List of Theorems and Formulas (Dirichlet Series)

01 General Dirichlet Series & Power Series

Definition 1.1.1 (General Dirichlet Series)

Let R be a real number set. And let $\sigma, t \in R$ and $\lambda_n \in R$, $\lambda_n < \lambda_{n+1}$ $\quad n=1,2,3,\cdots$.

When $s=\sigma+it$ and a_n are complex numbers, we call the following series **General Dirichlet Series**.

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

Theorem 1.1.2

Let Dirichlet serie $f(s)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$. And assume $f(s)$ is convergent at $s=s_c=\sigma_c+t_c i$. Then

1. $f(s)$ converges uniformly in $|Arg(s-s_c)| \le \theta < \dfrac{\pi}{2}$.

2. $f(s)$ converges for any $s=\sigma+ti \quad s.t. \quad \sigma > \sigma_c$.

This σ_c is called the **line of convergence**. By convention, $\sigma_c=\infty$ if $f(s)$ converges nowhere and $\sigma_c=-\infty$ if $f(s)$ converges everywhere on the complex plane.

How to calculate σ_c

1. When $\sum_{k=1}^{n} a_k$ is divergent, $\qquad \sigma_c = \limsup_{n\to\infty} \dfrac{\log|a_1+a_2+\cdots+a_n|}{\lambda_n}$

2. When $\sum_{k=1}^{n} a_k$ is convergent, $\qquad \sigma_c = \limsup_{n\to\infty} \dfrac{\log|a_{n+1}+a_{n+2}+a_{n+3}\cdots|}{\lambda_n}$

Theorem 1.1.3 (Holomorphy)

If Dirichlet serie $f(s)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ $(s=\sigma+ti)$ converges for $\sigma>\sigma_c$,

$f(s)$ is **holomophic** at $\sigma>\sigma_c$. And the derivative of $f(s)$ is given as follows.

$$f^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \lambda_n^k a_n e^{-\lambda_n s}$$

Theorem 1.1.4 (Uniqueness)

Let two Dirichlet series are as follows.

$$f(s)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \quad , \quad g(s)=\sum_{n=1}^{\infty} b_n e^{-\lambda_n s}$$

If both are convergent in a certain domain and $f(s)=g(s)$ holds at there, $a_n=b_n$ for $n=1,2,3,\cdots$.

02 Dirichlet Series & Logarithmic Power Series

Definition 2.1.1 (Ordinary Dirichlet Series)

When $s, a_n \ (n=1,2,3,\cdots)$ are complex numbers, we call the following **Ordinary Dirichlet Series**.

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{a_1}{1^s} + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + \cdots$$

Theorem 2.1.2

When σ, t are real numbers and $f(s) = \sum_{n=1}^{\infty} a_n / n^s \ (s = \sigma + ti)$ is Dirichlet series, one of the followings holds.

1. $f(s)$ converges for arbitrary s .

2. $f(s)$ diverges for arbitrary s .

3. There exist a certain real number σ_c such that $f(s)$ converges for s $s.t.$ $\sigma > \sigma_c$ and $f(s)$ diverges for s $s.t.$ $\sigma < \sigma_c$.

This σ_c is called the **line of convergence**. By convention, $\sigma_c = \infty$ if $f(s)$ converges nowhere and $\sigma_c = -\infty$ if $f(s)$ converges everywhere on the complex plane.

How to calculate σ_c

1. When $\sum_{k=1}^{n} a_k$ is divergent, $$\sigma_c = \limsup_{n \to \infty} \frac{\log |a_1 + a_2 + \cdots + a_n|}{\log n}$$

2. When $\sum_{k=1}^{n} a_k$ is convergent, $$\sigma_c = \limsup_{n \to \infty} \frac{\log |a_{n+1} + a_{n+2} + a_{n+3} \cdots |}{\log n}$$

Example 1 p- series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

Then,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} 1 = n$$

Since this is divergent,

$$\sigma_c = \limsup_{n \to \infty} \frac{\log |n|}{\log n} = 1$$

Example 2 Dirichlet Eta series

$$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + - \cdots$$

Then,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (-1)^{k-1} = 1 \ or \ 0$$

Since this is divergent,

$$\sigma_c = \limsup_{n \to \infty} \frac{\log |1 \ or \ 0|}{\log n} = \lim_{n \to \infty} \frac{\log |1|}{\log n} = 0$$

03 Complementary Series of Dirichlet Series

Formula 3.1.1 (Laurent Expansion of $\zeta(z)$)

When $\zeta(z)$ is Riemann zeta function, the following expression holds on whole complex plane.

$$\zeta(z) = \frac{1}{z-1} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^r}{r!}$$

$$= \frac{1}{z-1} + \gamma_0 - \gamma_1 \frac{(z-1)^1}{1!} + \gamma_2 \frac{(z-1)^2}{2!} - \gamma_3 \frac{(z-1)^3}{3!} + - \cdots$$

Where, γ_r is Stieltjes constant defined by the following expression.

$$\gamma_r = \lim_{n \to \infty}\left\{ \sum_{k=1}^{n} \frac{(\log k)^r}{k} - \frac{(\log n)^{r+1}}{r+1} \right\}$$

Formula 3.1.2 (Taylor Expansion of $\eta(z)$)

When $\eta(z) = \sum_{r=1}^{\infty}(-1)^{r-1} e^{-z\log r}$ is Dirichlet eta series and γ_s $s = 0, 1, 2 \cdots$ are Stieltjes constants, the following expression holds on whole complex plane.

$$\eta(z) = \log 2 + \sum_{r=1}^{\infty}(-1)^r \left\{ \frac{\log^{r+1} 2}{r+1} - \sum_{s=0}^{r-1}\binom{r}{s}\gamma_s (\log 2)^{r-s} \right\} \frac{(z-1)^r}{r!}$$

$$= \log 2 - \left(\frac{\log^2 2}{2} - \gamma_0 \log^1 2 \right)\frac{(z-1)^1}{1!}$$

$$+ \left(\frac{\log^3 2}{3} - \gamma_0 \log^2 2 - 2\gamma_1 \log^1 2 \right)\frac{(z-1)^2}{2!}$$

$$- \left(\frac{\log^4 2}{4} - \gamma_0 \log^3 2 - 3\gamma_1 \log^2 2 - 3\gamma_2 \log^1 2 \right)\frac{(z-1)^3}{3!} + - \cdots$$

Formula 3.1.3 (Laurent Expansion of $\lambda(z)$)

When $\lambda(z) = \sum_{r=1}^{\infty} e^{-z\log(2r-1)}$ is Dirichlet lambda series and γ_s $s = 0, 1, 2 \cdots$ are Stieltjes constants, the following expression holds on whole complex plane.

$$\lambda(z) = \frac{1}{2(z-1)} + \frac{\gamma_0 + \log 2}{2} + \frac{1}{2}\sum_{r=1}^{\infty}(-1)^r \left\{ \gamma_r + \frac{\log^{r+1} 2}{r+1} - \sum_{s=0}^{r-1}\binom{r}{s}\gamma_s (\log 2)^{r-s} \right\} \frac{(z-1)^r}{r!}$$

$$= \frac{1}{2(z-1)} + \frac{\gamma_0 + \log 2}{2}$$

$$- \frac{1}{2}\left(\gamma_1 + \frac{\log^2 2}{2} - \gamma_0 \log^1 2 \right)\frac{(z-1)^1}{1!}$$

$$+ \frac{1}{2}\left(\gamma_2 + \frac{\log^3 2}{3} - \gamma_0 \log^2 2 - 2\gamma_1 \log^1 2 \right)\frac{(z-1)^2}{2!}$$

$$- \frac{1}{2}\left(\gamma_3 + \frac{\log^4 2}{4} - \gamma_0 \log^3 2 - 3\gamma_1 \log^2 2 - 3\gamma_2 \log^1 2 \right)\frac{(z-1)^3}{3!} + - \cdots$$

Formula 3.1.4 (Taylor Expansion of $\beta(z)$)

When $\beta(z) = \sum_{r=1}^{\infty}(-1)^{r-1} e^{-z\log(2r-1)}$ is Dirichlet beta series, the following expression holds on the whole complex plane.

$$\beta(z) = \frac{1}{4}\sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^r\binom{r}{s}(\log 4)^{r-s}\left\{\gamma_s\left(\frac{1}{4}\right)-\gamma_s\left(\frac{3}{4}\right)\right\}\frac{(z-1)^r}{r!}$$

Where, $\gamma_r(a)$ is Generalized Stieltjes constant defined by the following expression.

$$\gamma_r(a) = \lim_{m\to\infty}\left\{\sum_{k=0}^{m}\frac{\log^r(k+a)}{k+a} - \frac{\log^{r+1}(m+a)}{r+1}\right\}$$

Formula 3.4.1 (Taylor Expansion of $\eta(z)$)

When $\eta(z) = \sum_{s=1}^{\infty}(-1)^{s-1}e^{-z\log s}$ is Dirichlet eta series, the following expression holds for a constant c except zeros of $\eta(z)$.

$$\eta(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{\log^r(s+1)}{(s+1)^c}\frac{(z-c)^r}{r!}$$

Where, $0^0 = 1$.

Formula 3.4.1' (Higher Derivative of $\eta(z)$)

When $\eta(z)$ is Dirichlet eta function, $\eta^{(n)}(z)$ is the n th order derivative and γ_s $s=0,1,2\cdots$ are Stieltjes constants, the following expressions hold for z such that $Re(z)>0$.

$$\eta^{(n)}(z) = (-1)^n\sum_{s=1}^{\infty}(-1)^{s-1}\frac{\log^n s}{s^z} \qquad n=0,1,2,\cdots$$

Where, $0^0 = 1$.

Especially, when $z=1$,

$$\sum_{s=1}^{\infty}(-1)^{s-1}\frac{\log^n s}{s} = \frac{\log^{n+1}2}{n+1} - \sum_{s=0}^{n-1}\binom{n}{s}\gamma_s(\log 2)^{n-s} \qquad n=1,2,3,\cdots$$

Example

$$\frac{\log^1 1}{1}-\frac{\log^1 2}{2}+\frac{\log^1 3}{3}-\frac{\log^1 4}{4}+-\cdots = \frac{\log^2 2}{2} - \gamma_0\log^1 2$$

$$\frac{\log^2 1}{1}-\frac{\log^2 2}{2}+\frac{\log^2 3}{3}-\frac{\log^2 4}{4}+-\cdots = \frac{\log^3 2}{3} - \gamma_0\log^2 2 - 2\gamma_1\log^1 2$$

Formula 3.4.2 (Half Double Taylor Series)

When $\eta(z) = \sum_{r=1}^{\infty}(-1)^{r-1}e^{-z\log r}$ is Dirichlet eta series, the following expression holds for a constant c except zeros of $\eta(z)$.

$$\eta(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^r\frac{\log^s(r-s+1)}{(r-s+1)^c}\frac{(z-c)^s}{s!}$$

Where, $0^0 = 1$.

