List of Theorems and Formulas (Infinite-degree Equation)

01 Power of Infinite Series

1.1 Multiple Cauchy Product

Formula 1.1.1 (Multiple Cauchy product of infinite series)

The following expressions hold for two or more convergent infinite series.

$$\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty} b_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r-s} b_s$$

$$\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty} b_r\right)\left(\sum_{r=0}^{\infty} c_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s} b_{s-t} c_t$$

$$\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty} b_r\right)\left(\sum_{r=0}^{\infty} c_r\right)\left(\sum_{r=0}^{\infty} d_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} a_{r-s} b_{s-t} c_{t-u} d_u$$

$$\vdots$$

$$\left(\sum_{r=0}^{\infty} a_{1,r}\right)\left(\sum_{r=0}^{\infty} a_{2,r}\right)\cdots\left(\sum_{r=0}^{\infty} a_{n,r}\right) = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{1,r_1-r_2} a_{2,r_2-r_3}\cdots a_{n-1,r_{n-1}-r_n} a_{n,r_n}$$

Especially,

$$\left(\sum_{r=0}^{\infty} a_r\right)\left\{\sum_{r=0}^{\infty} (-1)^r a_r\right\} = \sum_{r=0}^{\infty}\sum_{s=0}^{2r} (-1)^s a_s a_{2r-s}$$

Formula 1.2.1 (Multiple Cauchy product of power series)

The following expressions hold for two or more convergent power series.

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{r=0}^{\infty} b_r z^r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r-s} b_s z^r$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{r=0}^{\infty} b_r z^r\right)\left(\sum_{r=0}^{\infty} c_r z^r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s} b_{s-t} c_t z^r$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{r=0}^{\infty} b_r z^r\right)\left(\sum_{r=0}^{\infty} c_r z^r\right)\left(\sum_{r=0}^{\infty} d_r z^r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} a_{r-s} b_{s-t} c_{t-u} d_u z^r$$

$$\vdots$$

$$\left(\sum_{r=0}^{\infty} a_{1,r} z^r\right)\left(\sum_{r=0}^{\infty} a_{2,r} z^r\right)\cdots\left(\sum_{r=0}^{\infty} a_{n,r} z^r\right) = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{1,r_1-r_2} a_{2,r_2-r_3}\cdots a_{n-1,r_{n-1}-r_n} a_{n,r_n} z^{r_1}$$

Especially,

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)\left\{\sum_{r=0}^{\infty} (-1)^r a_r z^r\right\} = \sum_{r=0}^{\infty}\sum_{s=0}^{2r} (-1)^s a_s a_{2r-s} z^{2r}$$

1.2 Power of Infinite Series (Part1)

Formula 1.2.1 (Power of infinite series)

The following expressions hold for convergent infinite series.

$$\left(\sum_{r=0}^{\infty} a_r\right)^2 = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r-s} a_s$$

$$\left(\sum_{r=0}^{\infty} a_r\right)^3 = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s} a_{s-t} a_t$$

$$\left(\sum_{r=0}^{\infty} a_r\right)^4 = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} a_{r-s} a_{s-t} a_{t-u} a_u$$

$$\vdots$$

$$\left(\sum_{r=0}^{\infty} a_r\right)^n = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{r_1-r_2}a_{r_2-r_3}\cdots a_{r_{n-1}-r_n}a_{r_n}$$

Formula 1.2.2 (Power of power series)

The following expressions hold for convergent power series.

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^2 = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r-s}a_s z^r$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^3 = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s}a_{s-t}a_t z^r$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^4 = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} a_{r-s}a_{s-t}a_{t-u}a_u z^r$$

$$\vdots$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^n = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{r_1-r_2}a_{r_2-r_3}\cdots a_{r_{n-1}-r_n}a_{r_n} z^r$$

1.3 Power of Infinite Series (Part2)

The following two formulas are important for simplifying the expressions in the below sections.

Formula 1.3.1 (Power of infinite series)

The following expressions hold for convergent infinite series.

$$\left(\sum_{r=0}^{\infty} a_r\right)^2 = \sum_{r=0}^{\infty} a_r^2 + 2\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s$$

$$\left(\sum_{r=0}^{\infty} a_r\right)^3 = \sum_{r=0}^{\infty} a_r^3 + 3\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s\right) - 3\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty} a_r a_s a_t$$

$$\left(\sum_{r=0}^{\infty} a_r\right)^4 = 2\sum_{r=0}^{\infty} a_r^4 - \left(\sum_{r=0}^{\infty} a_r^2\right)^2 + 4\left(\sum_{r=0}^{\infty} a_r\right)^2\left(\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s\right)$$
$$- 8\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty} a_r a_s a_t\right) + 8\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty} a_r a_s a_t a_u$$

Formula 1.3.2 (Power of power series)

The following expressions hold for convergent power series.

