_{r=1}^{n} \frac{1}{r}\left(\frac{x-1}{x}\right)^r + \frac{1}{n+1}\left(\frac{x-1}{\xi}\right)^{n+1} \qquad 1<\xi<x$$

$$= \sum_{r=1}^{\infty} \frac{1}{r}\left(\frac{x-1}{x}\right)^r \qquad x\geqq\frac{1}{2}$$

Example

$$\log 3 = \frac{1}{1}\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \cdots$$

Theorem 1.2.4 (Generalized Binomial Theorem)

$$(1+x)^\alpha = \sum_{r=0}^{n}\binom{\alpha}{r}x^r + \binom{\alpha}{n+1}x^{n+1}(1+\xi)^{\alpha-(n+1)} \qquad 0<\xi<x$$

$$= \sum_{r=0}^{\infty}\binom{\alpha}{r}x^r \qquad |x|<1$$

$$= x^\alpha\sum_{r=0}^{\infty}\binom{\alpha}{r}\frac{1}{x^r} \qquad \left\{= \sum_{r=0}^{\infty}\binom{\alpha}{\alpha-r}x^{\alpha-r}\right\} \qquad |x|>1$$

$$(1+x)^\alpha = \sum_{r=0}^{\infty}(-1)^r\binom{-\alpha}{r}\left(\frac{x}{1+x}\right)^r \qquad \left\{= \sum_{r=0}^{\infty}\binom{\alpha-1+r}{\alpha-1}\left(\frac{x}{1+x}\right)^r\right\} \qquad x>-\frac{1}{2}$$

Example

$$(1+x)^\pi = 1 + \binom{\pi}{1}x^1 + \binom{\pi}{2}x^2 + \binom{\pi}{3}x^3 + \cdots \qquad |x|<1$$

$$= x^\pi\left\{1 + \binom{\pi}{1}\frac{1}{x^1} + \binom{\pi}{2}\frac{1}{x^2} + \binom{\pi}{3}\frac{1}{x^3} + \cdots\right\} \qquad x>1$$

$$= 1 + \binom{-\pi}{1}\left(-\frac{x}{1+x}\right)^1 + \binom{-\pi}{2}\left(-\frac{x}{1+x}\right)^2 + \cdots \qquad x>-\frac{1}{2}$$

Taylor Expansion of Power Function

$$x^\alpha = \sum_{r=0}^{\infty}\binom{\alpha}{r}(x-1)^r \qquad 0<x<2$$

$$x^\alpha = (x-1)^\alpha\sum_{r=0}^{\infty}\binom{\alpha}{r}\frac{1}{(x-1)^r} \qquad x>2$$

$$x^\alpha = \sum_{r=0}^{\infty}\binom{-\alpha}{r}\left(\frac{1}{x}-1\right)^r \qquad x>\frac{1}{2}$$

Example

$$x^e = 1 + \binom{e}{1}(x-1)^1 + \binom{e}{2}(x-1)^2 + \binom{e}{3}(x-1)^3 + \cdots \qquad 0<x<2$$

$$= (x-1)^e\left\{1 + \binom{e}{1}\frac{1}{(x-1)^1} + \binom{e}{2}\frac{1}{(x-1)^2} + \binom{e}{3}\frac{1}{(x-1)^3} + \cdots\right\} \qquad x>2$$

$$= 1+\binom{-e}{1}\left(\frac{1}{x}-1\right)^1+\binom{-e}{2}\left(\frac{1}{x}-1\right)^2+\binom{-e}{3}\left(\frac{1}{x}-1\right)^3+\cdots \qquad x>\frac{1}{2}$$

Taylor Expansion of x

$$x=\sum_{r=0}^{\infty}\left(\frac{x-1}{x}\right)^r \qquad x>\frac{1}{2}$$

$$x=\sum_{r=1}^{\infty}\left(\frac{x}{1+x}\right)^r \qquad x>-\frac{1}{2}$$

Examples

$$2=1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\cdots \qquad \left(=\frac{1}{1-1/2}\right)$$

$$=\frac{2}{3}+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^4+\cdots \qquad \left(=\frac{2}{3}\frac{1}{1-2/3}\right)$$

$$3=1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^4+\cdots \qquad \left(=\frac{1}{1-2/3}\right)$$

$$=\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\left(\frac{3}{4}\right)^4+\cdots \qquad \left(=\frac{3}{4}\frac{1}{1-3/4}\right)$$

Taylor Expansion of Fractional Function

$$\frac{1}{x^{\alpha}}=\sum_{r=0}^{\infty}\binom{-\alpha}{r}(x-1)^r \qquad 0<x<2$$

$$\frac{1}{x^{\alpha}}=\sum_{r=0}^{\infty}\binom{\alpha}{r}\left(\frac{1}{x}-1\right)^r \qquad x>\frac{1}{2}$$

Example

$$\frac{1}{x^{\sqrt{2}}}=1+\binom{\sqrt{2}}{1}\left(\frac{1}{x}-1\right)^1+\binom{\sqrt{2}}{2}\left(\frac{1}{x}-1\right)^2+\binom{\sqrt{2}}{3}\left(\frac{1}{x}-1\right)^3+\cdots \qquad x>\frac{1}{2}$$

02 Multiple Series & Exponetial Function

Formula 2.1.0 (Multiple Series & Half Multiple Series)

When a multiple series $\sum_{r_1,r_2,\cdots,r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n}$ is absolutely convergent, the following expressions hold.

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{r,s} = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r-s,s}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} a_{r,s,t} = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s,s-t,t}$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{r_1-r_2,r_2-r_3,\cdots,r_{n-1}-r_n,r_n}$$

Note

In short, we should just perform the following operations to $\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_{n-1},r_n}$ ..

Replace r_{n-1} with $r_{n-1}-r_n$, and replace the 1st ∞ with r_{n-1} from the right.

Replace r_{n-2} with $r_{n-2}-r_{n-1}$, and replace the 2nd ∞ with r_{n-2} from the right.

$\vdots$

Replace r_1 with r_1-r_2 , and replace the $(n-1)$th ∞ with r_1 from the right.

Formula 2.1.1

When m is a non-negative integer, the following expressions hold.

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a^r b^s \frac{x^{m+r+s}}{(m+r+s)!} = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a^{r-s} b^s \frac{x^{m+r}}{(m+r)!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} a^r b^s c^t \frac{x^{m+r+s+t}}{(m+r+s+t)!} = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a^{r-s} b^{s-t} c^t \frac{x^{m+r}}{(m+r)!}$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\prod_{k=1}^{n} a_k^{r_k} \frac{x^{m+\sum_{k=1}^{n} r_k}}{\left(m+\sum_{k=1}^{n} r_k\right)!} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_1^{r_1-r_2} a_2^{r_2-r_3}\cdots a_n^{r_n} \frac{x^{m+r_1}}{(m+r_1)!}$$

Formula 2.1.2

$$\sum_{s=0}^{r} 2^s = \frac{2^{1+r}-1}{2-1}$$

$$\sum_{s=0}^{r}\sum_{t=0}^{s} 2^{s-t} 3^t = 3\cdot\frac{3^{1+r}-1}{3-1} - 2\cdot\frac{2^{1+r}-1}{2-1}$$

$$\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} 2^{s-t} 3^{t-u} 4^u = 8\cdot\frac{4^{1+r}-1}{4-1} - 9\cdot\frac{3^{1+r}-1}{3-1} + 2\cdot\frac{2^{1+r}-1}{2-1}$$

$$\vdots$$

$$\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\cdots\sum_{r_n=0}^{r_{n-1}} 2^{r_2-r_3} 3^{r_3-r_4}\cdots n^{r_n} = \sum_{s=0}^{n-2} (-1)^s \frac{{}_nC_s\,(n-s)^{n-1}(n-s-1)}{n!}\,\frac{(n-s)^{1+r_1}-1}{n-s-1}$$

Example

$$\sum_{s=0}^{7}\sum_{t=0}^{s}\sum_{u=0}^{t}2^{s-t}3^{t-u}4^{u} = 8\cdot\frac{4^{1+7}-1}{4-1} - 9\cdot\frac{3^{1+7}-1}{3-1} + 2\cdot\frac{2^{1+7}-1}{2-1} = 145750$$

Formula 2.1.3

$$\sum_{s=0}^{r}2^{s} = \frac{2^{2+r}-2\cdot 1^{2+r}+0^{2+r}}{2!}$$

$$\sum_{s=0}^{r}\sum_{t=0}^{s}2^{s-t}3^{t} = \frac{3^{3+r}-3\cdot 2^{3+r}+3\cdot 1^{3+r}-0^{3+r}}{3!}$$

$$\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}2^{s-t}3^{t-u}4^{u} = \frac{4^{4+r}-4\cdot 3^{4+r}+6\cdot 2^{4+r}-4\cdot 1^{4+r}+0^{4+r}}{4!}$$

$$\vdots$$

$$\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\cdots\sum_{r_n=0}^{r_{n-1}}2^{r_2-r_3}3^{r_3-r_4}\cdots n^{r_n} = \frac{1}{n!}\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}(n-s)^{n+r_1}$$

Example

$$\sum_{s=0}^{4}\sum_{t=0}^{s}2^{s-t}3^{t} = 2^0 3^0+\left(2^1 3^0+2^0 3^1\right)+\left(2^2 3^0+2^1 3^1+2^0 3^2\right)+\left(2^3 3^0+2^2 3^1+2^1 3^2+2^0 3^3\right)$$

$$= \frac{3^{3+4}-3\cdot 2^{3+4}+3\cdot 1^{3+4}-0^{3+4}}{3!} = 301$$

Formula 2.1.4

$$\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}(n-s)^{n-1} = 0 \qquad n=1,2,3,\cdots$$

$$\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}(n-s)^{n} = n! \qquad n=0,1,2,\cdots$$

$$\sum_{s=0}^{n}(-1)^{s}{}_{n}C_{s}(n-s)^{n+1} = {}_{n+1}C_{2}\,n! \qquad n=1,2,3,\cdots$$

Example $n=2$

$$\sum_{s=0}^{2}(-1)^{s}{}_{2}C_{s}(2-s)^{2-1} = {}_{2}C_{0}2^1 - {}_{2}C_{1}1^1 + {}_{2}C_{2}0^1 = 0$$

$$\sum_{s=0}^{2}(-1)^{s}{}_{2}C_{s}(2-s)^{2} = {}_{2}C_{0}2^2 - {}_{2}C_{1}1^2 + {}_{2}C_{2}0^2 = 2!$$

$$\sum_{s=0}^{2}(-1)^{s}{}_{2}C_{s}(2-s)^{2+1} = {}_{2}C_{0}2^3 - {}_{2}C_{1}1^3 + {}_{2}C_{2}0^3 = {}_{3}C_{2}\,2!$$

Formula 2.1.5

$$\left(e^{x}-1\right)^{1} = \sum_{r=0}^{\infty}\left(1^{1+r}-0^{1+r}\right)\frac{x^{1+r}}{(1+r)!}$$

$$\left(e^{x}-1\right)^{2} = \sum_{r=0}^{\infty}\left(2^{2+r}-2\cdot 1^{2+r}+0^{2+r}\right)\frac{x^{2+r}}{(2+r)!}$$

$$\left(e^{x}-1\right)^{3} = \sum_{r=0}^{\infty}\left(3^{3+r}-3\cdot 2^{3+r}+3\cdot 1^{3+r}-0^{3+r}\right)\frac{x^{3+r}}{(3+r)!}$$

$$\vdots$$

$$\left(e^{x}-1\right)^{n} = \sum_{r=0}^{\infty}\sum_{s=0}^{n}(-1)^{n-s}{}_{n}C_{s}s^{n+r}\frac{x^{n+r}}{(n+r)!}$$

Formula 2.2.1

When m is a non-negative integer, the following expressions hold.

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^{m+r+s}}{(m+r+s)!} = \sum_{r=0}^{\infty}\frac{{}_{1+r}C_{1}\,x^{m+r}}{(m+r)!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{m+r+s+t}}{(m+r+s+t)!} = \sum_{r=0}^{\infty}\frac{{}_{2+r}C_{2}\,x^{m+r}}{(m+r)!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\frac{x^{m+r+s+t+u}}{(m+r+s+t+u)!} = \sum_{r=0}^{\infty}\frac{{}_{3+r}C_{3}\,x^{m+r}}{(m+r)!}$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{x^{m+\sum_{k=1}^{n} r_k}}{\left(m+\sum_{k=1}^{n} r_k\right)!} = \sum_{r=0}^{\infty}\frac{{}_{n-1+r}C_{n-1}\,x^{m+r}}{(m+r)!}$$

Formula 2.2.2

$$\frac{1x^1}{1!}+\frac{2x^2}{2!}+\frac{3x^3}{3!}+\frac{4x^4}{4!}+\cdots = e^x\cdot\frac{x^1}{1!}$$

$$\frac{1x^2}{2!}+\frac{3x^3}{3!}+\frac{6x^4}{4!}+\frac{10x^5}{5!}+\cdots = e^x\cdot\frac{x^2}{2!}$$

$$\frac{1x^3}{3!}+\frac{4x^4}{4!}+\frac{10x^5}{5!}+\frac{20x^6}{6!}+\cdots = e^x\cdot\frac{x^3}{3!}$$

$$\vdots$$

$$\sum_{r=0}^{\infty}\frac{{}_{n+r}C_{n}\,x^{n+r}}{(n+r)!} = e^x\cdot\frac{x^n}{n!}$$

Formula 2.2.3

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^{1+r+s}}{(1+r+s)!} = e^x\cdot\frac{x^1}{1!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{2+r+s+t}}{(2+r+s+t)!} = e^x\cdot\frac{x^2}{2!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\frac{x^{3+r+s+t+u}}{(3+r+s+t+u)!} = e^x\cdot\frac{x^3}{3!}$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{x^{n-1+\sum_{k=1}^{n} r_k}}{\left(n-1+\sum_{k=1}^{n} r_k\right)!} = e^x\cdot\frac{x^{n-1}}{(n-1)!}$$

Formula 2.2.4

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}\frac{x^{n-m+\sum_{k=1}^{n+1} r_k}}{\left(n-m+\sum_{k=1}^{n+1} r_k\right)!} = e^x\sum_{r=0}^{n}\binom{m}{r}\frac{x^{n-r}}{(n-r)!} \qquad (n \geqq m)$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}\frac{x^{n+m+\sum_{k=1}^{n+1}r_k}}{\left(n+m+\sum_{k=1}^{n+1}r_k\right)!} = e^x\sum_{r=0}^{n}\binom{-m}{r}\frac{x^{n-r}}{(n-r)!} - \sum_{s=0}^{m-1}\binom{-m+s}{n}\frac{x^s}{s!}$$

Example: $n=2,\ m=1,2$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{0+r+s+t}}{(0+r+s+t)!} = e^x\sum_{r=0}^{2}\binom{2}{r}\frac{x^{2-r}}{(2-r)!} = e^x\left(1\cdot\frac{x^2}{2!} + 2\cdot\frac{x^1}{1!} + 1\cdot\frac{x^0}{0!}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{1+r+s+t}}{(1+r+s+t)!} = e^x\sum_{r=0}^{2}\binom{1}{r}\frac{x^{2-r}}{(2-r)!} = e^x\left(1\cdot\frac{x^2}{2!} + 1\cdot\frac{x^1}{1!}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{2+r+s+t}}{(2+r+s+t)!} = e^x\cdot\frac{x^2}{2!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{3+r+s+t}}{(3+r+s+t)!} = e^x\sum_{r=0}^{2}\binom{-1}{r}\frac{x^{2-r}}{(2-r)!} - \sum_{s=0}^{1-1}\binom{-1+s}{2}\frac{x^s}{s!}$$

$$= e^x\left(\frac{x^2}{2!}-\frac{x^1}{1!}+\frac{x^0}{0!}\right) - \frac{x^0}{0!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{4+r+s+t}}{(4+r+s+t)!} = e^x\sum_{r=0}^{2}\binom{-2}{r}\frac{x^{2-r}}{(2-r)!} - \sum_{s=0}^{2-1}\binom{-2+s}{2}\frac{x^s}{s!}$$

$$= e^x\left(1\cdot\frac{x^2}{2!}-2\cdot\frac{x^1}{1!}+3\cdot\frac{x^0}{0!}\right) - \left(3\cdot\frac{x^0}{0!}+1\cdot\frac{x^1}{1!}\right)$$

Formula 2.2.4'

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^{2+r+s}}{(2+r+s)!} = e^x\left(\frac{x^1}{1!}-\frac{x^0}{0!}\right) + 1$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{3+r+s+t}}{(3+r+s+t)!} = e^x\left(\frac{x^2}{2!}-\frac{x^1}{1!}+\frac{x^0}{0!}\right) - 1$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\frac{x^{4+r+s+t+u}}{(4+r+s+t+u)!} = e^x\left(\frac{x^3}{3!}-\frac{x^2}{2!}+\frac{x^1}{1!}-\frac{x^0}{0!}\right) + 1$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{x^{n+\sum_{k=1}^{n}r_k}}{\left(n+\sum_{k=1}^{n}r_k\right)!} = e^x\sum_{r=0}^{n-1}(-1)^r\frac{x^{n-1-r}}{(n-1-r)!} + (-1)^n$$

Formula 2.3.1

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{x^{m+r+s}}{(m+r+s)!} = \sum_{r=0}^{\infty}(-1)^r\frac{{}_{1+r}C_1\,x^{m+r}}{(m+r)!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{x^{m+r+s+t}}{(m+r+s+t)!} = \sum_{r=0}^{\infty}(-1)^r\frac{{}_{2+r}C_2\,x^{m+r}}{(m+r)!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}(-1)^{r+s+t+u}\frac{x^{m+r+s+t+u}}{(m+r+s+t+u)!} = \sum_{r=0}^{\infty}(-1)^r\frac{{}_{3+r}C_3\,x^{m+r}}{(m+r)!}$$

⋮

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\frac{x^{m+\sum_{k=1}^{n} r_k}}{\left(m+\sum_{k=1}^{n} r_k\right)!} = \sum_{r=0}^{\infty}(-1)^r\frac{{}_{n-1+r}C_{n-1}\,x^{m+r}}{(m+r)!}$$

Formula 2.3.2

$$\frac{1x^1}{1!}-\frac{2x^2}{2!}+\frac{3x^3}{3!}-\frac{4x^4}{4!}+-\cdots = \frac{1}{e^x}\frac{x^1}{1!}$$

$$\frac{1x^2}{2!}-\frac{3x^3}{3!}+\frac{6x^4}{4!}-\frac{10x^5}{5!}+-\cdots = \frac{1}{e^x}\frac{x^2}{2!}$$

$$\frac{1x^3}{3!}-\frac{4x^4}{4!}+\frac{10x^5}{5!}-\frac{20x^6}{6!}+-\cdots = \frac{1}{e^x}\frac{x^3}{3!}$$

⋮

$$\sum_{r=0}^{\infty}(-1)^r\frac{{}_{n+r}C_n\,x^{n+r}}{(n+r)!} = \frac{1}{e^x}\frac{x^n}{n!}$$

Formula 2.3.3

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{x^{1+r+s}}{(1+r+s)!} = \frac{1}{e^x}\frac{x^1}{1!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{x^{2+r+s+t}}{(2+r+s+t)!} = \frac{1}{e^x}\frac{x^2}{2!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}(-1)^{r+s+t+u}\frac{x^{3+r+s+t+u}}{(3+r+s+t+u)!} = \frac{1}{e^x}\frac{x^3}{3!}$$

⋮

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\frac{x^{n-1+\sum_{k=1}^{n} r_k}}{\left(n-1+\sum_{k=1}^{n} r_k\right)!} = \frac{1}{e^x}\frac{x^{n-1}}{(n-1)!}$$

Formula 2.3.4

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}(-1)^{\sum_{k=1}^{n+1} r_k}\frac{x^{n-m+\sum_{k=1}^{n+1} r_k}}{\left(n-m+\sum_{k=1}^{n+1} r_k\right)!} = \frac{(-1)^m}{e^x}\sum_{r=0}^{n}\binom{m}{r}\frac{(-1)^r x^{n-r}}{(n-r)!}$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}(-1)^{\sum_{k=1}^{n+1} r_k}\frac{x^{n+m+\sum_{k=1}^{n+1} r_k}}{\left(n+m+\sum_{k=1}^{n+1} r_k\right)!} = \frac{(-1)^m}{e^x}\sum_{r=0}^{n}\binom{-m}{r}\frac{(-1)^r x^{n-r}}{(n-r)!} - (-1)^m\sum_{s=0}^{m-1}\binom{-m+s}{n}\frac{(-1)^s x^s}{s!}$$

Example: n**=2,** m**=1**

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{x^{1+r+s+t}}{(1+r+s+t)!}=\frac{(-1)^1}{e^x}\sum_{r=0}^{2}\binom{1}{r}\frac{(-1)^r x^{2-r}}{(2-r)!}=-\frac{1}{e^x}\left(1\cdot\frac{x^2}{2!}-1\cdot\frac{x^1}{1!}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{x^{2+r+s+t}}{(2+r+s+t)!}=\frac{1}{e^x}\frac{x^2}{2!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{x^{3+r+s+t}}{(3+r+s+t)!}=\frac{(-1)^1}{e^x}\sum_{r=0}^{1}\binom{-1}{r}\frac{(-1)^r x^{2-r}}{(2-r)!}-(-1)^1\sum_{s=0}^{1-1}\binom{s-1}{2}\frac{(-1)^s x^s}{s!}$$

$$=-\frac{1}{e^x}\left(\frac{x^2}{2!}+\frac{x^1}{1!}+\frac{x^0}{0!}\right)+\frac{x^0}{0!}$$

Formula 2.3.4'

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{x^{2+r+s}}{(2+r+s)!}=1-\frac{1}{e^x}\left(\frac{x^1}{1!}+\frac{x^0}{0!}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{x^{3+r+s+t}}{(3+r+s+t)!}=1-\frac{1}{e^x}\left(\frac{x^2}{2!}+\frac{x^1}{1!}+\frac{x^0}{0!}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}(-1)^{r+s+t+u}\frac{x^{4+r+s+t+u}}{(4+r+s+t+u)!}=1-\frac{1}{e^x}\left(\frac{x^3}{3!}+\frac{x^2}{2!}+\frac{x^1}{1!}+\frac{x^0}{0!}\right)$$

$\vdots$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{x^{n+\sum_{k=1}^{n}r_k}}{\left(n+\sum_{k=1}^{n}r_k\right)!}=1-\frac{1}{e^x}\sum_{r=0}^{n-1}\frac{x^{n-1-r}}{(n-1-r)!}$$

Formula 2.4.1

$$\sum_{r=0}^{\infty}\frac{1^r}{(1+r)!}x^{1+r}=\frac{1}{1!}\left(e^x-1\right)^1$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{1^r2^s}{(2+r+s)!}x^{2+r+s}=\frac{1}{2!}\left(e^x-1\right)^2$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r2^s3^t}{(3+r+s+t)!}x^{3+r+s+t}=\frac{1}{3!}\left(e^x-1\right)^3$$

$\vdots$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+\sum_{k=1}^{n}r_k\right)!}x^{n+\sum_{k=1}^{n}r_k}=\frac{1}{n!}\left(e^x-1\right)^n$$

Formula 2.4.2

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n-m+\sum_{k=1}^{n}r_k\right)!}x^{n-m+\sum_{k=1}^{n}r_k}=\frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\,(n-r)^{m-1}e^{(n-r)x}$$

for $n\geq m$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+m+\sum_{k=1}^{n}r_k\right)!}x^{n+m+\sum_{k=1}^{n}r_k}=\frac{1}{n!}\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^m}e^{(n-s)x}$$

$$-\frac{1}{n!}\sum_{r=0}^{m}\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^{m-r}}\frac{x^r}{r!}$$

Example: $n=3$, $m=1,2$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r2^s3^t}{(1+r+s+t)!}x^{1+r+s+t}=\frac{1}{(3-1)!}\sum_{r=0}^{3-1}(-1)^r{}_{3-1}C_r(3-r)^{2-1}e^{(3-r)x}$$

$$=\frac{1}{2!}\left(3^1e^{3x}-2\cdot2^1e^{2x}+1^1e^x\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r2^s3^t}{(2+r+s+t)!}x^{2+r+s+t}=\frac{1}{(3-1)!}\sum_{r=0}^{3-1}(-1)^r{}_{3-1}C_r(3-r)^{1-1}e^{(3-r)x}$$

$$=\frac{1}{2!}\left(e^{3x}-2e^{2x}+e^x\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r2^s3^t}{(3+r+s+t)!}x^{3+r+s+t}=\frac{1}{3!}\left(e^x-1\right)^3$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r2^s3^t}{(4+r+s+t)!}x^{4+r+s+t}=\frac{1}{3!}\sum_{s=0}^{3-1}\frac{(-1)^s{}_3C_s}{(3-s)^1}e^{(3-s)x}-\frac{1}{3!}\sum_{r=0}^{1}\sum_{s=0}^{3-1}\frac{(-1)^s{}_3C_s}{(3-s)^{1-r}}\frac{x^r}{r!}$$

$$=\frac{1}{3!}\left(\frac{1}{3^1}e^{3x}-\frac{3}{2^1}e^{2x}+\frac{3}{1^1}e^x\right)$$

$$-\frac{1}{3!}\left\{\left(\frac{1}{3^1}-\frac{3}{2^1}+\frac{3}{1^1}\right)\frac{x^0}{0!}+\left(\frac{1}{3^0}-\frac{3}{2^0}+\frac{3}{1^0}\right)\frac{x^1}{1!}\right\}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r2^s3^t}{(5+r+s+t)!}x^{5+r+s+t}=\frac{1}{3!}\sum_{s=0}^{3-1}\frac{(-1)^s{}_3C_s}{(3-s)^2}e^{(3-s)x}-\frac{1}{3!}\sum_{r=0}^{2}\sum_{s=0}^{3-1}\frac{(-1)^s{}_3C_s}{(3-s)^{2-r}}\frac{x^r}{r!}$$

$$=\frac{1}{3!}\left(\frac{1}{3^2}e^{3x}-\frac{3}{2^2}e^{2x}+\frac{3}{1^2}e^x\right)$$

$$-\frac{1}{3!}\left\{\left(\frac{1}{3^2}-\frac{3}{2^2}+\frac{3}{1^2}\right)\frac{x^0}{0!}+\left(\frac{1}{3^1}-\frac{3}{2^1}+\frac{3}{1^1}\right)\frac{x^1}{1!}+\left(\frac{1}{3^0}-\frac{3}{2^0}+\frac{3}{1^0}\right)\frac{x^2}{2!}\right\}$$

Formula 2.4.3

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n-m+\sum_{k=1}^{n}r_k\right)!}(\log x)^{n-m+\sum_{k=1}^{n}r_k}=\frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r(n-r)^{m-1}x^{(n-r)}$$

$$\text{for } n\geq m$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+m+\sum_{k=1}^{n}r_k\right)!}(\log x)^{n+m+\sum_{k=1}^{n}r_k}=\frac{1}{n!}\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^m}x^{(n-s)}$$

$$-\frac{1}{n!}\sum_{r=0}^{m}\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^{m-r}}\frac{(\log x)^r}{r!}$$

