09 Maclaurin Series of Completed Riemann Zeta

9.1 Maclaurin Series of $\xi(z)$

Lemma 9.1.1 (Maclaurin series of gamma function)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials ,

$$\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3-z}{2}\right) = \sum_{r=0}^{\infty}(-1)^r \frac{g_r(3/2)}{2^r r!}z^r \qquad z \neq 3, 5, 7, \cdots \tag{1.g}$$

$$= 1 + \frac{g_1(3/2)}{2^1 1!}z^1 + \frac{g_2(3/2)}{2^2 2!}z^2 + \frac{g_3(3/2)}{2^3 3!}z^3 + \cdots$$

Where,

$$g_r\left(\frac{3}{2}\right) = \begin{cases} 1 & r = 0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{r-1}\left(\frac{3}{2}\right)\right) & r = 1,2,3,\cdots \end{cases}$$

Proof

Formula 12.1.1 in **" 12 Series Expansion of Gamma Function & the Reciprocal " (A la Carte)** was as follows. When $\Gamma(z)$ is gamma function, $\psi_n(z)$ is polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomial,

$$\Gamma(z) = \Gamma(a) + \sum_{n=1}^{\infty}\frac{c_n(a)}{n!}(z-a)^n \qquad a \neq 0, -1, -2, -3, \cdots$$

Where,

$$c_n(a) = \Gamma(a)\sum_{k=1}^{n} B_{n,k}(\psi_0(a), \psi_1(a), \ldots, \psi_{n-1}(a)) \qquad n=1,2,3,\cdots$$

Using this, $\Gamma\{(3-z)/2\}$ can be expanded in Taylor series as follows.

$$\Gamma\left(\frac{3-z}{2}\right) = \Gamma(a) + \sum_{r=1}^{\infty}\frac{\Gamma^{(r)}(a)}{r!}\left(\frac{3-z}{2} - a\right)^r \qquad a \neq 3,5,7,\cdots$$

$$\Gamma^{(r)}(a) = \Gamma(a)\sum_{k=1}^{r} B_{r,k}(\psi_0(a), \psi_1(a), \ldots, \psi_{r-1}(a)) \qquad r=1,2,3,\cdots$$

Putting $a = 3/2$,

$$\Gamma\left(\frac{3-z}{2}\right) = \Gamma\left(\frac{3}{2}\right) + \sum_{r=1}^{\infty}\frac{\Gamma^{(r)}(3/2)}{r!}\left(-\frac{z}{2}\right)^r \tag{1.w}$$

$$\Gamma^{(r)}\left(\frac{3}{2}\right) = \Gamma\left(\frac{3}{2}\right)\sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{r-1}\left(\frac{3}{2}\right)\right) \qquad r=1,2,3,\cdots$$

Here, putting

$$g_r\left(\frac{3}{2}\right) = \begin{cases} 1 & r = 0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{r-1}\left(\frac{3}{2}\right)\right) & r = 1,2,3,\cdots \end{cases}$$

Then,

$$\Gamma^{(r)}\left(\frac{3}{2}\right) = \Gamma\left(\frac{3}{2}\right)g_r\left(\frac{3}{2}\right) \qquad r=0,1,2,\cdots$$

Substituting this for (1.w) ,

$$\Gamma\left(\frac{3-z}{2}\right) = \Gamma\left(\frac{3}{2}\right) + \Gamma\left(\frac{3}{2}\right)\sum_{r=1}^{\infty}(-1)^r \frac{g_r(3/2)}{2^r r!}\left(\frac{z}{2}\right)^r$$

i.e.

$$\Gamma\left(\frac{3-z}{2}\right) = \frac{\sqrt{\pi}}{2}\sum_{r=0}^{\infty}(-1)^r \frac{g_r(3/2)}{2^r r!} z^r \qquad \left\{ \because \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \right\}$$

Multiplying both sides by $2/\sqrt{\pi}$, we obtain the desired expression.