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^2 = \sum_{r=0}^{\infty} a_r^2 z^{2r} + 2\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s z^{r+s}$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^3 = \sum_{r=0}^{\infty} a_r^3 z^{3r} + 3\left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s z^{r+s}\right) - 3\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty} a_r a_s a_t z^{r+s+t}$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^4 = 2\sum_{r=0}^{\infty} a_r^4 z^{4r} - \left(\sum_{r=0}^{\infty} a_r^2 z^{2r}\right)^2 + 4\left(\sum_{r=0}^{\infty} a_r z^r\right)^2\left(\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s z^{r+s}\right)$$
$$- 8\left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty} a_r a_s a_t z^{r+s+t}\right) + 8\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty} a_r a_s a_t a_u z^{r+s+t+u}$$

02 Infinite-degree Equation with Integers as Roots

Symbols used in this chapter

In this chapter, $\psi_r(z)$ is the polygamma function, $B_{r,k}(f_1, f_2, \dots)$ are Bell polynomials, γ is Euler-Mascheroni constant and a_r, b_r are the following constants.

$$a_r = \sum_{k=1}^{r} (-1)^k B_{r,k}\left(\psi_0(1), \psi_1(1), \dots, \psi_{r-1}(1)\right) \qquad r=1,2,3,\cdots$$

$$b_r = \sum_{k=1}^{r} (-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \dots, \psi_{r-1}\left(\frac{1}{2}\right)\right) \qquad r=1,2,3,\cdots$$

2.1 Infinite-degree Equation with Integers as Roots (Part1)

Formula 2.1.1

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)e^{-\frac{z}{r}} = 1 + \sum_{r=1}^{\infty}(-1)^r\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s \gamma^{r-s}}{s!\,(r-s)!}\right\}z^r$$

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{r}\right)e^{\frac{z}{r}} = 1 + \sum_{r=1}^{\infty}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s \gamma^{r-s}}{s!\,(r-s)!}\right\}z^r$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r}\right)e^{-\frac{z}{2r}} = 1 + \sum_{r=1}^{\infty}\frac{(-1)^r}{2^r}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s \gamma^{r-s}}{s!\,(r-s)!}\right\}z^r$$

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{2r}\right)e^{\frac{z}{2r}} = 1 + \sum_{r=1}^{\infty}\frac{1}{2^r}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s \gamma^{r-s}}{s!\,(r-s)!}\right\}z^r$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)e^{-\frac{z}{2r-1}} = 1+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!\,(r-s)!}\right\}z^r$$

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{2r-1}\right)e^{\frac{z}{2r-1}} = 1+\sum_{r=1}^{\infty}\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!\,(r-s)!}\right\}z^r$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)e^{-\frac{z}{2r}} = 1+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{\gamma^r}{(2r)!!}+\sum_{s=1}^{r}\frac{(-1)^s b_s \gamma^{r-s}}{(2s)!!\,(2r-2s)!!}\right\}z^r$$

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{2r-1}\right)e^{\frac{z}{2r}} = 1+\sum_{r=1}^{\infty}\left\{\frac{\gamma^r}{(2r)!!}+\sum_{s=1}^{r}\frac{(-1)^s b_s \gamma^{r-s}}{(2s)!!\,(2r-2s)!!}\right\}z^r$$

2.2 Infinite-degree Equation with Integers as Roots (Part2)

Formula 2.2.1

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)\left(1-\frac{z}{r}\right) = \sum_{r=0}^{\infty}\frac{(-1)^r \pi^{2r}}{(2r+1)!}z^{2r} \qquad \left(=\frac{\sin \pi z}{\pi z}\right)$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r}\right)\left(1-\frac{z}{2r}\right) = \sum_{r=0}^{\infty}\frac{(-1)^r \pi^{2r}}{2^{2r}(2r+1)!}z^{2r} \qquad \left(=\frac{\sin(\pi z/2)}{\pi z/2}\right)$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)\left(1-\frac{z}{2r-1}\right) = \sum_{r=0}^{\infty}\frac{(-1)^r \pi^{2r}}{(4r)!!}z^{2r} \qquad \left(=\cos\frac{\pi z}{2}\right)$$

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)\left(1-\frac{z}{2r}\right)=1+\frac{b_1-a_1}{2!!}z^1$$
$$+\sum_{r=2}^{\infty}\left\{\frac{b_r+(-1)^r a_r}{(2r)!!}+\sum_{s=1}^{r-1}\frac{(-1)^{r-s}b_s a_{r-s}}{(2s)!!\{2(r-s)\}!!}\right\}z^r$$
$$\prod_{r=1}^{\infty}\left(1-\frac{z}{2r-1}\right)\left(1+\frac{z}{2r}\right)=1+\frac{a_1-b_1}{2!!}z^1$$
$$+\sum_{r=2}^{\infty}\left\{\frac{a_r+(-1)^r b_r}{(2r)!!}+\sum_{s=1}^{r-1}\frac{(-1)^{r-s}a_s b_{r-s}}{(2s)!!\{2(r-s)\}!!}\right\}z^r$$