The first few terms are as follows. As is expected, it is difficult to call this Taylor series.

$$\eta(z) = \frac{\log^0 1}{1^c}\frac{(z-c)^0}{0!}$$

$$-\frac{\log^0 2}{2^c}\frac{(z-c)^0}{0!} - \frac{\log^1 1}{1^c}\frac{(z-c)^1}{1!}$$

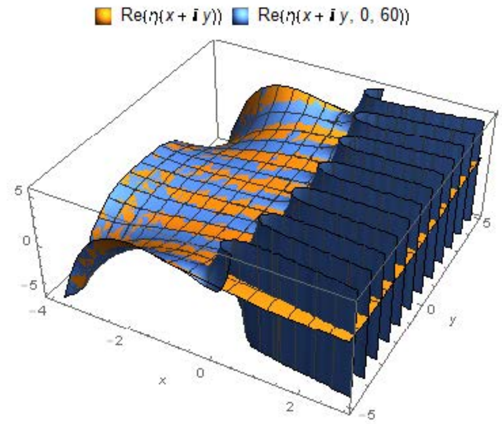
$$+\frac{log^0 3}{3^c}\frac{(z-c)^0}{0!}+\frac{log^1 2}{2^c}\frac{(z-c)^1}{1!}+\frac{log^2 1}{1^c}\frac{(z-c)^2}{2!}$$

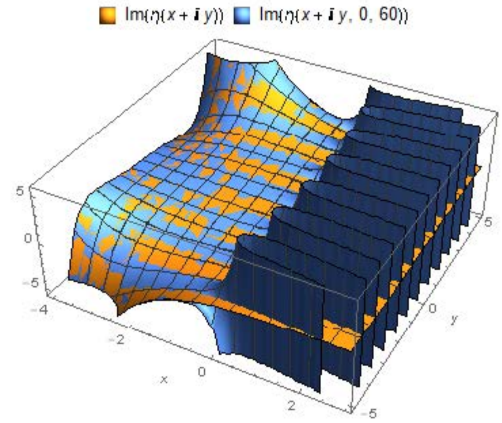
$$-\frac{log^0 4}{4^c}\frac{(z-c)^0}{0!}-\frac{log^1 3}{3^c}\frac{(z-c)^1}{1!}-\frac{log^2 2}{2^c}\frac{(z-c)^2}{2!}-\frac{log^3 1}{1^c}\frac{(z-c)^3}{3!}$$

$$\vdots$$

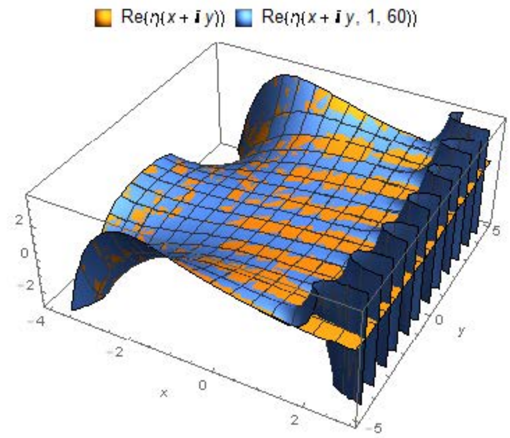
When $c=0\,,z=x+iy$, both sides are illustrated as follows. The left figure is a real part and the right figure is an imaginary part. In both figures, the left-hand side is orange and the right-hand side is blue.

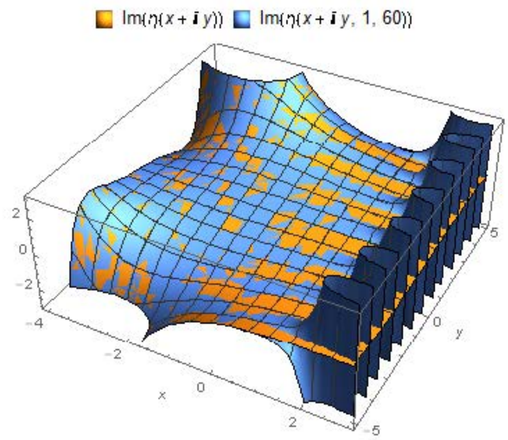
Surprisingly, the opposite side of the figure in Formula 3.4.1 is drawn. That is, **this formula expresses the half plane on the left side of the line of convergence.**





Even more surprisingly, if the real and imaginary parts of both sides are drawn at $c=1$, it becomes as follows.





The line of convergence moved from $x=0$ to $x=2$. As the result of drawing variously, I found that **the line of convergence in this formula is movable and is $x=2c$.**

Definition 3.7.0

When the function $f(z)$ is expressed with 2 series of functions on a domain D and the complement $\overline{D}$ respectively,

$$f(z) = \begin{cases} \sum_r a_r(z) & z \in D \\ \sum_r b_r(z) & z \in \overline{D} \end{cases}$$

we call the series $\sum b_r(z)$ as complementary series of a series $\sum a_r(z)$.

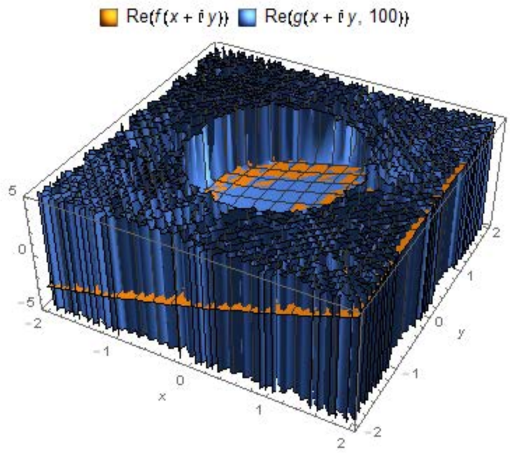
According to the definition, we can see that Formulas 3.4.2 is example of complementary series in ordinary Dirichlet series. In this section, complementary series in power series and general Dirichlet series are illustrated.

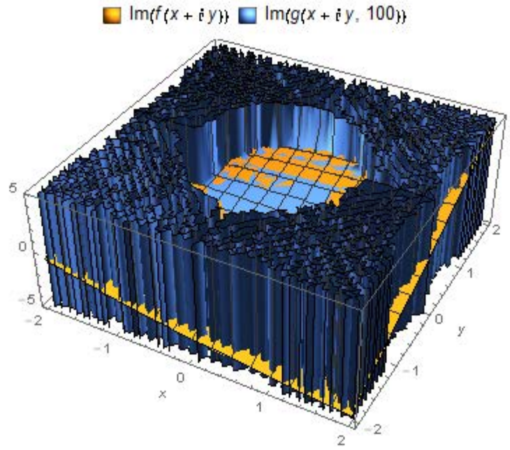
Example 1 Complementary Series of Binomial series

Binomial function is expanded to power series called binary series as follows.

$$(1+z)^{\alpha} = \sum_{r=0}^{\infty} \binom{\alpha}{r} z^{r} \qquad |z| \le 1 \qquad (\ |z| = 1 \text{ is allowed at } \alpha > 0\ ) \tag{7.1}$$

When $\alpha = \sqrt{2}$, $z = x + iy$, both sides are illustrated as follows. The left figure is a real part and the right figure is an imaginary part. In both figures, the left-hand side is orange and the right-hand side is blue. Since the convergence radius is 1 , it is drawn according to the textbook.

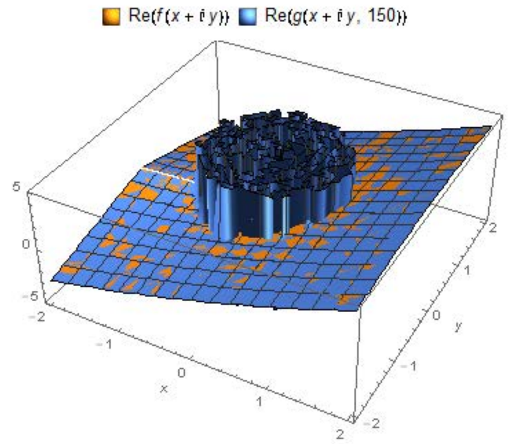


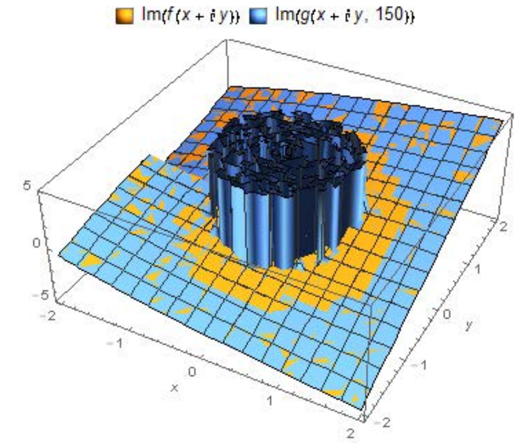


On the other hand, according to Theorem 3.2.1 in " **03 Generalized Multinomial Theorem** " (SuperCalculus) , this function is also expanded to series as follows.

$$(1+z)^{\alpha} = \sum_{r=0}^{\infty} \binom{\alpha}{\alpha - r} z^{\alpha - r} \qquad |z| > 1 \tag{7.1'}$$

When $\alpha = \sqrt{2}$, if the real part and the imaginary part of both sides are illustrated, it is as follows.





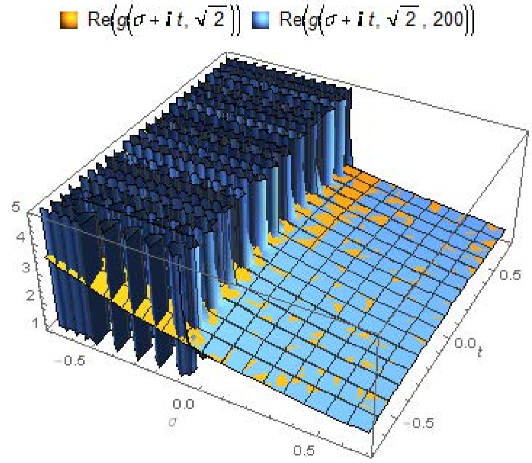
This time, the outside of the convergence circle $|z| = 1$ is drawn. Then, we can see that (7.1') is complementary series of (7.1) .

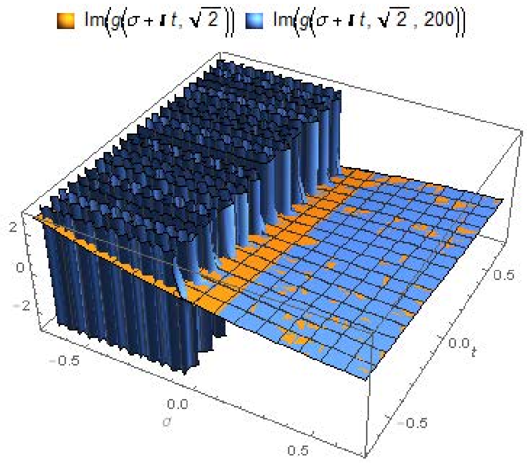
Example 2 Complementary Series of Binomial Dirichlet series

According to " **01 General Dirichlet Series & Power Series** ", a power series is easily converted to the general Dirichlet series by the variable transform $z = e^{-s}$. First, if this is applied to (7.1) ,

$$\left(1+e^{-s}\right)^{\alpha} = \sum_{r=0}^{\infty}\binom{\alpha}{r}e^{-rs} \qquad \left|e^{-s}\right| \le 1 \qquad (\ |e^{-s}| = 1 \text{ is allowed at } \alpha > 0\ ) \tag{7.2}$$

When $\alpha=\sqrt{2}$, $s=\sigma+it$, if the real part and the imaginary part of both sides are illustrated, it is as follows. In both figures, the left-hand side is orange and the right-hand side is blue. Since the convergence area is $\sigma \ge 0$ from $1 \ge \left|e^{-s}\right| = e^{-\sigma}$, It is drawn so in both figures.