Both sides of (1.g) are drawn as follows. The left side is blue and the right side is orange. There are singular points at $z = 3, 5, 7, \cdots$, so the convergence radius is 3.

```
Tblψ[r_, z_] := Table[PolyGamma[k, z], {k, 0, r - 1}]
g_r_[3/2] := If[r == 0, 1, Sum[BellY[r, k, Tblψ[r, 3/2]], {k, 1, r}]]
f_Γ[z_, m_] := Sum[(-1)^r g_r[3/2]/(2^r r!) z^r, {r, 0, m}]
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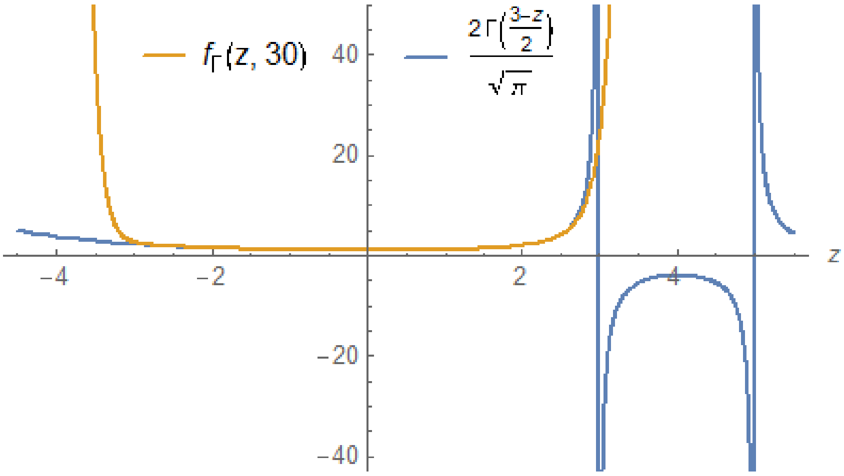

Lemma 9.1.2 (Maclaurin series of holomorphized Riemann zeta)

When $\zeta(z)$ is the Riemann zeta function and γ_r is Stieltjes constant, the following expression holds on the whole complex plane.

$$-z\zeta(1-z) = \sum_{r=0}^{\infty} c_r z^r \tag{1.z}$$

$$= 1 - \frac{\gamma_0}{0!}z^1 - \frac{\gamma_1}{1!}z^2 - \frac{\gamma_2}{2!}z^3 - \frac{\gamma_3}{3!}z^4 - \cdots$$

Where,

$$c_r = \begin{cases} 1 & r = 0 \\ -\dfrac{\gamma_{r-1}}{(r-1)!} & r = 1, 2, 3, \cdots \end{cases}$$

Proof

When γ_r is Stieltjes constant, it is known that the following expression holds on the whole complex plane except $z=1$.

$$\zeta(z) = \frac{1}{z-1} + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \gamma_r (z-1)^r$$

Multiplying both sides by $z-1$,

$$(z-1)\zeta(z) = 1 + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \gamma_r (z-1)^{r+1} = 1 + \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\gamma_{r-1}}{(r-1)!} (z-1)^r$$

Replacing z with $1-z$,

$$(1-z-1)\zeta(1-z) = 1 + \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\gamma_{r-1}}{(r-1)!} (1-z-1)^r$$

$$= 1 + \sum_{r=1}^{\infty} (-1)^{2r-1} \frac{\gamma_{r-1}}{(r-1)!!} z^r$$

i.e.

$$-z\zeta(1-z) = 1 - \sum_{r=1}^{\infty} \frac{\gamma_{r-1}}{(r-1)!} z^r$$

In this expression, however γ_{-1} may be defined, 1 cannot be included into Σ. So, defining a new coefficient c_r including the denominator as in the proviso of the lemma, we obtain the desired expressions.

Q.E.D.

Both sides of (1.z) are drawn as follows. The left side is blue and the right side is orange. Both sides are exactly overlapped and the left side (blue) is invisible. There is no singularity, so the convergence radius is ∞.

```
γs_ := StieltjesGamma[s]        cr_ := If[r == 0, 1, - γr-1/(r - 1)!]