Formula 2.4.3'

$$\sum_{r=0}^{\infty}\frac{1^r}{(1+r)!}(\log x)^{1+r} = \frac{1}{1!}(x-1)^1$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{1^r 2^s}{(2+r+s)!}(\log x)^{2+r+s} = \frac{1}{2!}(x-1)^2$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r 2^s 3^t}{(3+r+s+t)!}(\log x)^{3+r+s+t} = \frac{1}{3!}(x-1)^3$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n} k^{r_k}}{\left(n+\sum_{k=1}^{n} r_k\right)!}(\log x)^{n+\sum_{k=1}^{n} r_k} = \frac{1}{n!}(x-1)^n$$

Formula 2.5.1

$$\sum_{r=0}^{\infty}(-1)^r\frac{1^r}{(1+r)!}x^{1+r} = \frac{1}{1!}\left(1-\frac{1}{e^x}\right)^1$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{1^r 2^s}{(2+r+s)!}x^{2+r+s} = \frac{1}{2!}\left(1-\frac{1}{e^x}\right)^2$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r 2^s 3^t}{(3+r+s+t)!}x^{3+r+s+t} = \frac{1}{3!}\left(1-\frac{1}{e^x}\right)^3$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\frac{\prod_{k=1}^{n} k^{r_k}}{\left(n+\sum_{k=1}^{n} r_k\right)!}x^{n+\sum_{k=1}^{n} r_k} = \frac{1}{n!}\left(1-\frac{1}{e^x}\right)^n$$

Formula 2.5.2

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\frac{\prod_{k=1}^{n} k^{r_k}}{\left(n-m+\sum_{k=1}^{n} r_k\right)!}x^{\sum_{k=1}^{n} r_k} = \frac{(-1)^{n-m}}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\frac{(n-r)^{m-1}}{e^{(n-r)x}}$$

$$\text{for } n \ge m$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\frac{\prod_{k=1}^{n} k^{r_k}}{\left(n+m+\sum_{k=1}^{n} r_k\right)!}x^{\sum_{k=1}^{n} r_k}$$

$$= \frac{(-1)^{n+m}}{n!}\left\{\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^m e^{(n-s)x}} - \sum_{r=0}^{m}\sum_{s=0}^{n-1}\frac{(-1)^{r+s}{}_nC_s}{(n-s)^{m-r}}\frac{x^r}{r!}\right\}$$

Example: $n=3$, $m=1$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r 2^s 3^t}{(2+r+s+t)!}x^{2+r+s+t} = \frac{(-1)^{3-1}}{(3-1)!}\sum_{r=0}^{3-1}(-1)^r{}_{3-1}C_r\frac{(3-r)^{1-1}}{e^{(3-r)x}}$$

$$= \frac{1}{2!}\left(\frac{1}{e^{3x}}-\frac{2}{e^{2x}}+\frac{1}{e^x}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r 2^s 3^t}{(3+r+s+t)!}x^{3+r+s+t}=\frac{(-1)^{3-0}}{3!}\left(\frac{1}{e^x}-1\right)^3$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r 2^s 3^t}{(4+r+s+t)!}x^{4+r+s+t}$$

$$=\frac{(-1)^{3+1}}{3!}\left\{\sum_{s=0}^{3-1}\frac{(-1)^s{}_3C_s}{(3-s)^1 e^{(3-s)x}}-\sum_{r=0}^{1}\sum_{s=0}^{3-1}\frac{(-1)^{r+s}{}_3C_s}{(3-s)^{1-r}}\frac{x^r}{r!}\right\}$$

$$=\frac{1}{3!}\left(\frac{1}{3^1e^{3x}}-\frac{3}{2^1e^{2x}}+\frac{3}{1^1e^x}\right)-\frac{1}{3!}\left\{\left(\frac{1}{3^1}-\frac{3}{2^1}+\frac{3}{1^1}\right)\frac{x^0}{0!}-\left(\frac{1}{3^0}-\frac{3}{2^0}+\frac{3}{1^0}\right)\frac{x^1}{1!}\right\}$$

Formula 2.5.3

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n-m+\sum_{k=1}^{n}r_k\right)!}(\log x)^{\sum_{k=1}^{n}r_k}=\frac{(-1)^{n-m}}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\frac{(n-r)^{m-1}}{x^{(n-r)}}$$

$$for\ n\ge m$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+m+\sum_{k=1}^{n}r_k\right)!}(\log x)^{\sum_{k=1}^{n}r_k}$$

$$=\frac{(-1)^{n+m}}{n!}\left\{\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^m x^{(n-s)}}-\sum_{r=0}^{m}\sum_{s=0}^{n-1}\frac{(-1)^{r+s}{}_nC_s}{(n-s)^{m-r}}\frac{(\log x)^r}{r!}\right\}$$

Formula 2.5.3'

$$\sum_{r=0}^{\infty}(-1)^r\frac{1^r}{(1+r)!}(\log x)^{1+r}=\frac{1}{1!}\left(1-\frac{1}{x}\right)^1$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{1^r2^s}{(2+r+s)!}(\log x)^{2+r+s}=\frac{1}{2!}\left(1-\frac{1}{x}\right)^2$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r2^s3^t}{(3+r+s+t)!}(\log x)^{3+r+s+t}=\frac{1}{3!}\left(1-\frac{1}{x}\right)^3$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+\sum_{k=1}^{n}r_k\right)!}(\log x)^{n+\sum_{k=1}^{n}r_k}=\frac{1}{n!}\left(1-\frac{1}{x}\right)^n$$

Formula 2.6.1 (Euler-Mascheroni Constant)

When γ is the Euler-Mascheroni Constant $(=0.57721566\cdots)$, the following expressions hold.

$$1-\gamma=\sum_{r=2}^{\infty}\sum_{s=2}^{\infty}\frac{1}{sr^s}=\sum_{r=2}^{\infty}\sum_{s=2}^{r}\frac{1}{s(2+r-s)^s}=\sum_{s=2}^{\infty}\frac{\zeta(s)-1}{s}$$

$$\gamma=\sum_{r=1}^{\infty}\sum_{s=2}^{\infty}\frac{(-1)^s}{s\,r^s}=\sum_{r=2}^{\infty}\sum_{s=2}^{r}\frac{(-1)^s}{s(1+r-s)^s}=\sum_{s=2}^{\infty}(-1)^s\frac{\zeta(s)}{s}$$

03 Higher Calculus of Binomial Identity

Formula 3.1.1 (zakii)

$$\sum_{s=0}^{n}(-1)^s {}_nC_s s^k = 0 \qquad k=0,1,\cdots,n-1$$

$$\sum_{s=0}^{n}(-1)^s {}_nC_s s^n = (-1)^n n!$$

Formula 3.2.1

$$\sum_{s=0}^{n}\frac{(-1)^s}{s+1}{}_nC_s = \frac{0!}{n+1}$$

$$\sum_{s=0}^{n}\frac{(-1)^s}{s+2}{}_nC_s = \frac{1!}{(n+1)(n+2)}$$

$$\sum_{s=0}^{n}\frac{(-1)^s}{s+3}{}_nC_s = \frac{2!}{(n+1)(n+2)(n+3)}$$

$$\vdots$$

$$\sum_{s=0}^{n}\frac{(-1)^s}{s+m}{}_nC_s = \frac{(m-1)!}{(n+1)(n+2)\cdots(n+m)} \qquad \{\, = B(1+n,m)\,\}$$

c.f.

$$\sum_{r=0}^{n-1}\frac{(-1)^r}{m+r}{}_{n-1}C_r = B(m,n)$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r}{p+r}\binom{q-1}{r} = B(p,q)$$

By-products (Partial Fraction Decomposition)

The following expression holds for $s \neq -1,-2,-3,\cdots$.

$$\prod_{t=1}^{n}\frac{1}{s+t} = \frac{1}{(n-1)!}\sum_{t=1}^{n}\frac{(-1)^{t-1}}{s+t}\binom{n-1}{t-1}$$

04 Euler-Maclaurin Summation Formula

4.1 & Bernoulli Polynomial

Bernoulli Number

(1) Definition

Bernoulli numbers B_k $(k=1,2,3,\cdots)$ are defined as coefficients of the following equation.

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}x^k$$

(2) Calculation Method

$$B_n=\sum_{k=0}^{n}\frac{1}{k+1}\sum_{r=0}^{k}(-1)^r{}_kC_r\,r^n$$

If the first few are written down, it is as follows.

$$B_0=\frac{1}{1}{}_0C_0\,0^0$$

$$B_1=\frac{1}{1}{}_0C_0\,0^1+\frac{1}{2}\left({}_1C_0\,0^1-{}_1C_1\,1^1\right)$$

$$B_2=\frac{1}{1}{}_0C_0\,0^2+\frac{1}{2}\left({}_1C_0\,0^2-{}_1C_1\,1^2\right)+\frac{1}{3}\left({}_2C_0\,0^2-{}_2C_1\,1^2+{}_2C_2\,2^2\right)$$

$$\vdots$$

Bernoulli Polynomial

(1) Definition

If B_n are Bernoulli numbers, polynomial $B_n(x)$ that satisfies the following 3 expressions is called ***Bernoulli Polynomial***.

$$B_0(x)=1$$

$$\frac{d}{dx}B_n(x)=nB_{n-1}(x)\qquad(n\geqq1)$$

$$\int_0^1 B_n(x)dx=0\qquad(n\geqq1)$$

(2) Properties

$$B_n(x)=n\int_0^x B_{n-1}(t)dt+B_n\qquad(n\geqq1)$$

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k=\sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}$$

$$B_n(0)=B_n$$

$$B_n(1)=B_n(0)\qquad(n\geqq2)$$

$$B_n(x+1)-B_n(x)=nx^{n-1}\qquad(n\geqq1)$$

$$B_n(1-x)=(-1)^nB_n(x)\qquad(n\geqq1)$$

For arbitrary natural number m and interval $[0,1]$,

$$\left|B_{2m}(x)\right|\leq\left|B_{2m}\right|$$

$$\left|B_{2m+1}(x)\right|\leq(2m+1)\left|B_{2m}\right|$$

Examples

$$B_1(x)=x-\frac{1}{2}\,,\qquad B_2(x)=x^2-x+\frac{1}{6}\,,\qquad B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x\,,$$

$$B_4(x)=x^4-2x^3+x^2-\frac{1}{30}\,,\qquad B_5(x)=x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x\,,$$

$$B_6(x)=x^6-3x^5+\frac{5}{2}x^4-\frac{1}{2}x^2+\frac{1}{42}\,,\qquad B_7(x)=x^7-\frac{7}{2}x^6+\frac{7}{2}x^5-\frac{7}{2}x^3+\frac{1}{6}x\,,$$

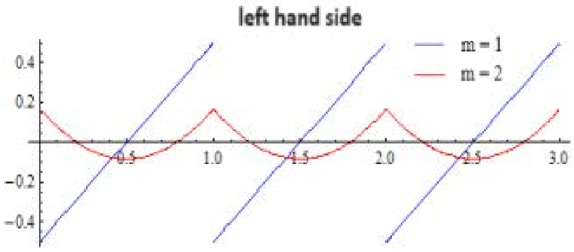
Formula 4.1.5 (Fourier Expansion of Bernoulli Polynomial)

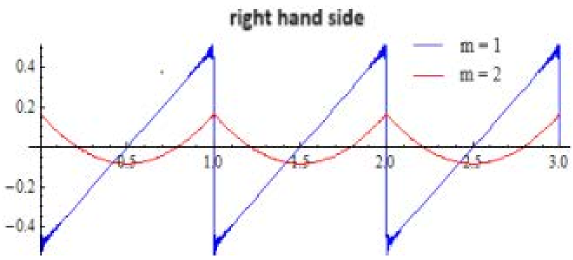
When m is a natural number, $\lfloor x\rfloor$ is a floor function and B_m are Bernoulli numbers ,

$$B_m(x-\lfloor x\rfloor) = -2m!\sum_{s=1}^{\infty}\frac{1}{(2\pi s)^m}cos\left(2\pi sx-\frac{m\pi}{2}\right) \qquad x \geqq 0$$

Example

If the left side and the right side are illustrated for $m=1,2$, it is as follows. Blue is $m=1$ and Red is $m=2$.





4.2 Euler-Maclaurin Summation Formula

Formula 4.2.1 ($k=a\sim b-1$, B_r)

When $f(x)$ is a function of class C^m on a closed interval $[a,b]$, $\lfloor x\rfloor$ is the floor function, B_r are Bernoulli numbers and $B_n(x)$ are Bernoulli polynomials, the following expression holds.

$$\sum_{k=a}^{b-1}f(k) = \int_a^b f(x)dx + \sum_{r=1}^{m}\frac{B_r}{r!}\left\{f^{(r-1)}(b)-f^{(r-1)}(a)\right\} + R_m$$

$$R_m = \frac{(-1)^{m+1}}{m!}\int_a^b B_m(x-\lfloor x\rfloor)f^{(m)}(x)dx$$

$$= (-1)^m 2\int_a^b\left\{\sum_{s=1}^{\infty}\frac{1}{(2\pi s)^m}cos\left(2\pi sx-\frac{m\pi}{2}\right)\right\}f^{(m)}(x)dx$$

When $\lim_{m\to\infty}|R_m|=\infty$ m is a even number $s.t.$ $\dfrac{\left|f^{(m)}(x)\right|}{(2\pi)^m} = minimum\ \ for\ x\in[a,b]$

Formula 4.2.2 ($k=a\sim b-1$, B_{2r})

When $f(x)$ is a function of class C^{2m} on a closed interval $[a,b]$, $\lfloor x\rfloor$ is the floor function, B_r are Bernoulli numbers and $B_n(x)$ are Bernoulli polynomials, the following expression holds.

$$\sum_{k=a}^{b-1}f(k) = \int_a^b f(x)dx - \frac{1}{2}\{f(b)-f(a)\} + \sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left\{f^{(2r-1)}(b)-f^{(2r-1)}(a)\right\} + R_{2m}$$

$$R_{2m} = -\frac{1}{(2m)!}\int_a^b B_{2m}(x-\lfloor x\rfloor)f^{(2m)}(x)\,dx$$
$$= (-1)^m 2\int_a^b \left\{ \sum_{s=1}^{\infty} \frac{\cos(2\pi sx)}{(2\pi s)^{2m}} \right\} f^{(2m)}(x)\,dx$$

When $\lim_{m\to\infty} |R_{2m}| = \infty$, m is natural number $s.t.$ $\frac{|f^{(2m)}(x)|}{(2\pi)^{2m}} = minimum$ $for\ x\in[a,b]$

Formula 4.2.2' ($k = a \sim b$, B_{2r})

When $f(x)$ is a function of class C^{2m} on a closed interval $[a,b]$, $\lfloor x\rfloor$ is the floor function,

B_r are Bernoulli numbers and $B_n(x)$ are Bernoulli polynomials, the following expression holds.

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)dx + \frac{1}{2}\{f(b)+f(a)\} + \sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left\{f^{(2r-1)}(b) - f^{(2r-1)}(a)\right\} + R_{2m}$$
$$R_{2m} = -\frac{1}{(2m)!}\int_a^b B_{2m}(x-\lfloor x\rfloor)f^{(2m)}(x)\,dx$$
$$= (-1)^m 2\int_a^b \left\{ \sum_{s=1}^{\infty} \frac{\cos(2\pi sx)}{(2\pi s)^{2m}} \right\} f^{(2m)}(x)\,dx$$

When $\lim_{m\to\infty} |R_{2m}| = \infty$, m is natural number $s.t.$ $\frac{|f^{(2m)}(x)|}{(2\pi)^{2m}} = minimum$ $for\ x\in[a,b]$

4.3 Sum of Elementary Sequence

Formula 4.3.1 (Sum of Arithmetic Sequence)

$$\sum_{k=0}^{n-1}(a+kd) = \frac{n}{2}\{2a+(n-1)d\}$$

Formula 4.3.2 (Sum of Geometric Sequence)

$$\sum_{k=0}^{n-1} r^k = (r^n-1)\sum_{s=0}^{m}\frac{B_s}{s!}(\log r)^{s-1} + R_m$$
$$R_m = (-1)^{m+1}\frac{(\log r)^m}{m!}\int_0^n B_m(x-\lfloor x\rfloor)\, r^x dx$$
$$\sum_{k=0}^{n-1} r^k = (r^n-1)\sum_{s=0}^{\infty}\frac{B_s}{s!}(\log r)^{s-1} = \frac{r^n-1}{r-1}$$

Formula 4.3.3 (Jacob Bernoulli)

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1}\sum_{r=0}^{m}\binom{m+1}{r}B_r n^{m+1-r} = \frac{1}{m+1}\{B_{m+1}(n) - B_{m+1}(0)\}$$

Example: $m=3$, $n=101$

$$\sum_{k=0}^{101-1} k^3 = 0^3+1^3+2^3+\cdots+100^3 = 25502500$$
$$\frac{1}{3+1}\{B_{3+1}(101)-B_{3+1}(0)\} = \frac{1}{4}\left(\frac{3060299999}{30}+\frac{1}{30}\right) = 25502500$$

Formula 4.3.4 (Sum of alternative integer powers of natural numbers)

$$\sum_{k=0}^{n-1}(-1)^{k-1}k^m = \frac{1}{m+1}\sum_{r=0}^{m}\binom{m+1}{r}B_r\cdot\left(n^{m+1-r} - 2^{m+1}\left\lceil\frac{n}{2}\right\rceil^{m+1-r}\right)$$

$$= \frac{1}{m+1}\left\{B_{m+1}(n) - B_{m+1} - 2^{m+1}\left\{B_{m+1}\left(\left\lceil\frac{n}{2}\right\rceil\right) - B_{m+1}\right\}\right\}$$

Example: $m=3$, $n=101$

$$\sum_{k=0}^{101-1}(-1)^{k-1}k^3 = 1^3-2^3+3^3-4^3+\cdots+99^3-100^3 = -507500$$

$$\frac{1}{3+1}\left\{B_{3+1}(101) - B_{3+1} - 2^{3+1}\left\{B_{3+1}\left(\left\lceil\frac{101}{2}\right\rceil\right) - B_{3+1}\right\}\right\} = -507500$$

Formula 4.3.4' (Sum of alternating squares of natural numbers)

$$\sum_{k=1}^{n}(-1)^{k-1}k^2 = (-1)^{n-1}\sum_{k=1}^{n}k = (-1)^{n-1}\frac{n(n+1)}{2}$$

Example: $n=999$

$$\sum_{k=1}^{999}(-1)^{k-1}k^2 = 1^2-2^2+3^2-4^2+-\cdots+999^2 = 499500$$

$$(-1)^{999-1}(1+2+3+\cdots+999) = \frac{999\cdot 1000}{2} = 499500$$

Formula 4.3.5s (Sum of Sine Sequence)

$$\sum_{k=0}^{n-1}\sin k = -\frac{\sin n}{2} - (\cos n - 1)\left\{1 + \sum_{r=1}^{m}(-1)^r\frac{B_{2r}}{(2r)!}\right\} + R_{2m}$$

$$R_{2m} = \frac{(-1)^{m+1}}{(2m)!}\int_0^n B_{2m}(x-\lfloor x\rfloor)\sin x\,dx$$

$$\sum_{k=0}^{n-1}\sin k = -\frac{\sin n}{2} - (\cos n - 1)\left(\frac{1}{2}\cot\frac{1}{2}\right) = \sin\frac{n-1}{2}\sin\frac{n}{2} / \sin\frac{1}{2}$$

Formula 4.3.5c (Sum of Cosine Sequence)

$$\sum_{k=0}^{n-1}\cos k = \sin n\cdot\left\{1 + \sum_{r=1}^{m}(-1)^r\frac{B_{2r}}{(2r)!}\right\} - \frac{1}{2}(\cos n - 1) + R_{2m}$$

$$R_{2m} = \frac{(-1)^{m+1}}{(2m)!}\int_0^n B_{2m}(x-\lfloor x\rfloor)\cos x\,dx$$

$$\sum_{k=0}^{n-1}\cos k = \sin n\cdot\frac{1}{2}\cot\frac{1}{2} - \frac{1}{2}(\cos n - 1) = \cos\frac{n-1}{2}\sin\frac{n}{2} / \sin\frac{1}{2}$$

4.4 Sum of Harmonic Sequence & Euler-Mascheroni Constant

Formula 4.4.1 (Sum of Harmonic Sequence)

When γ is Euler-Mascheroni Constant and m is a natural number greater than or equal to 2 ,

$$\sum_{k=1}^{n-1}\frac{1}{k} = \gamma + \log n - \frac{1}{2n} - \sum_{r=1}^{m}\frac{B_{2r}}{2r\cdot n^{2r}} + \int_n^{\infty}\frac{B_{2m}(x-\lfloor x\rfloor)}{x^{2m+1}}dx$$

Example: $\sum_{k=1}^{100} 1/k$

When $m=2$,

$$\gamma + \log 101 - \frac{1}{2\cdot 101} - \left(\frac{B_2}{2\cdot 101^2} + \frac{B_4}{4\cdot 11^4}\right) = 5.18737751763962\cdots$$

This all digits (14 digits below the decimal point) are significant digits.

Calculation of Euler-Mascheroni Constant

$$\gamma = \sum_{k=1}^{n-1}\frac{1}{k} - \log n + \frac{1}{2n} + \sum_{r=1}^{m}\frac{B_{2r}}{2r\cdot n^{2r}} - \int_n^\infty \frac{B_{2m}(x-\lfloor x\rfloor)}{x^{2m+1}}dx$$

Where, $m=n$. Then, the number of significant digits is roughly given by $2n-?$.

Example: $m=n=10$

$$\sum_{k=1}^{10-1}\frac{1}{k} - \log 10 + \frac{1}{2\cdot 10} + \sum_{r=1}^{10}\frac{B_{2r}}{2r\cdot 10^{2r}} = 0.577215664901532860\cdots$$

This all digits (18 digits below the decimal point) are significant digits.

4.5 Sum of Zeta Sequence & Zeta Function

Formula 4.5.1 (Sum of Zeta Sequence)

When $\zeta(p)$ is the Riemann Zeta Function and $B(p,q)$ is the beta function, the following expression hols for $p\neq 1$.

$$\sum_{k=1}^{n-1}\frac{1}{k^p} = \zeta(p) + \frac{1}{1-p}\sum_{r=0}^{m}\binom{1-p}{r}B_r n^{1-p-r} + R_m$$

$$R_m = \frac{1}{mB(m,p)}\int_n^\infty \frac{B_m(x-\lfloor x\rfloor)}{x^{p+m}}dx$$

Where, m is an even number s.t. $\lceil p\rceil \leqq m < \infty$.

Example: $\sum_{k=1}^{100} 1/k^{1.1}$

When $m = 1.1\!\uparrow = 2$,

$$\sum_{k=1}^{101-1}\frac{1}{k^{1.1}} \fallingdotseq \zeta(1.1) + \frac{1}{1-1.1}\sum_{r=0}^{2}\binom{1-1.1}{r}B_r n^{1-1.1-r} = 4.278024023\cdots$$

This all digits (9 digits below the decimal point) are significant digits.

Calculation of Riemann Zeta Function

Transposing the terms of (1.1) , we obtain as follows.

$$\zeta(p) = \sum_{k=1}^{n-1}\frac{1}{k^p} - \frac{1}{1-p}\sum_{r=0}^{m}\binom{1-p}{r}B_r n^{1-p-r} - \frac{1}{mB(m,p)}\int_n^\infty \frac{B_m(x-\lfloor x\rfloor)}{x^{p+m}}dx$$

Where, $m=n$, Then, the number of significant digits is roughly given by $2n-?$

Example: $\zeta(1.3)$

When $m=n=10$,

$$\sum_{k=1}^{10-1}\frac{1}{k^{1.3}} - \frac{1}{1-1.3}\sum_{r=0}^{10}\binom{1-1.3}{r}B_r 10^{1-1.3r} = 3.9319492118095\cdots$$

This all digits (13 digits after the decimal point) are significant digits.

4.6 Sum of real number powers of natural numbers

Formula 4.6.1

When $\zeta(p)$ is the Riemann Zeta Function and $B(p,q)$ is the beta function, the following expression hols for $p \neq -1$.

$$\sum_{k=1}^{n-1} k^p = \zeta(-p) + \frac{1}{1+p}\sum_{r=0}^{m}\binom{1+p}{r}B_r n^{1+p-r} + \frac{1}{m\,B(m,-p)}\int_n^{\infty}\frac{B_m(x-\lfloor x\rfloor)}{x^{-p+m}}dx$$

Where, m is an even number s.t. $\lceil p\rceil \leqq m < \infty$.

Example 1 : $\sum_{k=1}^{100} k^{0.1}$

When $m=2$,

$$\sum_{k=1}^{101-1} k^{0.1} \fallingdotseq \zeta(-0.1)+\frac{1}{1+0.1}\sum_{r=0}^{2}\binom{1+0.1}{r}B_r 101^{1+0.1-r} = 144.456549944\cdots$$

This all digits (9 digits below the decimal point) are significant digits.

4.7 Sum of alternative real powers

Formula 4.7.1 (Sum of alternative positive powers of natural numbers)

When $\zeta(p)$ is the Riemann Zeta Function and $B(p,q)$ is the beta function, the following expression holsfor $p \neq -1$.

$$\sum_{k=0}^{n-1}(-1)^{k-1}k^p = \left(1-2^{1+p}\right)\zeta(-p)+\frac{1}{1+p}\sum_{r=0}^{m}\binom{1+p}{r}B_r\left(n^{1+p-r}-2^{1+p}\left\lceil\frac{n}{2}\right\rceil^{1+p-r}\right)+R_m$$

$$R_m = \frac{1}{m\,B(m,-p)}\left\{\int_n^{\infty}\frac{B_m(x-\lfloor x\rfloor)}{x^{-p+m}}dx - 2^{1+p}\int_{\lceil n/2\rceil}^{\infty}\frac{B_m(x-\lfloor x\rfloor)}{x^{-p+m}}dx\right\}$$

Where, m is an even number s.t. $\lceil p\rceil \leqq m < \infty$.

Example : $p=0.6$, $n=1001$

```
fl[p_, n_] := Sum[(-1)^(k-1) k^p, {k, 0, n-1}]

fr[p_, n_, m_] := (1 - 2^(1+p)) Zeta[-p]
      + 1/(1+p) Sum[Binomial[1 + p, r] BernoulliB[r] (n^(1+p-r) - 2^(1+p) Ceiling[n/2]^(1+p-r)), {r, 0, m}]

SetPrecision[{fl[0.6, 1001], fr[0.6, 1001, 4]}, 15]
{-31.2026520696209, -31.2026520696212}
```

In this case, $m=4$ gives the best approximation (11 digits below the decimal point) .

Formula 4.7.2 (Sum of Eta Sequence)

When $\zeta(p)$ is the Riemann Zeta Function , $\eta(p)$ is the Dirichlet Eta Function and $B(p,q)$ is the Beta Function, the following expression hols for $p \neq -1$.

$$\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k^p} = \left(1-2^{1-p}\right)\zeta(p)+\frac{1}{1-p}\sum_{r=0}^{m}\binom{1-p}{r}B_r\left(n^{1-p-r}-2^{1-p}\left\lceil\frac{n}{2}\right\rceil^{1-p-r}\right)+R_m$$

$$R_m = \frac{1}{m\,B(m,p)}\left\{\int_n^\infty \frac{B_m(x-\lfloor x\rfloor)}{x^{p+m}}dx - 2^{1-p}\int_{\lceil n/2\rceil}^\infty \frac{B_m(x-\lfloor x\rfloor)}{x^{p+m}}dx\right\}$$

Where, m is an even number s.t. $\lceil p\rceil \leqq m < \infty$.