2.3 Infinite-degree Equation with Imaginary Integers as Roots (Part1)

Formula 2.3.1

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{ir}\right)e^{-\frac{z}{ir}}=1+\sum_{r=1}^{\infty}i^r\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r$$
$$\prod_{r=1}^{\infty}\left(1-\frac{z}{ir}\right)e^{\frac{z}{ir}}=1+\sum_{r=1}^{\infty}(-i)^r\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r$$
$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2ir}\right)e^{-\frac{z}{2ir}}=1+\sum_{r=1}^{\infty}\frac{i^r}{2^r}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r$$
$$\prod_{r=1}^{\infty}\left(1-\frac{z}{2ir}\right)e^{\frac{z}{2ir}}=1+\sum_{r=1}^{\infty}\frac{(-i)^r}{2^r}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!\,(r-s)!}\right\}z^r$$
$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{i(2r-1)}\right\}e^{-\frac{z}{i(2r-1)}}=1+\sum_{r=1}^{\infty}i^r\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!(r-s)!}\right\}z^r$$
$$\prod_{r=1}^{\infty}\left\{1-\frac{z}{i(2r-1)}\right\}e^{\frac{z}{i(2r-1)}}=1+\sum_{r=1}^{\infty}(-i)^r\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!(r-s)!}\right\}z^r$$
$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{i(2r-1)}\right\}e^{-\frac{z}{2ir}}=1+\sum_{r=1}^{\infty}i^r\left\{\frac{\gamma^r}{(2r)!!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\gamma^{r-s}}{(2s)!!(2r-2s)!!}\right\}z^r$$
$$\prod_{r=1}^{\infty}\left\{1-\frac{z}{i(2r-1)}\right\}e^{\frac{z}{2ir}}=1+\sum_{r=1}^{\infty}(-i)^r\left\{\frac{\gamma^r}{(2r)!!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\gamma^{r-s}}{(2s)!!(2r-2s)!!}\right\}z^r$$

2.4 Infinite-degree Equation with Imaginary Integers as Roots (Part2)

Formula 2.4.1

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{ir}\right)\left(1-\frac{z}{ir}\right)=\sum_{r=0}^{\infty}\frac{\pi^{2r}}{(2r+1)!}z^{2r}\qquad\left(=\frac{\sinh \pi z}{\pi z}\right)$$
$$\prod_{r=1}^{\infty}\left(1+\frac{z}{2ir}\right)\left(1-\frac{z}{2ir}\right)=\sum_{r=0}^{\infty}\frac{\pi^{2r}}{2^{2r}(2r+1)!}z^{2r}\qquad\left(=\frac{\sinh(\pi z/2)}{\pi z/2}\right)$$
$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{i(2r-1)}\right\}\left\{1-\frac{z}{i(2r-1)}\right\}=\sum_{r=0}^{\infty}\frac{\pi^{2r}}{(4r)!!}z^{2r}\qquad\left(=\cosh\frac{\pi z}{2}\right)$$

$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{i(2r-1)}\right\}\left(1-\frac{z}{2ir}\right)=1+i\frac{a_1-b_1}{2!!}z$$
$$+\sum_{r=2}^{\infty}i^r\left\{\frac{a_r+(-1)^r b_r}{(2r)!!}+\sum_{s=1}^{r-1}\frac{(-1)^{r-s}a_s b_{r-s}}{(2s)!!\{2(r-s)\}!!}\right\}z^r$$
$$\prod_{r=1}^{\infty}\left\{1-\frac{z}{i(2r-1)}\right\}\left(1+\frac{z}{2ir}\right)=1+i\frac{b_1-a_1}{2!!}z$$
$$+\sum_{r=2}^{\infty}i^r\left\{\frac{b_r+(-1)^r a_r}{(2r)!!}+\sum_{s=1}^{r-1}\frac{(-1)^{r-s}b_s a_{r-s}}{(2s)!!\{2(r-s)\}!!}\right\}z^r$$

2.5 Infinite-degree Equation with Square Roots of Integers

Formula 2.5.1 (Infinite-degree Equation with Square Roots of Positive Integers)

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{\sqrt{r}}\right)\left(1-\frac{z}{\sqrt{r}}\right)e^{\frac{z^2}{r}}=1+\sum_{r=1}^{\infty}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!(r-s)!}\right\}z^{2r}$$
$$\prod_{r=1}^{\infty}\left(1+\frac{z}{\sqrt{2r}}\right)\left(1-\frac{z}{\sqrt{2r}}\right)e^{\frac{z^2}{2r}}=1+\sum_{r=1}^{\infty}\frac{1}{2^r}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!(r-s)!}\right\}z^{2r}$$
$$\prod_{r=1}^{\infty}\left(1+\frac{z}{\sqrt{2r-1}}\right)\left(1-\frac{z}{\sqrt{2r-1}}\right)e^{\frac{z^2}{2r-1}}=1+\sum_{r=1}^{\infty}\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!(r-s)!}\right\}z^{2r}$$
$$\prod_{r=1}^{\infty}\left(1+\frac{z}{\sqrt{2r-1}}\right)\left(1-\frac{z}{\sqrt{2r-1}}\right)e^{\frac{z^2}{2r}}=1+\sum_{r=1}^{\infty}\left\{\frac{\gamma^r}{(2r)!!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\gamma^{r-s}}{(2s)!!(2r-2s)!!}\right\}z^{2r}$$

Formula 2.5.2 (Infinite-degree Equation with Square Roots of Negative Integers)