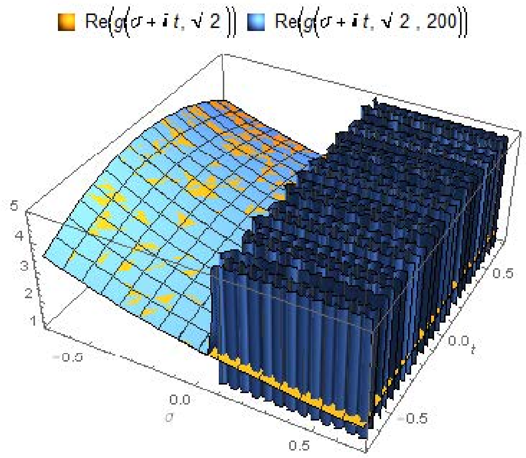


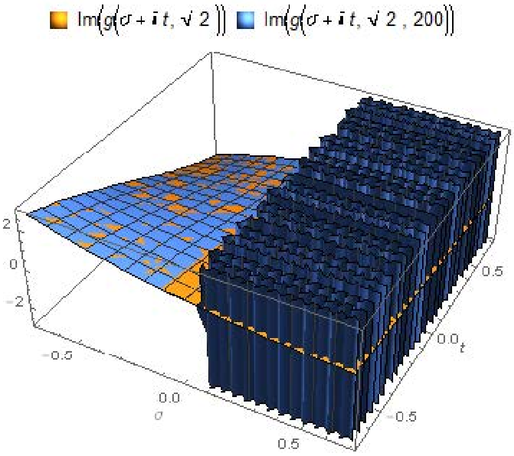


Next, if the variable transform $z=e^{-s}$ is applied to (7.1') ,

$$\left(1+e^{-s}\right)^{\alpha} = \sum_{r=0}^{\infty}\binom{\alpha}{\alpha-r}e^{-(\alpha-r)s} \qquad \left|e^{-s}\right| > 1 \tag{7.2'}$$

When $\alpha=\sqrt{2}$, $s=\sigma+it$, if the real part and the imaginary part of both sides are illustrated, it is as follows. In both figures, the left-hand side is orange and the right-hand side is blue. Since the convergence area is $\sigma < 0$ from $1 < \left|e^{-s}\right| = e^{-\sigma}$, It is drawn so in both figures. Then, we can see that (7.2') is complementary series of (7.2) .





If Example1 and Example 2 are seen, Formula 3.4.2 may not be surprising. However, these examples are very rare. It is usual that the complementarity series of arbitrary series is not obtained easily.

Nevertheless, **in ordinary alternative Dirichlet series**, it seems that **the complementary series is easily obtained only by expanding each terms of the series to Taylor series and rearranging it along the diagonal.** This is a great surprise. Further, the line of convergence is movable freely. The termwise calculus is easy. We can treat it almost like a Taylor series It looks promising as a tool for various purposes.

04 Absolute Value of Dirichlet Eta Function

Definition

Dirichlet Eta Function $\eta(z)$ is defined in the half plane $Re\{\eta(z)\}>0$ as follows.

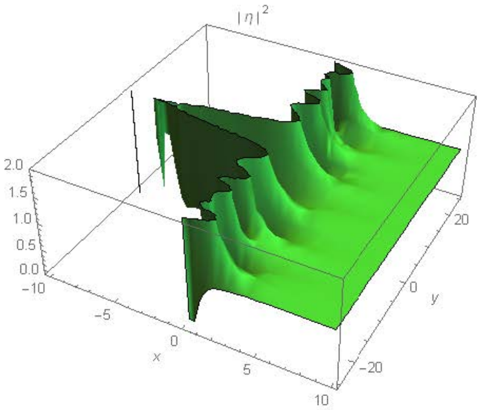
$$\eta(z) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^z}$$

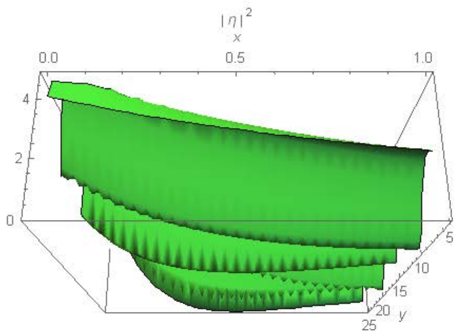
Squared Absolute Value of Dirichlet Eta

Squared absolute value of Dirichlet eta function is

$$g(x,y) = |\eta(x,y)|^2$$

This is a real-valued function with two variables. And it is shown in the figure as follows.





In the left figure, dents are observed along $x=1/2$ and $x=1$. The right figure is a view of the left figure from the bottom. We can see that zeros of $\eta(z)$ are located in two lines along $x=1/2$ and $x=1$ .The zeros on the $x=1/2$ correspond to the zeros of $\zeta(z)$ function and the zeros on the $x=1$ are η specific zeros. On the other hand, there is no zero on the $x=0$.

Expression of Squared Absolute Value by Series

Squared absolute value of Dirichlet eta function $|\eta(x,y)|^2$ is expressed as follows.

Formula 4.3.2

When $\eta(x,y)$ is the Dirichlet Eta Function,

$$|\eta(x,y)|^2 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{(-1)^{r+s}}{(rs)^x} \cos\left(y \log\frac{s}{r}\right) \quad \{ := g(x,y) \}$$

Theorems at Zeros

Theorem 4.4.0

When $\eta(x,y)$ is Dirichlet Eta Function, if $\eta(a,b)=0$,

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{(-1)^{r+s}}{(rs)^a} \cos\left(b \log\frac{s}{r}\right) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{(-1)^{r+s}}{(rs)^a} \sin\left(b \log\frac{s}{r}\right) = 0$$

Theorem 4.4.1

When $\eta(x,y)$ is Dirichlet Eta Function, if $\eta(a,b)=0$,

$$\sum_{s=1}^{\infty} \frac{(-1)^{r+s}}{(rs)^a} \cos\left(b \log\frac{s}{r}\right) = \sum_{s=1}^{\infty} \frac{(-1)^{r+s}}{(rs)^a} \sin\left(b \log\frac{s}{r}\right) = 0 \qquad for\ r=1,2,3,\cdots$$

Corollary 4.4.1'

When $\eta(x,y)$ is Dirichlet Eta Function, if $\eta(a,b)=0$,

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\cos\left(b\log\frac{s}{r}\right) = \sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\sin\left(b\log\frac{s}{r}\right) = 0 \qquad for\ r=1,2,3,\cdots$$

Corollary 4.4.1''

When $\eta(x,y)$ is Dirichlet Eta Function, if $\eta(a,b)=0$, the following expressions hold for arbitrary real number θ.

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\cos(b\log s+\theta) = \sum_{s=1}^{\infty}\frac{(-1)^s}{s^a}\sin(b\log s+\theta) = 0$$

Theorem 4.4.2

When $\eta(x,y)$ is Dirichlet Eta Function and $c(r)$ is arbitrary real valued function, if $\eta(a,b)=0$, the followings hold.

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{c(r)}{(rs)^a}\cos\left(b\log\frac{s}{r}\right) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{c(r)}{(rs)^a}\sin\left(b\log\frac{s}{r}\right) = 0$$

Partial Derivatives of Squared Absolute Value

Formula 4.5.1 (First order Partial Derivatives)

When squared absolute value of Dirichlet eta function is

$$g(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \qquad \left(=|\eta(x,y)|^2\right)$$

The 1st order partial derivatives are givern as follows.

$$g_x = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right)$$

$$g_y = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\sin\left(y\log\frac{s}{r}\right)$$

Formula 4.5.2 (Second order Partial Derivatives)

When squared absolute value of Dirichlet eta function is

$$g(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \qquad \left(=|\eta(x,y)|^2\right)$$

The 2nd order partial derivatives are givern as follows.

$$g_{xx} = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r\log s+\log^2 r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right)$$

$$g_{xy} = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log^2 r}{(rs)^x}\sin\left(y\log\frac{s}{r}\right) \qquad (=g_{yx})$$

$$g_{yy} = 2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r\log s-\log^2 r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right)$$

Theorem 4.7.1 ($g(x,y)$ and $g_x(x,y)$)

Let real valued function with two variables $g(x,y)$, $g_x(x,y)$ are as follows respectively.

$$g(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) \qquad \left\{=|\eta(x,y)|^2\right\}$$

$$g_x(x,y) = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right)$$

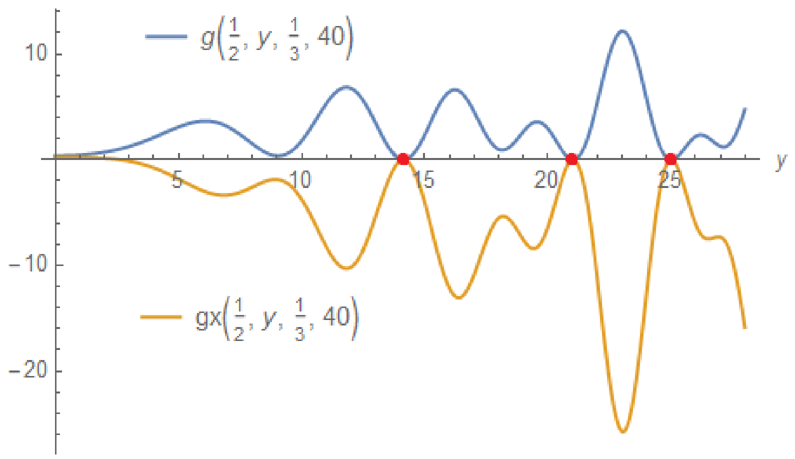
Then, when (a,b) is a zero of Dirichlet eta function $\eta(x,y)$,

(1) b is the common root of $g(a,y)=0$, $g_x(a,y)=0$.

(2) b is at least a multiple root in both $g(a,y)=0$, $g_x(a,y)=0$.

(3) $g(a,y)$ and $g_x(a,y)$ are almost symmetric with respect to the y -axis.

g and g_x on the critical line (Red point is b)



From the above, the following hypothesis is obtained, which is equivalent to the Riemann hypothesis.

Hypothesis 4.7.5

When $\eta(x,y)$ is the Dirichlet eta function on the complex plane, the following inequality holds.

$$g_x(x,y) = -2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s}\frac{\log r}{(rs)^x}\cos\left(y\log\frac{s}{r}\right) < 0 \quad \text{for} \quad \begin{array}{l} 0 < x < 1/2 \\ y \ge 3 \end{array}$$

05 Split of Dirichlet Series

Definition 5.1.0 (Basic Split)

When we can create positive or negative term series $A_k\ (k=1,2,\cdots m)$ by choosing the terms from the k th term $(k=1,2,\cdots m)$ with $(m-1)$ skipping in a series, We call this a **basic m -split** (or simply **m -split**).