fζ[z_, m_] := Sum[cr z^r, {r, 0, m}]
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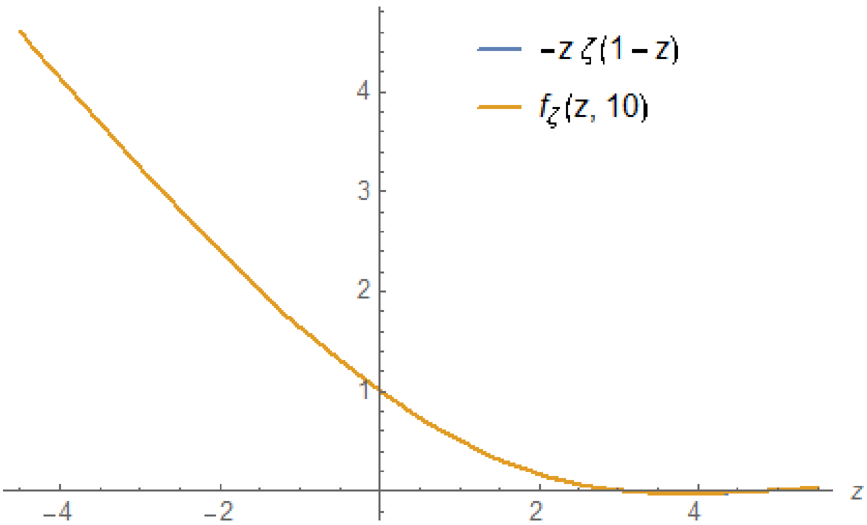


Theorem 9.1.3 (Maclaurin series of $\xi(z)$)

Let completed Riemann zeta be

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) \tag{1.1}$$

Then, the following expression holds on the whole complex plane.

$$\xi(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{(-1)^{s-t}g_{s-t}(3/2)}{2^{s-t}(s-t)!}c_t z^r \tag{1.2}$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{3}{2}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r}B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{r-1}\left(\frac{3}{2}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ -\frac{\gamma_{r-1}}{(r-1)!} & r=1,2,3,\cdots \end{cases}$$

Proof

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

$$= \xi(1-z) = -z(1-z)\pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z)$$

$$= -\pi^{\frac{z}{2}}\frac{2}{\sqrt{\pi}}\frac{1-z}{2}\Gamma\left(\frac{1-z}{2}\right)\{z\zeta(1-z)\}$$

i.e.

$$\xi(z) = \pi^{\frac{z}{2}}\left\{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3-z}{2}\right)\right\}\{-z\zeta(1-z)\} \tag{1.1'}$$

As first, Maclaurin series of $\pi^{z/2}$ is

$$\pi^{\frac{z}{2}} = \sum_{r=0}^{\infty}\frac{\log^r\pi}{2^r r!}z^r \tag{1.p}$$

From Lemma 9.1.1 and Lemma 9.1.2 , the two subsequent functions are as follows.

$$\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3-z}{2}\right) = \sum_{r=0}^{\infty}(-1)^r\frac{g_r(3/2)}{2^r r!}z^r \tag{1.g}$$

$$-z\zeta(1-z) = \sum_{r=0}^{\infty}c_r z^r \tag{1.z}$$

Where,

$$g_r\left(\frac{3}{2}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r}B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{r-1}\left(\frac{3}{2}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ -\dfrac{\gamma_{r-1}}{(r-1)!} & r=1,2,3,\cdots \end{cases}$$

Substituting these for (1.1') ,

$$\xi(z) = \left(\sum_{r=0}^{\infty} \frac{\log^r \pi}{2^r r!} z^r \right) \left(\sum_{r=0}^{\infty} (-1)^r \frac{g_r(3/2)}{2^r r!} z^r \right) \left(\sum_{r=0}^{\infty} c_r z^r \right)$$

According to Formula 1.1.2 in **" 01 Power of Infinite Series " (Infinite-degree Equation)** , the product of the 3 power series is expressed by the following equation.

$$\left(\sum_{r=0}^{\infty} a_r z^r \right) \left(\sum_{r=0}^{\infty} b_r z^r \right) \left(\sum_{r=0}^{\infty} c_r z^r \right) = \sum_{r=0}^{\infty} \sum_{s=0}^{r} \sum_{t=0}^{s} a_{r-s} b_{s-t} c_t z^r$$

So, putting

$$a_r = \frac{\log^r \pi}{2^r r!} \quad , \quad b_r = (-1)^r \frac{g_r(3/2)}{2^r r!}$$

we obtain

$$\xi(z) = \sum_{r=0}^{\infty} \sum_{s=0}^{r} \sum_{t=0}^{s} \frac{\log^{r-s} \pi}{2^{r-s}(r-s)!} \frac{(-1)^{s-t} g_{s-t}(3/2)}{2^{s-t}(s-t)!} c_t z^r \tag{1.2}$$

Where,

$$g_r\left(\frac{3}{2}\right) = \begin{cases} 1 & r=0 \\ \displaystyle\sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \cdots, \psi_{r-1}\left(\frac{3}{2}\right) \right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ -\dfrac{\gamma_{r-1}}{(r-1)!} & r=1,2,3,\cdots \end{cases}$$

Q.E.D.