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{i\sqrt{r}}\right)\left(1-\frac{z}{i\sqrt{r}}\right)e^{-\frac{z^2}{r}}=1+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!(r-s)!}\right\}z^{2r}$$
$$\prod_{r=1}^{\infty}\left(1+\frac{z}{i\sqrt{2r}}\right)\left(1-\frac{z}{i\sqrt{2r}}\right)e^{-\frac{z^2}{2r}}=1+\sum_{r=1}^{\infty}\frac{(-1)^r}{2^r}\left\{\frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s a_s\gamma^{r-s}}{s!(r-s)!}\right\}z^{2r}$$
$$\prod_{r=1}^{\infty}\left(1+\frac{z}{i\sqrt{2r-1}}\right)\left(1-\frac{z}{i\sqrt{2r-1}}\right)e^{-\frac{z^2}{2r-1}}$$
$$=1+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!(r-s)!}\right\}z^{2r}$$
$$\prod_{r=1}^{\infty}\left(1+\frac{z}{i\sqrt{2r-1}}\right)\left(1-\frac{z}{i\sqrt{2r-1}}\right)e^{-\frac{z^2}{2r}}=1+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{\gamma^r}{(2r)!!}+\sum_{s=1}^{r}\frac{(-1)^s b_s\gamma^{r-s}}{(2s)!!(2r-2s)!!}\right\}z^{2r}$$

2.6 Infinite-degree Equation with Square Numbers as Roots

Formula 2.6.1 (Infinite-degree Equation with Square Numbers as Roots)

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{r^2}\right)=\sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(2r+1)!}z^r \qquad \left(=\frac{\sin(\pi\sqrt{z})}{\pi\sqrt{z}}\right)$$
$$\prod_{r=1}^{\infty}\left\{1-\frac{z}{(2r)^2}\right\}=\sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{2^{2r}(2r+1)!}z^r \qquad \left(=\frac{\sin(\pi\sqrt{z/4})}{\pi\sqrt{z/4}}\right)$$

$$\prod_{r=1}^{\infty}\left\{1-\frac{z}{(2r-1)^2}\right\} = \sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(4r)!!}z^r \qquad \left(=\cos\frac{\pi\sqrt{z}}{2}\right)$$

Formula 2.6.2 (Infinite-degree Equation with Negative Square Numbers as Roots)

$$\prod_{r=1}^{\infty}\left(1+\frac{z}{r^2}\right) = \sum_{r=0}^{\infty}\frac{\pi^{2r}}{(2r+1)!}z^r \qquad \left(=\frac{\sinh\left(\pi\sqrt{z}\right)}{\pi\sqrt{z}}\right)$$

$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{(2r)^2}\right\} = \sum_{r=0}^{\infty}\frac{\pi^{2r}}{2^{2r}(2r+1)!}z^r \qquad \left(=\frac{\sinh\left(\pi\sqrt{z/4}\right)}{\pi\sqrt{z/4}}\right)$$

$$\prod_{r=1}^{\infty}\left\{1+\frac{z}{(2r-1)^2}\right\} = \sum_{r=0}^{\infty}\frac{\pi^{2r}}{(4r)!!}z^r \qquad \left(=\cosh\frac{\pi\sqrt{z}}{2}\right)$$

03 Vieta's Formulas in Infinite-degree Equation

3.1 Properties of Infinite-degree Equation

(1) Fundamental theorem of algebra does not hold generally.

(2) Vieta's Formulas does not hold generally.

(3) Roots of an infinite-degree equation with rational coefficients are not algebraic numbers generally

3.2 Vieta's Formulas (Part1)

Formula 3.2.1 (Vieta's Formulas)

Assume that a function $f(z)$ on the complex plane has zeros $z_1, z_2, z_3, z_4, \cdots$ and is completely factored as follows.

$$f(z) = \left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\left(1-\frac{z}{z_3}\right)\left(1-\frac{z}{z_4}\right)\cdots$$

Then, $f(z)$ is expanded to a power series as follows.

$$f(z) = 1+a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

Where,

$$a_1 = -\sum_{r_1=1}^{\infty} \frac{1}{z_{r_1}}$$

$$a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{1}{z_{r_1} z_{r_2}}$$

$$a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{1}{z_{r_1} z_{r_2} z_{r_3}}$$

$$a_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty} \frac{1}{z_{r_1} z_{r_2} z_{r_3} z_{r_3}}$$

$$\vdots$$

$$a_n = (-1)^n \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} \frac{1}{z_{r_1} z_{r_2}\cdots z_{r_n}}$$

Formula 3.2.2

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} \frac{1}{(r_1 r_2 \cdots r_n)^2} = \frac{\pi^{2n}}{(2n+1)!}$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} \frac{1}{\{(2r_1-1)(2r_2-1)\cdots(2r_n-1)\}^2} = \frac{\pi^{2n}}{(4n)!!}$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} \frac{(-1)^{r_1+r_2+\cdots+r_n}}{r_1 r_2 \cdots r_n} = \frac{\alpha_n+(-1)^n\beta_n}{(2n)!!} + \sum_{s=1}^{n-1}\frac{(-1)^{n-s}\alpha_s\beta_{n-s}}{(2s)!!\{2(n-s)\}!!}$$

Where, $\psi_r(z)$ is the polygamma function, $B_{r,k}(f_1, f_2, \ldots)$ are Bell polynomials and

$$\alpha_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) \qquad r=1,2,3,\cdots$$

$$\beta_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{r-1}\left(\frac{1}{2}\right)\right) \qquad r=1,2,3,\cdots$$