Formula 5.1.1

Let $\zeta(n,z)$ be Hurwitz zeta function, $\psi_n(z)$ be polygamma function, and Riemann zeta series $\zeta(n)$ and the m -split series A_k are as follows respectively.

$$\zeta(n) = \sum_{r=1}^{\infty}\frac{1}{r^n}$$

$$A_k = \sum_{r=0}^{\infty}\frac{1}{(mr+k)^n} \qquad k=1,2,\cdots,m$$

Then, the following expressions hold for $k=1,2,\cdots,m$.

$$\sum_{r=0}^{\infty}\frac{1}{(mr+k)^n} = \frac{1}{m^n}\zeta\left(n,\frac{k}{m}\right) = \frac{(-1)^n}{m^n(n-1)!}\psi_{n-1}\left(\frac{k}{m}\right)$$

$$= \frac{(-1)^n}{m^n(n-1)!}\int_0^{\infty}\frac{t^{n-1}e^{-\frac{k}{m}t}}{1-e^{-t}}dt$$

Formula 5.1.2

Let $\zeta(n,z)$ be Hurwitz zeta function, $\psi_n(z)$ be polygamma function, and Dirichlet lambda series $\lambda(n)$ and the m -split series A_k are as follows respectively.

$$\lambda(n) = \sum_{r=0}^{\infty}\frac{1}{(2r+1)^n}$$

$$A_k = \sum_{r=0}^{\infty}\frac{1}{(2mr+2k-1)^n} \qquad k=1,2,\cdots,m$$

Then, the following expressions hold for $k=1,2,\cdots,m$.

$$\sum_{r=0}^{\infty}\frac{1}{(2mr+2k-1)^n} = \frac{1}{(2m)^n}\zeta\left(n,\frac{2k-1}{2m}\right) = \frac{(-1)^n}{(2m)^n(n-1)!}\psi_{n-1}\left(\frac{2k-1}{2m}\right)$$

$$= \frac{1}{(2m)^n(n-1)!}\int_0^{\infty}\frac{t^{n-1}e^{-\frac{2k-1}{2m}t}}{1-e^{-t}}dt$$

Example1 Three-split of $\lambda(3)$

$$\lambda(3) = 1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\frac{1}{9^3}+\frac{1}{11^3}+\frac{1}{13^3}+\frac{1}{15^3}+\frac{1}{17^3}+\frac{1}{19^3}+\cdots = 1.0517997$$

The three-split is as follows.

$$A_1 = 1+\frac{1}{7^3}+\frac{1}{13^3}+\frac{1}{19^3}+\cdots = \frac{-1}{6^3\,2!}\psi_1\left(\frac{1}{6}\right) = \frac{182\zeta(3)+4\sqrt{3}\pi^3}{6^3\,2!}$$

$$= 1.0036855$$

$$A_2 = \frac{1}{3^3} + \frac{1}{9^3} + \frac{1}{15^3} + \frac{1}{21^3} + \cdots = \frac{-1}{6^3\,2!}\psi_1\left(\frac{3}{6}\right) = \frac{14\zeta(3)}{6^3\,2!} = 0.0389555$$

$$A_3 = \frac{1}{5^3} + \frac{1}{11^3} + \frac{1}{17^3} + \frac{1}{23^3} + \cdots = \frac{-1}{6^3\,2!}\psi_1\left(\frac{5}{6}\right) = \frac{182\zeta(3) - 4\sqrt{3}\pi^3}{6^3\,2!}$$
$$= 0.0091587$$

Example2 Four-split of $\zeta(2)$

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \cdots$$
$$= 1.6649341$$

The four-split is as follows.

$$A_1 = 1 + \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} + \cdots = \frac{1}{4^2\,1!}\psi_1\left(\frac{1}{4}\right) = \frac{\pi^4 + 8\beta(2)}{4^2\,1!} = 1.0748331$$

$$A_2 = \frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \frac{1}{14^2} + \cdots = \frac{1}{4^2\,1!}\psi_1\left(\frac{2}{4}\right) = \frac{\pi^2}{2\times 4^2\,1!} = 0.3084251$$

$$A_3 = \frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{15^2} + \cdots = \frac{1}{4^2\,1!}\psi_1\left(\frac{3}{4}\right) = \frac{\pi^4 - 8\beta(2)}{4^2\,1!} = 0.1588675$$

$$A_4 = \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{12^2} + \frac{1}{16^2} + \cdots = \frac{1}{4^2\,1!}\psi_1\left(\frac{4}{4}\right) = \frac{\pi^2}{6\times 4^2\,1!} = 0.1028084$$

Definition 5.5.0 (Composite Split)

Composing fewer m series from the basic n-split series is called **composite m-split** .

Example1 Composite 2-split of $\lambda(3)$

Dirichlet lambda series $\lambda(3)$ split into basic 3 series.

$$A_1 = 1 + \frac{1}{7^3} + \frac{1}{13^3} + \frac{1}{19^3} + \cdots = \sum_{r=0}^{\infty}\frac{1}{(6r+1)^3}$$

$$A_2 = \frac{1}{3^3} + \frac{1}{9^3} + \frac{1}{15^3} + \frac{1}{21^3} + \cdots = \sum_{r=0}^{\infty}\frac{1}{(6r+3)^3}$$

$$A_3 = \frac{1}{5^3} + \frac{1}{11^3} + \frac{1}{17^3} + \frac{1}{23^3} + \cdots = \sum_{r=0}^{\infty}\frac{1}{(6r+5)^3}$$

$$\lambda(3) = A_1 + A_2 + A_3$$

There are the following 3 combinations in composing the series A_1, A_2, A_3 into 2 series.

$$A_1 + \left(A_2 + A_3\right) \quad , \quad A_2 + \left(A_1 + A_3\right) \quad , \quad A_3 + \left(A_1 + A_2\right)$$

These are as follows.

$A_1 + \left(A_2 + A_3\right)$

$$a_1 = 1 + \frac{1}{7^3} + \frac{1}{13^3} + \frac{1}{19^3} + \frac{1}{25^3} + \frac{1}{31^3} + \cdots = \frac{182\zeta(3) + 4\sqrt{3}\pi^3}{6^3\,2!}$$

$$a_2 = \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{9^3} + \frac{1}{11^3} + \frac{1}{15^3} + \frac{1}{17^3} + \cdots = \frac{196\zeta(3) - 4\sqrt{3}\pi^3}{6^3\,2!}$$

$A_2 + (A_1 + A_3)$

$$a_1 = \frac{1}{3^3} + \frac{1}{9^3} + \frac{1}{15^3} + \frac{1}{21^3} + \frac{1}{27^3} + \cdots = \frac{14\zeta(3)}{6^3\,2!}$$

$$a_2 = 1 + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} + \frac{1}{13^3} + \frac{1}{17^3} + \cdots = \frac{2\times 182\zeta(3)}{6^3\,2!}$$

$A_3 + (A_1 + A_2)$

$$a_1 = \frac{1}{5^3} + \frac{1}{11^3} + \frac{1}{17^3} + \frac{1}{23^3} + \frac{1}{29^3} + \cdots = \frac{182\zeta(3) - 4\sqrt{3}\pi^3}{6^3\,2!}$$

$$a_2 = 1 + \frac{1}{3^3} + \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{13^3} + \frac{1}{15^3} + \cdots = \frac{196\zeta(3) + 4\sqrt{3}\pi^3}{6^3\,2!}$$

Example1 Composite 2-split of Madhava–Leibniz series

Madhava–Leibniz series $\beta(1)\ (=\pi/4)$ is as follows.

$$\beta(1) = 1 - \frac{1}{3^1} + \frac{1}{5^1} - \frac{1}{7^1} + \frac{1}{9^1} - \frac{1}{11^1} + \frac{1}{13^1} - \frac{1}{15^1} + \frac{1}{17^1} - + \cdots$$

The possible composite 2-split are:as follows.

Split 1

$$\begin{cases} a_{11} = 1 - \dfrac{1}{7^1} + \dfrac{1}{9^1} - \dfrac{1}{15^1} + \dfrac{1}{17^1} - \dfrac{1}{23^1} + - \cdots = \dfrac{\pi}{8}\left(\sqrt{2}+1\right) \\ a_{12} = \dfrac{1}{3^1} - \dfrac{1}{5^1} + \dfrac{1}{11^1} - \dfrac{1}{13^1} + \dfrac{1}{19^1} - \dfrac{1}{21^1} + - \cdots = \dfrac{\pi}{8}\left(\sqrt{2}-1\right) \end{cases}$$

$$\beta(1) = a_{11} - a_{12}$$

Split 2

$$\begin{cases} a_{21} = 1 - \dfrac{1}{3^1} + \dfrac{1}{9^1} - \dfrac{1}{11^1} + \dfrac{1}{17^1} - \dfrac{1}{19^1} + - \cdots = \dfrac{\pi}{8} - \dfrac{\sqrt{2}}{4}\ln\left(\sqrt{2}-1\right) \\ a_{22} = \dfrac{1}{5^1} - \dfrac{1}{7^1} + \dfrac{1}{13^1} - \dfrac{1}{15^1} + \dfrac{1}{21^1} - \dfrac{1}{23^1} + - \cdots = \dfrac{\pi}{8} + \dfrac{\sqrt{2}}{4}\ln\left(\sqrt{2}-1\right) \end{cases}$$

$$\beta(1) = a_{21} + a_{22}$$

06 Reflection Split of Dirichlet Series

Formula 6.1.2 (Polygamma Reflection Formula)

When $\psi_n(z)$ $(n=1,2,3,\cdots)$ is the Polygamma function, the following expressions hold.

$$\psi_{2n-1}(z)+\psi_{2n-1}(1-z) = -\pi\frac{d^{2n-1}}{dz^{2n-1}}\cot(\pi z)$$

$$\psi_{2n-2}(z)-\psi_{2n-2}(1-z) = -\pi\frac{d^{2n-2}}{dz^{2n-2}}\cot(\pi z)$$

Definition 6.1.3' (Restrictive)

(1) Let Dirichlet lambda series $\lambda(2n)$ be

$$\lambda(2n) = 1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{9^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}+\cdots$$

When this series is split as follows, we call this **reflection** m **-split of Dilichlet lambda series**.

$$a_k = \sum_{r=0}^{\infty}\left\{\frac{1}{(4mr+2k-1)^{2n}}+\frac{1}{(4mr+4m-2k+1)^{2n}}\right\} \qquad k=1,2,\cdots,m$$

$$a_1+a_2+\cdots+a_m = \lambda(2n)$$

(2) Let Dirichlet beta series $\beta(2n-1)$ be

$$\beta(2n-1) = 1-\frac{1}{3^{2n-1}}+\frac{1}{5^{2n-1}}-\frac{1}{7^{2n-1}}+\frac{1}{9^{2n-1}}-\frac{1}{11^{2n-1}}+\frac{1}{13^{2n-1}}-+\cdots$$

When this series is split as follows, we call this **reflection** m **-split of Dilichlet beta series**.

$$a_k = (-1)^{k-1}\sum_{r=0}^{\infty}\left\{\frac{1}{(4mr+2k-1)^{2n-1}}-\frac{1}{(4mr+4m-2k+1)^{2n-1}}\right\} \qquad k=1,2,\cdots,m$$

$$a_1+a_2+\cdots+a_m = \beta(2n-1)$$

Formula 6.1.4 (Reflection Split of Dirichlet Series)

The sum of reflection split series a_k in Definition 6.1.3' is given by the following expression.