Especially, when $n \to \infty$,

$$\eta(p) = \left(1-2^{1-p}\right)\zeta(p)$$

Examples : $p=1.7$, $n=1001$, $n=\infty$

$$\text{fl}[p\_, n\_] := \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k^p}$$

$$\text{fr}[p\_, n\_, m\_] := \left(1-2^{1-p}\right)\text{Zeta}[p] + \frac{1}{1-p}\sum_{r=0}^{m}\text{Binomial}[1-p, r]\,\text{BernoulliB}[r]\left(n^{1-p-r} - 2^{1-p}\,\text{Ceiling}\left[\frac{n}{2}\right]^{1-p-r}\right)$$

```
SetPrecision[{fl[1.7, 1001]        , fr[1.7, 1001, 4]}, 15]
{0.789721725383435                 , 0.789721725383434  }

SetPrecision[{fl[1.7, ∞]           , fr[1.7, ∞, 4]}, 30]
{0.789725693648715920680558610911  , 0.789725693648715920680558610911 }
```

When $n=1001$, the right side is calculated up to $m=4$, but 14 digits after the decimal point are significant figures.

05 Generalized Bernoulli Polynomials and Numbers

When p is a real number, the generalized Bernoulli Polynomial is given by the following.

$$B_p(x) = -2\Gamma(1+p)\sum_{s=1}^{\infty}\frac{1}{(2\pi s)^p}\cos\left(2\pi sx-\frac{\pi p}{2}\right) \qquad 0\leqq x\leqq 1$$

When p is a real number, the generalizedl Bernoulli Numberl is given by the following.

$$B_p = \begin{cases} -\dfrac{2\Gamma(1+p)}{(2\pi)^p}\cos\dfrac{p\pi}{2}\cdot\zeta(p) & p\neq 1,-1,-2,-3,\cdots \\ -1/2 & p=1 \\ -p\zeta(1-p) & p=-1,-2,-3\cdots \end{cases}$$

Formula 5.3.1 (Generalized Bernoulli's Formula)

$$\sum_{k=0}^{n-1}k^p = \frac{1}{p+1}\sum_{r=0}^{p}\binom{p+1}{r}B_r n^{p+1-r} = \frac{1}{p+1}\left\{B_{p+1}(n)-B_{p+1}(0)\right\} \qquad p\neq -1$$

Formula 5.3.2'

When $p\neq 0,-1,-2,-3,\cdots$

$$\sum_{k=0}^{n-1}k^p \fallingdotseq \frac{1}{p+1}\sum_{r=0}^{m}\binom{p+1}{r}B_r n^{p+1-r} + \zeta(-p)$$

Where $|p|\uparrow+1 \leqq m < \infty$

By-products

$$B_{p+1}(0) = -(p+1)\zeta(-p)$$

$$B_{p+1}(n) = \sum_{r=0}^{m}\binom{p+1}{r}B_r n^{p+1-r} + \frac{p+1}{m B(m,-p)}\int_n^{\infty}\frac{B_m(x-\lfloor x\rfloor)}{x^{-p+m}}dx$$

Formula 5.4.1 (Generalized Bernoulli Number & Zeta Function)

When $\zeta(p), B_p$ denote the Riemann zeta function and the generalized Bernoulli number, the followings hold true.

$$\zeta(p) = -\frac{B_{1-p}}{1-p} \qquad p\neq 1,0$$

$$\zeta(1-p) = -\frac{B_p}{p} \qquad p\neq 0,1$$

$$B_p = \frac{p}{1-p}\frac{2\Gamma(p)}{(2\pi)^p}\cos\frac{p\pi}{2}\cdot B_{1-p} \qquad p\neq 1,0,-1,-2,\cdots$$

$$B_{1-p} = \frac{1-p}{p}\frac{2\Gamma(1-p)}{(2\pi)^{1-p}}\sin\frac{p\pi}{2}\cdot B_p \qquad p\neq 0,1,2,3,\cdots$$

$$\zeta(p) = \frac{2\Gamma(1-p)}{(2\pi)^{1-p}}\sin\frac{p\pi}{2}\cdot\zeta(1-p) \qquad p\neq 0,1,2,3,\cdots$$

$$\zeta(1-p) = \frac{2\Gamma(p)}{(2\pi)^p}\cos\frac{p\pi}{2}\cdot\zeta(p) \qquad p\neq 1,0,-1,-2,\cdots$$

06 Superellipse (Lamé curve)

6.1 Equations of a superellipse

A superellipse (horizontally long) is expressed as follows.

Implicit Equation

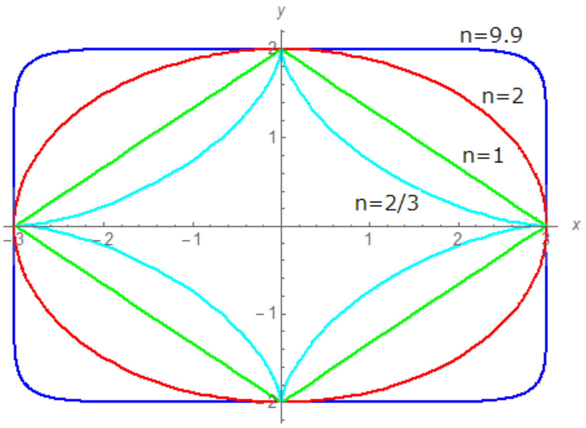
$$\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1 \qquad 0 < b \le a \quad , \; n > 0$$

Explicit Equation

$$y = \pm b\left(1 - \left|\frac{x}{a}\right|^n\right)^{\frac{1}{n}} \qquad 0 < b \le a \quad , \; n > 0$$

When $a=3,\ b=2$, the superellipses for $n=9.9,\ 2,\ 1$ *and* $2/3$ are drawn as follows.

Fig1



6.2 Area of a superellipse

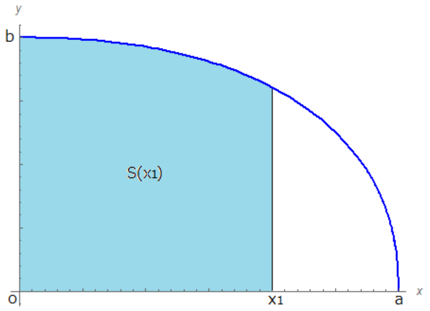
Formula 6.2.1

When n,a,b $(b \le a)$ are positive numbers respectively and $\Gamma(z)$ is the gamma function, the area S of the ellipse of degree n is given by the following expression.

$$S = 4ab\left\{\Gamma\left(1+\frac{1}{n}\right)\right\}^2 / \,\Gamma\left(1+\frac{2}{n}\right) = 4ab\sum_{r=0}^{\infty}\binom{1/n}{r}\frac{(-1)^r}{nr+1}$$

6.3 A part of area of a superellipse

In this section, we calculate the area of the light-blue portion of the following figure.



Formula 6.3.1

When $n,a,b\ (b \le a)$ are positive numbers respectively, the area $s(x)$ from 0 to x of the ellipse of degree n in the 1st quadrant is given by the following expression.

$$s(x) = b\sum_{r=0}^{\infty}\binom{1/n}{r}\frac{(-1)^r}{a^{nr}}\frac{x^{nr+1}}{nr+1}$$

By-Products

$$S = 4ab\sum_{r=0}^{\infty}\binom{1/n}{r}\frac{(-1)^r}{nr+1}$$

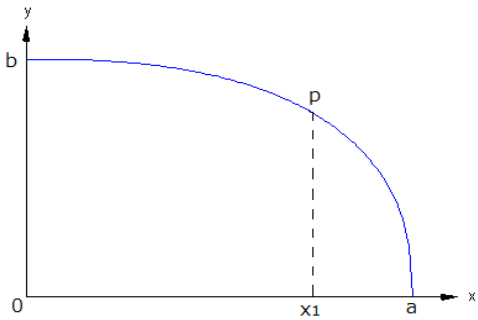
$$\left\{\Gamma\left(1+\frac{1}{n}\right)\right\}^2 / \Gamma\left(1+\frac{2}{n}\right) = \sum_{r=0}^{\infty}\binom{1/n}{r}\frac{(-1)^r}{nr+1}$$

6.4 Arc length of a superellipse

6.4.1 Arc length of an oblong superellipse

In this sub-section, we calculate the length $\overset{\frown}{bP}$ of the following oblong superellipse.

$$y = b\left(1-\frac{x^n}{a^n}\right)^{\frac{1}{n}} \qquad 0 < b \le a \quad , \ n > 0 \tag{1.0}$$



Formula 6.4.1

Let $n,a,b\ (b \le a)$ are positive numbers, x be a number s.t. $0 < x \le a\left\{1+\left(\frac{b}{a}\right)^{\frac{n}{n-1}}\right\}^{-\frac{1}{n}}$ and $(z)_s$ be Pochhammer's symbol. Then, the arc length $l(x)$ from 0 to x of (1.0) is given by the followings.

$$l(x) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\binom{\frac{2(n-1)r}{n}-1+s}{\frac{2(n-1)r}{n}-1}\frac{b^{2r}}{a^{2nr+ns}}\frac{x^{2(n-1)r+ns+1}}{2(n-1)r+ns+1}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\left(\frac{2(n-1)r}{n}\right)_s\frac{1}{s!}\frac{b^{2r}}{a^{2nr+ns}}\frac{x^{2(n-1)r+ns+1}}{2(n-1)r+ns+1}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{1/2}{r-s}\left(\frac{2(n-1)(r-s)}{n}\right)_s\frac{1}{s!}\frac{b^{2(r-s)}}{a^{2n(r-s)+ns}}\frac{x^{2(n-1)(r-s)+ns+1}}{2(n-1)(r-s)+ns+1}$$

6.4.2 Arc length of a longwise superellipse

In Formula 6.4.1, we cannot calculate the arc length $\overset{\frown}{aP_1}$ of an oblong superellipse. However, this problem is easily solvable. That is, let us replace x and y in (1.0) , as follows.

$$y = a\left(1 - \frac{x^n}{b^n}\right)^{\frac{1}{n}} \qquad 0 < b \le a \quad , \; n > 0 \tag{2.0}$$

And we calculate by the following formula.

Formula 6.4.2

Let $n,a,b\ (b \le a)$ are positive numbers, x be a number s.t. $0 < x \le b\left\{1 + \left(\frac{a}{b}\right)^{\frac{n}{n-1}}\right\}^{-\frac{1}{n}}$ and $(z)_s$ be Pochhammer's symbol. Then, the arc length $l(x)$ from 0 to x of (2.0) is given by the followings.

$$l(x) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\binom{\frac{2(n-1)r}{n}-1+s}{\frac{2(n-1)r}{n}-1}\frac{a^{2r}}{b^{2nr+ns}}\frac{x^{2(n-1)r+ns+1}}{2(n-1)r+ns+1} \tag{2.1}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\left(\frac{2(n-1)r}{n}\right)_s\frac{1}{s!}\frac{a^{2r}}{b^{2nr+ns}}\frac{x^{2(n-1)r+ns+1}}{2(n-1)r+ns+1} \tag{2.1'}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{1/2}{r-s}\left(\frac{2(n-1)(r-s)}{n}\right)_s\frac{1}{s!}\frac{a^{2(r-s)}}{b^{2n(r-s)+ns}}\frac{x^{2(n-1)(r-s)+ns+1}}{2(n-1)(r-s)+ns+1} \tag{2.1''}$$

Note

In the longwise superellipse, the convergence speed of the double series (2.1) (or (2.1')) is slow. It is about 1/100 of the convergence speed of the diagonal series (2.1") .

6.5 Peripheral length of a superellipse

Formula 6.5.1

When $n,a,b\ (b \le a)$ are positive numbers respectively, the peripheral length L of the ellipse of degree n is given by the following expressions.

$$L = 4\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\binom{\frac{2(n-1)r}{n}-1+s}{\frac{2(n-1)r}{n}-1}\frac{b^{2r}a^{1-2r}A^{-\frac{2(n-1)r+ns+1}{n}} + a^{2r}b^{1-2r}B^{-\frac{2(n-1)r+ns+1}{n}}}{2(n-1)r+ns+1}$$

$$= 4\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\left(\frac{2(n-1)r}{n}\right)_s\frac{1}{s!}\frac{b^{2r}a^{1-2r}A^{-\frac{2(n-1)r+ns+1}{n}} + a^{2r}b^{1-2r}B^{-\frac{2(n-1)r+ns+1}{n}}}{2(n-1)r+ns+1}$$

$$= 4\sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{1/2}{r-s}\left(\frac{2(n-1)(r-s)}{n}\right)_s\frac{1}{s!}\frac{1}{2(n-1)(r-s)+ns+1}\times$$
$$\left\{b^{2(r-s)}a^{1-2(r-s)}A^{-\frac{2(n-1)(r-s)+ns+1}{n}} + a^{2(r-s)}b^{1-2(r-s)}B^{-\frac{2(n-1)(r-s)+ns+1}{n}}\right\} \tag{1.1''}$$

Where, $A = 1 + \left(\frac{b}{a}\right)^{\frac{n}{n-1}}$, $B = 1 + \left(\frac{a}{b}\right)^{\frac{n}{n-1}}$

Example: Peripheral length of the ellipse of degree 2.5, a =3, b =2

We calculate this peripheral length by numerical integral and by the above formula respectively. Since the numerical integral can not be accurately calculated, We add two integration values in the previous section, and quadruple it.

Formula (1.1") is the only choice for the convergence speed. As the result of calculating $\sum\sum$ to 45, both were consistent until 9 digits below the decimal point.

Numerical Integral

```
4 × ( 2.708999727549167452512418 + 1.413871105491204061256779 )
16.491483332161486055O7679
```

Diagonal Series

```
a = 3; b = 2; n = 2.5;
```

$$A = 1 + \left(\frac{b}{a}\right)^{\frac{n}{n-1}};$$

```
SetPrecision[A, 10]
1.508761886
```

$$B = 1 + \left(\frac{a}{b}\right)^{\frac{n}{n-1}};$$

```
SetPrecision[B, 10]
2.965556046
```

$$Ld[m\_] := 4 \sum_{r=0}^{m} \sum_{s=0}^{r} \text{Binomial}\left[\frac{1}{2}, r-s\right] \text{Pochhammer}\left[\frac{2(n-1)(r-s)}{n}, s\right] \frac{1}{s!} \times \frac{b^{2(r-s)} a^{1-2(r-s)} A^{-\frac{2(n-1)(r-s)+ns+1}{n}} + a^{2(r-s)} b^{1-2(r-s)} B^{-\frac{2(n-1)(r-s)+ns+1}{n}}}{2(n-1)(r-s)+ns+1}$$

```
SetPrecision[Ld[45], 12]
16.4914833320
```

07 New Formula for the Sum of Powers

7.3 Expression with binomial coefficients of a power (Part2)

Formula 7.3.1

When m,n are natural numbers, the following expression holds.

$$n^m = \sum_{r=0}^{m-1} {}_mD_{r+1}\,{}_{n+r}C_m$$

Where, ${}_mD_r$ $r=1,2,\cdots,m$ are **Eulerian Numbers** which are given by

$${}_mD_r = \sum_{s=0}^{r-1} (-1)^s\,{}_{m+1}C_s (r-s)^m \qquad m=1,2,3,\cdots$$

Formula 7.3.2

When m is a natural number and x is a positive number, the following expression holds.

$$x^m = \sum_{r=0}^{m-1} {}_mD_{r+1} \binom{x+r}{m}$$

7.4 New formula for the sum of powers

Formula 7.4.1

When m,n are natural numbers, the following expression holds.

$$\sum_{k=1}^{n} k^m = \sum_{r=1}^{m} {}_mD_r\,{}_{n+r}C_{m+1}$$

Example: $\sum_{k=1}^{100} k^5$

$${}_{101}C_6 + 26\,{}_{102}C_6 + 66\,{}_{103}C_6 + 26\,{}_{104}C_6 + {}_{105}C_6 = 171708332500$$

$$\frac{2\cdot 100^6 + 6\cdot 100^5 + 5\cdot 100^4 - 100^2}{12} = 171708332500$$

Formula 7.4.2

When m is a natural number and x is a positive number, the following expression holds.

$$\sum_{k=0}^{x} k^m = \sum_{r=1}^{m} {}_mD_r \binom{x+r}{m+1}$$

Example: $\sum_{k=0}^{0.9} k^4$

$$\binom{1.9}{5} + 11\binom{2.9}{5} + 11\binom{3.9}{5} + \binom{4.9}{5} = 0.659148$$

$$\frac{6\cdot 0.9^5 + 15\cdot 0.9^4 + 10\cdot 0.9^3 - 0.9^1}{30} = 0.659148$$

7.5 Formula for the sum of powers of real numbers

Formula 7.5.1

When m,n are natural numbers and a,b are real numbers, the following expression holds.

$$\sum_{k=1}^{n} (ak+b)^m = b^m\,{}_nC_1 + \sum_{r=1}^{m} {}_mC_r\, a^r b^{m-r} \sum_{s=1}^{r} {}_rD_s\,{}_{n+s}C_{r+1}$$

Example 1

$$\sum_{k=1}^{50}(\pi k-e)^4 = e^4{}_{50}C_1 - 4\pi e^3{}_{51}C_2$$
$$+ 6\pi^2 e^2({}_{51}C_3 + {}_{52}C_3)$$
$$- 4\pi^3 e({}_{51}C_4 + 4\,{}_{52}C_4 + {}_{53}C_4)$$
$$+ \pi^4({}_{51}C_5 + 11\,{}_{52}C_5 + 11\,{}_{53}C_5 + {}_{54}C_5)$$
$$= 50e^4 - 4\cdot 1275\pi e^3 + 6(20825 + 22100)\pi^2 e^2$$
$$- 4(249900 + 4\cdot 270725 + 292825)\pi^3 e$$
$$+ (2349060 + 11\cdot 2598960 + 11\cdot 2869685 + 3162510)\pi^4$$

i.e.

$$\sum_{k=1}^{50}(\pi k-e)^4 = 50e^4 - 5100\pi e^3 + 257550\pi^2 e^2 - 6502500\pi^3 e + 65666665\pi^4$$

Example 2

$$\sum_{k=1}^{100}(\sqrt{2}k+\sqrt{3})^3 = (\sqrt{3})^3{}_{100}C_1 + 3\sqrt{2}(\sqrt{3})^2{}_{101}C_2$$
$$+ 3(\sqrt{2})^2\sqrt{3}({}_{101}C_3 + {}_{102}C_3)$$
$$+ (\sqrt{2})^3({}_{101}C_4 + 4\,{}_{102}C_4 + {}_{103}C_4)$$
$$= 100\cdot 3\sqrt{3} + 5050\cdot 9\sqrt{2} + (166650 + 171700)\cdot 6\sqrt{3}$$
$$+ (4082925 + 4\times 4249575 + 4421275)\cdot 2\sqrt{2}$$

i.e.

$$\sum_{k=1}^{100}(\sqrt{2}k+\sqrt{3})^3 = 51050450\sqrt{2} + 2030400\sqrt{3}$$

Note: Calculation method one by one for Eulerian Numbers

Eulerian Number can be calculated also by the algorithm one by one as follows.

		Calculating formula
2	1 1	
	2,2	
3	1 4 1	4 = 1×2 + 1×2
	3,2 2,3	
4	1 11 11 1	11 = 1×3 + 4×2
	4,2 3,3 2,4	
5	1 26 66 26 1	26 = 1×4 + 11×2 , 66 = 11×3 + 11×3
	5,2 4,3 3,4 2,5	
6	1 57 302 302 57 1	57 = 1×5 + 26×2 , 302 = 26×4 + 66×3
	⋮	⋮

08 Taylor series and Maclaurin series

8.1 Case without a singular point

Theorem 8.1.1

When a function $f(z)$ is holomorphic in the whole domain D, the following expression holds for any $a \in D$.

$$\sum_{s=r}^{\infty} \frac{f^{(s)}(a)}{s!}(-1)^{s-r}{}_sC_r a^{s-r} = \frac{f^{(r)}(0)}{r!} \qquad for\ \ r=0,1,2,\cdots$$

Example 1 $f(z)=\cos z$

$$\sum_{s=r}^{\infty} \frac{(-1)^{s-r}}{s!}\cos\left(a+\frac{s\pi}{2}\right){}_sC_r a^{s-r} = \frac{1}{r!}\cos\frac{r\pi}{2} \qquad for\ r=0,1,2,\cdots$$

Special values

When $r=0$,

$$\frac{a^0\cos a}{0!}+\frac{a^1\sin a}{1!}-\frac{a^2\cos a}{2!}-\frac{a^3\sin a}{3!}++--\cdots = 1$$

e.g.

$$\frac{1}{0!}\left(\frac{\pi}{4}\right)^0+\frac{1}{1!}\left(\frac{\pi}{4}\right)^1-\frac{1}{2!}\left(\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(\frac{\pi}{4}\right)^3++--\cdots = \sqrt{2}$$

When $r=1$,

$$\frac{a^1\cos a}{1!}+\frac{a^2\sin a}{2!}-\frac{a^3\cos a}{3!}-\frac{a^4\sin a}{4!}++--\cdots = \sin a$$

e.g.

$$\frac{1}{1!}\left(\frac{\pi}{4}\right)^1+\frac{1}{2!}\left(\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(\frac{\pi}{4}\right)^3-\frac{1}{4!}\left(\frac{\pi}{4}\right)^4++--\cdots = 1$$

Example 2 $\sinh z$

$$\sum_{s=r}^{\infty} \frac{e^a-(-1)^{-s}e^{-a}}{2}\frac{(-1)^{s-r}}{s!}{}_sC_r a^{s-r} = \frac{1-(-1)^{-r}}{2}\frac{1}{r!} \qquad for\ \ r=0,1,2,\cdots$$

When $r=0$,

$$\frac{a^0\sinh a}{0!}-\frac{a^1\cosh a}{1!}+\frac{a^2\sinh a}{2!}-\frac{a^3\cosh a}{3!}+-\cdots = 0$$

When $r=1$

$$\frac{a^0\cosh a}{0!}-\frac{a^1\sinh a}{1!}+\frac{a^2\cosh a}{2!}-\frac{a^3\sinh a}{3!}+-\cdots = 1$$

Example 3 $(1+z)e^z$

$$e^a\sum_{s=r}^{\infty} \frac{s+1+a}{s!}(-1)^{s-r}{}_sC_r a^{s-r} = \frac{r+1}{r!} \qquad for\ \ r=0,1,2,\cdots$$

When $r=0$,

$$\frac{(1-a)a^0}{0!}+\frac{(2-a)a^1}{1!}+\frac{(3-a)a^2}{2!}+\frac{(4-a)a^3}{3!}+\cdots = e^a$$

When $r=1$,

$$\frac{(2-a)a^0}{0!}+\frac{(3-a)a^1}{1!}+\frac{(4-a)a^2}{2!}+\frac{(5-a)a^3}{3!}+\cdots=2e^a$$

Generally,

$$\frac{(b-a)a^0}{0!}+\frac{(b-a+1)a^1}{1!}+\frac{(b-a+2)a^2}{2!}+\frac{(b-a+3)a^3}{3!}+\cdots=be^a$$

8.2 Case with a singular point

Theorem 8.2.1

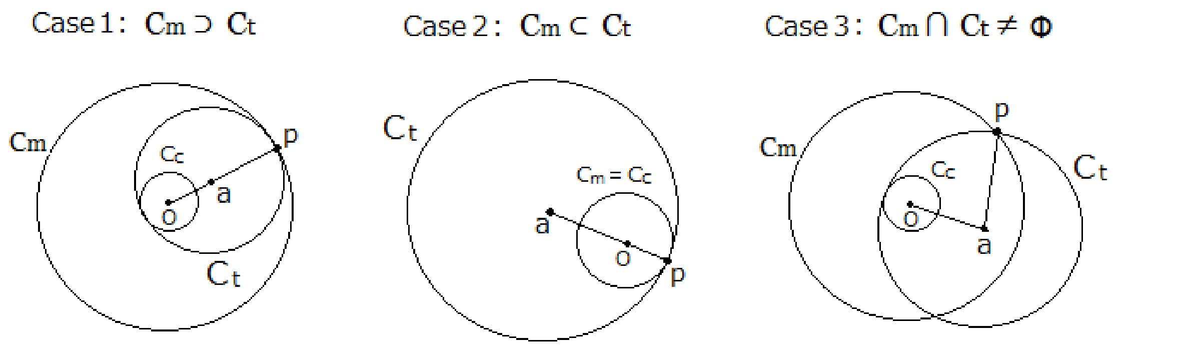
When a function $f(z)$ is holomorphic in a domain D except a singular point p, the following expression holds for a s.t. $|a|<|p-a|$

$$\sum_{s=r}^{\infty}\frac{f^{(s)}(a)}{s!}(-1)^{s-r}{}_sC_r\,a^{s-r}=\frac{f^{(r)}(0)}{r!}\qquad for\ \ r=0,1,2,\cdots$$

Where, the singular point p is assumed to be nearest to the origin O and the point a.

Three cases where singularity exists

In domain D, let the origin be O, the singular point of the function $f(z)$ be p, the convergence circle of the Maclaurin series be C_m, and the convergence circle of the Taylor series be C_t. Then, the following 3 cases exist.



8.3 Example of Case1 : $C_m \supset C_t$

This is a case where the convergence circle C_m of the Maclaurin series includes the convergence circle C_t of the Taylor series. We consider the following function as the example.

$$f(z)=tanh^{-1}z$$

Then, the following equations have to hold.

$$tanh^{-1}a+\sum_{s=1}^{\infty}\frac{(-1)^s}{2s}\left\{\frac{1}{(1-a)^s}+\frac{(-1)^{s-1}}{(1+a)^s}\right\}a^s=tanh^{-1}0\qquad r=0$$

$$\sum_{s=r}^{\infty}\frac{(-1)^{s-r}}{2s}\left\{\frac{1}{(1-a)^s}+\frac{(-1)^{s-1}}{(1+a)^s}\right\}{}_sC_r\,a^{s-r}=\frac{1+(-1)^{r-1}}{2r}\qquad r=1,2,3,\cdots$$

The following formula is derived from the first equation.