3.3 Vieta's Formulas (Part2)

Formula 3.3.1

Assume that a function $f(z)$ on the complex plane has zeros $z_1, z_2, z_3, z_4, \cdots$ and is incompletely factored as follows.

$$f(z) = \prod_{r=1}^{\infty}\left(1-\frac{z}{z_r}\right)e^{\frac{z}{z_r}}$$

Then, $f(z)$ is expanded to a power series as follows.

$$f(z) = 1+c_1 z^1+c_2 z^2+c_3 z^3+c_4 z^4+\cdots$$

$$c_1 = \frac{a_1 a_1^0}{0!}-\frac{a_0 a_1^1}{1!}$$

$$c_2 = \frac{a_2 a_1^0}{0!}-\frac{a_1 a_1^1}{1!}+\frac{a_0 a_1^2}{2!}$$

$$c_3 = \frac{a_3 a_1^0}{0!}-\frac{a_2 a_1^1}{1!}+\frac{a_1 a_1^2}{2!}-\frac{a_0 a_1^3}{3!}$$

$$\vdots$$

$$c_r = \sum_{s=0}^{r}(-1)^s \frac{a_{r-s} a_1^s}{s!}$$

Where,

$$a_0 = 1 \ , \ a_1 = -\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}} \ , \ a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1} z_{r_2}} \ , \ a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{z_{r_1} z_{r_2} z_{r_3}} \ , \ \cdots$$

Especially, $c_1 \sim c_4$ are expressed briefly as follows.

$$c_1 = 0 \ , \ c_2 = -\frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{z_r^2} \ , \ c_3 = -\frac{1}{3}\sum_{r=1}^{\infty}\frac{1}{z_r^3} \ , \ c_4 = -\frac{1}{8}\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}^4}+\frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}^2 z_{r_2}^2}$$

Formula 3.3.2

Let $\zeta(z)$ be the Riemann zeta function, $\psi_r(z)$ be the polygamma function, $B_{r,k}(f_1, f_2, \ldots)$ are Bell polynomials, γ be Euler-Mascheroni constant and α_r, a_r are the following constants.

$$\alpha_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) \qquad r=1,2,3,\cdots$$

$$a_0 = 1 \ , \ a_1 = -\sum_{r_1=1}^{\infty}\frac{1}{r_1} \ , \ a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{r_1 r_2} \ , \ a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{r_1 r_2 r_3} \ , \ \cdots$$

Then, the following expression holds.

$$\sum_{s=0}^{r}(-1)^s \frac{a_{r-s} a_1^s}{s!} = \frac{\gamma^r}{r!}+\sum_{s=1}^{r}\frac{(-1)^s \alpha_s \gamma^{r-s}}{s!(r-s)!} \qquad r=1,2,3,\cdots$$

Especially,

$$-\frac{\zeta(2)}{2} = \frac{\gamma^2}{2!}-\frac{\alpha_1\gamma^1}{1!1!}+\frac{\alpha_2\gamma^0}{2!0!}$$

$$-\frac{\zeta(3)}{3} = \frac{\gamma^3}{3!}-\frac{\alpha_1\gamma^2}{1!2!}+\frac{\alpha_2\gamma^1}{2!1!}-\frac{\alpha_3\gamma^0}{3!0!}$$

$$-\frac{\zeta(4)}{8}+\frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{r_1^2 r_2^2} = \frac{\gamma^4}{4!}-\frac{\alpha_1\gamma^3}{1!3!}+\frac{\alpha_2\gamma^2}{2!2!}-\frac{\alpha_3\gamma^1}{3!1!}+\frac{\alpha_4\gamma^0}{4!0!}$$

Formula 3.3.3

Let $\lambda(z)$ be the Dirichlet lamda function, $\psi_r(z)$ be the polygamma function, $B_{r,k}(f_1, f_2, \cdots)$ are Bell polynomials, γ be Euler-Mascheroni constant and β_r, a_r are the following constants.

$$\beta_r = \sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \cdots, \psi_{r-1}\left(\frac{1}{2}\right)\right) \quad r=1,2,3,\cdots$$

$$a_0 = 1 \quad , \quad a_1 = -\sum_{r_1=1}^{\infty}\frac{1}{2r_1-1} \quad , \quad a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{(2r_1-1)(2r_2-1)}$$

$$a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{(2r_1-1)(2r_2-1)(2r_3-1)} \quad , \cdots$$

Then, the following expression holds.

$$\sum_{s=0}^{r}(-1)^s \frac{a_{r-s}\, a_1^s}{s\,!} = \frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r\,!} + \sum_{s=1}^{r}\frac{(-1)^s \beta_s \left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!\,(r-s)\,!}$$