(1) For Dirichlet lambda series $\lambda(2n)$,

$$a_k = -\frac{\pi}{(4m)^{2n}(2n-1)!}\frac{d^{2n-1}}{dz^{2n-1}}\cot(\pi z)\Bigg|_{\frac{2k-1}{4m}} \qquad k=1,2,\cdots,m$$

(2) For Dirichlet beta series $\beta(2n-1)$,

$$a_k = \frac{(-1)^{k-1}\pi}{(4m)^{2n-1}(2n-2)!}\frac{d^{2n-2}}{dz^{2n-2}}\cot(\pi z)\Bigg|_{\frac{2k-1}{4m}} \qquad k=1,2,\cdots,m$$

Theorem 6.1.5

The sum of reflection split series a_k in Definition 6.1.3' is the spcial value of elementary function

Example1 Reflection 2-split of $\lambda(2n)$

$$\lambda(2n) = 1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{9^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}+\cdots$$

The reflection 2-split is as follows.

$$1+\frac{1}{7^{2n}}+\frac{1}{9^{2n}}+\frac{1}{15^{2n}}+\frac{1}{17^{2n}}+\cdots = -\frac{\pi}{8^{2n}(2n-1)!}\left.\frac{d^{2n-1}}{dz^{2n-1}}cot(\pi z)\right|_{1/8}$$

$$\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}+\frac{1}{19^{2n}}+\cdots = -\frac{\pi}{8^{2n}(2n-1)!}\left.\frac{d^{2n-1}}{dz^{2n-1}}cot(\pi z)\right|_{3/8}$$

When $n=2$

$$a_1 = 1+\frac{1}{7^4}+\frac{1}{9^4}+\frac{1}{15^4}+\frac{1}{17^4}+\cdots = \frac{\pi^4(16+11\sqrt{2})}{3072} = 1.000610446\cdots$$

$$a_2 = \frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{11^4}+\frac{1}{13^4}+\frac{1}{19^4}+\cdots = \frac{\pi^4(16-11\sqrt{2})}{3072} = 0.01406758545\cdots$$

Example2 Reflection 2-split of $\beta(2n-1)$

$$\beta(2n-1) = 1-\frac{1}{3^{2n-1}}+\frac{1}{5^{2n-1}}-\frac{1}{7^{2n-1}}+\frac{1}{9^{2n-1}}-\frac{1}{11^{2n-1}}+\frac{1}{13^{2n-1}}-+\cdots$$

The reflection 2-split is as follows.

$$1-\frac{1}{7^{2n-1}}+\frac{1}{9^{2n-1}}-\frac{1}{15^{2n-1}}+-\cdots = \frac{\pi}{8^{2n-1}(2n-2)!}\left.\frac{d^{2n-2}}{dz^{2n-2}}cot(\pi z)\right|_{1/8}$$

$$-\left(\frac{1}{3^{2n-1}}-\frac{1}{5^{2n-1}}+\frac{1}{11^{2n-1}}-\frac{1}{13^{2n-1}}+-\cdots\right) = -\frac{\pi}{8^{2n-1}(2n-2)!}\left.\frac{d^{2n-2}}{dz^{2n-2}}cot(\pi z)\right|_{3/8}$$

When $n=3$

$$a_1 = 1-\frac{1}{7^5}+\frac{1}{9^5}-\frac{1}{15^5}+\frac{1}{17^5}-\frac{1}{23^5}+-\cdots = \frac{\pi^5(80+57\sqrt{2})}{49152} = 0.9999568$$

$$a_2 = -\left(\frac{1}{3^5}-\frac{1}{5^5}+\frac{1}{11^5}-\frac{1}{13^5}+\frac{1}{19^5}-\frac{1}{21^5}+-\cdots\right) = \frac{\pi^5(80-57\sqrt{2})}{49152} = -0.0037989$$

Example3 Reflection 3-split of $\beta(2n-1)$

$$\beta(2n-1) = 1-\frac{1}{3^{2n-1}}+\frac{1}{5^{2n-1}}-\frac{1}{7^{2n-1}}+\frac{1}{9^{2n-1}}-\frac{1}{11^{2n-1}}+\frac{1}{13^{2n-1}}-+\cdots$$

The reflection 3-split is as follows.

$$1-\frac{1}{11^{2n-1}}+\frac{1}{13^{2n-1}}-\frac{1}{23^{2n-1}}+\frac{1}{25^{2n-1}}+-\cdots = -\frac{\pi}{12^{2n-1}(2n-2)!}\left.\frac{d^{2n-2}}{dz^{2n-2}}cot(\pi z)\right|_{1/12}$$

$$-\left(\frac{1}{3^{2n-1}}-\frac{1}{9^{2n-1}}+\frac{1}{15^{2n-1}}-\frac{1}{21^{2n-1}}+-\cdots\right) = \frac{\pi}{12^{2n-1}(2n-2)!}\left.\frac{d^{2n-2}}{dz^{2n-2}}cot(\pi z)\right|_{3/12}$$

$$\frac{1}{5^{2n-1}}-\frac{1}{7^{2n-1}}+\frac{1}{17^{2n-1}}-\frac{1}{19^{2n-1}}+-\cdots = -\frac{\pi}{12^{2n-1}(2n-2)!}\left.\frac{d^{2n-2}}{dz^{2n-2}}cot(\pi z)\right|_{5/12}$$

When $n=3$

$$1-\frac{1}{11^5}+\frac{1}{13^5}-\frac{1}{23^5}+\frac{1}{25^5}-\frac{1}{35^5}+-\cdots = \frac{\pi^5(305+176\sqrt{3})}{186624} = 0.9999964$$

$$-\left(\frac{1}{3^5}-\frac{1}{9^5}+\frac{1}{15^5}-\frac{1}{21^5}+\frac{1}{27^5}-\frac{1}{33^5}+-\cdots\right) = -\frac{5\pi^5}{373248} = -0.0040994$$

$$\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{17^5}-\frac{1}{19^5}+\frac{1}{29^5}-\frac{1}{31^5}+-\cdots = \frac{\pi^5(305-176\sqrt{3})}{186624} = 0.0002608$$

Algebraic Solvability of Reflection Split

Formula 6.4.1 (Trigonometric Reflection Formula)

For natural number n and $k=1,2,3,\cdots,n-1$, the following expressions hold.

$$\cos\frac{k\pi}{n}=\frac{1}{2}\left\{(-1)^{\frac{k}{n}}-(-1)^{1-\frac{k}{n}}\right\}\quad,\quad \sin\frac{k\pi}{n}=\frac{1}{2i}\left\{(-1)^{\frac{k}{n}}+(-1)^{1-\frac{k}{n}}\right\}$$

Corollary 6.4.1

For natural number n and $k=1,2,3,\cdots,n-1$, the following expressions hold.

$$\cot\frac{k\pi}{n}=\frac{i\left\{(-1)^{\frac{k}{n}}-(-1)^{1-\frac{k}{n}}\right\}}{(-1)^{\frac{k}{n}}+(-1)^{1-\frac{k}{n}}}\quad,\quad \csc\frac{k\pi}{n}=\frac{2i}{(-1)^{\frac{k}{n}}+(-1)^{1-\frac{k}{n}}}$$

Theorem 6.4.2 (Algebraic Solvability)

The sum of reflection split series a_k in Definition 6.1.3' is expressed with addition, multiplication and rational powers, except circle ratio π .

Example Reflect 7-split of $\beta(3)$

$$\beta(3)=1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\frac{1}{13^3}-+\cdots$$

The reflection 7-split is as follows.

$$1-\frac{1}{27^3}+\frac{1}{29^3}-\frac{1}{55^3}+\frac{1}{57^3}-\frac{1}{83^3}+-\cdots=-\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{1}{28}}-(-1)^{\frac{27}{28}}\right\}}{\left\{(-1)^{\frac{1}{28}}+(-1)^{\frac{27}{28}}\right\}^3}$$

$$-\left(\frac{1}{3^3}-\frac{1}{25^3}+\frac{1}{31^3}-\frac{1}{53^3}+\frac{1}{59^3}-\frac{1}{81^3}+-\cdots\right)=\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{3}{28}}-(-1)^{\frac{25}{28}}\right\}}{\left\{(-1)^{\frac{3}{28}}+(-1)^{\frac{25}{28}}\right\}^3}$$

$$\frac{1}{5^3}-\frac{1}{23^3}+\frac{1}{33^3}-\frac{1}{51^3}+\frac{1}{61^3}-\frac{1}{79^3}+-\cdots=-\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{5}{28}}-(-1)^{\frac{23}{28}}\right\}}{\left\{(-1)^{\frac{5}{28}}+(-1)^{\frac{23}{28}}\right\}^3}$$

$$-\left(\frac{1}{7^3}-\frac{1}{21^3}+\frac{1}{35^3}-\frac{1}{49^3}+\frac{1}{63^3}-\frac{1}{77^3}+-\cdots\right)=\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{7}{28}}-(-1)^{\frac{21}{28}}\right\}}{\left\{(-1)^{\frac{7}{28}}+(-1)^{\frac{21}{28}}\right\}^3}$$

$$\frac{1}{9^3}-\frac{1}{19^3}+\frac{1}{37^3}-\frac{1}{47^3}+\frac{1}{65^3}-\frac{1}{75^3}+-\cdots=-\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{9}{28}}-(-1)^{\frac{19}{28}}\right\}}{\left\{(-1)^{\frac{9}{28}}+(-1)^{\frac{19}{28}}\right\}^3}$$

$$-\left(\frac{1}{11^3}-\frac{1}{17^3}+\frac{1}{39^3}-\frac{1}{45^3}+\frac{1}{67^3}-\frac{1}{73^3}+-\cdots\right)=\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{11}{28}}-(-1)^{\frac{17}{28}}\right\}}{\left\{(-1)^{\frac{11}{28}}+(-1)^{\frac{17}{28}}\right\}^3}$$

$$\frac{1}{13^3}-\frac{1}{15^3}+\frac{1}{41^3}-\frac{1}{43^3}+\frac{1}{69^3}-\frac{1}{71^3}+-\cdots = -\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{13}{28}}-(-1)^{\frac{15}{28}}\right\}}{\left\{(-1)^{\frac{13}{28}}+(-1)^{\frac{15}{28}}\right\}^3}$$

Reflection $m2^n$ -split of Dirichlet Series

Formula 6.5.1

When T_n is Chebyshev Polynomial of the 1st kind, the following expressions hold for natural numbers m, k.

$$\cos\frac{(2k-1)\pi}{m} = T_{2k-1}\left(\cos\frac{\pi}{m}\right) \quad , \quad \sin\frac{(2k-1)\pi}{m} = (-1)^{k-1}T_{2k-1}\left(\sin\frac{\pi}{m}\right)$$

Formula 6.5.2

The following expressions hold for natural number n.

$$\begin{cases} \cos\dfrac{\pi}{2^n} = \dfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} \\ \sin\dfrac{\pi}{2^n} = \dfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} \end{cases} \qquad (n-1)\text{-}nests$$

$$\begin{cases} \cos\dfrac{\pi}{3\cdot 2^n} = \dfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{3}}}} \\ \sin\dfrac{\pi}{3\cdot 2^n} = \dfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{3}}}} \end{cases} \qquad n\text{-}nests$$

$$\begin{cases} \cos\dfrac{\pi}{5\cdot 2^n} = \dfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\dfrac{1+\sqrt{5}}{2}}}}} \\ \sin\dfrac{\pi}{5\cdot 2^n} = \dfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\dfrac{1+\sqrt{5}}{2}}}}} \end{cases} \qquad n\text{-}nests$$

Theorem 6.5.3

The sum of reflection split series a_k $k=1,2,\cdots,2^n$ in Definition 6.1.3' is expressed with addition, multiplication and the nested radical of 2, except circle ratio π.