Formula 8.3.1

$$tanh^{-1}a=\sum_{s=1}^{\infty}\frac{1}{2s}\left\{\left(\frac{a}{1+a}\right)^s-(-1)^s\left(\frac{a}{1-a}\right)^s\right\}\qquad |Re(a)|<\frac{1}{2}$$

Examples

$$tanh^{-1}\frac{1}{2} = \frac{1}{2}\left(\frac{1}{3^1}+1\right)+\frac{1}{4}\left(\frac{1}{3^2}-1\right)+\frac{1}{6}\left(\frac{1}{3^3}+1\right)+\frac{1}{8}\left(\frac{1}{3^4}-1\right)+\cdots$$

$$tanh^{-1}\frac{1}{3} = \frac{1}{2}\left(\frac{1}{4^1}+\frac{1}{2^1}\right)+\frac{1}{4}\left(\frac{1}{4^2}-\frac{1}{2^2}\right)+\frac{1}{6}\left(\frac{1}{4^3}+\frac{1}{2^3}\right)+\frac{1}{8}\left(\frac{1}{4^4}-\frac{1}{2^4}\right)+\cdots$$

8.4 Example of Case2 : $C_m \subset C_t$

This is a case where the convergence circle C_m of the Maclaurin series is included in the convergence circle C_t of the Taylor series. We consider the following function as the example.

$$f(z)= \frac{5}{5-z}$$

Therefore, the following equation has to hold.

$$\sum_{s=r}^{\infty}\frac{5}{(5-a)^{s+1}}(-1)^{s-r}{}_sC_r a^{s-r} = \frac{1}{5^r} \qquad \begin{cases} Re(a) < 5/2 \\ r = 0,1,2,\cdots \end{cases}$$

Now, changing the convergence radius 5 to 1 and replacing $a/(a-1)$ with $1/x$, we obtain the following formula.

Formula 8.4.1 (Geometric Series with coefficients)

$$\sum_{s=r}^{\infty}(-1)^{-r}\frac{{}_sC_r}{x^s} = -\left(\frac{1}{1-x}\right)^r\frac{x}{1-x} \qquad \begin{cases} |x| > 1 \\ r = 0,1,2,\cdots \end{cases}$$

Especially, when $r=1$,

$$\sum_{s=1}^{\infty}\frac{s}{x^s} = \frac{x}{(1-x)^2} \qquad |x|>1$$

Example 1

$$\frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\cdots = \frac{2}{1^2}$$

$$\frac{1}{3^1}+\frac{2}{3^2}+\frac{3}{3^3}+\frac{4}{3^4}+\cdots = \frac{3}{2^2}$$

$\vdots$

Example 2

$$\frac{1}{2^1}-\frac{2}{2^2}+\frac{3}{2^3}-\frac{4}{2^4}+-\cdots = \frac{2}{3^2}$$

$$\frac{1}{3^1}-\frac{2}{3^2}+\frac{3}{3^3}-\frac{4}{3^4}+-\cdots = \frac{3}{4^2}$$

$\vdots$

Example 3

$$\frac{1}{1.3^1}+\frac{2}{1.3^2}+\frac{3}{1.3^3}+\frac{4}{1.3^4}+\cdots = \frac{1.3}{0.3^2}$$

$$\frac{1}{1.3^1}-\frac{2}{1.3^2}+\frac{3}{1.3^3}-\frac{4}{1.3^4}+-\cdots = \frac{1.3}{2.3^2}$$

$$\frac{1}{(1+i)^1}+\frac{2}{(1+i)^2}+\frac{3}{(1+i)^3}+\frac{4}{(1+i)^4}+\cdots=\frac{1+i}{i^2}$$

$$\frac{1}{(1+i)^1}-\frac{2}{(1+i)^2}+\frac{3}{(1+i)^3}-\frac{4}{(1+i)^4}+-\cdots=\frac{1+i}{(2+i)^2}$$

8.5 Example of Case3 : $C_m \cap C_t \neq \phi$

This is a case where the convergence circle C_m of the Maclaurin series and the convergence circle C_t of the Taylor series are overlapping partially. We consider the following function as the example.

$$f(z)=\frac{1}{1+z^2}$$

Then, the following equation has to hold.

$$\sum_{s=r}^{\infty}(-1)^s\left(1+a^2\right)^{-\frac{s+1}{2}}\sin\left\{(s+1)\cot^{-1}a\right\}(-1)^{s-r}{}_sC_r\,a^{s-r}=(-1)^r\sin\frac{(r+1)\pi}{2}$$

$$r=0,1,2,\cdots$$

Here, setting $a=1/\sqrt{3}$, $r=0,1$, we obtaint the following formula.

Formula 8.5.1 (Alternating series of powers of 1/2)

$$\frac{1}{2^0}+\frac{1}{2^1}-\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^6}+\frac{1}{2^7}-\frac{1}{2^9}-\frac{1}{2^{10}}++--\cdots=\frac{4}{3}$$

$$\frac{0}{2^0}+\frac{1}{2^1}-\frac{3}{2^3}-\frac{4}{2^4}+\frac{6}{2^6}+\frac{7}{2^7}-\frac{9}{2^9}-\frac{10}{2^{10}}++--\cdots=0$$

8.6 Sum of Stieltjes constants

Using Theorem 8.1.1, the following formula can be proven.

Formula 8.6.1 (*O. Marichev*)

When γ_s are Stieltjes constants and $\zeta^{(n)}(0)$ is the n-th order differential coefficient of Riemann zeta function, The following expression holds.

$$\sum_{s=0}^{\infty}\frac{\gamma_{s+n}}{s!}=(-1)^n\left\{n!+\zeta^{(n)}(0)\right\}\qquad n=0,1,2,\cdots$$

Special values

The following special values are known about Riemann zeta function.

$$\zeta^{(0)}(0)=-\frac{1}{2}\ \ (=\zeta(0))\quad,\quad \zeta^{(1)}(0)=-\frac{\log 2\pi}{2}\ \ (\textit{Glaisher-Kinkelin constant})$$

Therefore,

$$\sum_{s=0}^{\infty}\frac{\gamma_s}{s!}=(-1)^0\left\{1!+\zeta^{(0)}(0)\right\}=\frac{1}{2}$$

$$\sum_{s=0}^{\infty}\frac{\gamma_{s+1}}{s!}=(-1)^1\left\{1!+\zeta^{(1)}(0)\right\}=\frac{\log 2\pi}{2}-1=-0.0810614667\cdots$$

09 Absolute Value of Gamma Function

9.1 Some infinite products

Formula 9.1.1

$$\prod_{r=1}^{\infty}\left(1+\frac{y}{r+x}\right)\left(1-\frac{y}{r-x}\right)=\frac{\Gamma(1+x)\Gamma(1-x)}{\Gamma(1+x+y)\Gamma(1-x-y)}$$

$$\prod_{r=1}^{\infty}\left(1+\frac{y}{r-x}\right)\left(1-\frac{y}{r+x}\right)=\frac{\Gamma(1+x)\Gamma(1-x)}{\Gamma(1+x-y)\Gamma(1-x+y)}$$

$$\prod_{r=1}^{\infty}\left(1+\frac{y}{r+x}\right)\left(1-\frac{y}{r+x}\right)=\frac{\Gamma^2(1+x)}{\Gamma(1+x+y)\Gamma(1+x-y)}$$

$$\prod_{r=1}^{\infty}\left(1+\frac{y}{r-x}\right)\left(1-\frac{y}{r-x}\right)=\frac{\Gamma^2(1-x)}{\Gamma(1-x+y)\Gamma(1-x-y)}$$

Formula 9.1.1'

$$\prod_{r=0}^{\infty}\left(1+\frac{y}{r+x}\right)\left(1-\frac{y}{r-x}\right)=\frac{\Gamma(x)\Gamma(-x)}{\Gamma(x+y)\Gamma(-x-y)}$$

$$\prod_{r=0}^{\infty}\left(1+\frac{y}{r-x}\right)\left(1-\frac{y}{r+x}\right)=\frac{\Gamma(x)\Gamma(-x)}{\Gamma(x-y)\Gamma(-x+y)}$$

$$\prod_{r=0}^{\infty}\left(1+\frac{y}{r+x}\right)\left(1-\frac{y}{r+x}\right)=\frac{\Gamma^2(x)}{\Gamma(x+y)\Gamma(x-y)}$$

$$\prod_{r=0}^{\infty}\left(1+\frac{y}{r-x}\right)\left(1-\frac{y}{r-x}\right)=\frac{\Gamma^2(-x)}{\Gamma(-x+y)\Gamma(-x-y)}$$

9.2 Square of the Absolute Value

Formula 9.2.1

When x,y are real numbers and $\Gamma(x+iy)$ is the gamma function in the complex plane,

$$|\Gamma(x+iy)|^2=\Gamma^2(x)\prod_{r=0}^{\infty}\left\{1+\left(\frac{y}{r+x}\right)^2\right\}^{-1}$$

i.e.

$$|\Gamma(x+iy)|=|\Gamma(x)|\prod_{r=0}^{\infty}\left\{1+\left(\frac{y}{r+x}\right)^2\right\}^{-1/2}$$

9.3 Partial Derivative with respect to y

Formula 9.3.1

When $\Gamma(z)$ $(z=x+iy)$ is the gamma function in the complex plane C,

$$\frac{\partial|\Gamma(z)|^2}{\partial y}=-2\Gamma^2(x)\left[\prod_{r=0}^{\infty}\left\{1+\left(\frac{y}{r+x}\right)^2\right\}^{-1}\right]\sum_{s=0}^{\infty}\frac{y}{(s+x)^2+y^2}$$

Theorem 9.3.2

When $\Gamma(z)$ $(z=x+iy)$ is the gamma function in the complex plane C,

For any x, if $y>0$ then $|\Gamma(z)|^2$ is monotonic decreasing with respect to y,

if $y<0$ then $|\Gamma(z)|^2$ is monotonic increasing with respect to y.

9.4 Partial Derivative with respect to x

Formula 9.4.2

When $\Gamma(z)$ $(z=x+iy)$ is the gamma function in the complex plane C and $\psi(x)$ is the digamma function,

$$\frac{\partial|\Gamma(z)|^2}{\partial x} = 2\Gamma^2(x)\left[\prod_{r=0}^{\infty}\left\{1+\left(\frac{y}{r+x}\right)^2\right\}^{-1}\right]\cdot\left[\psi(x)+\sum_{s=0}^{\infty}\frac{1}{s+x}\frac{y^2}{(s+x)^2+y^2}\right]$$

Theorem 9.4.3

When $\Gamma(z)$ $(z=x+iy)$ is the gamma function in the complex plane C ,

For any y, if $x>1.461632144968\cdots$ then $|\Gamma(z)|^2$ is monotonic increasing with respect to x .

Theorem 9.4.3'

When $\Gamma(z)$ $(z=x+iy)$ is the gamma function in the complex plane C ,

If $x \geqq 0$ & $|y| > 1.047662675461731\cdots$, $|\Gamma(z)|^2$ is monotonic increasing with respect to x .

10 Convergence Acceleration & Summation Method by Double Series of Functions

10.1 Double Series of Functions & Summation Method

Theorem 10.1.3

Let $b(s)$ be a real function, D_1, D_2 are small domains, $f_1(z)$, $f_2(z)$ are series of functions defined as follows.

$$b(s) = \sum_{r=s}^{\infty} b_{rs} = 1 \qquad s=0,1,2,\cdots$$

$$f_1(z) = \sum_{s=0}^{\infty} a_s(z) \qquad z \in D_1$$

$$f_2(z) = \sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z)$$

(1) When $f_1(z)$ is absolutely convergent,

i if $f_2(z)$ is convergent in the D_1,

$$\sum_{s=0}^{\infty} a_s(z) = \sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z) \qquad z \in D_1$$

ii if $f_2(z)$ is divergent in the D_1,

$$\sum_{s=0}^{\infty} a_s(z) = "\sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z)" \qquad z \in D_1$$

where, "S(z)" means that S(z) is interpreted as convergent. (Same as below.)

(2) When $f_2(z)$ is absolutely convergent in the D_1,

i if $f_1(z)$ is convergent,

$$\sum_{s=0}^{\infty} a_s(z) = \sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z) \qquad z \in D_1$$

ii if $f_1(z)$ is divergent,

$$"\sum_{s=0}^{\infty} a_s(z)" = \sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z) \qquad z \in D_1$$

(3) When $f_2(z)$ is absolutely convergent in the D_2 $(\neq D_1)$,

$$"\sum_{s=0}^{\infty} a_s(z)" = \sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z) \qquad z \in D_2$$

10.2 Accelerator

The following equations were obtained in the previous section.

$$\sum_{s=0}^{\infty} a_s(z) = \sum_{r=0}^{\infty} \sum_{s=0}^{r} b_{rs} a_s(z) \tag{2.1}$$

$$b(s) = \sum_{r=s}^{\infty} b_{rs} = 1 \qquad for\ s=0,1,2,\cdots \tag{b}$$

When (b) converges faster than the left side of (2.1), by transforming this like the right side of (1.1), we canaccelerate the convergence of the left side. In this case, we can call the series (b) **accelerator** .

The accelerator is desired fast-convergence more. Altthough this is considered variously, I recommend the following.

Formula 10.2.1 (Knopp's accelerator)

$$b(s) = \sum_{r=s}^{\infty} \frac{q^{r-s}}{(q+1)^{r+1}} \binom{r}{s} = 1 \qquad for \quad \begin{matrix} s = 0,1,2,\cdots \\ q > 0 \end{matrix}$$

10.3 Knopp Transformation & Double Series of Functions

Transformation of a series using Knopp's accelerator is called **Knopp Transformation**.

Lemma 10.3.1

Let q be positive number, $f_1(z)=\sum_{s=0}^{\infty}a_s(z)$ be series of functions in a small domain D_1 and Knopp transformation be

$$f_2(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}a_s(z)$$

Then, if $f_1(z)$ is bounded, $f_2(z)$ is absolutely convergent,

Theorem 10.3.2 (Knopp Transformation)

Let q be a positive number, D_1, D_2 are small domains, $f_1(z), f_2(z)$ are series of functions defined asfollows.

$$f_1(z) = \sum_{s=0}^{\infty}a_s(z) \qquad z\in D_1$$

$$f_2(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}a_s(z)$$

(1) When $f_1(z)$ is convergent,

$$\sum_{s=0}^{\infty}a_s(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}a_s(z) \qquad z\in D_1$$

(2) When $f_2(z)$ is absolutely convergent in the D_1, if $f_1(z)$ is divergent,

$$"\sum_{s=0}^{\infty}a_s(z)" = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}a_s(z) \qquad z\in D_1$$

where, "S(z)" means that S(z) is interpreted as convergent. (Same as below.)

(3) When $f_2(z)$ is absolutely convergent in the D_2 $(\neq D_1)$,

$$"\sum_{s=0}^{\infty}a_s(z)" = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}a_s(z) \qquad z\in D_2$$

10.4 Acceleration of Power Series

10.4.1 Acceleration of Mercator Series

$$\log(z+1) = \sum_{s=1}^{\infty}(-1)^{s-1}\frac{z^s}{s} = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}(-1)^{s-1}\frac{z^s}{s} \qquad |z|\leq 1$$

This transformation produces significant acceleration and asymptotic effects around the convergence circle.

10.4.2 Acceleration of Madhava Series

Madhava series (Gregory–Leibniz series) is as follows.

$$\frac{\pi}{4} = \sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1} = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}\frac{(-1)^s}{2s+1}$$

Especially when $q=1$,

$$\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{1}{2^{r+1}}\binom{r}{s}\frac{(-1)^s}{2s+1} = \frac{1}{2^1}\frac{1}{1}+\frac{1}{2^2}\left(\frac{1}{1}-\frac{1}{3}\right)+\frac{1}{2^3}\left(\frac{1}{1}-\frac{2}{3}+\frac{1}{5}\right)+\frac{1}{2^4}\left(\frac{1}{1}-\frac{3}{3}+\frac{3}{5}-\frac{1}{7}\right)+\cdots$$

$$= \frac{1}{2^1}\frac{0!!}{1!!}+\frac{1}{2^2}\frac{2!!}{3!!}+\frac{1}{2^3}\frac{4!!}{5!!}+\frac{1}{2^4}\frac{6!!}{7!!}+\cdots = \frac{\pi}{4}$$

10.5 Acceleration of Fourier Series

$$-\log\left(2\sin\frac{z}{2}\right) = \sum_{s=1}^{\infty}\frac{\cos(sz)}{s} = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}\frac{\cos(sz)}{s} \qquad \begin{matrix} 0<Re(z)<2\pi \\ Im(z)=0 \end{matrix}$$

The Fourier series converges only in the real interval, but this transformation extends the convergence region to the complex area on both sides of the interval.

10.6 Acceleration of Dirichlet Series

$$\eta(z) = "\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^z}" = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}\frac{(-1)^{s-1}}{s^z} \qquad z \in C$$

This conversion provides a large acceleration effect near the convergence axis. Furthermore, an analytical continuation occurs beyond the convergence axis. As the result, when $z = -1, -3$,

$$"1^1 - 2^1 + 3^1 - 4^1 +- \cdots" = \frac{1}{4}$$

$$"1^3 - 2^3 + 3^3 - 4^3 +- \cdots" = -\frac{1}{8}$$

10.7 Application to Divergent Series

10.7.1 Application to Oscillating Series

$$\frac{1}{2}\cot\frac{z}{2} = "\sum_{s=1}^{\infty}\sin(sz)" = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}\sin(sz) \qquad \begin{matrix} 0<Re(z)<2\pi \\ Im(z)=0 \end{matrix}$$

Although this Fourier series oscillates (diverges), the summation method is applied by this transformation.
As the result, when $z = \pi/4, \pi/3, \pi/2, 1$,

$$"\frac{1}{\sqrt{2}}+1+\frac{1}{\sqrt{2}}+0-\frac{1}{\sqrt{2}}-1-\frac{1}{\sqrt{2}}-0 \;++++---- \cdots" = \frac{1}{2}\cot\frac{\pi}{8} = \frac{1+\sqrt{2}}{2}$$

$$"\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+0-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}-0 \;+++--- \cdots" = \frac{1}{2}\cot\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$"1 + 0 - 1 - 0 \;++-- \cdots" = \frac{1}{2}\cot\frac{\pi}{4} = \frac{1}{2}$$

$$"\sin 1+\sin 2+\sin 3+\sin 4+\cdots" = \frac{1}{2}\cot\frac{1}{2}$$

10.7.2 Application to Divergent Alternating Series

$$\sum_{s=1}^{\infty}(-1)^s\frac{(2s)!}{(2s-2)!} = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}(-1)^s\frac{(2s)!}{(2s-2)!} = \frac{1}{2}$$

$$\sum_{s=2}^{\infty}(-1)^s\frac{(2s)!}{(2s-3)!} = \sum_{r=2}^{\infty}\sum_{s=2}^{r}\frac{q^{r-s}}{(q+1)^{r+1}}\binom{r}{s}(-1)^s\frac{(2s)!}{(2s-3)!} = 0$$

Since these domains are all natural numbers, they must be interpreted as follows according to Theorem 10.3.2.(2)

$$"1\cdot 2 - 3\cdot 4 + 5\cdot 6 - 7\cdot 8 +- \cdots" = -\frac{1}{2}$$

$$"2\cdot 3\cdot 4 - 4\cdot 5\cdot 6 + 6\cdot 7\cdot 8 - 8\cdot 9\cdot 10 +- \cdots" = 0$$

12 Series Expansion of Gamma Function & the Reciprocal

Formula 12.1.0 (Masayuki Ui)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials ,

$$\frac{d^n}{dz^n}\Gamma(z) = \Gamma(z)\sum_{k=1}^{n} B_{n,k}\left(\psi_0(z), \psi_1(z), \ldots, \psi_{n-1}(z)\right)$$

$$\frac{d^n}{dz^n}\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z)}\sum_{k=1}^{n}(-1)^k B_{n,k}\left(\psi_0(z), \psi_1(z), \ldots, \psi_{n-1}(z)\right)$$

Formula 12.1.1 (Taylor expansion)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials , the following expression holds for a s.t. $a \neq 0, -1, -2, -3, \cdots$.

$$\Gamma(z) = \Gamma(a) + \sum_{n=1}^{\infty}\frac{c_n(a)}{n!}(z-a)^n$$

where,

$$c_n(a) = \Gamma(a)\sum_{k=1}^{n} B_{n,k}\left(\psi_0(a), \psi_1(a), \ldots, \psi_{n-1}(a)\right) \qquad n=1, 2, 3, \cdots$$

Example: Taylor expansion around 2

$$\Gamma(z) = 1 + \frac{\psi_0(2)}{1!}(z-2) + \frac{\psi_0(2)^2+\psi_1(2)}{2!}(z-2)^2 + \frac{\psi_0(2)^3+3\psi_0(2)\psi_1(2)+\psi_2(2)}{3!}(z-2)^3 + \cdots$$

Formula 12.1.2 (Taylor expansion)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials , the following expression holds for a s.t. $a \neq 0, -1, -2, -3, \cdots$.

$$\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(a)} + \sum_{n=1}^{\infty}\frac{c_n(a)}{n!}(z-a)^n$$

where,

$$c_n(a) = \frac{1}{\Gamma(a)}\sum_{k=1}^{n}(-1)^k B_{n,k}\left(\psi_0(a), \psi_1(a), \ldots, \psi_{n-1}(a)\right)$$

Example: Taylor expansion around 2

$$\frac{1}{\Gamma(z)} = 1 - \frac{\psi_0(2)}{1!}(z-2) + \frac{\psi_0(2)^2-\psi_1(2)}{2!}(z-2)^2 - \frac{\psi_0(2)^3-3\psi_0(2)\psi_1(2)+\psi_2(2)}{3!}(z-2)^3 + - \cdots$$

Formula 12.2.1 (Laurent expansion)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials ,

$$\Gamma(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{c_n}{n!}z^{n-1}$$

where,

$$c_n = \sum_{k=1}^{n} B_{n,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)\right) \qquad n=1, 2, 3, \cdots$$

i.e.

$$\Gamma(z) = \frac{1}{z} + \frac{\psi_0(1)}{1!} + \frac{\psi_0(1)^2+\psi_1(1)}{2!}z^1 + \frac{\psi_0(1)^3+3\psi_0(1)\psi_1(1)+\psi_2(1)}{3!}z^2 + \cdots$$

Formula 12.2.2 (reciprocal Laurent expansion)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials ,

$$\frac{1}{\Gamma(z)} = z + \sum_{n=1}^{\infty} \frac{c_n}{n!} z^{n+1}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1) \right) \qquad n=1,2,3,\cdots$$

i.e.

$$\frac{1}{\Gamma(z)} = z - \frac{\psi_0(1)}{1!} z^2 + \frac{\psi_0(1)^2 - \psi_1(1)}{2!} z^3 - \frac{\psi_0(1)^3 - 3\psi_0(1)\psi_1(1) + \psi_2(1)}{3!} z^4 + - \cdots$$

Formula 12.3.1 (Maclaurin Expansion)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials ,

$$\frac{1}{\Gamma(1+z)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n$$

$$\frac{1}{\Gamma(1-z)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{n!} z^n$$

$$\frac{1}{\Gamma(1+z/2)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!} z^n$$

$$\frac{1}{\Gamma(1-z/2)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{2^n n!} z^n$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1) \right) \qquad n=1,2,3,\cdots$$

Formula 12.3.2 (Maclaurin Expansion)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials ,

$$\frac{\sqrt{\pi}}{\Gamma\{(1+z)/2\}} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!} z^n$$

$$\frac{\sqrt{\pi}}{\Gamma\{(1-z)/2\}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{2^n n!} z^n$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{n-1}\left(\frac{1}{2}\right) \right) \qquad n=1,2,3,\cdots$$

Formula 12.3.3 (Maclaurin Expansion)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials ,

$$\frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{2^n n!} z^n$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{n-1}\left(\frac{3}{2}\right) \right) \qquad n=1,2,3,\cdots$$

Formula 12.4.1 (Taylor expansion around 1)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \dots)$ are Bell polynomials ,

$$\frac{1}{\Gamma(z)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n!} (z-1)^n$$

$$\frac{1}{\Gamma(2-z)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{n!} (z-1)^n$$

$$\frac{1}{\Gamma\{(1+z)/2\}} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!} (z-1)^n$$

$$\frac{1}{\Gamma\{(3-z)/2\}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{2^n n!} (z-1)^n$$

$$\frac{1}{\Gamma(1-z)} = -(z-1) - \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{n!} (z-1)^{n+1}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(1), \psi_1(1), \dots, \psi_{n-1}(1) \right) \qquad n=1,2,3,\cdots$$

Formula 12.4.2 (Taylor expansion around 1)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \dots)$ are Bell polynomials ,

$$\frac{1}{\Gamma(1+z)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n!} (z-1)^n$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(2), \psi_1(2), \dots, \psi_{n-1}(2) \right) \qquad n=1,2,3,\cdots$$

Formula 12.5.1 (Taylor expansion around 1)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \dots)$ are Bell polynomials ,

$$\frac{\sqrt{\pi}}{\Gamma(z/2)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!} (z-1)^n$$

$$\frac{\sqrt{\pi}}{\Gamma(1-z/2)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{2^n n!} (z-1)^n$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \dots, \psi_{n-1}\left(\frac{1}{2}\right) \right) \qquad n=1,2,3,\cdots$$

Formula 12.5.2 (Taylor expansion around 1)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \dots)$ are Bell polynomials ,

$$\frac{\sqrt{\pi}}{2\Gamma(1+z/2)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!} (z-1)^n$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \dots, \psi_{n-1}\left(\frac{3}{2}\right) \right) \qquad n=1,2,3,\cdots$$

13 Convergence Acceleration of Multiple Series

13.1 Series Acceleration Method

If multiple series are converted to a single series in some way, an acceleration method can be applied. Since this is similar to a series circuit in an electric circuit, we call this ***series acceleration method***.

Formula 2.1.0 (reprint)

(0) When a multiple series $\sum_{r_1,r_2,\cdots,r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n}$ is absolutely convergent, the following expression holds.

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{r_1-r_2,r_2-r_3,\cdots,r_{n-1}-r_n,r_n}$$

(1) When a multiple series $\sum_{r_1,r_2,\cdots,r_n=1}^{\infty} a_{r_1,r_2,\cdots,r_n}$ is absolutely convergent, the following expression holds.

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\cdots\sum_{r_n=1}^{\infty} a_{r_1,r_2,\cdots,r_n} = \sum_{r_1=1}^{\infty}\sum_{r_2=1}^{r_1}\cdots\sum_{r_n=1}^{r_{n-1}} a_{1+r_1-r_2,1+r_2-r_3,\cdots,1+r_{n-1}-r_n,r_n}$$

How to convert multiple series to semi-multiple series (reprint)

In short, we should just perform the following operations to $\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_{n-1},r_n}$ ..

Replace r_{n-1} with $r_{n-1}-r_n$, and replace the 1st ∞ with r_{n-1} from the right.

Replace r_{n-2} with $r_{n-2}-r_{n-1}$, and replace the 2nd ∞ with r_{n-2} from the right.

⋮

Replace r_1 with r_1-r_2 , and replace the $(n-1)$th ∞ with r_1 from the right.

When the subscripts starts from 1 (i.e. (1)) , $+1$ is only added.

Examples

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-1)^{r+s+t}x^{2r+2s+2t+3}}{(2r+1)(2s+1)(2t+1)} = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^{r}x^{2r+3}}{(2r-2s+1)(2s-2t+1)(2t+1)}$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{r_3=1}^{\infty}\sum_{r_4=1}^{\infty}\frac{(-1)^{r_1+r_2+r_3+r_4}}{r_1 r_2 r_3 r_4} = \sum_{r_1=1}^{\infty}\sum_{r_2=1}^{r_1}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3}\frac{(-1)^{1+r_1}}{(1+r_1-r_2)(1+r_2-r_3)(1+r_3-r_4)r_4}$$

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) = \sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{(-1)^{1+r}}{\{(1+r-s)s\}^x}\cos\left(y\log\frac{s}{1+r-s}\right)$$

Theorem 13.1.2 (Series Acceleration Method)

(0) When a multiple series of functions $\sum_{r_1,r_2,\cdots,r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n}(z)$ is absolutely convergent in the domain D ,

the following expression holds for arbitrary positive number q .