The first few of (1.2) are

$$\begin{aligned} \xi(z) = 1 &+ \left(\frac{\log^1 \pi}{2^1 1!} - \frac{g_1(3/2)}{2^1 1!} - \frac{\gamma_0}{0!} \right) z^1 \\ &+ \left(\frac{\log^2 \pi}{2^2 2!} + \frac{g_2(3/2)}{2^2 2!} - \frac{\gamma_1}{1!} \right. \\ &\qquad \left. - \frac{\log^1 \pi}{2^1 1!} \frac{g_1(3/2)}{2^1 1!} + \frac{g_1(3/2)}{2^1 1!} \frac{\gamma_0}{0!} - \frac{\log^1 \pi}{2^1 1!} \frac{\gamma_0}{0!} \right) z^2 \\ &+ \left(\frac{\log^3 \pi}{2^3 3!} - \frac{g_3(3/2)}{2^3 3!} - \frac{\gamma_2}{2!} \right. \\ &\qquad - \frac{\log^2 \pi}{2^2 2!} \frac{g_1(3/2)}{2^1 1!} - \frac{\log^2 \pi}{2^2 2!} \frac{\gamma_0}{0!} - \frac{g_2(3/2)}{2^2 2!} \frac{\gamma_0}{0!} \\ &\qquad + \frac{\log^1 \pi}{2^1 1!} \frac{g_2(3/2)}{2^2 2!} - \frac{\log^1 \pi}{2^1 1!} \frac{\gamma_1}{1!} + \frac{g_1(3/2)}{2^1 1!} \frac{\gamma_1}{1!} \end{aligned}$$

$$+\frac{\log^1\pi}{2^1 1!}\frac{g_1(3/2)}{2^1 1!}\frac{\gamma_0}{0!}\Bigg)z^3$$

$$+$$

$$\vdots$$

$$= 1. - 0.0230957\,z + 0.0233439\,z^2 - 0.000497984\,z^3 + 0.000253182\,z^4$$
$$-5.05025\times10^{-6}\,z^5 + 1.72099\times10^{-6}\,z^6 - 3.23784\times10^{-8}\,z^7 + 8.31597\times10^{-9}\,z^8$$
$$\vdots$$

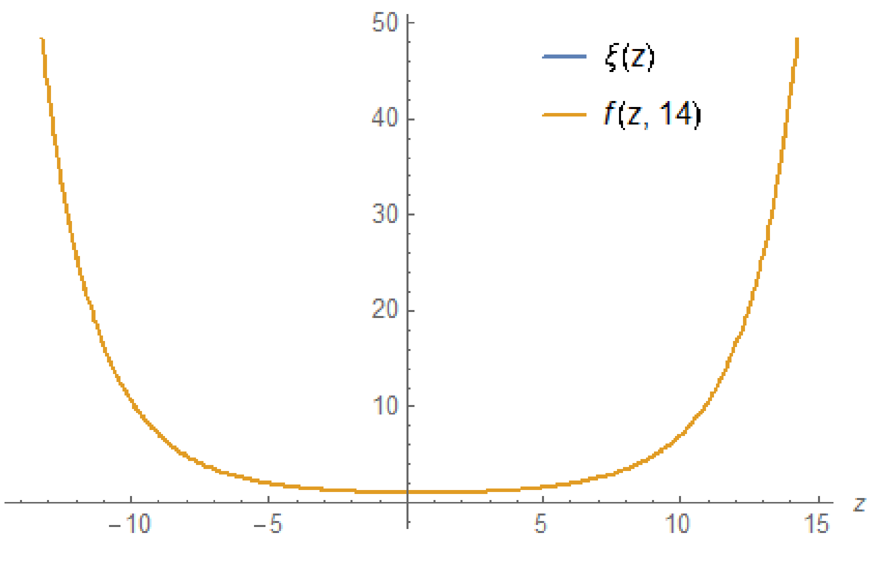
Both sides of (1.2) are drawn as follows. The left side is blue and the right side is orange. Although the right side is calculated up to z^{14}, both sides are exactly overlapped and the left side (blue) is invisible.
The singular points of the gamma function (1.g) are offset by the trivial zeros of Riemann zeta (1.z) and are disappeared. so the convergence radius is ∞ .

```
Tblψ[r_, z_] := Table[PolyGamma[k, z], {k, 0, r - 1}]
```