Corollary 6.5.3

(1) The sum of reflection split series a_k $k=1,2,\cdots,3\cdot 2^n$ is expressed with addition, multiplication and the nested radical of 2 and 3, except circle ratio π.

(2) The sum of reflection split series a_k $k=1,2,\cdots,5\cdot 2^n$ is expressed with addition, multiplication and the nested radical of 2 and 5, except circle ratio π.

Example 1: Reflection 8-split of $\lambda(2)$

Dirichlet lambda series $\lambda(2)$ is

$$\lambda(2) = 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{13^2}+\cdots$$

The reflection 8-split is as follows.

$$a_1 = 1 + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{63^2} + \frac{1}{65^2} + \frac{1}{95^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{61^2} + \frac{1}{67^2} + \frac{1}{93^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2+\sqrt{2-\sqrt{2}}}\right)}$$

$$a_3 = \frac{1}{5^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{59^2} + \frac{1}{69^2} + \frac{1}{91^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2-\sqrt{2-\sqrt{2}}}\right)}$$

$$a_4 = \frac{1}{7^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{57^2} + \frac{1}{71^2} + \frac{1}{89^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2-\sqrt{2+\sqrt{2}}}\right)}$$

$$a_5 = \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{41^2} + \frac{1}{55^2} + \frac{1}{73^2} + \frac{1}{87^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2-\sqrt{2+\sqrt{2}}}\right)}$$

$$a_6 = \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{43^2} + \frac{1}{53^2} + \frac{1}{75^2} + \frac{1}{85^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2-\sqrt{2-\sqrt{2}}}\right)}$$

$$a_7 = \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{45^2} + \frac{1}{51^2} + \frac{1}{77^2} + \frac{1}{83^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2+\sqrt{2-\sqrt{2}}}\right)}$$

$$a_8 = \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{47^2} + \frac{1}{49^2} + \frac{1}{79^2} + \frac{1}{81^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}$$

$$\lambda(2) = a_1 + a_2 + a_3 + \cdots + a_8$$

Example 2: Reflection 6-split of $\beta(3)$

Dirichlet beta series $\beta(3)$ is

$$\beta(3) = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{15^3} + - \cdots$$

The reflection 6-split is as follows.

$$a_1 = 1 - \frac{1}{23^3} + \frac{1}{25^3} - \frac{1}{47^3} + \frac{1}{49^3} - \frac{1}{71^3} + - \cdots$$
$$= \frac{\left(56+39\sqrt{2}+32\sqrt{3}+23\sqrt{6}\right)\pi^3}{6912}$$

$$a_2 = -\left(\frac{1}{3^3} - \frac{1}{21^3} + \frac{1}{27^3} - \frac{1}{45^3} + \frac{1}{51^3} - \frac{1}{69^3} + - \cdots\right) = \frac{\left(-4-3\sqrt{2}\right)\pi^3}{6912}$$

$$a_3 = \frac{1}{5^3} - \frac{1}{19^3} + \frac{1}{29^3} - \frac{1}{43^3} + \frac{1}{53^3} - \frac{1}{67^3} + - \cdots$$
$$= \frac{\left(56-39\sqrt{2}-32\sqrt{3}+23\sqrt{6}\right)\pi^3}{6912}$$

$$a_4 = -\left(\frac{1}{7^3} - \frac{1}{17^3} + \frac{1}{31^3} - \frac{1}{41^3} + \frac{1}{55^3} - \frac{1}{65^3} + - \cdots\right)$$
$$= \frac{\left(56+39\sqrt{2}-32\sqrt{3}-23\sqrt{6}\right)\pi^3}{6912}$$

$$a_5 = \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{33^3} - \frac{1}{39^3} + \frac{1}{57^3} - \frac{1}{63^3} + - \cdots = \frac{\left(-4+3\sqrt{2}\right)\pi^3}{6912}$$

$$a_6 = -\left(\frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{35^3} - \frac{1}{37^3} + \frac{1}{59^3} - \frac{1}{61^3} + - \cdots\right)$$

$$= \frac{\left(56-39\sqrt{2}+32\sqrt{3}-23\sqrt{6}\right)\pi^3}{6912}$$

$$a_1 + a_2 + \cdots + a_6 = \beta(3)$$

Reflection $m2^n$-split of $\lambda(2)$

Dirichlet lambda series $\lambda(2)$ is as follows.

$$\lambda(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots$$

If 2-split, 4-split, 6-split, 8-split and 10-split of this are calculated, it is as follows.

2-split

$$a_1 = 1 + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{23^2} + \cdots = \frac{\pi^2}{16\left(2-\sqrt{2}\right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{21^2} + \cdots = \frac{\pi^2}{16\left(2+\sqrt{2}\right)}$$

4-split

$$a_1 = 1 + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{47^2} + \cdots = \frac{\pi^2}{64\left(2-\sqrt{2+\sqrt{2}}\right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{45^2} + \cdots = \frac{\pi^2}{64\left(2-\sqrt{2-\sqrt{2}}\right)}$$

$$a_3 = \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{43^2} + \cdots = \frac{\pi^2}{64\left(2+\sqrt{2-\sqrt{2}}\right)}$$

$$a_4 = \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{41^2} + \cdots = \frac{\pi^2}{64\left(2+\sqrt{2+\sqrt{2}}\right)}$$

6-split

$$a_1 = 1 + \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{47^2} + \frac{1}{49^2} + \frac{1}{71^2} + \cdots = \frac{\pi^2}{144\left(2-\sqrt{2+\sqrt{3}}\right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{45^2} + \frac{1}{51^2} + \frac{1}{69^2} + \cdots = \frac{\pi^2}{3^2\,16\left(2-\sqrt{2}\right)}$$

$$a_3 = \frac{1}{5^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{43^2} + \frac{1}{53^2} + \frac{1}{67^2} + \cdots = \frac{\pi^2}{144\left(2-\sqrt{2-\sqrt{3}}\right)}$$

$$a_4 = \frac{1}{7^2} + \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{41^2} + \frac{1}{55^2} + \frac{1}{65^2} + \cdots = \frac{\pi^2}{144\left(2+\sqrt{2-\sqrt{3}}\right)}$$

$$a_5 = \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{33^2} + \frac{1}{39^2} + \frac{1}{57^2} + \frac{1}{63^2} + \cdots = \frac{\pi^2}{3^2\,16\left(2+\sqrt{2}\right)}$$

$$a_6 = \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{35^2} + \frac{1}{37^2} + \frac{1}{59^2} + \frac{1}{61^2} + \cdots = \frac{\pi^2}{144\left(2+\sqrt{2+\sqrt{3}}\right)}$$

Where, a_2, a_5 are $1/3^2$ times the reflection 2-split series of $\lambda(2)$.

8-split

$$a_1 = 1 + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{63^2} + \frac{1}{65^2} + \frac{1}{95^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{61^2} + \frac{1}{67^2} + \frac{1}{93^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2+\sqrt{2-\sqrt{2}}}\right)}$$

$$a_3 = \frac{1}{5^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{59^2} + \frac{1}{69^2} + \frac{1}{91^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2-\sqrt{2-\sqrt{2}}}\right)}$$

$$a_4 = \frac{1}{7^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{57^2} + \frac{1}{71^2} + \frac{1}{89^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2-\sqrt{2+\sqrt{2}}}\right)}$$

$$a_5 = \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{41^2} + \frac{1}{55^2} + \frac{1}{73^2} + \frac{1}{87^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2-\sqrt{2+\sqrt{2}}}\right)}$$

$$a_6 = \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{43^2} + \frac{1}{53^2} + \frac{1}{75^2} + \frac{1}{85^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2-\sqrt{2-\sqrt{2}}}\right)}$$

$$a_7 = \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{45^2} + \frac{1}{51^2} + \frac{1}{77^2} + \frac{1}{83^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2+\sqrt{2-\sqrt{2}}}\right)}$$

$$a_8 = \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{47^2} + \frac{1}{49^2} + \frac{1}{79^2} + \frac{1}{81^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}$$

10-split

$$a_1 = 1 + \frac{1}{39^2} + \frac{1}{41^2} + \frac{1}{79^2} + \frac{1}{81^2} + \frac{1}{119^2} + \cdots = \frac{\pi^2}{400\left(2-\sqrt{2+\sqrt{2+\frac{1+\sqrt{5}}{2}}}\right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{37^2} + \frac{1}{43^2} + \frac{1}{77^2} + \frac{1}{83^2} + \frac{1}{117^2} + \cdots = \frac{\pi^2}{400\left(2-\sqrt{2+\sqrt{2+\frac{1-\sqrt{5}}{2}}}\right)}$$

$$a_3 = \frac{1}{5^2} + \frac{1}{35^2} + \frac{1}{45^2} + \frac{1}{75^2} + \frac{1}{85^2} + \frac{1}{115^2} + \cdots = \frac{\pi^2}{5^2\,16\left(2-\sqrt{2}\right)}$$

$$a_4 = \frac{1}{7^2} + \frac{1}{33^2} + \frac{1}{47^2} + \frac{1}{73^2} + \frac{1}{87^2} + \frac{1}{113^2} + \cdots = \frac{\pi^2}{400\left(2-\sqrt{2-\sqrt{2+\frac{1-\sqrt{5}}{2}}}\right)}$$

$$a_5 = \frac{1}{9^2} + \frac{1}{31^2} + \frac{1}{49^2} + \frac{1}{71^2} + \frac{1}{89^2} + \frac{1}{111^2} + \cdots = \frac{\pi^2}{400\left(2-\sqrt{2-\sqrt{2+\frac{1+\sqrt{5}}{2}}}\right)}$$

$$a_6 = \frac{1}{11^2} + \frac{1}{29^2} + \frac{1}{51^2} + \frac{1}{69^2} + \frac{1}{91^2} + \frac{1}{109^2} + \cdots = \frac{\pi^2}{400\left(2+\sqrt{2-\sqrt{2+\frac{1+\sqrt{5}}{2}}}\right)}$$

$$a_7 = \frac{1}{13^2} + \frac{1}{27^2} + \frac{1}{53^2} + \frac{1}{67^2} + \frac{1}{93^2} + \frac{1}{107^2} + \cdots = \frac{\pi^2}{400\left(2+\sqrt{2-\sqrt{2+\frac{1-\sqrt{5}}{2}}}\right)}$$

$$a_8 = \frac{1}{15^2} + \frac{1}{25^2} + \frac{1}{55^2} + \frac{1}{65^2} + \frac{1}{95^2} + \frac{1}{105^2} + \cdots = \frac{\pi^2}{5^2\,16\left(2+\sqrt{2}\right)}$$

$$a_9 = \frac{1}{17^2} + \frac{1}{23^2} + \frac{1}{57^2} + \frac{1}{63^2} + \frac{1}{97^2} + \frac{1}{103^2} + \cdots = \frac{\pi^2}{400\left(2+\sqrt{2+\sqrt{2+\frac{1-\sqrt{5}}{2}}}\right)}$$

$$a_{10} = \frac{1}{19^2} + \frac{1}{21^2} + \frac{1}{59^2} + \frac{1}{61^2} + \frac{1}{99^2} + \frac{1}{101^2} + \cdots = \frac{\pi^2}{400\left(2+\sqrt{2+\sqrt{2+\frac{1+\sqrt{5}}{2}}}\right)}$$

Where, a_2 , a_5 are $1/5^2$ times the reflection 2-split series of $\lambda(2)$.