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n}(z) = \sum_{k=0}^{\infty}\sum_{r_1=0}^{k}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}}\frac{q^{k-r_1}}{(q+1)^{k+1}}\binom{k}{r_1} a_{r_1-r_2,r_2-r_3,\cdots,r_{n-1}-r_n,r_n}(z)$$

(1) When a multiple series of functions $\sum_{r_1,r_2,\cdots,r_n=1}^{\infty} a_{r_1,r_2,\cdots,r_n}(z)$ is absolutely convergent in the domain D ,

the following expression holds for arbitrary positive number q .

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\cdots\sum_{r_n=1}^{\infty} a_{r_1,r_2,\cdots,r_n}(z) = \sum_{k=1}^{\infty}\sum_{r_1=1}^{k}\sum_{r_2=1}^{r_1}\cdots\sum_{r_n=1}^{r_{n-1}}\frac{q^{k-r_1}}{(q+1)^{k+1}}\binom{k}{r_1} a_{1+r_1-r_2,1+r_2-r_3,\cdots,1+r_{n-1}-r_n,r_n}(z)$$

Example 1

$$\left(\tan^{-1}x\right)^3 = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-1)^{r+s+t}x^{2r+2s+2t+3}}{(2r+1)(2s+1)(2t+1)}$$

$$= \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{q^{k-r}}{(q+1)^{k+1}}\binom{k}{r}\frac{(-1)^r x^{2r+3}}{(2r-2s+1)(2s-2t+1)(2t+1)}$$

Example 2

$$(\log 2)^4 = \sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{r_3=1}^{\infty}\sum_{r_4=1}^{\infty}\frac{(-1)^{r_1+r_2+r_3+r_4}}{r_1 r_2 r_3 r_4}$$

$$= \sum_{k=1}^{\infty}\sum_{r_1=1}^{k}\sum_{r_2=1}^{r_1}\sum_{r_3=1}^{r_2}\sum_{r_4=1}^{r_3}\frac{q^{k-r_1}}{(q+1)^{k+1}}\binom{k}{r_1}\frac{(-1)^{1+r_1}}{(1+r_1-r_2)(1+r_2-r_3)(1+r_3-r_4)r_4}$$

Example 3

$$|\eta(x,y)|^2 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right)$$

$$= \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{s=1}^{r}\frac{q^{k-r}}{(q+1)^{k+1}}\binom{k}{r}\frac{(-1)^{1+r}}{\{(1+r-s)s\}^x}\cos\left(y\log\frac{s}{1+r-s}\right)$$

13.2 Parallel Acceleration Method

In this section, I present the method of accelerating as it is, without rearranging a multiple series. Since this is similar to a parallel circuit in an electric circuit, we call this ***parallel acceleration method***.

Formula 13.2.1 (Parallel Accelerator)

$$b(r_1,r_2,\cdots,r_n) = \sum_{k=r_1+r_2+\cdots+r_n}^{\infty}\frac{q^{k-r_1-r_2-\cdots-r_n}}{(q+1)^{k+1}}\binom{k}{r_1+r_2+\cdots+r_n} = 1 \quad \text{for} \quad \begin{array}{l} r_s = 0,1,2,\cdots \\ s = 1,2,\cdots,n \\ q > 0 \end{array}$$

Proposition 13.2.2 (Parallel Acceleration Method)

(0) When a multiple series of functions $\sum_{r_1,r_2,\cdots,r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n}(z)$ is absolutely convergent in the domain D,

the following expression holds for arbitrary positive number q .

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}a_{r_1,r_2,\cdots,r_n}(z) = \sum_{k=0}^{\infty}\sum_{r_1=0}^{k}\sum_{r_2=0}^{k}\cdots\sum_{r_n=0}^{k}\frac{q^{k-r_1-r_2-\cdots-r_n}}{(q+1)^{k+1}}\binom{k}{r_1+r_2+\cdots+r_n}a_{r_1,r_2,\cdots,r_n}(z)$$

(1) When a multiple series of functions $\sum_{r_1,r_2,\cdots,r_n=1}^{\infty} a_{r_1,r_2,\cdots,r_n}(z)$ is absolutely convergent in the domain D,

the following expression holds for arbitrary positive number q .

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\cdots\sum_{r_n=1}^{\infty}a_{r_1,r_2,\cdots,r_n}(z) = \sum_{k=1}^{\infty}\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\cdots\sum_{r_n=1}^{k}\frac{q^{k-r_1-r_2-\cdots-r_n}}{(q+1)^{k+1}}\binom{k}{r_1+r_2+\cdots+r_n}a_{r_1,r_2,\cdots,r_n}(z)$$

Example 1

$$\left(\tan^{-1}x\right)^3 = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-1)^{r+s+t}x^{2r+2s+2t+3}}{(2r+1)(2s+1)(2t+1)}$$

$$= \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{k}\sum_{t=0}^{k}\frac{q^{k-r-s-t}}{(q+1)^{k+1}}\binom{k}{r+s+t}\frac{(-1)^{r+s+t}x^{2r+2s+2t+3}}{(2r+1)(2s+1)(2t+1)}$$

Example 2

$$(\log 2)^4 = \sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{r_3=1}^{\infty}\sum_{r_4=1}^{\infty}\frac{(-1)^{r_1+r_2+r_3+r_4}}{r_1 r_2 r_3 r_4}$$

$$= \sum_{k=1}^{\infty}\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\sum_{r_3=1}^{k}\sum_{r_4=1}^{k}\frac{q^{k-r_1-r_2-r_3-r_4}}{(q+1)^{k+1}}\binom{k}{r_1+r_2+r_3+r_4}\frac{(-1)^{r_1+r_2+r_3+r_4}}{r_1 r_2 r_3 r_4}$$

Example 3

$$|\eta(x,y)|^2 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right)$$

$$= \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{s=1}^{k}\frac{q^{k-r-s}}{(q+1)^{k+1}}\binom{k}{r+s}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right)$$

Speed comparison between serial acceleration method and parallel acceleration method.

When the upper limit m of Σ of n -multipl series is the same, the serial acceleration method is at most $n!$ times faster than the parallel acceleration method. For quadruple series or higher, it is better to use the serial acceleration method for calculations and the parallel acceleration method for explanations. For triple series or less, it is better to use the parallel acceleration method for both calculation and explanation.

14 Taylor Expansion by Real Part & Imaginary Part

Lemma 14.1.0

When x, y are real numbers and r is a non-negative integer, the following expressions hold.

$$(x+iy)^r = \sum_{s=0}^{\lceil \frac{r-1}{2} \rceil} (-1)^s \binom{r}{2s} x^{r-2s} y^{2s} + i \sum_{s=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^s \binom{r}{2s+1} x^{r-2s-1} y^{2s+1}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

Formula 14.1.1

Suppose that a complex function $f(z)$ $(z = x+iy)$ is expanded around a real number a into a Taylor series with real coefficients as follows.

$$f(z) = \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!} (z-a)^r$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$

$$u(x,y) = \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!} \sum_{s=0}^{\lceil \frac{r-1}{2} \rceil} (-1)^s \binom{r}{2s} (x-a)^{r-2s} y^{2s}$$

$$v(x,y) = \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!} \sum_{s=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^s \binom{r}{2s+1} (x-a)^{r-2s-1} y^{2s+1}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

Example

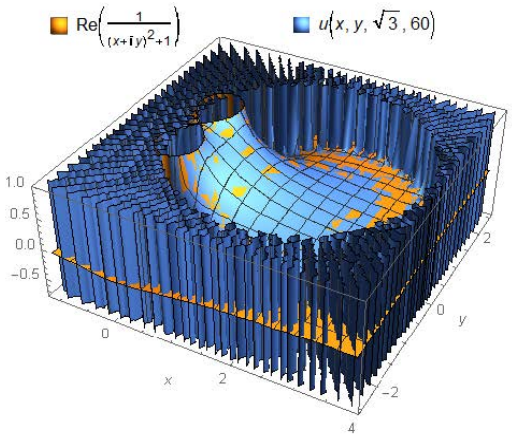
$$f(z) = \frac{1}{z^2+1} = \sum_{r=0}^{\infty} \frac{(-1)^r r!}{(a^2+1)^{(r+1)/2}} \sin\left\{(r+1)\cot^{-1} a\right\} \frac{(z-a)^r}{r!}$$

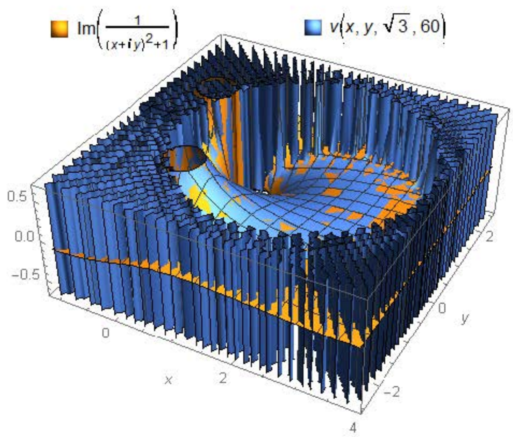
$$u(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r \sin\left\{(r+1)\cot^{-1} a\right\}}{(a^2+1)^{(r+1)/2}} \sum_{s=0}^{\lceil \frac{r-1}{2} \rceil} (-1)^s \binom{r}{2s} (x-a)^{r-2s} y^{2s}$$

$$v(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r \sin\left\{(r+1)\cot^{-1} a\right\}}{(a^2+1)^{(r+1)/2}} \sum_{s=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^s \binom{r}{2s+1} (x-a)^{r-2s-1} y^{2s+1}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

When $a = \sqrt{3}$, both sides are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. Convergence circles with radius 2 are observed.





Formula 14.1.2

Suppose that a complex function $f(z)$ $(z=x+iy)$ is expanded around a real number a into a Taylorseries with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(s)}(a) \frac{(z-a)^s}{s!}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} f^{(2r+s)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} f^{(2r+s+1)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$.

Example

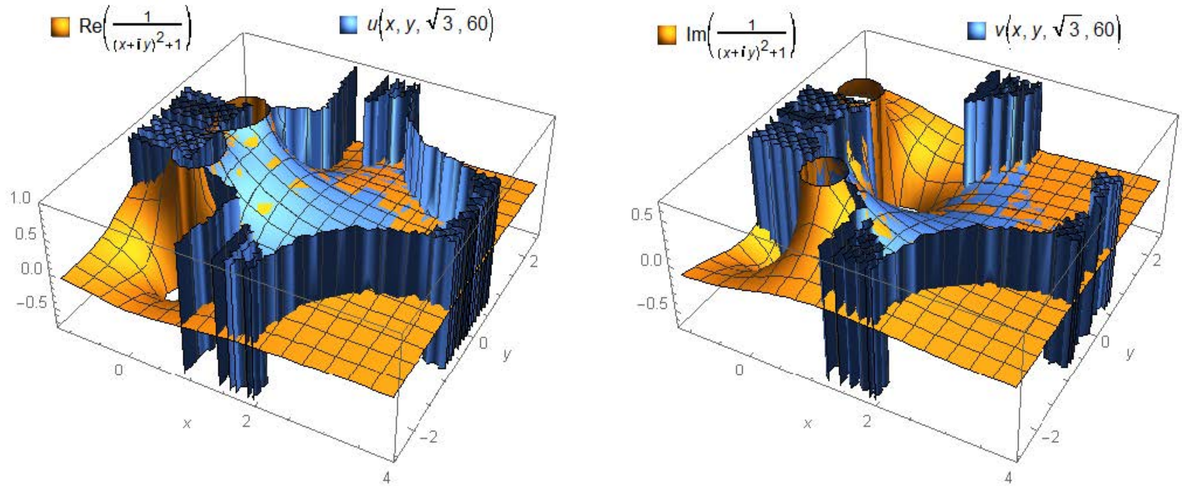
$$f(z) = \frac{1}{z^2+1} = \sum_{r=0}^{\infty} \frac{(-1)^r r!}{(a^2+1)^{(r+1)/2}} \sin\{(r+1)\cot^{-1} a\} \frac{(z-a)^r}{r!}$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s (2r+s)!}{(a^2+1)^{(2r+s+1)/2}} \sin\{(2r+s+1)\cot^{-1} a\} \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s (2r+s+1)!}{(a^2+1)^{(2r+s+2)/2}} \sin\{(2r+s+2)\cot^{-1} a\} \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$.

When $a = \sqrt{3}$, both sides are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side.



These convergence areas are squares inscribed in a convergence circle of radius 2. And, both expressions are series within the square, and are asymptotic expansions outside the square.

For odd and even functions, Formula 14.1.2 becomes as follows.

Formula 14.1.2 ' (Odd Function)

Suppose a complex function $f(z)$ $(z=x+iy)$ is expanded into a Maclaurin series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(2s+1)}(0)\frac{z^{2s+1}}{(2s+1)!}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$.

Formula 14.1.2 " (Even Function)

Suppose a complex function $f(z)$ $(z = x + iy)$ is expanded into a Maclaurin series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(2s)}(0)\frac{z^{2s}}{(2s)!}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$.

15 Taylor Series of Elementary Functions by Real & Imaginary Parts

The followings hold for elementary functions. Where, $0^0 = 1$.

1/(1−z)

$$\frac{1}{1-z} = \sum_{s=0}^{\infty} s!\frac{z^s}{s!} \qquad |z| < 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s)!\frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+1)!\frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

1/(1+z)

$$\frac{1}{1+z} = \sum_{s=0}^{\infty} s!\frac{(-1)^s z^s}{s!} \qquad |z| < 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s)!\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+1)!\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

z^p $(p,a \geqq 0)$

$$z^p = \sum_{s=0}^{\infty}\frac{\Gamma(1+p)}{\Gamma(1+p-s)}a^{p-s}\frac{(z-a)^s}{s!} \qquad |z| < |a|$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\Gamma(1+p)}{\Gamma(p-2r-s+1)}a^{p-2r-s}\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\Gamma(1+p)}{\Gamma(p-2r-s)}a^{p-2r-s-1}\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

a^z $(a \geq 0)$

$$a^z = \sum_{s=0}^{\infty}\log^s a\,\frac{z^s}{s!} \qquad |z| < \infty$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\log^{2r+s} a\,\frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\log^{2r+s+1} a\,\frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Especially when $a = e$ $(=2.71828\cdots)$,

$$e^z = \sum_{s=0}^{\infty}\frac{z^s}{s!} \qquad |z| < \infty$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

log z $(a \geq 0)$

$$\log z = \log a - \sum_{s=1}^{\infty} \frac{(s-1)!}{a^s} \frac{(-1)^s (z-a)^s}{s!} \qquad |z-a| \leq a,\ z \neq 0$$

$$u(x,y) = \log a - \sum_{s=1}^{\infty} \frac{(s-1)!}{a^s} \frac{(-1)^s (x-a)^s}{s!} - \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} \frac{(2r+s-1)!}{a^{2r+s}} \frac{(-1)^s (x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(2r+s)!}{a^{2r+s+1}} \frac{(-1)^s (x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

log **(1+*z*)**

$$\log(1+z) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1} z^s}{s} \qquad |z| \leq 1,\ z \neq -1$$

$$u(x,y) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1} x^s}{s} - \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} (2r+s-1)! \frac{(-1)^s x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (2r+s)! \frac{(-1)^s x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

log **(1-*z*)**

$$\log(1-z) = -\sum_{s=1}^{\infty} \frac{z^s}{s} \qquad |z| \leq 1,\ z \neq 1$$

$$u(x,y) = -\sum_{s=1}^{\infty} \frac{x^s}{s} - \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} (2r+s-1)! \frac{x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (2r+s)! \frac{x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

sin z

$$\sin z = \sum_{s=0}^{\infty} (-1)^s \frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \infty$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \frac{x^{2s+1}}{(2s+1)!} \frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \frac{x^{2s}}{(2s)!} \frac{y^{2r+1}}{(2r+1)!}$$

cos z

$$\cos z = \sum_{s=0}^{\infty} (-1)^s \frac{z^{2s}}{(2s)!} \qquad |z| < \infty$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \frac{x^{2s}}{(2s)!} \frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{s+1} \frac{x^{2s+1}}{(2s+1)!} \frac{y^{2r+1}}{(2r+1)!}$$

tan z

$$\tan z = \sum_{s=0}^{\infty}(-1)^{s}\frac{2^{2s+2}\left(2^{2s+2}-1\right)B_{2s+2}}{2s+2}\frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \frac{\pi}{2}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s}\frac{2^{2r+2s+2}\left(2^{2r+2s+2}-1\right)B_{2r+2s+2}}{2r+2s+2}\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s}\frac{2^{2r+2s+2}\left(2^{2r+2s+2}-1\right)B_{2r+2s+2}}{2r+2s+2}\frac{x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!}$$

cot z

$$\cot z = \frac{1}{z} + \sum_{s=0}^{\infty}(-1)^{s+1}\frac{2^{2s+2}}{2s+2}B_{2s+2}\frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \pi$$

$$u(x,y) = \frac{x}{x^2+y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}}{2r+2s+2}B_{2r+2s+2}\frac{(-1)^{s}x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = -\frac{y}{x^2+y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}}{2r+2s+2}B_{2r+2s+2}\frac{(-1)^{s}x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!}$$

z cot z

$$z\cot z = \sum_{s=0}^{\infty}(-1)^{s}2^{2s}B_{2s}\frac{z^{2s}}{(2s)!} \qquad |z| < \pi$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s}2^{2r+2s}B_{2r+2s}\frac{x^{2s}}{(2s)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s+1}2^{2r+2s+2}B_{2r+2s+2}\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r+1}}{(2r+1)!}$$

sec z

$$\sec z = \sum_{s=0}^{\infty}(-1)^{s}E_{2s}\frac{z^{2s}}{(2s)!} \qquad |z| < \frac{\pi}{2}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s}E_{2r+2s}\frac{x^{2s}}{(2s)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s+1}E_{2r+2s+2}\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r+1}}{(2r+1)!}$$

csc z

$$\csc z = \frac{1}{z} + \sum_{s=0}^{\infty}(-1)^{s}\frac{2^{2s+2}-2}{2s+2}B_{2s+2}\frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \pi$$

$$u(x,y) = \frac{x}{x^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}-2}{2r+2s+2}B_{2r+2s+2}\frac{(-1)^{s}x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = -\frac{y}{x^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}-2}{2r+2s+2}B_{2r+2s+2}\frac{(-1)^{s}x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!}$$

z csc z

$$z\csc z = 1 - \sum_{s=1}^{\infty}(-1)^s(2^{2s}-2)B_{2s}\frac{z^{2s}}{(2s)!} \qquad |z| < \pi$$

$$u(x,y) = 1 - \sum_{s=1}^{\infty}(-1)^s(2^{2s}-2)B_{2s}\frac{x^{2s}}{(2s)!} - \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}(-1)^s(2^{2r+2s}-2)B_{2r+2s}\frac{x^{2s}}{(2s)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s(2^{2r+2s+2}-2)B_{2r+2s+2}\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r+1}}{(2r+1)!}$$

sinh z

$$\sinh z = \sum_{s=0}^{\infty}\frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \infty$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

cosh z

$$\cosh z = \sum_{s=0}^{\infty}\frac{z^{2s}}{(2s)!} \qquad |z| < \infty$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

tanh z

$$\tanh z = \sum_{s=0}^{\infty}\frac{2^{2s+2}\left(2^{2s+2}-1\right)B_{2s+2}}{2s+2}\frac{z^{2s+1}}{(2s+1)!} \qquad |x| < \frac{\pi}{2}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}\left(2^{2r+2s+2}-1\right)B_{2r+2s+2}}{2r+2s+2}\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}\left(2^{2r+2s+2}-1\right)B_{2r+2s+2}}{2r+2s+2}\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

coth z

$$\coth z = \frac{1}{z} + \sum_{s=0}^{\infty}\frac{2^{2s+2}}{2s+2}B_{2s+2}\frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \pi$$

$$u(x,y) = \frac{x}{x^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}}{2r+2s+2}B_{2r+2s+2}\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\frac{y}{x^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}}{2r+2s+2}B_{2r+2s+2}\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

z coth z

$$z\,coth\,z = \sum_{s=0}^{\infty} 2^{2s} B_{2s} \frac{z^{2s}}{(2s)!} \qquad |z| < \pi$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} 2^{2r+2s} B_{2r+2s} \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} 2^{2r+2s+2} B_{2r+2s+2} \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

sech z

$$sech\,z = \sum_{s=0}^{\infty} E_{2s} \frac{z^{2s}}{(2s)!} \qquad |z| < \frac{\pi}{2}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} E_{2r+2s} \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} E_{2r+2s+2} \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

csch z

$$csch\,z = \frac{1}{z} - \sum_{s=0}^{\infty} \frac{2^{2s+2}-2}{2s+2} B_{2s+2} \frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \pi$$

$$u(x,y) = \frac{x}{x^2+y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{2^{2r+2s+2}-2}{2r+2s+2} B_{2r+2s+2} \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\frac{y}{x^2+y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{2^{2r+2s+2}-2}{2r+2s+2} B_{2r+2s+2} \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

z csch z

$$z\,csch\,z = 1 - \sum_{s=1}^{\infty} (2^{2s}-2) B_{2s} \frac{z^{2s}}{(2s)!} \qquad |z| < \pi$$

$$u(x,y) = 1 - \sum_{s=1}^{\infty} (2^{2s}-2) B_{2s} \frac{x^{2s}}{(2s)!} - \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} (2^{2r+2s}-2) B_{2r+2s} \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (2^{2r+2s+2}-2) B_{2r+2s+2} \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

sin⁻¹z

$$sin^{-1} z = \sum_{s=0}^{\infty} \{(2s-1)!!\}^2 \frac{z^{2s+1}}{(2s+1)!} \qquad |z| \le 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \{(2r+2s-1)!!\}^2 \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \{(2r+2s-1)!!\}^2 \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

cos $^{-1}$***z***

$$\cos^{-1}x = \frac{\pi}{2} - \sum_{s=0}^{\infty}\{(2s-1)!!\}^2\frac{z^{2s+1}}{(2s+1)!} \qquad |z| \le 1$$

$$u(x,y) = \frac{\pi}{2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\{(2r+2s-1)!!\}^2\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\{(2r+2s-1)!!\}^2\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

tan $^{-1}$***z***

$$\tan^{-1}x = \sum_{s=0}^{\infty}(-1)^s(2s)!\frac{x^{2s+1}}{(2s+1)!} \qquad |z| \le 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s(2r+2s)!\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s(2r+2s)!\frac{x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!}$$

cot $^{-1}$***z***

$$\cot^{-1}z = sign\{Re(z)\}\frac{\pi}{2} - \sum_{s=0}^{\infty}(-1)^s(2s)!\frac{x^{2s+1}}{(2s+1)!} \qquad |z| \le 1$$

$$u(x,y) = sign(x)\frac{\pi}{2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s(2r+2s)!\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s(2r+2s)!\frac{x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!}$$

sinh $^{-1}$***z***

$$\sinh^{-1}z = \sum_{s=0}^{\infty}(-1)^s\{(2s-1)!!\}^2\frac{z^{2s+1}}{(2s+1)!} \qquad |z| \le 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\{(2r+2s-1)!!\}^2\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\{(2r+2s-1)!!\}^2\frac{x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!}$$

tanh $^{-1}$***z***

$$\tanh^{-1}z = \sum_{s=0}^{\infty}(2s)!\frac{z^{2s+1}}{(2s+1)!} \qquad |z| \le 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+2s)!\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+2s)!\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

16 Split of Power Series

Formula 16.2.1 (n-split)

Suppose that the function $f(z)$ is expanded into a power series on the domain D as follows.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

And let the n-split series $f(k,n,z)$ $k=0,1,2,\cdots,n-1$ are as follows.

$$f(0,n,z) = \sum_{r=0}^{\infty} a_{nr+0} z^{nr+0} = a_0 z^0 + a_n z^n + a_{2n} z^{2n} + a_{3n} z^{3n} + \cdots$$

$$f(1,n,z) = \sum_{r=0}^{\infty} a_{nr+1} z^{nr+1} = a_1 z^1 + a_{n+1} z^{n+1} + a_{2n+1} z^{2n+1} + a_{3n+1} z^{3n+1} + \cdots$$

$$f(2,n,z) = \sum_{r=0}^{\infty} a_{nr+2} z^{nr+2} = a_2 z^2 + a_{n+2} z^{n+2} + a_{2n+2} z^{2n+2} + a_{3n+2} z^{3n+2} + \cdots$$

$$\vdots$$

$$f(n-1,n,z) = \sum_{r=0}^{\infty} a_{nr+n-1} z^{nr+n-1} = a_{n-1} z^{n-1} + a_{2n-1} z^{2n-1} + a_{3n-1} z^{3n-1} + a_{4n-1} z^{4n-1} + \cdots$$

Then, the following expressions hold for $n = 2,3,4,\cdots$, $k = 0,1,2,\cdots,n-1$.

$$f(k,n,z) = \frac{f(z) - \lambda_n (-1)^k f(-z)}{n} + \frac{1}{n} \sum_{s=1}^{\lfloor n/2 \rfloor} \left[(-1)^{-\frac{2sk}{n}} f\left\{ (-1)^{\frac{2s}{n}} z \right\} + (-1)^{\frac{2sk}{n}} f\left\{ (-1)^{-\frac{2s}{n}} z \right\} \right]$$

Where, $\lambda_n = \{1+(-1)^n\}/2$, $\lfloor x \rfloor$ is the floor function.

Formula 16.2.2 (alternating n-split)

Suppose that the function $f(z)$ is expanded into a power series on the domain D as follows.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

And let the alternating n-split series $\underline{f}(k,n,z)$ $k=0,1,2,\cdots,n-1$ are as follows.

$$\underline{f}(0,n,z) = \sum_{r=0}^{\infty} (-1)^r a_{nr+0} z^{nr+0} = a_0 z^0 - a_n z^n + a_{2n} z^{2n} - a_{3n} z^{3n} + - \cdots$$

$$\underline{f}(1,n,z) = \sum_{r=0}^{\infty} (-1)^r a_{nr+1} z^{nr+1} = a_1 z^1 - a_{n+1} z^{n+1} + a_{2n+1} z^{2n+1} - a_{3n+1} z^{3n+1} + - \cdots$$

$$\underline{f}(2,n,z) = \sum_{r=0}^{\infty} (-1)^r a_{nr+2} z^{nr+2} = a_2 z^2 - a_{n+2} z^{n+2} + a_{2n+2} z^{2n+2} - a_{3n+2} z^{3n+2} + - \cdots$$

$$\vdots$$

$$\underline{f}(n-1,n,z) = \sum_{r=0}^{\infty} (-1)^r a_{nr+n-1} z^{nr+n-1} = a_{n-1} z^{n-1} - a_{2n-1} z^{2n-1} + a_{3n-1} z^{3n-1} - a_{4n-1} z^{4n-1} + - \cdots$$

Then, following expressions hold for $n = 2,3,4,\cdots$, $k = 0,1,2,\cdots,n-1$.

$$\underline{f}(k,n,z) = \underline{\lambda}_n \frac{(-1)^k f(-z)}{n} + \frac{1}{n} \sum_{s=1}^{\lfloor n/2 \rfloor} \left[(-1)^{-\frac{(2s-1)k}{n}} f\left\{ (-1)^{\frac{2s-1}{n}} z \right\} + (-1)^{\frac{(2s-1)k}{n}} f\left\{ (-1)^{-\frac{2s-1}{n}} z \right\} \right]$$

Where, $\underline{\lambda}_n = \{1-(-1)^n\}/2$, $\lfloor x \rfloor$ is the floor function.