$$g_{r\_}\left[\frac{3}{2}\right] := \text{If}\left[r == 0,\ 1,\ \sum_{k=1}^{r}\text{BellY}\left[r,\ k,\ \text{Tbl}\psi\left[r,\ \frac{3}{2}\right]\right]\right]$$

$$\gamma_{s\_} := \text{StieltjesGamma}[s] \qquad c_{r\_} := \text{If}\left[r == 0,\ 1,\ -\frac{\gamma_{r-1}}{(r-1)!}\right]$$

$$f[z\_,\ m\_] := \sum_{r=0}^{m}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{\text{Log}[\pi]^{r-s}}{2^{r-s}\,(r-s)!}\,\frac{(-1)^{s-t}\,g_{s-t}[3/2]}{2^{s-t}\,(s-t)!}\,c_t\,z^r$$



However, very high calculation precision is required at a point far from the origin. The following is the function value at the first zero point $1/2+i\,14.1347\cdots$ of $\xi(z)$. It is calculated up to z^{36} at 64 digit precision and is barely matched up to the 7th decimal place.

```
SetPrecision[{ξ[ZetaZero[1]], f[ZetaZero[1], 36]}, 64]
{0, 3.68380076348383 × 10^-8 - 1.65454062318318 9× 10^-8 i}
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9.2 Maclaurin Series of $\Xi(z)$

Lemma 9.2.1 (Maclaurin series of gamma function)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \dots)$ are Bell polynomials ,

$$\Gamma\left(\frac{5}{4}+\frac{z}{2}\right) = \Gamma\left(\frac{5}{4}\right)\sum_{r=0}^{\infty}\frac{g_r(5/4)}{2^r r!}z^r \qquad z \ne -\frac{5}{2}, -\frac{9}{2}, -\frac{13}{2}, \cdots \tag{2.g}$$

Where,

$$g_r\left(\frac{5}{4}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{5}{4}\right), \psi_1\left(\frac{5}{4}\right), \dots, \psi_{r-1}\left(\frac{5}{4}\right)\right) & r = 1,2,3,\cdots \end{cases}$$

Proof

Formula 12.1.1 in **" 12 Series Expansion of Gamma Function & the Reciprocal " (A la Carte)** was as follows. When $\Gamma(z)$ is gamma function, $\psi_n(z)$ is polygamma function and $B_{n,k}(f_1, f_2, \dots)$ is Bell polynomial,

$$\Gamma(z) = \Gamma(a) + \sum_{n=1}^{\infty}\frac{c_n(a)}{n!}(z-a)^n \qquad a \ne 0, -1, -2, -3, \cdots$$

Where,

$$c_n(a) = \Gamma(a)\sum_{k=1}^{n} B_{n,k}(\psi_0(a), \psi_1(a), \dots, \psi_{n-1}(a)) \qquad n=1,2,3,\cdots$$

Using this, $\Gamma(5/4+z/2)$ can be expanded in Taylor series as follows.

$$\Gamma\left(\frac{5}{4}+\frac{z}{2}\right) = \Gamma(a) + \sum_{r=1}^{\infty}\frac{\Gamma^{(r)}(a)}{r!}\left(\frac{5}{4}+\frac{z}{2}-a\right)^r \qquad a \ne -\frac{5}{2}, -\frac{9}{2}, -\frac{13}{2}, \cdots$$

$$\Gamma^{(r)}(a) = \Gamma(a)\sum_{k=1}^{r} B_{r,k}(\psi_0(a), \psi_1(a), \dots, \psi_{r-1}(a)) \qquad r=1,2,3,\cdots$$

Putting $a = 5/4$,

$$\Gamma\left(\frac{5}{4}+\frac{z}{2}\right) = \Gamma\left(\frac{5}{4}\right) + \sum_{r=1}^{\infty}\frac{\Gamma^{(r)}(5/4)}{r!}\left(\frac{z}{2}\right)^r \tag{2.w}$$

$$\Gamma^{(r)}\left(\frac{5}{4}\right) = \Gamma\left(\frac{5}{4}\right)\sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{5}{4}\right), \psi_1\left(\frac{5}{4}\right), \dots, \psi_{r-1}\left(\frac{5}{4}\right)\right) \quad r=1,2,3,\cdots$$