Especially,

$$-\frac{\lambda(2)}{2} = \frac{\left(\frac{\gamma}{2}+\log 2\right)^2}{2\,!} + \sum_{s=1}^{2}\frac{(-1)^s \beta_s \left(\frac{\gamma}{2}+\log 2\right)^{2-s}}{(2s)!!\,(2-s)\,!}$$

$$-\frac{\lambda(3)}{3} = \frac{\left(\frac{\gamma}{2}+\log 2\right)^3}{3\,!} + \sum_{s=1}^{3}\frac{(-1)^s \beta_s \left(\frac{\gamma}{2}+\log 2\right)^{3-s}}{(2s)!!\,(3-s)\,!}$$

$$-\frac{\lambda(4)}{8} + \frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{(2r_1-1)^2(2r_2-1)^2} = \frac{\left(\frac{\gamma}{2}+\log 2\right)^4}{4\,!} + \sum_{s=1}^{4}\frac{(-1)^s \beta_s \left(\frac{\gamma}{2}+\log 2\right)^{4-s}}{(2s)!!\,(4-s)\,!}$$

3.4 Infinite-degree Equation with Real Coefficients

Definition

When the function $f(z)$ defined in the domain D satisfies

$$f\left(\overline{z}\right) = \overline{f(z)} \qquad z \in D$$

we say that the $f(z)$ has **complex conjugate property**. ($\overline{z}$ denotes the conjugate complex number of z)

Theorem 3.4.1

If a_k $(k=0,1,2,\cdots)$ are real numbers in function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then $f\left(\overline{z}\right) = \overline{f(z)}$.

Corollary 3.4.1

When a_k $(k=0,1,2,\cdots)$ are real numbers in the infinite-degree equation $\sum_{k=0}^{\infty} a_k z^k = 0$,

if z_0 is the root, $\overline{z_0}$ is also the root.

3.5 Vieta's Formulas (Part3)

Formula 3.5.1 (Infinite-degree Equation with Conjugate Complex Roots)

Assume that the function $f(z)$ on the complex plane has zeros $z_k = x_k \pm i y_k$, $y_k \neq 0$ $(k=1,2,3,\cdots)$ and is completely factored as follows.

$$f(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right) = \prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)$$

Then, $f(z)$ is expanded to a power series as follows.

$$f(z) = 1+a_1z^1+a_2z^2+a_3z^3+a_4z^4+\cdots$$

Where,

$$a_1 = -\sum_{r_1=1}^{\infty}\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}$$

$$a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^2 x_{r_1}x_{r_2}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)} + \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2}$$

$$a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^3 x_{r_1}x_{r_2}x_{r_3}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

$$a_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{2^4 x_{r_1}x_{r_2}x_{r_3}x_{r_4}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)\left(x_{r_4}^2+y_{r_4}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^2\left(x_{r_1}x_{r_2}+x_{r_1}x_{r_3}+x_{r_2}x_{r_3}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

$\vdots$

$$a_{2n-1} = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-1}x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-2}=r_{2n-3}+1}^{\infty}\frac{2^{2n-3}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-3}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-2}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-2}}^2+y_{r_{2n-2}}^2\right)}$$

$\vdots$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}+\cdots+x_{r_n}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

$$a_{2n} = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty}\frac{2^{2n}x_{r_1}x_{r_2}\cdots x_{r_{2n}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n}}^2+y_{r_{2n}}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-2}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-1}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$\vdots$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

Corollary 3.5.1 (Infinite-degree Equation with Conjugate Imaginary Roots)

When an infinite-degree equation

$$1+a_1z^1+a_2z^2+a_3z^3+a_4z^4+\cdots = 0$$

has the roots $z_k = \pm iy_k$, $y_k \ne 0$ ($k=1,2,3,\cdots$) and is completely factored by these roots,

$$a_{2r-1} = 0 \quad (r=1,2,3,\cdots)$$

$$a_2 = \sum_{r_1=1}^{\infty} \frac{1}{y_{r_1}^2}$$

$$a_4 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2}$$

$$a_6 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2 y_{r_3}^2}$$

$$a_8 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2 y_{r_3}^2 y_{r_4}^2}$$

$$\vdots$$

3.6 Vieta's Formulas (Part4)

Formula 3.6.1 (Infinite-degree Equation with Conjugate Complex Roots)

Assume that the function $f(z)$ on the complex plane has zeros $z_k = x_k \pm iy_k$, $y_k \ne 0$ $(k=1,2,3,\cdots)$ and is incompletely factored as follows.

$$f(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} = \prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)e^{\frac{2x_r z}{x_r^2+y_r^2}}$$

Then, $f(z)$ is expanded to a power series as follows.

$$f(z) = 1+c_1z^1+c_2z^2+c_3z^3+c_4z^4+\cdots$$

$$c_r = \sum_{s=0}^{r} (-1)^s \frac{a_{r-s}a_1^s}{s!}$$

Where,

$$a_0 = 1$$

$$a_1 = -\sum_{r_1=1}^{\infty} \frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}$$

$$a_2 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^2 x_{r1}x_{r2}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)} + \sum_{r_1=1}^{\infty} \frac{2^0}{x_{r1}^2+y_{r1}^2}$$

$$a_3 = -\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{2^3 x_{r1}x_{r2}x_{r3}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)} - \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$

$$a_4 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} \frac{2^4 x_{r1}x_{r2}x_{r3}x_{r4}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)\left(x_{r4}^2+y_{r4}^2\right)}$$

$$+ \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{2^2\left(x_{r1}x_{r2}+x_{r1}x_{r3}+x_{r2}x_{r3}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$