16.3 Two-split of Power Series

Example 1 Two-split of Exponential Series

$$f(z) = 1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\cdots = e^z \qquad \text{(Series to be splited)}$$

$$f(0,2,z) = 1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\frac{z^8}{8!}+\frac{z^{10}}{10!}+\cdots = \frac{e^z+e^{-z}}{2} = \cosh z$$

$$f(1,2,z) = \frac{z^1}{1!}+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+\frac{z^9}{9!}+\frac{z^{11}}{11!}+\cdots = \frac{e^z-e^{-z}}{2} = \sinh z$$

Example 2 Two-split of Exponential Generating Series

$$f(z) = 1-\frac{z}{2}+\frac{z^2}{12}-\frac{z^4}{720}+\frac{z^6}{30240}-\frac{z^8}{1209600}+-\cdots = \frac{z}{e^z-1} \qquad \text{(Series to be splited)}$$

$$f(0,2,z) = 1+\frac{z^2}{12}-\frac{z^4}{720}+\frac{z^6}{30240}-\frac{z^8}{1209600}+-\cdots = \frac{1}{2}\left(\frac{z}{e^z-1}+\frac{-z}{e^{-z}-1}\right) = \frac{z}{2}\coth\frac{z}{2}$$

$$f(1,2,z) = -\frac{z}{2} = \frac{1}{2}\left(\frac{z}{e^z-1}-\frac{-z}{e^{-z}-1}\right) = -\frac{z}{2}$$

Example $\underline{1}$ Alternating two-split of Exponential Series

$$f(z) = 1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\cdots = e^z \qquad \text{(Series to be splited)}$$

$$\underline{f}(0,2,z) = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\frac{z^8}{8!}-\frac{z^{10}}{10!}+-\cdots = \frac{e^{iz}+e^{-iz}}{2} = \cos z$$

$$\underline{f}(1,2,z) = \frac{z^1}{1!}-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\frac{z^9}{9!}-\frac{z^{11}}{11!}+-\cdots = \frac{e^{iz}-e^{-iz}}{2i} = \sin z$$

Example $\underline{2}$ Alternating two-split of Exponential Generating Series

$$f(z) = 1-\frac{z}{2}+\frac{z^2}{12}-\frac{z^4}{720}+\frac{z^6}{30240}-\frac{z^8}{1209600}+-\cdots = \frac{z}{e^z-1} \qquad \text{(Series to be splited)}$$

$$\underline{f}(0,2,z) = 1-\frac{z^2}{12}-\frac{z^4}{720}-\frac{z^6}{30240}-\frac{z^8}{1209600}-\cdots = \frac{1}{2}\left(\frac{iz}{e^{iz}-1}+\frac{-iz}{e^{-iz}-1}\right) = \frac{z}{2}\cot\frac{z}{2}$$

$$\underline{f}(1,2,z) = -\frac{z}{2} = \frac{1}{2i}\left(\frac{iz}{e^{iz}-1}-\frac{-iz}{e^{-iz}-1}\right) = -\frac{z}{2}$$

16.3.3 Two-split of Even and Odd Functions

When $f(z)$ is an even function or an odd function, it cannot be splited into two by 16.3.1 .
In such a case, it can be splited into two by the following formula.

(1) When $f(z)$ is an even function,

$$f(0,2,z) = \frac{f(z)+f(iz)}{2} \quad , \quad f(1,2,z) = \frac{f(z)-f(iz)}{2}$$

(2) When $f(z)$ is an odd function,

$$f(0,2,z) = \frac{f(z)+i^{-1}f(iz)}{2} \quad , \quad f(1,2,z) = \frac{f(z)-i^{-1}f(iz)}{2}$$

Example 1 $f(z) = \cosh z$

$$1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\frac{z^8}{8!}+\frac{z^{10}}{10!}+\cdots = \cosh z \qquad \text{(Series to be splited)}$$

$$1+\frac{z^4}{4!}+\frac{z^8}{8!}+\frac{z^{12}}{12!}+\frac{z^{16}}{16!}+\frac{z^{20}}{20!}+\cdots = \frac{\cosh z + \cos z}{2}$$

$$\frac{z^2}{2!}+\frac{z^6}{6!}+\frac{z^{10}}{10!}+\frac{z^{14}}{14!}+\frac{z^{18}}{18!}+\cdots = \frac{\cosh z - \cos z}{2}$$

Especially when $z=1$,

$$1+\frac{1}{4!}+\frac{1}{8!}+\frac{1}{12!}+\frac{1}{16!}+\frac{1}{20!}+\cdots = \frac{\cosh 1 + \cos 1}{2} = 1.04169147\cdots$$

$$\frac{1}{2!}+\frac{1}{6!}+\frac{1}{10!}+\frac{1}{14!}+\frac{1}{18!}+\cdots = \frac{\cosh 1 - \cos 1}{2} = 0.50138916\cdots$$

Example 2 $f(z) = \sinh z$

$$\frac{z^1}{1!}+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+\frac{z^9}{9!}+\frac{z^{11}}{11!}+\cdots = \sinh z \qquad \text{(Series to be splited)}$$

$$\frac{z^1}{1!}+\frac{z^5}{5!}+\frac{z^9}{9!}+\frac{z^{13}}{13!}+\frac{z^{17}}{17!}+\cdots = \frac{\sinh z + \sin z}{2}$$

$$\frac{z^3}{3!}+\frac{z^7}{7!}+\frac{z^{11}}{11!}+\frac{z^{15}}{15!}+\frac{z^{19}}{19!}+\cdots = \frac{\sinh z - \sin z}{2}$$

Especially when $z=1$,

$$\frac{1}{1!}+\frac{1}{5!}+\frac{1}{9!}+\frac{1}{13!}+\frac{1}{17!}+\cdots = \frac{\sinh 1 + \sin 1}{2} = 1.00833608\cdots$$

$$\frac{1}{3!}+\frac{1}{7!}+\frac{1}{11!}+\frac{1}{15!}+\frac{1}{19!}+\cdots = \frac{\sinh 1 - \sin 1}{2} = 0.16686510\cdots$$

16.4 Three-split of Power Series

Example 1 Three-split of Exponential Series

$$1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\cdots = e^z \qquad \text{(Series to be splited)}$$

$$1+\frac{z^3}{3!}+\frac{z^6}{6!}+\frac{z^9}{9!}+\frac{z^{12}}{12!}+\frac{z^{15}}{15!}+\cdots = \frac{e^z}{3}+\frac{e^{(-1)^{2/3}z}+e^{(-1)^{-2/3}z}}{3} = \frac{e^z}{3}+\frac{2}{3\sqrt{e^z}}\cos\frac{\sqrt{3}\,z}{2}$$

$$\frac{z^1}{1!}+\frac{z^4}{4!}+\frac{z^7}{7!}+\frac{z^{10}}{10!}+\frac{z^{13}}{13!}+\frac{z^{16}}{16!}+\cdots = \frac{e^z}{3}+\frac{(-1)^{-2/3}e^{(-1)^{2/3}z}+(-1)^{2/3}e^{(-1)^{-2/3}z}}{3}$$

$$= \frac{e^z}{3}-\frac{1}{3\sqrt{e^z}}\cos\frac{\sqrt{3}\,z}{2}+\frac{1}{\sqrt{3e^z}}\sin\frac{\sqrt{3}\,z}{2}$$

$$\frac{z^2}{2!}+\frac{z^5}{5!}+\frac{z^8}{8!}+\frac{z^{11}}{11!}+\frac{z^{14}}{14!}+\frac{z^{17}}{17!}+\cdots = \frac{e^z}{3}+\frac{(-1)^{-4/3}e^{(-1)^{2/3}z}+(-1)^{4/3}e^{(-1)^{-2/3}z}}{3}$$

$$= \frac{e^z}{3}-\frac{1}{3\sqrt{e^z}}\cos\frac{\sqrt{3}\,z}{2}-\frac{1}{\sqrt{3e^z}}\sin\frac{\sqrt{3}\,z}{2}$$

Especially when $z=1$,

$$1+\frac{1}{3!}+\frac{1}{6!}+\frac{1}{9!}+\frac{1}{12!}+\frac{1}{15!}+\cdots = \frac{e}{3}+\frac{2\cos\left(\sqrt{3}/2\right)}{3\sqrt{e}} = 1.16805831\cdots$$

$$\frac{1}{1!}+\frac{1}{4!}+\frac{1}{7!}+\frac{1}{10!}+\frac{1}{13!}+\frac{z^{16}}{16!}+\cdots = \frac{e}{3}-\frac{\cos(\sqrt{3}/2)}{3\sqrt{e}}+\frac{\sin(\sqrt{3}/2)}{3\sqrt{e}} = 1.04186535\cdots$$

$$\frac{1}{2!}+\frac{1}{5!}+\frac{1}{8!}+\frac{1}{11!}+\frac{1}{14!}+\frac{1}{17!}+\cdots = \frac{e}{3}-\frac{\cos(\sqrt{3}/2)}{3\sqrt{e}}-\frac{\sin(\sqrt{3}/2)}{3\sqrt{e}} = 0.50835816\cdots$$

Example 2 Three-split of Logarithmic Series ($|z|<1,\ z\neq 1$)

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\cdots = -\log(1-z)$$ (Series to be splited)

$$\frac{z^3}{3}+\frac{z^6}{6}+\frac{z^9}{9}+\frac{z^{12}}{12}+\frac{z^{15}}{15}+\frac{z^{18}}{18}+\cdots = -\frac{\log(1-z)}{3} - \frac{\log\{1-(-1)^{2/3}z\}+\log\{1-(-1)^{-2/3}z\}}{3}$$

$$\frac{z^1}{1}+\frac{z^4}{4}+\frac{z^7}{7}+\frac{z^{10}}{10}+\frac{z^{13}}{13}+\frac{z^{16}}{16}+\cdots = -\frac{\log(1-z)}{3} - \frac{(-1)^{-2/3}\log\{1-(-1)^{2/3}z\}+(-1)^{2/3}\log\{1-(-1)^{-2/3}z\}}{3}$$

$$\frac{z^2}{2}+\frac{z^5}{5}+\frac{z^8}{8}+\frac{z^{11}}{11}+\frac{z^{14}}{14}+\frac{z^{17}}{17}+\cdots = -\frac{\log(1-z)}{3} - \frac{(-1)^{-4/3}\log\{1-(-1)^{2/3}z\}+(-1)^{4/3}\log\{1-(-1)^{-2/3}z\}}{3}$$

When $z=1/3$,

$$\frac{1}{3\cdot3^3}+\frac{1}{6\cdot3^6}+\frac{1}{9\cdot3^9}+\frac{1}{12\cdot3^{12}}+\frac{1}{15\cdot3^{15}}+\frac{1}{18\cdot3^{18}}+\cdots = \frac{1}{3}\log\frac{3}{2}-\frac{1}{3}\log\frac{13}{9} = 0.01258010\cdots$$

$$\frac{1}{1\cdot3}+\frac{1}{4\cdot3^4}+\frac{1}{7\cdot3^7}+\frac{1}{10\cdot3^{10}}+\frac{1}{13\cdot3^{13}}+\frac{1}{16\cdot3^{16}}+\cdots = \frac{1}{3}\log\frac{3}{2}+\frac{1}{6}\log\frac{13}{9}+\frac{1}{\sqrt{3}}\arctan\frac{\sqrt{3}}{7} = 0.33648681\cdots$$

$$\frac{1}{2\cdot3^2}+\frac{1}{5\cdot3^5}+\frac{1}{8\cdot3^8}+\frac{1}{11\cdot3^{10}}+\frac{1}{14\cdot3^{14}}+\frac{1}{17\cdot3^{17}}+\cdots = \frac{1}{3}\log\frac{3}{2}+\frac{1}{6}\log\frac{13}{9}-\frac{1}{\sqrt{3}}\arctan\frac{\sqrt{3}}{7} = 0.05639818\cdots$$

Example 1 Alternating three-split of Exponential Series

$$1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\cdots = e^z$$ (Series to be splited)

$$1-\frac{z^3}{3!}+\frac{z^6}{6!}-\frac{z^9}{9!}+\frac{z^{12}}{12!}-\frac{z^{15}}{15!}+\cdots = \frac{(-1)^0e^{-z}}{3}+\frac{e^{(-1)^{1/3}z}+e^{(-1)^{-1/3}z}}{3} = \frac{1}{3e^z}+\frac{2\sqrt{e^z}}{3}\cos\frac{\sqrt{3}\,z}{2}$$

$$\frac{z^1}{1!}-\frac{z^4}{4!}+\frac{z^7}{7!}-\frac{z^{10}}{10!}+\frac{z^{13}}{13!}-\frac{z^{16}}{16!}+\cdots = \frac{(-1)^1e^{-z}}{3}+\frac{(-1)^{-1/3}e^{(-1)^{1/3}z}+(-1)^{1/3}e^{(-1)^{-1/3}z}}{3} = -\frac{1}{3e^z}+\frac{\sqrt{e^z}}{3}\cos\frac{\sqrt{3}\,z}{2}+\sqrt{\frac{e^z}{3}}\sin\frac{\sqrt{3}\,z}{2}$$

$$\frac{z^2}{2!}-\frac{z^5}{5!}+\frac{z^8}{8!}-\frac{z^{11}}{11!}+\frac{z^{14}}{14!}-\frac{z^{17}}{17!}+\cdots = \frac{(-1)^2e^{-z}}{3}+\frac{(-1)^{-2/3}e^{(-1)^{1/3}z}+(-1)^{2/3}e^{(-1)^{-1/3}z}}{3}$$

$$= \frac{1}{3e^z} - \frac{\sqrt{e^z}}{3}\cos\frac{\sqrt{3}\,z}{2} + \sqrt{\frac{e^z}{3}}\sin\frac{\sqrt{3}\,z}{2}$$

Especially when $z=1$,

$$1-\frac{1}{3!}+\frac{1}{6!}-\frac{1}{9!}+\frac{1}{12!}-\frac{1}{15!}+-\cdots = \frac{1}{3e}+\frac{2}{3}\sqrt{e}\cos\frac{\sqrt{3}}{2} = 0.83471946\cdots$$

$$\frac{1}{1!}-\frac{1}{4!}+\frac{1}{7!}-\frac{1}{10!}+\frac{1}{13!}-\frac{1}{16!}+-\cdots = \frac{1}{3e}+\frac{1}{3}\sqrt{e}\cos\frac{\sqrt{3}}{2}+\sqrt{\frac{e}{3}}\sin\frac{\sqrt{3}}{2}$$

$$= 0.95853147\cdots$$

$$\frac{1}{2!}-\frac{1}{5!}+\frac{1}{8!}-\frac{1}{11!}+\frac{1}{14!}-\frac{1}{17!}+-\cdots = \frac{1}{3e}-\frac{1}{3}\sqrt{e}\cos\frac{\sqrt{3}}{2}+\sqrt{\frac{e}{3}}\sin\frac{\sqrt{3}}{2}$$

$$= 0.49169144\cdots$$

Example 2 Alternating three-split of Logarithmic Series ($|z| \le 1$)

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\cdots = -\log(1-z)$$ (Series to be splited)

$$-\frac{z^3}{3}+\frac{z^6}{6}-\frac{z^9}{9}+\frac{z^{12}}{12}-\frac{z^{15}}{15}+\frac{z^{18}}{18}-+\cdots = -\frac{\log(1+z)}{3} - \frac{\log\{1-(-1)^{1/3}z\}+\log\{1-(-1)^{-1/3}z\}}{3}$$

$$\frac{z^1}{1}-\frac{z^4}{4}+\frac{z^7}{7}-\frac{z^{10}}{10}+\frac{z^{13}}{13}-\frac{z^{16}}{16}+-\cdots = -\frac{(-1)^1\log(1+z)}{3} - \frac{(-1)^{-1/3}\log\{1-(-1)^{1/3}z\}+(-1)^{1/3}\log\{1-(-1)^{-1/3}z\}}{3}$$

$$\frac{z^2}{2}-\frac{z^5}{5}+\frac{z^8}{8}-\frac{z^{11}}{11}+\frac{z^{14}}{14}-\frac{z^{17}}{17}+-\cdots = -\frac{(-1)^2\log(1+z)}{3} - \frac{(-1)^{-2/3}\log\{1-(-1)^{1/3}z\}+(-1)^{2/3}\log\{1-(-1)^{-1/3}z\}}{3}$$

Especially when $z=1$,

$$-\frac{1}{3}+\frac{1}{6}-\frac{1}{9}+\frac{1}{12}-\frac{1}{15}+\frac{1}{18}-+\cdots = -\frac{\log 2}{3} = -0.23104906\cdots$$

$$\frac{1}{1}-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\frac{1}{13}-\frac{1}{16}+-\cdots = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}}+\log 2\right) = 0.83564884\cdots$$

$$\frac{1}{2}-\frac{1}{5}+\frac{1}{8}-\frac{1}{11}+\frac{1}{14}-\frac{1}{17}+-\cdots = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}}-\log 2\right) = 0.37355072\cdots$$

16.5 Four-split of Power Series

Example 1 Alternating four-split of Logarithmic Series ($|z| \le 1$)

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\cdots = -\log(1-z)$$ (Series to be splited)

Dividing this into four and substituting $z=1$ for each,

$$-\frac{1}{4}+\frac{1}{8}-\frac{1}{12}+\frac{1}{16}-\frac{1}{20}+\frac{1}{24}-+\cdots = -\frac{\log 2}{4} = -0.17328679\cdots$$

$$\frac{1}{1}-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}+\frac{1}{17}-\frac{1}{21}+-\cdots = \frac{\pi+2\operatorname{arccoth}\sqrt{2}}{4\sqrt{2}} = 0.86697298\cdots$$

$$\frac{1}{2}-\frac{1}{6}+\frac{1}{10}-\frac{1}{14}+\frac{1}{18}-\frac{1}{22}+-\cdots=\frac{\pi}{8} \qquad = 0.39269908\cdots$$

$$\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\frac{1}{19}-\frac{1}{23}+-\cdots=\frac{\pi-2arccoth\sqrt{2}}{4\sqrt{2}} \qquad = 0.24374774\cdots$$

Example 2 Alternating four-split of Exponential Series

$$1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\cdots = e^z \qquad \text{(Series to be splited)}$$

$$1-\frac{z^4}{4!}+\frac{z^8}{8!}-\frac{z^{12}}{12!}+\frac{z^{16}}{16!}-\frac{z^{20}}{20!}+-\cdots = \frac{e^{(-1)^{1/4}z}+e^{(-1)^{-1/4}z}}{4}+\frac{e^{(-1)^{3/4}z}+e^{(-1)^{-3/4}z}}{4}$$

$$=cosh\frac{z}{\sqrt{2}}\cdot cos\frac{z}{\sqrt{2}}$$

$$\frac{z^1}{1!}-\frac{z^5}{5!}+\frac{z^9}{9!}+\frac{z^{13}}{13!}+\frac{z^{17}}{17!}+\frac{z^{21}}{21!}+-\cdots = \frac{(-1)^{-1/4}e^{(-1)^{1/4}z}+(-1)^{1/4}e^{(-1)^{-1/4}z}}{4}$$

$$+\frac{(-1)^{-3/4}e^{(-1)^{3/4}z}+(-1)^{3/4}e^{(-1)^{-3/4}z}}{5}$$

$$=\frac{e^{z/\sqrt{2}}}{2\sqrt{2}}\left(cos\frac{z}{\sqrt{2}}+sin\frac{z}{\sqrt{2}}\right)-\frac{e^{-z/\sqrt{2}}}{2\sqrt{2}}\left(cos\frac{z}{\sqrt{2}}-sin\frac{z}{\sqrt{2}}\right)$$

$$\frac{z^2}{2!}-\frac{z^6}{6!}+\frac{z^{10}}{10!}-\frac{z^{14}}{14!}+\frac{z^{18}}{18!}-\frac{z^{22}}{22!}+-\cdots = \frac{(-1)^{-2/4}e^{(-1)^{1/4}z}+(-1)^{2/4}e^{(-1)^{-1/4}z}}{4}$$

$$+\frac{(-1)^{-6/4}e^{(-1)^{3/4}z}+(-1)^{6/4}e^{(-1)^{-3/4}z}}{4}$$

$$=sinh\frac{z}{\sqrt{2}}\cdot sin\frac{z}{\sqrt{2}}$$

$$\frac{z^3}{3!}-\frac{z^7}{7!}+\frac{z^{11}}{11!}-\frac{z^{15}}{15!}+\frac{z^{19}}{19}-\frac{z^{23}}{23!}+-\cdots = \frac{(-1)^{-3/4}e^{(-1)^{1/4}z}+(-1)^{3/4}e^{(-1)^{-1/4}z}}{4}$$

$$+\frac{(-1)^{-9/4}e^{(-1)^{3/4}z}+(-1)^{9/4}e^{(-1)^{-3/4}z}}{4}$$

$$=-\frac{e^{z/\sqrt{2}}}{2\sqrt{2}}\left(cos\frac{z}{\sqrt{2}}-sin\frac{z}{\sqrt{2}}\right)+\frac{e^{-z/\sqrt{2}}}{2\sqrt{2}}\left(cos\frac{z}{\sqrt{2}}+sin\frac{z}{\sqrt{2}}\right)$$

Especially when $z=1$,

$$1-\frac{1}{4!}+\frac{1}{8!}-\frac{1}{12!}+\frac{1}{16!}-\frac{1}{20!}+-\cdots = cos\frac{1}{\sqrt{2}}cosh\frac{1}{\sqrt{2}} \qquad = 0.95835813\cdots$$

$$\frac{1}{1!}-\frac{1}{5!}+\frac{1}{9!}+\frac{1}{13!}+\frac{1}{17!}-\frac{1}{21!}+-\cdots = \frac{1-i}{2\sqrt{2}}\left(sin\frac{1+i}{\sqrt{2}}+sinh\frac{1+i}{\sqrt{2}}\right) = 0.99166942\cdots$$

$$\frac{1}{2!}-\frac{1}{6!}+\frac{1}{10!}-\frac{1}{14!}+\frac{1}{18!}-\frac{1}{22!}+-\cdots = sin\frac{1}{\sqrt{2}}sinh\frac{1}{\sqrt{2}} \qquad = 0.49861138\cdots$$

$$\frac{1}{3!}-\frac{1}{7!}+\frac{1}{11!}-\frac{1}{15!}+\frac{1}{19}-\frac{1}{23!}+-\cdots = \frac{1+i}{2\sqrt{2}}\left(sin\frac{1+i}{\sqrt{2}}-sinh\frac{1+i}{\sqrt{2}}\right) = 0.16646827\cdots$$

16.6 Five-split of Power Series

Example 2 Five-split of Logarithmic Series ($|z|<1,\ z\neq 1$)

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\cdots = -\log(1-z)$$ (Series to be splited)

Substituting $z=1/2$ for this,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+\cdots = \log 2 = 0.69314718\cdots$$

Dividing this into 5 ,

$$\frac{1}{5\cdot 2^5}+\frac{1}{10\cdot 2^{10}}+\frac{1}{15\cdot 2^{15}}+\frac{1}{20\cdot 2^{20}}+\frac{1}{25\cdot 2^{25}}+\frac{1}{30\cdot 2^{30}}+\cdots = \log 2-\frac{\log 31}{5} = 0.00634973\cdots$$

$$\frac{1}{1\cdot 2}+\frac{1}{6\cdot 2^6}+\frac{1}{11\cdot 2^{11}}+\frac{1}{16\cdot 2^{16}}+\frac{1}{21\cdot 2^{21}}+\frac{1}{26\cdot 2^{26}}+\cdots = 0.50264953\cdots$$

$$\frac{1}{2\cdot 2^2}+\frac{1}{7\cdot 2^7}+\frac{1}{12\cdot 2^{12}}+\frac{1}{17\cdot 2^{17}}+\frac{1}{22\cdot 2^{22}}+\frac{1}{27\cdot 2^{27}}+\cdots = 0.12613687\cdots$$

$$\frac{1}{3\cdot 2^3}+\frac{1}{8\cdot 2^8}+\frac{1}{13\cdot 2^{13}}+\frac{1}{18\cdot 2^{18}}+\frac{1}{23\cdot 2^{23}}+\frac{1}{28\cdot 2^{28}}+\cdots = 0.04216455\cdots$$

$$\frac{1}{4\cdot 2^4}+\frac{1}{9\cdot 2^9}+\frac{1}{14\cdot 2^{14}}+\frac{1}{19\cdot 2^{19}}+\frac{1}{24\cdot 2^{24}}+\frac{1}{29\cdot 2^{29}}+\cdots = 0.01584647\cdots$$

Example 2 Alternating five-split of Logarithmic Series ($|z|\le 1$)

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\cdots = -\log(1-z)$$ (Series to be splited)

Dividing alternately this into 5 parts and substituting $z=1$ for each,

$$-\frac{1}{5}+\frac{1}{10}-\frac{1}{15}+\frac{1}{20}-\frac{1}{25}+\frac{1}{30}-+\cdots = -0.13862943\cdots = -\frac{\log 2}{5}$$

$$\frac{1}{1}-\frac{1}{6}+\frac{1}{11}-\frac{1}{16}+\frac{1}{21}-\frac{1}{26}+-\cdots = 0.88831357\cdots$$

$$\frac{1}{2}-\frac{1}{7}+\frac{1}{12}-\frac{1}{17}+\frac{1}{22}-\frac{1}{27}+-\cdots = 0.40690163\cdots$$

$$\frac{1}{3}-\frac{1}{8}+\frac{1}{13}-\frac{1}{18}+\frac{1}{23}-\frac{1}{28}+-\cdots = 0.25375156\cdots$$

$$\frac{1}{4}-\frac{1}{9}+\frac{1}{14}-\frac{1}{19}+\frac{1}{24}-\frac{1}{29}+-\cdots = 0.18064575\cdots$$

Note

It is possible to represent $(-1)^{m/n}$ $(m,n=1,2,3,\cdots)$ in the formula with elementary transcendental functions or radicals. However, in the case of 5 divisions or more, it becomes very complicated.