Here, putting

$$g_r\left(\frac{5}{4}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{5}{4}\right), \psi_1\left(\frac{5}{4}\right), \dots, \psi_{r-1}\left(\frac{5}{4}\right)\right) & r = 1,2,3,\cdots \end{cases}$$

Then,

$$\Gamma^{(r)}\left(\frac{5}{4}\right) = \Gamma\left(\frac{5}{4}\right) g_r \qquad r=0,1,2,\cdots$$

Substituting this for (2.w) ,

$$\Gamma\left(\frac{5}{4}+\frac{z}{2}\right) = \Gamma\left(\frac{5}{4}\right) + \sum_{r=1}^{\infty}\Gamma\left(\frac{5}{4}\right)\frac{g_r(5/4)}{r!}\left(\frac{z}{2}\right)^r = \Gamma\left(\frac{5}{4}\right)\sum_{r=0}^{\infty}\frac{g_r(5/4)}{2^r r!}z^r$$

There are singular points at $z = -5/2, -9/2, -13/2, \cdots$, so the convergence radius is $5/2$.

Lemma 9.2.2 (Maclaurin series of holomorphized Riemann zeta)

When $\zeta(z)$ is the Riemann zeta function and γ_r is Stieltjes constant, the following expression holds on the whole complex plane.

$$2\left(z-\frac{1}{2}\right)\zeta\left(z+\frac{1}{2}\right) = -\zeta\left(\frac{1}{2}\right)\sum_{r=0}^{\infty} c_r z^r \tag{2.z}$$

Where,

$$c_r = \begin{cases} 1 & r=0 \\ \dfrac{2}{\zeta(1/2)} \displaystyle\sum_{s=r}^{\infty} (-1)^r \frac{\gamma_{s-1}}{(s-1)!}\binom{s}{r}\left(\frac{1}{2}\right)^{s-r} & r=1,2,3,\cdots \end{cases}$$

Proof

Let holomorphized Riemann zeta be

$$(z-1)\zeta(z) = 1+\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\gamma_{r-1}}{(r-1)!}(z-1)^r$$

Replacing z with $z+1/2$,

$$\left(z-\frac{1}{2}\right)\zeta\left(z+\frac{1}{2}\right) = 1+\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\gamma_{r-1}}{(r-1)!}\left(z-\frac{1}{2}\right)^r \tag{2.t}$$

$$= 1+\frac{\gamma_0}{0!}\left(z-\frac{1}{2}\right)^1-\frac{\gamma_2}{1!}\left(z-\frac{1}{2}\right)^2+\frac{\gamma_3}{2!}\left(z-\frac{1}{2}\right)^3-\frac{\gamma_4}{3!}\left(z-\frac{1}{2}\right)^4+-\cdots$$

This is a Taylor expansion around $z=1/2$.

Since the function $(z-1)\zeta(z)$ is holomorphized, there is no singularity on the whole complex plane. So the convergence radius of the Taylor serie is ∞. Here, from the binomial theorem,

$$\left(z-\frac{1}{2}\right)^r = \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\left(\frac{1}{2}\right)^{r-s} z^s$$

Substituting this for (2.t) ,

$$\left(z-\frac{1}{2}\right)\zeta\left(z+\frac{1}{2}\right) = \sum_{r=0}^{\infty}(-1)^{r-1}\frac{\gamma_{r-1}}{(r-1)!}\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\left(\frac{1}{2}\right)^{r-s} z^s$$

$$= \sum_{r=0}^{\infty}\sum_{s=r}^{\infty}(-1)^{s-1}\frac{\gamma_{s-1}}{(s-1)!}(-1)^{s-r}\binom{s}{r}\left(\frac{1}{2}\right)^{s-r} z^r$$

i.e.

$$\left(z-\frac{1}{2}\right)\zeta\left(z+\frac{1}{2}\right) = \sum_{r=0}^{\infty}\left\{\sum_{s=r}^{\infty}(-1)^{r-1}\frac{\gamma_{s-1}}{(s-1)!}\binom{s}{r}\left(\frac{1}{2}\right)^{s-r}\right\} z^r$$