$\vdots$

$$a_{2n-1}=-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-1}x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-2}=r_{2n-3}+1}^{\infty}\frac{2^{2n-3}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-3}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-2}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-2}}^2+y_{r_{2n-2}}^2\right)}$$

$\vdots$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}+\cdots+x_{r_n}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

$$a_{2n}=\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty}\frac{2^{2n}x_{r_1}x_{r_2}\cdots x_{r_{2n}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n}}^2+y_{r_{2n}}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-2}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-1}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$\vdots$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

Especially, $c_1 \sim c_4$ are represented by a faster formula as follows.

$$c_1 = 0$$

$$c_2 = -\frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2+\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}$$

$$c_3 = -\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$

$$c_4 = -\frac{1}{8}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^4+\frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2\left(\frac{2x_{r2}}{x_{r2}^2+y_{r2}^2}\right)^2$$

$$-\left\{\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}\right\}\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}+\frac{1}{2}\left(\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^2\left(x_{r1}x_{r2}+x_{r1}x_{r3}+x_{r2}x_{r3}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)}+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$

Corollary 3.6.1 (Infinite-degree Equation with roots whose real part is 1/2)

Assume that the function $f(z)$ on the complex plane has zeros $z_k = 1/2 \pm iy_k$, $y_k \neq 0$ $(k=1,2,3,\cdots)$ and is incompletely factored as follows.

$$f(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} = \prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)e^{\frac{z}{1/4+y_r^2}}$$

Then, $f(z)$ is expanded to a power series as follows.

$$f(z) = 1+c_1 z^1+c_2 z^2+c_3 z^3+c_4 z^4+\cdots$$

$$c_r = \sum_{s=0}^{r} (-1)^s \frac{a_{r-s}\, a_1^s}{s!}$$

Where,

$$a_0 = 1$$

$$a_1 = -\sum_{r_1=1}^{\infty} \frac{1}{1/4+y_{r_1}^2}$$

$$a_2 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)} + \sum_{r_1=1}^{\infty} \frac{1}{1/4+y_{r_1}^2}$$

$$a_3 = -\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)}$$

$$- \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)}$$

$$a_4 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} \frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)\left(1/4+y_{r_4}^2\right)}$$

$$+ \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{3}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)}$$

$$+ \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)}$$

$$\vdots$$

Especially, $c_1 \sim c_4$ are represented by a faster formula as follows.

$$c_1 = 0$$

$$c_2 = -\frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^2 + \sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r_1}^2}$$

$$c_3 = -\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^3 + \sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^2$$

$$c_4 = -\frac{1}{8}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^4 + \sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^3 - \frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^2 \sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r_1}^2}$$

$$+\frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^2\left(\frac{1}{1/4+y_{r_2}^2}\right)^2 + \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)}$$

04 Sum of series equivalent to the Riemann hypothesis

4.1 Factorization of $-z\zeta(1-z)$

Formula 4.4.1 (Factorization around 0)

When γ is Euler-Mascheroni constant, $\zeta(z)$ is Riemann zeta function and the non-trivial zeros are $x_n+iy_n \quad n=1,2,3,\cdots$, the following expression holds.

$$-z\zeta(1-z) = \frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}} e^{\left(\log 2-1-\frac{\gamma}{2}\right)z} \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right) e^{\frac{2x_n z}{x_n^2+y_n^2}}$$

4.2 Maclaurin Expansion by Stieltjes Constants

Formula 4.2.1

When $\zeta(z)$ is Riemann zeta function, the following expression holds on whole complex plane.

$$-z\zeta(1-z) = 1 - \sum_{s=1}^{\infty}\frac{s\gamma_{s-1}}{s!}z^s \tag{2.1}$$

Where, γ_s is Stieltjes constant defined by the following expression.

$$\gamma_s = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{(\log k)^s}{k} - \frac{(\log n)^{s+1}}{s+1}\right\}$$

4.3 Maclaurin Expansion by Hadamard Product

Let $\psi_n(z)$ be the polygamma function and a_n , b_n , $c_n \quad n=1,2,3$ are as follow respectively,

$$a_1 = \frac{1}{2!!}\psi_0\left(\frac{3}{2}\right) \quad , \quad a_2 = \frac{1}{4!!}\left\{\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right\}$$

$$, \quad a_3 = \frac{1}{6!!}\left\{\psi_0^3\left(\frac{3}{2}\right)-3\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right)+\psi_2\left(\frac{3}{2}\right)\right\}$$

$$b_1 = \frac{1}{1!}\left(\log 2-1-\frac{\gamma}{2}\right)^1 , \quad b_2 = \frac{1}{2!}\left(\log 2-1-\frac{\gamma}{2}\right)^2 , \quad b_3 = \frac{1}{3!}\left(\log 2-1-\frac{\gamma}{2}\right)^3$$

$$c_1 = 0 \quad , \quad c_2 = -\frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2 + \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2} \quad ,$$

$$c_3 = -\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3 + \sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1(x_{r1}+x_{r2})}{(x_{r1}^2+y_{r1}^2)(x_{r2}^2+y_{r2}^2)}$$

Then,

$$\begin{aligned}-z\zeta(1-z) = 1 &+ z^1(a_1+b_1) \\ &+ z^2(a_2+b_2+c_2+a_1b_1) \\ &+ z^3(a_3+b_3+c_3+a_2b_1+b_2a_1+c_2a_1+c_2b_1) \\ &+ \\ &\vdots\end{aligned} \tag{3.0}$$

Coefficients of the 1st, 2nd, 3rd degree

Comparing Formula 4.2.1 and (3.0) , we obtain the following formula.