Remark

The denominators are not rationalized purposely. If this expression is used, the sum of each split series is determined by the combination of $+$ and $-$ in the nested square roots. However, in 6-split and 10-split, it is impossible to represent all the sums with only these combinations. Interestingly, the shortags are filled up by reflection 2-split series of $\lambda(2)$. So, the following thorem holds.

Theorem 6.6.1

Let p be a prime number greater than 2 and reflection $2p$ -split series a_k of $\lambda(2)$ are as follows.

$$a_k = \sum_{r=0}^{\infty}\left\{\frac{1}{(8pr+2k-1)^2} + \frac{1}{(8pr+8p-2k+1)^2}\right\} \qquad k=1,2,\cdots,2p$$

$$a_1 + a_2 + \cdots + a_{2p} = \lambda(2)$$

Then, following expressions hold.

$$a_{\frac{p+1}{2}} = \frac{\pi^2}{p^2 16\left(2-\sqrt{2}\right)} \quad , \quad a_{\frac{3p+1}{2}} = \frac{\pi^2}{p^2 16\left(2+\sqrt{2}\right)}$$

Example: 14-split

$$a_4 = \frac{1}{7^2} + \frac{1}{49^2} + \frac{1}{63^2} + \frac{1}{105^2} + \frac{1}{119^2} + \frac{1}{161^2} + \cdots = \frac{\pi^2}{7^2\,16\left(2-\sqrt{2}\right)}$$

$$a_{11} = \frac{1}{21^2} + \frac{1}{35^2} + \frac{1}{77^2} + \frac{1}{91^2} + \frac{1}{133^2} + \frac{1}{147^2} + \cdots = \frac{\pi^2}{7^2\,16\left(2+\sqrt{2}\right)}$$

07 Zeros of p-series

Formula 7.1.1

The following expression holds for the Riemann Zeta Function $\zeta(z)\ \ (z = x + iy)$.

$$\zeta(z) = \sum_{s=1}^{\infty} \frac{1}{s^z} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots \qquad x > 0\ ,\ x \neq 1$$

Proof

$$\zeta(z) = \frac{1}{1-2^{1-z}}\eta(z) = \left(\frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \frac{1}{5^z} - \frac{1}{6^z} + - \cdots\right) \bigg/ \left(\frac{1}{1^z} - \frac{2}{2^z}\right) \qquad x > 0\ ,\ x \neq 1$$

When this is calculated by hand,

$$\frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots$$

$$\frac{1}{1^z} - \frac{2}{2^z}\ \Big)\ \frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \frac{1}{5^z} - \frac{1}{6^z} + - \cdots$$

$$\frac{1}{1^z} - \frac{2}{2^z}$$

$$\frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \frac{1}{5^z} - \frac{1}{6^z} + - \cdots$$

$$\frac{1}{2^z} - \frac{2}{4^z}$$

$$\frac{1}{3^z} + \frac{1}{4^z} + \frac{1}{5^z} - \frac{1}{6^z} + \frac{1}{7^z} - \frac{1}{8^z} + - \cdots$$

$$\frac{1}{3^z} - \frac{2}{6^z}$$

$$\frac{1}{4^z} + \frac{1}{5^z} + \frac{1}{6^z} + \frac{1}{7^z} - \frac{1}{8^z} + \frac{1}{9^z} - \frac{1}{10^z} + - \cdots$$

$$\frac{1}{4^z} - \frac{2}{8^z}$$

$$\frac{1}{5^z} + \frac{1}{6^z} + \frac{1}{7^z} + \frac{1}{8^z} + \frac{1}{9^z} - \frac{1}{10^z} + \frac{1}{11^z} - \frac{1}{12^z} + - \cdots$$

$$\vdots$$

Convergence Acceleration of p-series

Formula 7.2.1ri (Knopp Transformation)

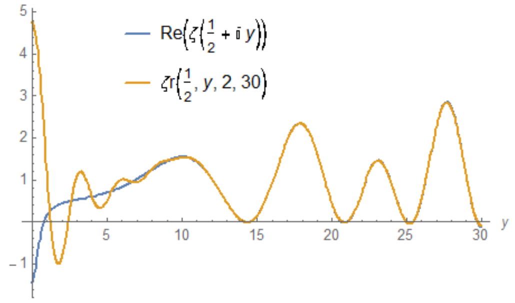
When the real and imaginary parts of Riemann zeta function $\zeta(x,y)$ are ζ_r , ζ_i respectively, the following expressions hold for $x > 0\ ,\ x \neq 1$.

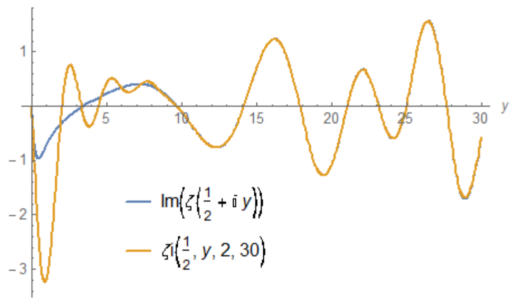
$$\zeta_r(x,y,q) = \sum_{k=1}^{\infty} \sum_{s=1}^{k} \frac{q^{k-s}}{(q+1)^{k+1}} \binom{k}{s} \frac{\cos(y \log s)}{s^x}$$

$$\zeta_i(x,y,q) = -\sum_{k=1}^{\infty} \sum_{s=1}^{k} \frac{q^{k-s}}{(q+1)^{k+1}} \binom{k}{s} \frac{\sin(y \log s)}{s^x}$$

Where, q is an arbitrary positive number.

The 2D figure on the critical line $x=1/2$ at $0 \leq y \leq 30$ is drawn as follows. The left is the real part and the right is the imaginary part. In both figures, blue is the left side and orange is the right side.



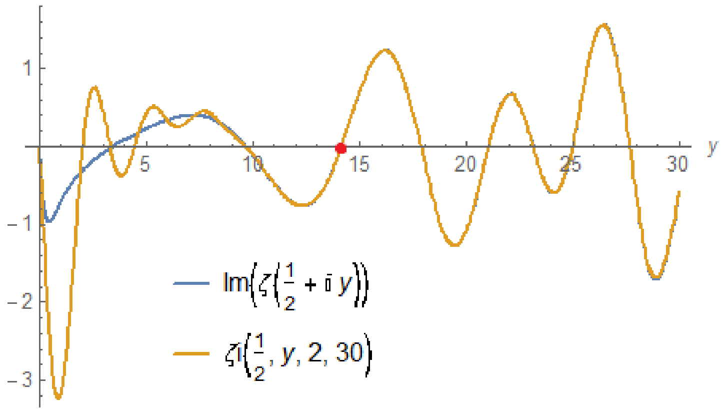


The differences between the p-series and $Re\{\zeta(1/2+iy)\}, Im\{\zeta(1/2+iy)\}$ are large where y is small, but the p-series approache $Re\{\zeta(1/2+iy)\}, Im\{\zeta(1/2+iy)\}$ quickly where y is large. And, they approach $Re\{\zeta(1/2+iy)\}, Im\{\zeta(1/2+iy)\}$ indefinitely as $|y|$ increases.

Zeros of p-series

The 1-st zero point

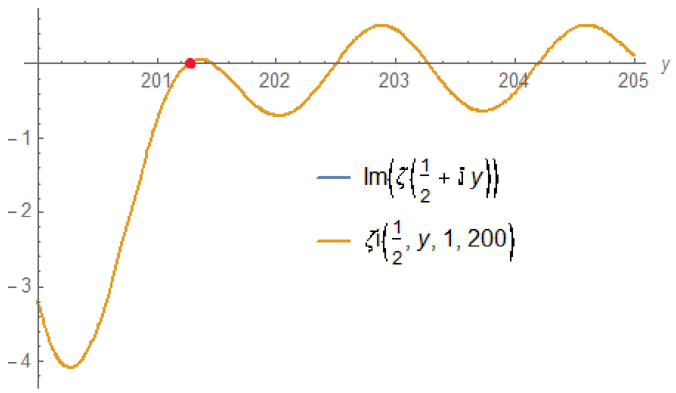
The imaginary part on the critical line $x=1/2$ at $0 \leq y \leq 30$ is drawn as follows. Blue is the left side and orange is the right side.



The 1-st zero of $\zeta(1/2+iy)$ coincides with the 2-nd uphill zero (red dot) of the imaginary part. So, when the zeros on both sides near $y=14$ were calculated, the significant 4 digits were obtained. But no better approximation was obtained.

The 80-th zero point

The imaginary part on the critical line $x=1/2$ at $200 \leq y \leq 205$ is drawn as follows. Blue is the left side and orange is the right side, but both sides overlap exactly and blue (left side) is not visible.



The 80-th zero point of $\zeta(1/2+iy)$ coincides with (perhaps) the 81-th uphill zero point (red dot) of the imaginary part.
So, when the zeros on both sides near $y=201$ were calculated, the following significant 15 digits were obtained.

```
SetPrecision[FindRoot[ζi[1/2, y, 1, 200], {y, 201}], 16]
{ y → 201.2647519437039 }

SetPrecision[Im[ZetaZero[80]], 16]
    201.2647519437038
```

Conclusion

At present, we must consider Formula 7.2.1ri as an asymptotic expansion in the critical strip $0<x<1$.
However, the calculation accuracy increases as the imaginary part $|y|$ of the independent variable increases.
As of 2004, the first 10 trillions on the critical line are known to satisfy the Riemann hypothesis.
Since the imaginary part $|y_r|$ after the10th trillion are very large, Formula 7.2.1ri is sufficiently useful even in the critical strip $0<x<1$.

09 Power Series of Riemann Zeta etc by Real & Imaginary Parts

In the following, $0^0 = 1$.