This has to be equal to the following Maclaurin series.

$$\left(z-\frac{1}{2}\right)\zeta\left(z+\frac{1}{2}\right) = \sum_{r=0}^{\infty} a_r z^r \tag{2.m}$$

Because if it is not so, it contradicts the uniqueness of the power series. Therefore,

$$a_r = \sum_{s=r}^{\infty}(-1)^{r-1}\frac{\gamma_{s-1}}{(s-1)!}\binom{s}{r}\left(\frac{1}{2}\right)^{s-r}$$

From (2.t), the 1st term $-\gamma_{-1}/(-1)!$ of a_0 has to be read as 1. That is,

$$a_r = \begin{cases} 1 - \sum_{s=1}^{\infty} \frac{\gamma_{s-1}}{(s-1)!}\left(\frac{1}{2}\right)^s & r=0 \\ \sum_{s=r}^{\infty} (-1)^{r-1} \frac{\gamma_{s-1}}{(s-1)!}\binom{s}{r}\left(\frac{1}{2}\right)^{s-r} & r=1,2,3,\cdots \end{cases}$$

Furthermore, from (2.m),

$$a_0 = 1 - \sum_{s=1}^{\infty} \frac{\gamma_{s-1}}{(s-1)!}\left(\frac{1}{2}\right)^s = -\frac{1}{2}\zeta\left(\frac{1}{2}\right)$$

Next,

$$\left(z-\frac{1}{2}\right)\zeta\left(z+\frac{1}{2}\right) = \sum_{r=0}^{\infty} a_r z^r = a_0 \sum_{r=0}^{\infty} \frac{a_r}{a_0} z^r = -\frac{1}{2}\zeta\left(\frac{1}{2}\right) \sum_{r=0}^{\infty} \frac{a_r}{-\frac{1}{2}\zeta\left(\frac{1}{2}\right)} z^r$$

Doubling both sides,

$$2\left(z-\frac{1}{2}\right)\zeta\left(z+\frac{1}{2}\right) = -\zeta\left(\frac{1}{2}\right) \sum_{r=0}^{\infty} \frac{-2a_r}{\zeta(1/2)} z^r$$

Then, putting

$$c_r = -\frac{2a_r}{\zeta(1/2)} = \begin{cases} 1 & r=0 \\ \frac{2}{\zeta(1/2)} \sum_{s=r}^{\infty} (-1)^r \frac{\gamma_{s-1}}{(s-1)!}\binom{s}{r}\left(\frac{1}{2}\right)^{s-r} & r=1,2,3,\cdots \end{cases}$$

we obtain

$$2\left(z-\frac{1}{2}\right)\zeta\left(z+\frac{1}{2}\right) = -\zeta\left(\frac{1}{2}\right) \sum_{r=0}^{\infty} c_r z^r$$

There is no singularity, so the convergence radius is ∞.

Theorem 9.2.3 (Maclaurin Series of $\Xi(z)$)

Let completed Riemann zeta be

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) \tag{2.1}$$

Then, the following expression holds on the whole complex plane.

$$\Xi(z) = \Xi(0)\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}(-1)^{r-s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{g_{s-t}(5/4)}{2^{s-t}(s-t)!}c_t z^r \tag{2.2}$$

$$\Xi(0) = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.9942415563\cdots$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{5}{4}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{5}{4}\right), \psi_1\left(\frac{5}{4}\right), \ldots, \psi_{r-1}\left(\frac{5}{4}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ \dfrac{2}{\zeta(1/2)} \displaystyle\sum_{s=r}^{\infty} (-1)^r \frac{\gamma_{s-1}}{(s-1)!} \binom{s}{r} \left(\frac{1}{2}\right)^{s-r} & r=1,2,3,\cdots \end{cases}$$

Proof

Let holomorphized Riemann Zeta be

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) \tag{2.1}$$