Formula 4.3.1

When γ is Euler-Mascheroni constant, γ_s is Stieltjes constant, $\psi_n(z)$ is the polygamma function and non-trivial zeros of Riemann zeta function are $x_n+iy_n \quad n=1,2,3,\cdots$, the following expressions hold.

$$-\gamma_0 = \frac{1}{2!!}\psi_0\left(\frac{3}{2}\right) + \log 2 - 1 - \frac{\gamma}{2}$$

$$-\gamma_1 = \frac{\gamma_0^2}{2} - \frac{1}{4!!}\psi_1\left(\frac{3}{2}\right) - \frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^2 + \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2}$$

$$-\frac{\gamma_2}{2} = \frac{\gamma_0^3}{3} + \gamma_0\gamma_1 + \frac{1}{6!!}\psi_2\left(\frac{3}{2}\right)$$

$$-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^3 + \sum_{r_1=1}^{\infty}\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1(x_{r_1}+x_{r_2})}{(x_{r_1}^2+y_{r_1}^2)(x_{r_2}^2+y_{r_2}^2)}$$

4.4 Proposition equivalent to the Riemann Hypothesis

Proposition 4.4.1

When γ_s is Stieltjes constant, $\psi_n(z)$ is the polygamma function and non-trivial zeros of Riemann zeta function are $1/2+iy_r \quad r=1,2,3,\cdots$, the following expressions hold.

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = \gamma_0 - \frac{1}{2}\log\pi + \frac{1}{2}\psi_0\left(\frac{3}{2}\right) \qquad (4.1_1)$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^2 = \gamma_0^2 + 2\gamma_0 + 2\gamma_1 - \log\pi + \psi_0\left(\frac{3}{2}\right) - \frac{1}{4}\psi_1\left(\frac{3}{2}\right) \qquad (4.1_2)$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^3 = \gamma_0^3 + 3\gamma_0^2 + 6\gamma_0 + 6\gamma_1 + 3\gamma_0\gamma_1 + \frac{3}{2}\gamma_2 - 3\log\pi$$

$$+3\psi_0\left(\frac{3}{2}\right) - \frac{3}{4}\psi_1\left(\frac{3}{2}\right) + \frac{1}{16}\psi_2\left(\frac{3}{2}\right) \qquad (4.1_3)$$

Numerical Calculation

When we take 20,000 zero points y_r on the critical line and calculate (4.1_1) ~ (4.1_3) using the formula manipulation software *Mathematica*, the results are as follows respectively.

```
γs_ := StieltjesGamma[s]; ψk_[p_] := PolyGamma[k, p];  yn_ := Im[ZetaZero[n]]
```

1st degree

$$\texttt{f1[m\_] :=} \sum_{r=1}^{m}\frac{1}{1/4+y_r^2} \qquad \texttt{g1 :=} \gamma_0 - \frac{1}{2}\texttt{Log}[\pi] + \frac{1}{2}\psi_0\left[\frac{3}{2}\right]$$

```
N[f1[20000]]          N[g1]
0.0230167             0.0230957
```

Both sides match up to 3 significant digits.

2nd degree

$$\texttt{f2[m\_] :=} \sum_{r=1}^{m}\left(\frac{1}{1/4+y_r^2}\right)^2 \qquad \texttt{g2 :=} \gamma_0^2 + 2\gamma_0 + 2\gamma_1 - \texttt{Log}[\pi] + \psi_0\left[\frac{3}{2}\right] - \frac{1}{4}\psi_1\left[\frac{3}{2}\right]$$

```
SetPrecision[f2[20000], 10]      SetPrecision[g2, 15]
0.0000371006364                  0.0000371006364
```

Both sides match up to 9 significant digits.

3rd degree

```
f3[m_] := Sum[(1/(1/4 + y_r^2))^3, {r, 1, m}]
```

$$f3[m\_] := \sum_{r=1}^{m} \left(\frac{1}{1/4 + y_r^{\,2}} \right)^3$$

$$g3 := \gamma_0^{\,3} + 3\gamma_0^{\,2} + 6\gamma_0 + 6\gamma_1 + 3\gamma_0\gamma_1 + \frac{3}{2}\gamma_2 - 3\,\mathrm{Log}[\pi] + 3\,\psi_0\left[\frac{3}{2}\right] - \frac{3}{4}\psi_1\left[\frac{3}{2}\right] + \frac{1}{16}\psi_2\left[\frac{3}{2}\right]$$

SetPrecision[f3[20000], 16]	SetPrecision[g3, 24]
$1.436778602886916 \times 10^{-7}$	$1.436778602886918 \times 10^{-7}$

Both sides match up to 15 significant digits. The reason that this number of digits does not reach 27 is probably due to the low calculation accuracy on the right side.

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Alien's Mathematics