Formula 9.1.1 (Maclaurin series of $\eta(z)$)

When the Dirichlet eta function is $\eta(z)$ ($z = x+iy$), the following expressions hold on the half plane $Re(z) > 0$

$$\eta(z) = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t \log^s(t+1)\frac{(-1)^s z^s}{s!}$$

$$Re\{\eta(z)\} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t \log^{2r+s}(t+1)\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$Im\{\eta(z)\} = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t \log^{2r+s+1}(t+1)\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Formula 9.2.1 (Maclaurin series of $\beta(z)$)

When the Dirichlet beta function is $\beta(z)$ ($z = x+iy$), the following expressions hold on the half plane $Re(z) > 0$

$$\beta(z) = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t \log^s(2t+1)\frac{(-1)^s z^s}{s!}$$

$$Re\{\beta(z)\} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t \log^{2r+s}(2t+1)\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$Im\{\beta(z)\} = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t \log^{2r+s+1}(2t+1)\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Formula 9.3.1 (Laurent series of $\zeta(z)$)

When the Riemann zeta function is $\zeta(z)$ ($z = x+iy$) and Stieltjes constansts are γ_s $s = 0, 1, 2, \cdots$,
the following expressions hold on the whole complex plane except $z = 1$.

$$\zeta(z) = \frac{1}{z-1} + \sum_{s=0}^{\infty}\gamma_s\frac{(-1)^s(z-1)^s}{s!}$$

$$Re\{\zeta(z)\} = \frac{x-1}{(x-1)^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\gamma_{2r+s}\frac{(-1)^s(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$Im\{\zeta(z)\} = -\frac{y}{(x-1)^2+y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\gamma_{2r+s+1}\frac{(-1)^s(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Formula 9.4.1 (Taylor series of $(z-1)\zeta(z)$)

When the Riemann zeta function is $\zeta(z)$ ($z = x+iy$) and Stieltjes constansts are γ_s $s = 0, 1, 2, \cdots$,
the following expressions hold on the whole complex plane.

$$(z-1)\zeta(z) = 1 - \sum_{s=1}^{\infty}s\,\gamma_{s-1}\frac{(-1)^s(z-1)^s}{s!}$$

$$Re\{(z-1)\zeta(z)\} = 1 - \sum_{s=1}^{\infty}s\,\gamma_{s-1}\frac{(-1)^s(x-1)^s}{s!} - \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}(2r+s)\,\gamma_{2r+s-1}\frac{(-1)^s(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$Im\{(z-1)\zeta(z)\} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+1)\,\gamma_{2r+s}\frac{(-1)^s(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

10 Dirichlet Series & Taylor Series

Abstract

(1) Dirichlet series can be converted to Taylor series within its convergence area.

(2) If the coefficients of the Dirichlet series and the center of the Taylor expansion are limited to real numbers, the Taylor series for each real part and imaginary part can be obtained.

(3) Hyperbolic functions are represented by finite or infinite General Dirichlet Series.

10.1 General Dirichlet Series & Taylor Series

Definition 10.1.0 (General Dirichlet Series)

When R is a real number set, $\lambda_t \in R$ $s.t.$ $\lambda_t < \lambda_{t+1}$ $t=1,2,3,\cdots$ and a_t are complex numbers, we call the following series **General Dirichlet Series**.

$$f(z) = \sum_{t=1}^{\infty} a_t e^{-\lambda_t z}$$

Formula 10.1.1 **(General Dirichlet Series ---> Taylor Series)**

When a function $f(z)$ that is holomorphic on domain D is expanded into a general Dirichlet series, the following expression holds for arbitrary complex numbers z, c belonging to the convergence area.

$$f(z) = \sum_{t=1}^{\infty} a_t e^{-\lambda_t z} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t} (-\lambda_t)^s \frac{(z-c)^s}{s!}$$

Example1 $coth z$ **- 1**

$$coth\, z - 1 = \sum_{t=1}^{\infty} 2\, e^{-2tz} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} 2e^{-2tc} (-2t)^s \frac{(z-c)^s}{s!}$$

Example2 $tanh z$ **- 1**

$$tanh\, z - 1 = \sum_{t=1}^{\infty} 2(-1)^t e^{-2tz} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} 2(-1)^t e^{-2tc} (-2t)^s \frac{(z-c)^s}{s!}$$

Example3 $csch z$

$$csch\, z = \sum_{t=1}^{\infty} 2e^{-(2t-1)z} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} 2e^{-c(2t-1)} \{-(2t-1)\}^s \frac{(z-c)^s}{s!}$$

Example4 $sech z$

$$sech\, z = \sum_{t=1}^{\infty} 2(-1)^{t-1} e^{-(2t-1)z} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} 2(-1)^{t-1} e^{-c(2t-1)} \{-(2t-1)\}^s \frac{(z-c)^s}{s!}$$

Formula 10.1.2 **(Taylor series by real & imaginary parts)**

When $f(z) = \sum_{t=1}^{\infty} a_t e^{-\lambda_t z}$ ($z = x+iy$) is general Dirichlet serie, u, v are real and imaginary parts of $f(z)$ and c, a_t $t=1,2,3,\cdots$ are arbitrary real numbers, the following expressions hold in the convergence area.

$$f(z) = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t} (-\lambda_t)^s \frac{(z-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t} (-\lambda_t)^{2r+s} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} a_t e^{-c\lambda_t} (-\lambda_t)^{2r+s+1} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

10.2 Finite General Dirichlet Series & Taylor Series

Formula 10.2.1 (Finite General Dirichlet Series)

When a function $f(z)$ that is holomorphic on domain D is expanded into a finite general Dirichlet series, the following expression holds for arbitrary complex numbers z, c

$$f(z) = \sum_{t=1}^{n} a_t e^{-\lambda_t z} = \sum_{s=0}^{\infty} \sum_{t=1}^{n} a_t e^{-c\lambda_t} (-\lambda_t)^s \frac{(z-c)^s}{s!}$$

Example1 $\cosh z$

$$\cosh z = \sum_{t=1}^{2} \frac{1}{2} e^{-(-1)^t z} = \sum_{s=0}^{\infty} \sum_{t=1}^{2} \frac{1}{2} e^{-c(-1)^t} \{-(-1)^t\}^s \frac{(z-c)^s}{s!}$$

Example2 $\sinh z$

$$\sinh z = \sum_{t=1}^{2} \frac{(-1)^{t-1}}{2} e^{-(-1)^t z} = \sum_{s=0}^{\infty} \sum_{t=1}^{2} \frac{(-1)^{t-1}}{2} e^{-c(-1)^t} \{-(-1)^t\}^s \frac{(z-c)^s}{s!}$$

Formula 10.2.2 (Taylor series by real & imaginary parts)

When $f(z) = \sum_{t=1}^{n} a_t e^{-\lambda_t z}$ ($z = x+iy$) is finite general Dirichlet serie, u, v are real and imaginary parts of $f(z)$ and c, a_t $t=1,2,3,\cdots$ are arbitrary real numbers, the following expressions hold.

$$f(z) = \sum_{s=0}^{\infty} \sum_{t=1}^{n} a_t e^{-c\lambda_t} (-\lambda_t)^s \frac{(z-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{n} a_t e^{-c\lambda_t} (-\lambda_t)^{2r+s} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{n} a_t e^{-c\lambda_t} (-\lambda_t)^{2r+s+1} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

10.3 Ordinary Dirichlet Series & Taylor Series

Definition 10.3.0 (Ordinary Dirichlet Series)

When z, a_n $(n=1,2,3,\cdots)$ are complex numbers, we call the following **Ordinary Dirichlet Series**.

$$f(z) = \sum_{t=1}^{\infty} \frac{a_t}{t^z} = \frac{a_1}{1^z} + \frac{a_2}{2^z} + \frac{a_3}{3^z} + \frac{a_4}{4^z} + \cdots$$

Formula 10.3.1 (Ordinary Dirichlet Series ---> Taylor Series)

When a function $f(z)$ that is holomorphic on domain D is expanded into a ordinary Dirichlet series, the following expression holds for arbitrary complex numbers z, c belonging to the convergence area.

$$f(z) = \sum_{t=1}^{\infty} \frac{a_t}{t^z} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{a_t}{t^c} (-\log t)^s \frac{(z-c)^s}{s!}$$

Example: Dirichlet Eta Series

$$\eta(z) = \sum_{t=1}^{\infty} \frac{(-1)^t}{t^z} = \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{(-1)^t}{t^c} (-\log t)^s \frac{(z-c)^s}{s!}$$

Formula 10.3.2 **(Taylor series by real & imaginary parts)**

When $f(z)=\sum_{t=1}^{\infty} a_t/t^z$ ($z=x+iy$) is ordinary Dirichlet serie, u,v are real and imaginary parts of $f(z)$ and c , a_t $t=1,2,3,\cdots$ are arbitrary real numbers, the following expressions hold in the convergence area.

$$f(z) = \sum_{s=0}^{\infty}\sum_{t=1}^{\infty}\frac{a_t}{t^c}(-\log t)^s\frac{(z-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{\infty}\frac{a_t}{t^c}(-\log t)^{2r+s}\frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{\infty}\frac{a_t}{t^c}(-\log t)^{2r+s+1}\frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Example: Dirichlet Beta Series

$$\beta(z) = \sum_{s=0}^{\infty}\sum_{t=1}^{\infty}\frac{(-1)^{t-1}}{(2t-1)^c}\{-\log(2t-1)\}^s\frac{(z-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{\infty}\frac{(-1)^{t-1}}{(2t-1)^c}\{-\log(2t-1)\}^{2r+s}\frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{\infty}\frac{(-1)^{t-1}}{(2t-1)^c}\{-\log(2t-1)\}^{2r+s+1}\frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

10.4 Finite Ordinary Dirichlet Series & Taylor Series

Formula 10.4.1 **(Finite Ordinary Dirichlet Series ---> Taylor Series)**

When a function $f(z)$ that is holomorphic on domain D is expanded into a ordinary Dirichlet series, the following expression holds for arbitrary complex numbers z,c belonging to the convergence area.

$$f(z) = \sum_{t=1}^{n}\frac{a_t}{t^z} = \sum_{s=0}^{\infty}\sum_{t=1}^{n}\frac{a_t}{t^c}(-\log t)^s\frac{(z-c)^s}{s!}$$

Example $a_t = (-1)^{t-1}$ **,** $n=6$

$$f(z) = \sum_{t=1}^{6}\frac{(-1)^{t-1}}{t^z} = \sum_{s=0}^{\infty}\sum_{t=1}^{6}\frac{(-1)^{t-1}}{t^c}(-\log t)^s\frac{(z-c)^s}{s!}$$

Formula 10.4.2 **(Taylor series by real & imaginary parts)**

When $f(z)=\sum_{t=1}^{n} a_t/t^z$ ($z=x+iy$) is finite ordinary Dirichlet serie, u,v are real and imaginary parts of $f(z)$ and c , a_t $t=1,2,3,\cdots$ are arbitrary real numbers, the following expressions hold.

$$f(z) = \sum_{s=0}^{\infty}\sum_{t=1}^{n}\frac{a_t}{t^c}(-\log t)^s\frac{(z-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{n}\frac{a_t}{t^c}(-\log t)^{2r+s}\frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{n}\frac{a_t}{t^c}(-\log t)^{2r+s+1}\frac{(x-c)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Example $\boldsymbol{a_t = (-1)^{t-1} t \ , n = 2}$

Finite ordinary Dirichlet series is

$$f(z) = \sum_{t=1}^{2} \frac{(-1)^{t-1} t}{t^z} \qquad \left(= \frac{1}{1^z} - \frac{2}{2^z} = 1 - 2^{1-z} \right)$$

Taylor series by real & imaginary parts are

$$f(z) = \sum_{s=0}^{\infty} \sum_{t=1}^{2} \frac{(-1)^{t-1} t}{t^c} (-\log t)^s \frac{(z-c)^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{2} \frac{(-1)^{t-1} t}{t^c} (-\log t)^{2r+s} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{2} \frac{(-1)^{t-1} t}{t^c} (-\log t)^{2r+s+1} \frac{(x-c)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

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