17 Expression of Polynomial with Real Coefficients by Real & Imaginary parts.

Formula 17.1.1

Let a be a real number and $f_n(z)$ $(z=x+iy)$ be a polynomial with real coefficients as follows.

$$f_n(z) = \sum_{r=0}^{n} \frac{f_n^{(r)}(a)}{r!}(z-a)^r$$

Then, the following expressions hold for the real and imaginary parts $u_n(x,y), v_n(x,y)$

$$u_n(x,y) = \sum_{r=0}^{n} \frac{f_n^{(r)}(a)}{r!} \sum_{s=0}^{\left\lceil \frac{r-1}{2} \right\rceil} (-1)^s \binom{r}{2s} (x-a)^{r-2s} y^{2s}$$

$$v_n(x,y) = \sum_{r=0}^{n} \frac{f_n^{(r)}(a)}{r!} \sum_{s=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^s \binom{r}{2s+1} (x-a)^{r-2s-1} y^{2s+1}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

Formula 17.1.2

Let a be a real number and $f_n(z)$ $(z=x+iy)$ be a polynomial with real coefficients as follows.

$$f_n(z) = \sum_{s=0}^{n} f_n^{(s)}(a) \frac{(z-a)^s}{s!}$$

Then, the following expressions hold for the real and imaginary parts $u_n(x,y), v_n(x,y)$

$$u_n(x,y) = \sum_{r=0}^{\left\lceil \frac{n-1}{2} \right\rceil} \sum_{s=0}^{n-2r} f_n^{(2r+s)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v_n(x,y) = \sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \sum_{s=0}^{n-2r-1} f_n^{(2r+s+1)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

Formula 17.1.2 ' (Odd Polynomial)

Let $f_{2n+1}(z)$ $(z=x+iy)$ be a polynomial with real coefficients as follows.

$$f_{2n+1}(z) = \sum_{s=0}^{n} f_{2n+1}^{(2s+1)}(0) \frac{z^{2s+1}}{(2s+1)!}$$

Then, the following expressions hold for the real and imaginary parts $u_{2n+1}(x,y), v_{2n+1}(x,y)$ (Where, $0^0 = 1$)

$$u_{2n+1}(x,y) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} f^{(2r+2s+1)}(0) \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v_{2n+1}(x,y) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} f^{(2r+2s+1)}(0) \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Formula 17.1.2 " (Even Polynomial)

Let $f_{2n}(z)$ $(z=x+iy)$ be a polynomial with real coefficients as follows.

$$f_{2n}(z) = \sum_{s=0}^{n} f_{2n}^{(2s)}(0) \frac{z^{2s}}{(2s)!}$$

Then, the following expressions hold for the real and imaginary parts $u_{2n}(x,y), v_{2n}(x,y)$. (Where, $0^0 = 1$)

$$u_{2n}(x,y) = \sum_{r=0}^{n}\sum_{s=0}^{n-r} f^{(2r+2s)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v_{2n}(x,y) = \sum_{r=0}^{n-1}\sum_{s=0}^{n-1-r} f^{(2r+2s+2)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Example1: Cyclotomic Equation

$$C_n(z) = \sum_{s=0}^{n} s!\frac{z^s}{s!} \qquad \left(= 1 + z + z^2 + \cdots + z^n \right)$$

$$u_n(x,y) = \sum_{r=0}^{\left\lceil \frac{n-1}{2} \right\rceil}\sum_{s=0}^{n-2r}(2r+s)!\frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v_n(x,y) = \sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor}\sum_{s=0}^{n-2r-1}(2r+s+1)!\frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

When $n=5$, these are expanded as follows.

$$u_5(x,y) = \sum_{r=0}^{\left\lceil \frac{5-1}{2} \right\rceil}\sum_{s=0}^{5-2r}(2r+s)!\frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$= \left(0!\frac{x^0}{0!} + 1!\frac{x^1}{1!} + 2!\frac{x^2}{2!} + 3!\frac{x^3}{3!} + 4!\frac{x^4}{4!} + 5!\frac{x^5}{5!}\right)\frac{y^0}{0!}$$

$$- \left(2!\frac{x^0}{0!} + 3!\frac{x^1}{1!} + 4!\frac{x^2}{2!} + 5!\frac{x^3}{3!}\right)\frac{y^2}{2!} + \left(4!\frac{x^0}{0!} + 5!\frac{x^1}{1!}\right)\frac{y^4}{4!}$$

$$= 1 + x + x^2 + x^3 + x^4 + x^5 - y^2 - 3xy^2 - 6x^2y^2 - 10x^3y^2 + y^4 + 5xy^4$$

$$v_5(x,y) = \sum_{r=0}^{\left\lfloor \frac{5-1}{2} \right\rfloor}\sum_{s=0}^{5-2r-1}(2r+s+1)!\frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

$$= \left(1!\frac{x^0}{0!} + 2!\frac{x^1}{1!} + 3!\frac{x^2}{2!} + 4!\frac{x^3}{3!} + 5!\frac{x^4}{4!}\right)\frac{y^1}{1!}$$

$$- \left(3!\frac{x^0}{0!} + 4!\frac{x^1}{1!} + 5!\frac{x^2}{2!}\right)\frac{y^3}{3!} + \left(5!\frac{x^0}{0!}\right)\frac{y^5}{5!}$$

$$= y + 2xy + 3x^2y + 4x^3y + 5x^4y - y^3 - 4xy^3 - 10x^2y^3 + y^5$$

Example2: Bernoulli Polynomial

$$B_n(z) = n!\sum_{s=0}^{n}\frac{B_{n-s}}{(n-s)!}\frac{z^s}{s!} \qquad \left(= \sum_{s=0}^{n}\binom{n}{s}B_{n-s}z^s \right)$$

$$u_n(x,y) = n!\sum_{r=0}^{\left\lceil \frac{n-1}{2} \right\rceil}\sum_{s=0}^{n-2r}\frac{B_{n-2r-s}}{(n-2r-s)!}\frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v_n(x,y) = n!\sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor}\sum_{s=0}^{n-2r-1}\frac{B_{n-2r-s-1}}{(n-2r-s-1)!}\frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

When $n=6$, these are expanded as follows.

$$u_6(x,y) = 6!\sum_{r=0}^{\left\lceil \frac{6-1}{2} \right\rceil}\sum_{s=0}^{6-2r}\frac{B_{6-2r-s}}{(6-2r-s)!}\frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$= 6!\left(\frac{B_6}{6!}\frac{x^0}{0!}+\frac{B_5}{5!}\frac{x^1}{1!}+\frac{B_4}{4!}\frac{x^2}{2!}+\frac{B_3}{3!}\frac{x^3}{3!}+\frac{B_2}{2!}\frac{x^4}{4!}+\frac{B_1}{1!}\frac{x^5}{5!}+\frac{B_0}{0!}\frac{x^6}{6!}\right)\frac{y^0}{0!}$$

$$- 6!\left(\frac{B_4}{4!}\frac{x^0}{0!}+\frac{B_3}{3!}\frac{x^1}{1!}+\frac{B_2}{2!}\frac{x^2}{2!}+\frac{B_1}{1!}\frac{x^3}{3!}+\frac{B_0}{0!}\frac{x^4}{4!}\right)\frac{y^2}{2!}$$

$$+ 6!\left(\frac{B_2}{2!}\frac{x^0}{0!}+\frac{B_1}{1!}\frac{x^1}{1!}+\frac{B_0}{0!}\frac{x^2}{2!}\right)\frac{y^4}{4!} - 6!\left(\frac{B_0}{0}\frac{x^0}{0}\right)\frac{y^6}{6!}$$

$$= 720\left(\frac{1}{30240}-\frac{x^2}{1440}+\frac{x^4}{288}-\frac{x^5}{240}+\frac{x^6}{720}\right.$$

$$\left.+\frac{y^2}{1440}-\frac{x^2 y^2}{48}+\frac{x^3 y^2}{24}-\frac{x^4 y^2}{48}+\frac{y^4}{288}-\frac{x y^4}{48}+\frac{x^2 y^4}{48}-\frac{y^6}{720}\right)$$

$$v_6(x,y) = 6!\sum_{r=0}^{\lfloor\frac{6-1}{2}\rfloor}\sum_{s=0}^{6-2r-1}\frac{B_{6-2r-s-1}}{(6-2r-s-1)!}\frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

$$= 6!\left(\frac{B_5}{5!}\frac{x^0}{0!}+\frac{B_4}{4!}\frac{x^1}{1!}+\frac{B_3}{3!}\frac{x^2}{2!}+\frac{B_2}{2!}\frac{x^3}{3!}+\frac{B_1}{1!}\frac{x^4}{4!}+\frac{B_0}{0!}\frac{x^5}{5!}\right)\frac{y^1}{1!}$$

$$- 6!\left(\frac{B_3}{3!}\frac{x^0}{0!}+\frac{B_2}{2!}\frac{x^1}{1!}+\frac{B_1}{1!}\frac{x^2}{2!}+\frac{B_0}{0!}\frac{x^3}{3!}\right)\frac{y^3}{3!} + 6!\left(\frac{B_1}{1!}\frac{x^0}{0!}+\frac{B_0}{0!}\frac{x^1}{1!}\right)\frac{y^5}{5!}$$

$$= 720\left(-\frac{x y}{720}+\frac{x^3 y}{72}-\frac{x^4 y}{48}+\frac{x^5 y}{120}-\frac{x y^3}{72}+\frac{x^2 y^3}{24}-\frac{x^3 y^3}{36}-\frac{y^5}{240}+\frac{x y^5}{120}\right)$$

18 Power Series with the signs of the terms inverted equidistantly

Formula 18.1.1

For integers $n=2,3,4,\cdots$, $k=0,1,2,\cdots$, suppose that the series $f(z)$ and the split series $f(k,n,z)$ are as follows respectively.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

$$f(k,n,z) = \sum_{r=0}^{\infty} a_{nr+k} z^{nr+k} = a_k z^k + a_{n+k} z^{n+k} + a_{2n+k} z^{2n+k} + a_{3n+k} z^{3n+k} + \cdots$$

Then, the series $g(k,n,z)$ in which the signs of the terms $a_{nr+k} z^{nr+k}$ $r=0,1,2,\cdots$ of $f(z)$ are inverted is

$$g(k,n,z) = \frac{n-2}{n} f(z) + \frac{2}{n}\left\{\lambda_n(-1)^k f(-z)\right\}$$
$$- \frac{2}{n}\sum_{s=1}^{\lfloor n/2 \rfloor}\left[(-1)^{-\frac{2sk}{n}} f\left\{(-1)^{\frac{2s}{n}} z\right\} + (-1)^{\frac{2sk}{n}} f\left\{(-1)^{-\frac{2s}{n}} z\right\}\right]$$

Where, $\lambda_n = \left\{1+(-1)^n\right\}/2$, $\lfloor x \rfloor$ is the floor function.

18.2 Sign Inversion at 3rd order Intervals

Example 1 Sign inversion of exponential series at 3rd-order intervals

Original series

$$1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\cdots = e^z$$

Series with inverted signs

$$-1+\frac{z^1}{1!}+\frac{z^2}{2!}-\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}-\frac{z^6}{6!}++-\cdots = \frac{e^z}{3} - \frac{2}{3}\left\{e^{(-1)^{2/3}z} + e^{(-1)^{-2/3}z}\right\}$$
$$= \frac{e^z}{3} - \frac{4}{3\sqrt{e^z}}\cos\frac{\sqrt{3}\,z}{2}$$

$$1-\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}-\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}-++\cdots = \frac{e^z}{3} - \frac{2}{3}\left\{(-1)^{-2/3} e^{(-1)^{2/3}z} + (-1)^{2/3} e^{(-1)^{-2/3}z}\right\}$$
$$= \frac{e^z}{3} + \frac{2}{3\sqrt{e^z}}\left(\cos\frac{\sqrt{3}\,z}{2} - \sqrt{3}\sin\frac{\sqrt{3}\,z}{2}\right)$$

$$1+\frac{z^1}{1!}-\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}-\frac{z^5}{5!}+\frac{z^6}{6!}+-+\cdots = \frac{e^z}{3} - \frac{2}{3}\left\{(-1)^{-4/3} e^{(-1)^{2/3}z} + (-1)^{4/3} e^{(-1)^{-2/3}z}\right\}$$
$$= \frac{e^z}{3} + \frac{2}{3\sqrt{e^z}}\left(\cos\frac{\sqrt{3}\,z}{2} + \sqrt{3}\sin\frac{\sqrt{3}\,z}{2}\right)$$

Especially when $z=1$,

$$-1+\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}-\frac{1}{6!}++-\cdots = \frac{e}{3} - \frac{4\cos\left(\sqrt{3}/2\right)}{3\sqrt{e}} = 0.38216520\cdots$$

$$1-\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}-++\cdots = \frac{e}{3} + \frac{2\cos\left(\sqrt{3}/2\right)}{3\sqrt{e}} - \frac{2\sin\left(\sqrt{3}/2\right)}{\sqrt{3e}}$$
$$= 0.63455111\cdots$$

$$1+\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}+-+\cdots = \frac{e}{3} + \frac{2\cos\left(\sqrt{3}/2\right)}{3\sqrt{e}} + \frac{2\sin\left(\sqrt{3}/2\right)}{\sqrt{3e}}$$
$$= 1.70156550\cdots$$

Example 2 Sign inversion of Logarithmic series at 3rd-order intervals ($|z| < 1,\ z \neq 1$)

Original series

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}+\cdots = -\log(1-z)$$

Series with inverted signs

$$\frac{z^1}{1}+\frac{z^2}{2}-\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}-\frac{z^6}{6}+\frac{z^7}{7}+-+\cdots = -\frac{1}{3}\log(1-z) + \frac{2}{3}\left[\log\left\{1-(-1)^{2/3}z\right\}+\log\left\{1-(-1)^{-2/3}z\right\}\right]$$

$$-\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}-\frac{z^7}{7}++-\cdots = -\frac{1}{3}\log(1-z) + \frac{2}{3}\left[(-1)^{-2/3}\log\left\{1-(-1)^{2/3}z\right\}+(-1)^{2/3}\log\left\{1-(-1)^{-2/3}z\right\}\right]$$

$$\frac{z^1}{1}-\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}-\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}-++\cdots = -\frac{1}{3}\log(1-z) + \frac{2}{3}\left[(-1)^{-4/3}\log\left\{1-(-1)^{2/3}z\right\}+(-1)^{4/3}\log\left\{1-(-1)^{-2/3}z\right\}\right]$$

When $z=1/2$,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}-\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}-\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+-+\cdots = \frac{1}{3}\log\frac{49}{8} = 0.60412625\cdots$$

$$-\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}-\frac{1}{7\cdot 2^7}++-\cdots = \frac{1}{3}\left(\log\frac{8}{7} - 2\sqrt{3}\arctan\frac{\sqrt{3}}{5}\right) = -0.34055118\cdots$$

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}-\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}-++\cdots = \frac{1}{3}\left(\log\frac{8}{7} + 2\sqrt{3}\arctan\frac{\sqrt{3}}{5}\right) = 0.42957211\cdots$$

18.3 Sign Inversion at 4th order Intervals

Example 1 Sign inversion of Binomial series at 4th-order intervals ($|z| < 1,\ z \neq 1$)

Original series

$$1+\frac{1!!}{2!!}z^1+\frac{3!!}{4!!}z^2+\frac{5!!}{6!!}z^3+\frac{7!!}{8!!}z^4+\frac{9!!}{10!!}z^5+\frac{11!!}{12!!}z^6+\frac{13!!}{14!!}z^7+\cdots = \frac{1}{\sqrt{1-z}}$$

Series with inverted signs

$$-1+\frac{1!!}{2!!}z^1+\frac{3!!}{4!!}z^2+\frac{5!!}{6!!}z^3-\frac{7!!}{8!!}z^4+\frac{9!!}{10!!}z^5+\frac{11!!}{12!!}z^6+\frac{13!!}{14!!}z^7-+++\cdots$$

$$=\frac{1}{2}\left(\frac{1}{\sqrt{1-z}}+\frac{1}{\sqrt{1+z}}\right)-\frac{1}{2}\left(\frac{1}{\sqrt{1-(-1)^{2/4}z}}+\frac{1}{\sqrt{1-(-1)^{-2/4}z}}\right) -\frac{1}{2}\left(\frac{1}{\sqrt{1-(-1)^{4/4}z}}+\frac{1}{\sqrt{1-(-1)^{-4/4}z}}\right)$$

$$1-\frac{1!!}{2!!}z^1+\frac{3!!}{4!!}z^2+\frac{5!!}{6!!}z^3+\frac{7!!}{8!!}z^4-\frac{9!!}{10!!}z^5+\frac{11!!}{12!!}z^6+\frac{13!!}{14!!}z^7+-++\cdots$$

$$=\frac{1}{2}\left(\frac{1}{\sqrt{1-z}}-\frac{1}{\sqrt{1+z}}\right)-\frac{1}{2}\left(\frac{(-1)^{-2/4}}{\sqrt{1-(-1)^{2/4}z}}+\frac{(-1)^{2/4}}{\sqrt{1-(-1)^{-2/4}z}}\right)$$

$$-\frac{1}{2}\left(\frac{(-1)^{-4/4}}{\sqrt{1-(-1)^{4/4}z}}+\frac{(-1)^{4/4}}{\sqrt{1-(-1)^{-4/4}z}}\right)$$

$$1+\frac{1!!}{2!!}z^1-\frac{3!!}{4!!}z^2+\frac{5!!}{6!!}z^3+\frac{7!!}{8!!}z^4+\frac{9!!}{10!!}z^5-\frac{11!!}{12!!}z^6+\frac{13!!}{14!!}z^7++-+\cdots$$

$$=\frac{1}{2}\left(\frac{1}{\sqrt{1-z}}+\frac{1}{\sqrt{1+z}}\right)-\frac{1}{2}\left(\frac{(-1)^{-4/4}}{\sqrt{1-(-1)^{2/4}z}}+\frac{(-1)^{4/4}}{\sqrt{1-(-1)^{-2/4}z}}\right)$$

$$-\frac{1}{2}\left(\frac{(-1)^{-8/4}}{\sqrt{1-(-1)^{4/4}z}}+\frac{(-1)^{8/4}}{\sqrt{1-(-1)^{-4/4}z}}\right)$$

$$1+\frac{1!!}{2!!}z^1+\frac{3!!}{4!!}z^2-\frac{5!!}{6!!}z^3+\frac{7!!}{8!!}z^4+\frac{9!!}{10!!}z^5+\frac{11!!}{12!!}z^6-\frac{13!!}{14!!}z^7+++-\cdots$$

$$=\frac{1}{2}\left(\frac{1}{\sqrt{1-z}}-\frac{1}{\sqrt{1+z}}\right)-\frac{1}{2}\left(\frac{(-1)^{-6/4}}{\sqrt{1-(-1)^{2/4}z}}+\frac{(-1)^{6/4}}{\sqrt{1-(-1)^{-2/4}z}}\right)$$

$$-\frac{1}{2}\left(\frac{(-1)^{-12/4}}{\sqrt{1-(-1)^{4/4}z}}+\frac{(-1)^{12/4}}{\sqrt{1-(-1)^{-4/4}z}}\right)$$

When $z=1/2$,

$$-1+\frac{1!!}{2^1 2!!}+\frac{3!!}{2^2 4!!}+\frac{5!!}{2^3 6!!}-\frac{7!!}{2^4 8!!}+\frac{9!!}{2^5 10!!}+\frac{11!!}{2^6 12!!}+\frac{13!!}{2^7 14!!}-+++\cdots = -0.62158357$$

$$=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{6}}-\sqrt{\frac{1}{\sqrt{5}}+\frac{2}{5}}$$

$$1-\frac{1!!}{2^1 2!!}+\frac{3!!}{2^2 4!!}+\frac{5!!}{2^3 6!!}+\frac{7!!}{2^4 8!!}-\frac{9!!}{2^5 10!!}+\frac{11!!}{2^6 12!!}+\frac{13!!}{2^7 14!!}+-++\cdots = 0.89806817$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{6}}-\sqrt{\frac{1}{\sqrt{5}}-\frac{2}{5}}$$

$$1+\frac{1!!}{2^1 2!!}-\frac{3!!}{2^2 4!!}+\frac{5!!}{2^3 6!!}+\frac{7!!}{2^4 8!!}+\frac{9!!}{2^5 10!!}-\frac{11!!}{2^6 12!!}+\frac{13!!}{2^7 14!!}++-+\cdots = 1.21930055$$

$$=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{6}}+\sqrt{\frac{1}{\sqrt{5}}+\frac{2}{5}}$$

$$1+\frac{1!!}{2^1 2!!}+\frac{3!!}{2^2 4!!}-\frac{5!!}{2^3 6!!}+\frac{7!!}{2^4 8!!}+\frac{9!!}{2^5 10!!}+\frac{11!!}{2^6 12!!}-\frac{13!!}{2^7 14!!}+++-\cdots = 1.33264196$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{6}}+\sqrt{\frac{1}{\sqrt{5}}-\frac{2}{5}}$$

Example 2 Sign inversion of Logarithmic series at 4th-order intervals ($|z|<1,\ z\neq 1$)

Original series

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}+\frac{z^8}{8}+\cdots = -\log(1-z)$$

Series with inverted signs

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}-\frac{z^8}{8}+++-\cdots = \arctan z+\frac{\log(1-iz)+\log(1+iz)}{2}$$

$$-\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}-\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}+\frac{z^8}{8}-+++\cdots = -\arctan z-\frac{\log(1-z)+\log(1+z)}{2}$$

$$\frac{z^1}{1}-\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}-\frac{z^6}{6}+\frac{z^7}{7}+\frac{z^8}{8}+-++\cdots = \arctan z - \frac{\log(1-iz)+\log(1+iz)}{2}$$

$$\frac{z^1}{1}+\frac{z^2}{2}-\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}-\frac{z^7}{7}+\frac{z^8}{8}++-+\cdots = \arctan z - \frac{\log(1-z)+\log(1+z)}{2}$$

When $z=1/2$,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}-\frac{1}{8\cdot 2^8}+++-\cdots = \frac{1}{2}\log\frac{15}{4} = 0.66087792\cdots$$

$$-\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}-\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}-+++\cdots = -\operatorname{arccot}2 + \log 2 - \frac{\log 3}{2}$$
$$= -0.31980657\cdots$$

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}-\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}+-++\cdots = \frac{1}{2}\log\frac{12}{5} = 0.43773436\cdots$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}-\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}-\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}++-+\cdots = \operatorname{arccot}2 + \log 2 - \frac{\log 3}{2}$$
$$= 0.60748864\cdots$$

Example 3 Sign inversion of exponential series at 4th-order intervals

Original series

$$1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\frac{z^7}{7!}+\frac{z^8}{8!}+\cdots = e^z$$

Series with inverted signs

$$-1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}-\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\frac{z^7}{7!}-\frac{z^8}{8!}+++-\cdots = e^z - \cosh z - \cos z$$

$$1-\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}-\frac{z^5}{5!}+\frac{z^6}{6!}+\frac{z^7}{7!}+\frac{z^8}{8!}-+++\cdots = e^z - \sinh z - \sin z$$

$$1+\frac{z^1}{1!}-\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}-\frac{z^6}{6!}+\frac{z^7}{7!}+\frac{z^8}{8!}+-++\cdots = e^z - \cosh z + \cos z$$

$$1+\frac{z^1}{1!}+\frac{z^2}{2!}-\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}-\frac{z^7}{7!}+\frac{z^8}{8!}++-+\cdots = e^z - \sinh z + \sin z$$

When $z=1$,

$$-1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}-\frac{1}{8!}+++-\cdots = e - \cosh 1 - \cos 1 = 0.63489888\cdots$$

$$1-\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}-+++\cdots = e - \sinh 1 - \sin 1 = 0.70160965\cdots$$

$$1+\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}-\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+-++\cdots = e - \cosh 1 + \cos 1 = 1.71550350\cdots$$

$$1+\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+\frac{1}{8!}++-+\cdots = e - \sinh 1 + \sin 1 = 2.38455162\cdots$$

18.4 Sign Inversion at 5th order Intervals

Original series

$$1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\frac{z^7}{7!}+\frac{z^8}{8!}+\frac{z^9}{9!}+\frac{z^{10}}{10!}+\cdots = e^z$$

Series with inverted signs (x=1/2)

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}-\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}+\frac{1}{9\cdot 2^9}-\frac{1}{10\cdot 2^{10}}++++-\cdots = \frac{2}{5}\log 31 - \log 2$$

$$= 0.68044770\cdots$$

$$-\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}-\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}+\frac{1}{9\cdot 2^9}+\frac{1}{10\cdot 2^{10}}-++++\cdots = -0.31215188\cdots$$

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}-\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}+\frac{1}{9\cdot 2^9}+\frac{1}{10\cdot 2^{10}}+-+++\cdots = 0.44087342\cdots$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}-\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}-\frac{1}{8\cdot 2^8}+\frac{1}{9\cdot 2^9}+\frac{1}{10\cdot 2^{10}}++-++\cdots = 0.60881807\cdots$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}-\frac{1}{9\cdot 2^9}+\frac{1}{10\cdot 2^{10}}+++-+\cdots = 0.66145422\cdots$$

Note

It is possible to represent $(-1)^{m/n}$ $(m,n = 1,2,3,\cdots)$ in the formula with elementary transcendental functions and radicals. However, in the case of the 5th order or higher, it becomes very complicated.

19 Composition of Trigonometric Functions

19.1 Basic Formula and its Application

Inverse tangent function with two variables

The arctangent function with two variables is defined as follows.

$$ArcTan[a,b] = \begin{cases} \tan^{-1}\dfrac{b}{a} & if\ a>0 \\ \tan^{-1}\dfrac{b}{a}+\pi & if\ a<0\ and\ b\geq 0 \\ \tan^{-1}\dfrac{b}{a}-\pi & if\ a<0\ and\ b<0 \\ +\dfrac{\pi}{2} & if\ a=0\ and\ b>0 \\ -\dfrac{\pi}{2} & if\ a=0\ and\ b<0 \\ undefined & if\ a=0\ and\ b=0 \end{cases}$$

Using an arctangent function with two variables, the composition formula for trigonometric functions can be simply written as follows.

Formula 19.1.1'

(1) Cosine representation

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta-\Phi)$$

(2) Sine representation

$$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\Phi)$$

Where, $\Phi = ArcTan[a,b]$

Special Values

$$\cos\theta \pm \sin\theta = \sqrt{2}\cos\left(\theta\mp\frac{\pi}{4}\right) \quad , \quad \sin\theta \pm \cos\theta = \sqrt{2}\sin\left(\theta\pm\frac{\pi}{4}\right)$$

Formula 19.1.2

(1) Cosine representation

$$\cos(A+B) + \sin(A-B) = 2\cos\left(A-\frac{\pi}{4}\right)\cos\left(B+\frac{\pi}{4}\right)$$

$$\cos(A+B) - \sin(A-B) = 2\cos\left(A+\frac{\pi}{4}\right)\cos\left(B-\frac{\pi}{4}\right)$$

$$\sin(A+B) + \cos(A-B) = 2\cos\left(A-\frac{\pi}{4}\right)\cos\left(B-\frac{\pi}{4}\right)$$

$$\sin(A+B) - \cos(A-B) = -2\cos\left(A+\frac{\pi}{4}\right)\cos\left(B+\frac{\pi}{4}\right)$$

(2) Sine representation

$$\cos(A+B) + \sin(A-B) = -2\sin\left(A+\frac{\pi}{4}\right)\sin\left(B-\frac{\pi}{4}\right)$$

$$\cos(A+B) - \sin(A-B) = -2\sin\left(A-\frac{\pi}{4}\right)\sin\left(B+\frac{\pi}{4}\right)$$
$$\sin(A+B) + \cos(A-B) = 2\sin\left(A+\frac{\pi}{4}\right)\sin\left(B+\frac{\pi}{4}\right)$$
$$\sin(A+B) - \cos(A-B) = -2\sin\left(A-\frac{\pi}{4}\right)\sin\left(B-\frac{\pi}{4}\right)$$

19.2 Recurrence Formula

Formula 19.2.2

$$a_1\cos(\theta+\phi_1) + a_2\cos(\theta+\phi_2) = A_2\cos\{(\theta+\phi_2)+\Phi_2\}$$
$$a_1\sin(\theta+\phi_1) + a_2\sin(\theta+\phi_2) = A_2\sin\{(\theta+\phi_2)+\Phi_2\}$$

Where,

$$A_2 = \sqrt{a_2^2+a_1^2+2a_2a_1\cos(\phi_1-\phi_2)}$$
$$\Phi_2 = ArcTan[a_2+a_1\cos(\phi_1-\phi_2)\ , a_1\sin(\phi_1-\phi_2)]$$

Formula 19.2.3

$$a_1\cos(\theta+\phi_1)+a_2\cos(\theta+\phi_2)+a_3\cos(\theta+\phi_3) = A_3\cos\{(\theta+\phi_3)+\Phi_3\}$$
$$a_1\sin(\theta+\phi_1)+a_2\sin(\theta+\phi_2)+a_3\sin(\theta+\phi_3) = A_3\sin\{(\theta+\phi_3)+\Phi_3\}$$

Where,

$$A_1 = a_1$$
$$\Phi_1 = 0$$
$$A_3 = \sqrt{a_3^2+A_2^2+2a_3A_2\cos(\phi_2-\phi_3+\Phi_2)}$$
$$\Phi_3 = ArcTan[a_3+A_2\cos(\phi_2-\phi_3+\Phi_2)\ , A_2\sin(\phi_2-\phi_3+\Phi_2)]$$

Formula 19.2.n

$$\sum_{r=1}^{n} a_r\cos(\theta+\phi_r) = A_n\cos\{(\theta+\phi_n)+\Phi_n\}$$
$$\sum_{r=1}^{n} a_r\sin(\theta+\phi_r) = A_n\sin\{(\theta+\phi_n)+\Phi_n\}$$

Where,

$$A_1 = a_1$$
$$\Phi_1 = 0$$
$$A_n = \sqrt{a_n^2+A_{n-1}^2+2a_nA_{n-1}\cos(\phi_{n-1}-\phi_n+\Phi_{n-1})}$$
$$\Phi_n = ArcTan[a_n+A_{n-1}\cos(\phi_{n-1}-\phi_n+\Phi_{n-1})\ , A_{n-1}\sin(\phi_{n-1}-\phi_n+\Phi_{n-1})]$$

Example: $\theta = 0\ ,\ a_r(x) = (-1)^{r-1}r^{-x},\ \phi_r(y) = y\log r\quad (r = 1, 2, \cdots, 6)$

$$\sum_{r=1}^{6}(-1)^{r-1}r^{-x}\cos(y\log r) = A_6(x,y)\cos(y\log 6+\Phi_6(x,y)) \tag{6c}$$
$$\sum_{r=1}^{6}(-1)^{r-1}r^{-x}\sin(y\log r) = A_6(x,y)\sin(y\log 6+\Phi_6(x,y)) \tag{6s}$$

I show the source code and the result of formula manipulation software ***Mathematica*** for drawing 3D figures of both sides of (6c). The left and right overlap exactly and look like spots.

a_r & ϕ_r

```
a_r_[x_] := (-1)^(r-1) r^-x        ϕ_r_[y_] := y Log[r]
```

u & v (Left hand side)

```
u[x_, y_, n_] := Sum[a_r[x] Cos[ϕ_r[y]], {r, 1, n}]        v[x_, y_, n_] := Sum[a_r[x] Sin[ϕ_r[y]], {r, 1, n}]
```

A_n & Φ_n (Recurrence formula)

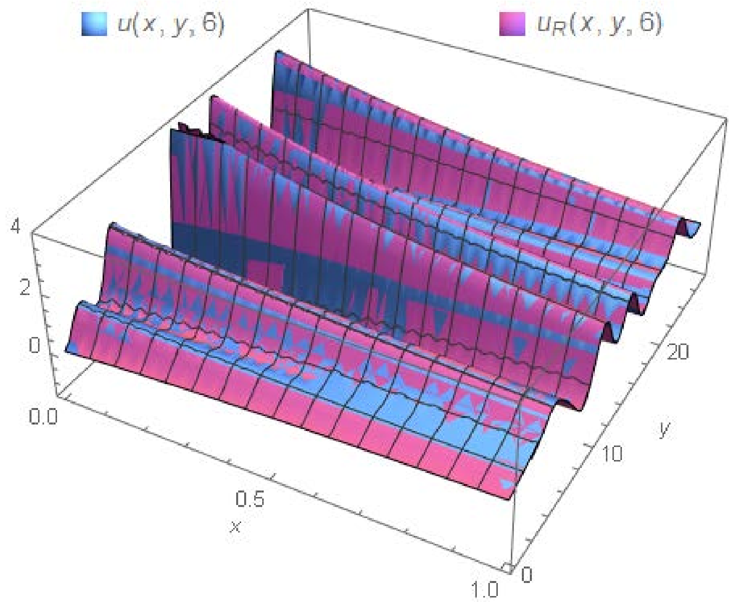
```
A_n_[x_, y_] :=
 If[n == 1, a_1[x], Sqrt[a_n[x]^2 + A_n-1[x, y]^2 + 2 a_n[x] A_n-1[x, y] Cos[ϕ_n-1[y] - ϕ_n[y] + Φ_n-1[x, y]]]]

Φ_n_[x_, y_] := If[n == 1, 0, ArcTan[a_n[x] + A_n-1[x, y] Cos[ϕ_n-1[y] - ϕ_n[y] + Φ_n-1[x, y]],
                                     A_n-1[x, y] Sin[ϕ_n-1[y] - ϕ_n[y] + Φ_n-1[x, y]]]]
```

u_R & v_R (Right hand side)

```
u_R[x_, y_, n_] := A_n[x, y] Cos[ϕ_n[y] + Φ_n[x, y]]
v_R[x_, y_, n_] := A_n[x, y] Sin[ϕ_n[y] + Φ_n[x, y]]
```

n = 6 (drawing)

```
dummy[x_, y_] := -10
Plot3D[{dummy[x, y], u[x, y, 6], u_R[x, y, 6]}, {x, 0, 1}, {y, 0, 27.3},
 AxesLabel → Automatic, PlotLegends → "Expressions", ClippingStyle → None,
 PlotStyle → {, , ColorData[96, 4]}, PlotRange → {-1.5, 4}]
```



19.3 Explicit Formula

Formula 19.3.2 (Trigonometric Polynomials)

$$c(\theta) = \sum_{r=1}^{n} a_r \cos(\theta+\phi_r) = A\cos(\theta+\Phi)$$

$$s(\theta) = \sum_{r=1}^{n} a_r \sin(\theta+\phi_r) = A\sin(\theta+\Phi)$$

Where,

$$A = \sqrt{\left(\sum_{r=1}^{n} a_r \cos\phi_r\right)^2 + \left(\sum_{r=1}^{n} a_r \sin\phi_r\right)^2} \qquad \left(= \sqrt{c^2(0)+s^2(0)}\right)$$

$$\Phi = ArcTan\left[\sum_{r=1}^{n} a_r \cos\phi_r \, , \, \sum_{r=1}^{n} a_r \sin\phi_r\right] \qquad \left(= ArcTan[c(0), s(0)]\right)$$

Formula 19.3.3 **(Trigonometric Series)**

$$c(\theta) = \sum_{r=1}^{\infty} a_r \cos(\theta+\phi_r) = A\cos(\theta+\Phi)$$

$$s(\theta) = \sum_{r=1}^{\infty} a_r \sin(\theta+\phi_r) = A\sin(\theta+\Phi)$$

Where,

$$A = \sqrt{\left(\sum_{r=1}^{\infty} a_r \cos\phi_r\right)^2 + \left(\sum_{r=1}^{\infty} a_r \sin\phi_r\right)^2} \qquad \left(= \sqrt{c^2(0)+s^2(0)}\right)$$

$$\Phi = ArcTan\left[\sum_{r=1}^{\infty} a_r \cos\phi_r \, , \, \sum_{r=1}^{\infty} a_r \sin\phi_r\right] \qquad \left(= ArcTan[c(0), s(0)]\right)$$

Example: $a_r(x) = (-1)^{r-1} r^{-x}$, $\phi_r(y) = y\log r$, $r = 1 \sim \infty$

$$c(\theta,x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^x} \cos(\theta + y\log r) = A(x,y)\cos\{\theta+\Phi(x,y)\} \quad \{= cr(\theta,x,y)\}$$

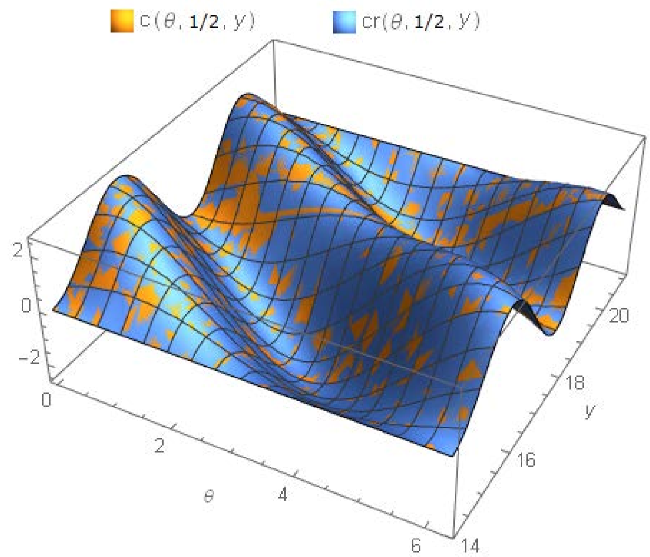
$$s(\theta,x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^x} \sin(\theta + y\log r) = A(x,y)\sin\{\theta+\Phi(x,y)\} \quad \{= sr(\theta,x,y)\}$$

Where,

$$A(x,y) = \sqrt{\{c(0,x,y)\}^2 + \{s(0,x,y)\}^2}$$

$$\Phi(x,y) = ArcTan[c(0,x,y), s(0,x,y)]$$

When $x = 1/2$, $c(\theta,x,y)$, $cr(\theta,x,y)$ are drawn in the 3D figure as follows. Orange is the left side and blue is the right side. The mottled pattern indicates that both sides overlap.



20 Series Expansion by Real & Imaginary Parts of Gamma Function

20.1 Taylor Expansion by Real & Imaginary Parts of Gamma Function & the Reciprocal

Formula 20.1.1 (Taylor Expansion of $\Gamma(z)$)

When $\Gamma(z)$ is gamma function $(z=x+iy)$, $\psi_n(z)$ is polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials and $u(x,y), v(x,y)$ are real and imaginary parts of $\Gamma(z)$, the followings hold for $a \neq 0,-1,-2,-3,\cdots$.

$$\Gamma(z) = \Gamma(a) + \sum_{s=1}^{\infty} c_s(a)\frac{(z-a)^s}{s!}$$

(The radius of convergence is the distance from a to the nearest singular point.)

$$u(x,y) = \Gamma(a) + \sum_{s=1}^{\infty} c_s(a)\frac{(x-a)^s}{s!} + \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} c_{2r+s}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} c_{2r+s+1}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

where,

$$c_n(a) = \Gamma(a)\sum_{k=1}^{n} B_{n,k}(\psi_0(a), \psi_1(a), \ldots, \psi_{n-1}(a)) \qquad n=1,2,3,\cdots$$

$$0^0 = 1.$$

Formula 20.1.2 (Taylor Expansion of $1/\Gamma(z)$)

When $\Gamma(z)$ is gamma function $(z=x+iy)$, $\psi_n(z)$ is polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials and $u(x,y), v(x,y)$ are real and imaginary parts of $1/\Gamma(z)$, the followings hold for $a \neq 0,-1,-2,-3,\cdots$.

$$\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(a)} + \sum_{s=1}^{\infty} c_s(a)\frac{(z-a)^s}{s!} \qquad |z-a| < \infty \tag{1.2}$$

$$u(x,y) = \frac{1}{\Gamma(a)} + \sum_{s=1}^{\infty} c_s(a)\frac{(x-a)^s}{s!} + \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} c_{2r+s}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} c_{2r+s+1}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

where,

$$c_n(a) = \frac{1}{\Gamma(a)}\sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(a), \psi_1(a), \ldots, \psi_{n-1}(a))$$

$$0^0 = 1.$$

20.2 Laurent Expansion by Real & Imaginary Parts of Gamma Function & the Reciprocal

Formula 20.2.1 (Laurent expansion of $\Gamma(z)$)

When $\Gamma(z)$ is gamma function $(z=x+iy)$, $\psi_n(z)$ is polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials and $u(x,y), v(x,y)$ are real and imaginary parts of $\Gamma(z)$, the followings hold for $a \neq 0,-1,-2,-3,\cdots$.

$$\Gamma(z) = \frac{1}{z} + \sum_{s=0}^{\infty}\frac{c_{s+1}}{s+1}\frac{z^s}{s!} \qquad |z| < 1$$

$$u(x,y) = \frac{x}{x^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{c_{2r+s+1}}{2r+s+1}\frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\frac{y}{x^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{c_{2r+s+2}}{2r+s+2}\frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

where,

$$c_n = \sum_{k=1}^{n} B_{n,k}\left(\psi_0(1), \psi_1(1), \dots, \psi_{n-1}(1)\right) \qquad n=1,2,3,\cdots$$

$$0^0 = 1.$$

Formula 20.2.2 (Laurent expansion of $1/\Gamma(z)$)

When $\Gamma(z)$ is gamma function $(z=x+iy)$, $\psi_n(z)$ is polygamma function, $B_{n,k}(f_1, f_2, \dots)$ is Bell polynomials and $u(x,y), v(x,y)$ are real and imaginary parts of $1/\Gamma(z)$, the followings hold for $a \ne 0,-1,-2,-3,\cdots$.

$$\frac{1}{\Gamma(z)} = z + \sum_{s=2}^{\infty} s c_{s-1} \frac{z^s}{s!} \qquad |z| < \infty$$

$$u(x,y) = x + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (2r+s)\, c_{2r+s-1} \frac{x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = y + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (2r+s+1)\, c_{2r+s} \frac{x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(1), \psi_1(1), \dots, \psi_{n-1}(1)\right) \qquad n=1,2,3,\cdots$$

$$0^0 = 1$$

Formula 20.3.1 (Maclaurin Expansion of $\Gamma(1+z)$)

When $\Gamma(z)$ is gamma function $(z=x+iy)$, $\psi_n(z)$ is polygamma function, $B_{n,k}(f_1, f_2, \dots)$ is Bell polynomials and $u(x,y), v(x,y)$ are real and imaginary parts of $\Gamma(1+z)$, the followings hold for $a \ne 0,-1,-2,-3,\cdots$.

$$\Gamma(1+z) = 1 + \sum_{s=1}^{\infty} c_s \frac{z^s}{s!} \qquad |z| < 1$$

$$u(x,y) = 1 + \sum_{s=1}^{\infty} c_s \frac{x^s}{s!} + \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} c_{2r+s} \frac{x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} c_{2r+s+1} \frac{x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

where,

$$c_n = \sum_{k=1}^{n} B_{n,k}\left(\psi_0(1), \psi_1(1), \dots, \psi_{n-1}(1)\right) \qquad n=1,2,3,\cdots$$

$$0^0 = 1$$

Formula 20.3.2 (Maclaurin Expansion of $1/\Gamma(1+z)$)

When $\Gamma(z)$ is gamma function $(z=x+iy)$, $\psi_n(z)$ is polygamma function, $B_{n,k}(f_1, f_2, \dots)$ is Bell polynomials and $u(x,y), v(x,y)$ are real and imaginary parts of $1/\Gamma(1+z)$, the followings hold for $a \ne 0,-1,-2,-3,\cdots$.

$$\frac{1}{\Gamma(1+z)} = 1 + \sum_{s=1}^{\infty} c_s \frac{z^s}{s!} \qquad |z| < \infty$$

$$u(x,y) = 1 + \sum_{s=1}^{\infty} c_s \frac{x^s}{s!} + \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} c_{2r+s} \frac{x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} c_{2r+s+1} \frac{x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(1), \psi_1(1), \dots, \psi_{n-1}(1) \right) \qquad n = 1, 2, 3, \cdots$$

$$0^0 = 1$$

Example: (numeric calculation)

According the formula, $1/\Gamma(1+z)$ is expanded to Maclaurin series. The polynomial $B_{n,k}(f_1, f_2, \dots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. The real and imaginary parts at $0.4+0.3i$ are calculated, and the function value and the series value are compared respectivery. the series are calculated up to 10 terms. The results are as follows.

```
Unprotect[Power]; Power[0, 0] = 1;
Tblψ[n_, z_] := Table[PolyGamma[k, z], {k, 0, n - 1}]
```

$$\mathtt{c[n\_]} := \sum_{k=1}^{n} (-1)^k\, \mathtt{BellY[n, k, Tbl\psi[n, 1]]}$$

$$\mathtt{g[z\_, m\_]} := 1 + \sum_{s=1}^{m} c[s]\, \frac{z^s}{s!}$$

$$\mathtt{u[x\_, y\_, m\_]} := 1 + \sum_{s=1}^{m} c[s]\, \frac{x^s}{s!} + \sum_{r=1}^{m} \sum_{s=0}^{m} c[2r+s]\, \frac{x^s}{s!}\, \frac{(-1)^r y^{2r}}{(2r)!}$$

$$\mathtt{v[x\_, y\_, m\_]} := \sum_{r=0}^{m} \sum_{s=0}^{m} c[2r+s+1]\, \frac{x^s}{s!}\, \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

$$\mathtt{N}\left[\left\{\mathtt{Re}\left[\frac{1}{\mathtt{Gamma}[1 + 0.4 + 0.3\, i]}\right], \mathtt{u[0.4, 0.3, 10]}\right\}\right]$$

{1.17946, 1.17946}

$$\mathtt{N}\left[\left\{\mathtt{Im}\left[\frac{1}{\mathtt{Gamma}[1 + 0.4 + 0.3\, i]}\right], \mathtt{v[0.4, 0.3, 10]}\right\}\right]$$

{0.0165836, 0.0165836}

The function value and the series value are exactly the same.

21 Taylor Expansion by Real & Imaginary Parts around a Complex Number

21.1 Taylor Expansion by Real & Imaginary Parts around a Complex Number

Theorem 21.1.1 (Taylor Expansion around a Complex Number)

Let $f(z)$ $(z=x+iy)$ be a complex function, $u(x,y)$, $v(x,y)$ are the real and imaginary parts. Further, let Re , Im are symbols representing the real & imaginary parts. Then, if $f(z)$ is holomorphic in the whole domain D , the following expressions hold for arbitrary point $(a,b)\in D$.

$$f(z) = \sum_{s=0}^{\infty} f^{(s)}(a+ib)\frac{\{z-(a+ib)\}^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{Re\left[f^{(2r+s)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!} - Im\left[f^{(2r+s+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{Im\left[f^{(2r+s)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!} + Re\left[f^{(2r+s+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

Where, $0^0=1$.

Theorem 21.1.2 (Taylor Expansion on the Vertical Line)

Let $f(z)$ $(z=x+iy)$ be a complex function, $u(x,y)$, $v(x,y)$ are the real & imaginary parts. Further, let Re , Im are symbols representing the real & imaginary parts. Then, if $f(z)$ is holomorphic in the whole domain D , the following expressions hold for arbitrary point $(a,b)\in D$.

$$f(a+iy) = \sum_{s=0}^{\infty}\frac{f^{(s)}(a+ib)}{s!}\{(a-a)+i(y-b)\}^s$$

$$u(a,y) = \sum_{r=0}^{\infty}(-1)^r\left\{Re\left[\frac{f^{(2r)}(a+ib)}{(2r)!}\right](y-b)^{2r} - Im\left[\frac{f^{(2r+1)}(a+ib)}{(2r+1)!}\right](y-b)^{2r+1}\right\}$$

$$v(a,y) = \sum_{r=0}^{\infty}(-1)^r\left\{Im\left[\frac{f^{(2r)}(a+ib)}{(2r)!}\right](y-b)^{2r} + Re\left[\frac{f^{(2r+1)}(a+ib)}{(2r+1)!}\right](y-b)^{2r+1}\right\}$$

Where, $0^0=1$.

21.2 Example 1: sin z

Expansion around $a+bi$

$$\sin z = \sum_{s=0}^{\infty}\sin\left(a+ib+\frac{s\pi}{2}\right)\frac{\{z-(a+ib)\}^s}{s!}$$

$$u(x,y) = \sum_{s=0}^{\infty}\sin\left(a+\frac{s\pi}{2}\right)\frac{(x-a)^s}{s!}\cdot\sum_{r=0}^{\infty}\left\{\cosh b\frac{(y-b)^{2r}}{(2r)!} + \sinh b\frac{(y-b)^{2r+1}}{(2r+1)!}\right\}$$

$$v(x,y) = \sum_{s=0}^{\infty}\cos\left(a+\frac{s\pi}{2}\right)\frac{(x-a)^s}{s!}\cdot\sum_{r=0}^{\infty}\left\{\sinh b\frac{(y-b)^{2r}}{(2r)!} + \cosh b\frac{(y-b)^{2r+1}}{(2r+1)!}\right\}$$

Where, $0^0=1$.

Expansion around b on the vertical line $x=a$

$$sin(a+iy) = \sum_{s=0}^{\infty} sin\left(a+ib+\frac{s\pi}{2}\right)\frac{\{(a-a)+i(y-b)\}^s}{s!}$$

$$u(a,y) = sin\,a \sum_{r=0}^{\infty}\left\{cosh\,b\,\frac{(y-b)^{2r}}{(2r)!} + sinh\,b\,\frac{(y-b)^{2r+1}}{(2r+1)!}\right\}$$

$$v(a,y) = cos\,a \sum_{r=0}^{\infty}\left\{sinh\,b\,\frac{(y-b)^{2r}}{(2r)!} + cosh\,b\,\frac{(y-b)^{2r+1}}{(2r+1)!}\right\}$$

Where, $0^0 = 1$.

21.4 Example 3: Dirichlet Eta Function

In this section, we take up Dirichlet Eta Function $\eta(z)$, which is defined by the following series.

$$\eta(z) = \sum_{s=1}^{\infty}(-1)^{s-1}e^{-z\log s} = \frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} +- \cdots$$

This function can be expanded to the Taylor series around any point on the complex plane. But of particular interest is the expansion on the critical line $x=1/2$ and on the boundary $x=1$ of the critical strip .

Expansion around $a+bi$

$$\eta(z) = \sum_{s=0}^{\infty}\eta^{(s)}(a+ib)\frac{\{z-(a+ib)\}^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{Re\left[\eta^{(2r+s)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!} - Im\left[\eta^{(2r+s+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{Im\left[\eta^{(2r+s)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!} + Re\left[\eta^{(2r+s+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

Where, $\eta^{(s)}(z) = \frac{z^{-s}}{\Gamma(1-s)} + (-1)^{-s}\sum_{k=2}^{\infty}\sum_{t=2}^{k}\frac{(-1)^{t-1}}{2^{k+1}}\binom{k}{t}\frac{\log^s t}{t^z}$, $0^0 = 1$.

Expansion around b on the vertical line $x=a$

$$\eta(a+iy) = \sum_{s=0}^{\infty}\frac{\eta^{(s)}(a+ib)}{s!}\{(a-a)+i(y-b)\}^s$$

$$u(a,y) = \sum_{r=0}^{\infty}(-1)^r\left\{Re\left[\frac{\eta^{(2r)}(a+ib)}{(2r)!}\right](y-b)^{2r} - Im\left[\frac{\eta^{(2r+1)}(a+ib)}{(2r+1)!}\right](y-b)^{2r+1}\right\}$$

$$v(a,y) = \sum_{r=0}^{\infty}(-1)^r\left\{Im\left[\frac{\eta^{(2r)}(a+ib)}{(2r)!}\right](y-b)^{2r} + Re\left[\frac{\eta^{(2r+1)}(a+ib)}{(2r+1)!}\right](y-b)^{2r+1}\right\}$$

Where, $0^0 = 1$.

Expansion around 10000 on the critical line $x=1/2$

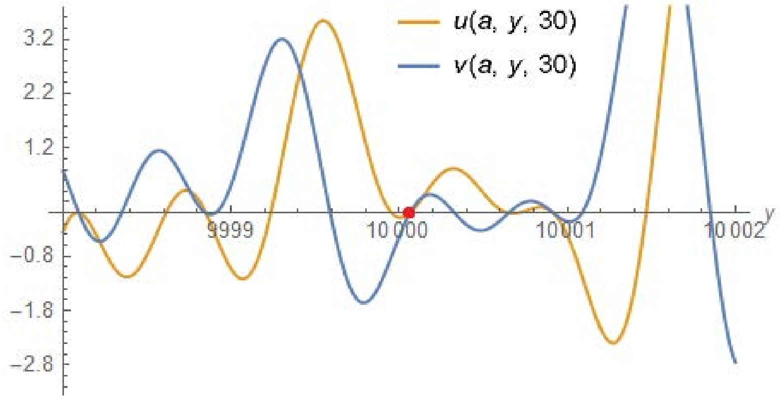
```
Array[c_ab, 101];
```

```
Clear[a, b]; a = 1 / 2; b = 10000;
N[Series[DirichletEta[z], {z, a + ⅈ b, 100}]];
Table[c_ab[n] = SeriesCoefficient[%, n], {n, 0, 100}];
```

$$u[a\_, y\_, m\_] := \sum_{r=0}^{m} (-1)^r \left(\mathrm{Re}[c_{ab}[2r]]\ (y-b)^{2r} - \mathrm{Im}[c_{ab}[2r+1]]\ (y-b)^{2r+1}\right)$$

$$v[a\_, y\_, m\_] := \sum_{r=0}^{m} (-1)^r \left(\mathrm{Im}[c_{ab}[2r]]\ (y-b)^{2r} + \mathrm{Re}[c_{ab}[2r+1]]\ (y-b)^{2r+1}\right)$$

```
Plot[{u[a, y, 30], v[a, y, 30]}, {y, 9998, 10002},
 AxesLabel → Automatic, PlotLegends → "Expressions", ClippingStyle → None,
 PlotRange → {-3.8, 3.8}, PlotStyle → {ColorData[97, 2], ColorData[97, 1]}]
```



Five non-trivial zeros are observed in this interval, but the zero point near $y = 10000$ are as follows.

```
SetPrecision[FindRoot[u[a, y, 5], {y, 10000.1}], 15]
{y → 10000.0653454145}
SetPrecision[FindRoot[v[a, y, 5], {y, 10000.1}], 15]
{y → 10000.0653454145}
SetPrecision[Im[ZetaZero[10143]], 14]
10000.0653454145
```

The 10143 th non-trivial zero point is obtained by calculation of only 5 terms.

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