This can be transformed as follows.

$$\Xi(z) = \pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left(\frac{5}{4}+\frac{z}{2}\right)\left\{2\left(z-\frac{1}{2}\right)\zeta\left(\frac{1}{2}+z\right)\right\} \tag{2.1'}$$

The 1st function can be expanded in Maclaurin series as follows.

$$\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)} = \frac{1}{\pi^{1/4}} \sum_{r=0}^{\infty} (-1)^r \frac{\log^r \pi}{2^r r!} z^r \tag{2.p}$$

From Lemma 9.2.1 and Lemma 9.2.2 , the two subsequent functions are as follows.

$$\Gamma\left(\frac{5}{4}+\frac{z}{2}\right) = \Gamma\left(\frac{5}{4}\right)\sum_{r=0}^{\infty} \frac{g_r(5/4)}{2^r r!} z^r \qquad z \ne -\frac{5}{2}, -\frac{9}{2}, -\frac{13}{2}, \cdots \tag{2.g}$$

$$2\left(z-\frac{1}{2}\right)\zeta\left(z+\frac{1}{2}\right) = -\zeta\left(\frac{1}{2}\right)\sum_{r=0}^{\infty} c_r z^r \tag{2.z}$$

Where,

$$g_r\left(\frac{5}{4}\right) = \begin{cases} 1 & r=0 \\ \displaystyle\sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{5}{4}\right), \psi_1\left(\frac{5}{4}\right), \cdots, \psi_{r-1}\left(\frac{5}{4}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ \dfrac{2}{\zeta(1/2)} \displaystyle\sum_{s=r}^{\infty} (-1)^r \frac{\gamma_{s-1}}{(s-1)!} \binom{s}{r} \left(\frac{1}{2}\right)^{s-r} & r=1,2,3,\cdots \end{cases}$$

Substitutin these for (2.1') ,

$$\Xi(z) = -\frac{1}{\pi^{1/4}}\Gamma\left(\frac{5}{4}\right)\zeta\left(\frac{1}{2}\right)\left(\sum_{r=0}^{\infty}(-1)^r\frac{\log^r\pi}{2^r r!}z^r\right)\left(\sum_{r=0}^{\infty}\frac{g_r(5/4)}{2^r r!}z^r\right)\left(\sum_{r=0}^{\infty}c_r z^r\right)$$

According to Formula 1.1.2 in **" 01 Power of Infinite Series " (Infinite-degree Equation)** , the product of the 3 power series is expressed by the following equation.

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{r=0}^{\infty} b_r z^r\right)\left(\sum_{r=0}^{\infty} c_r z^r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s} b_{s-t} c_t z^r$$

So, putting

$$a_r = \sum_{r=0}^{\infty} (-1)^r \frac{\log^r \pi}{2^r r!} z^r \quad , \quad b_r = \sum_{r=0}^{\infty} \frac{g_r(5/4)}{2^r r!} z^r$$

$$\Xi(0) = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.9942415563\cdots$$

we obtain

$$\Xi(z) = \Xi(0)\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}(-1)^{r-s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{g_{s-t}(5/4)}{2^{s-t}(s-t)!}c_t z^r \tag{2.2}$$

Where,

$$g_r\left(\frac{5}{4}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{5}{4}\right), \psi_1\left(\frac{5}{4}\right), \cdots, \psi_{r-1}\left(\frac{5}{4}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ \dfrac{2}{\zeta(1/2)}\sum_{s=r}^{\infty}(-1)^r\dfrac{\gamma_{s-1}}{(s-1)!}\dbinom{s}{r}\left(\dfrac{1}{2}\right)^{s-r} & r=1,2,3,\cdots \end{cases}$$

Q.E.D.

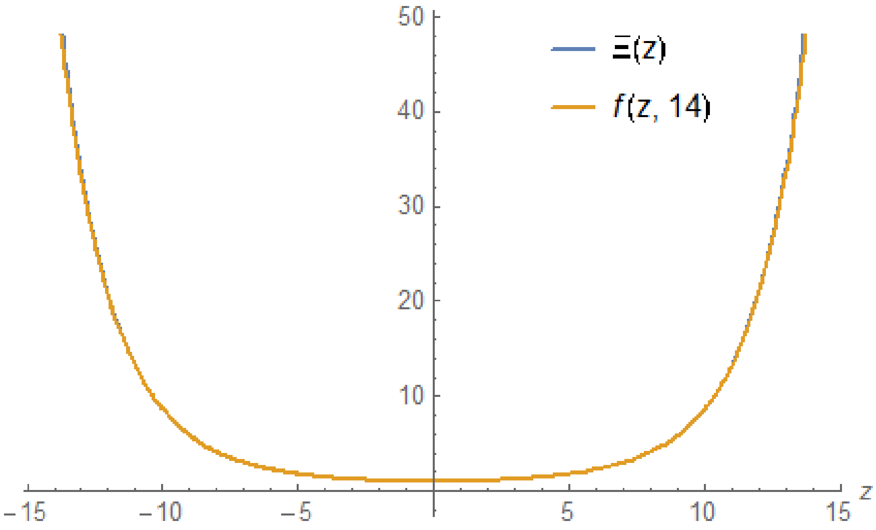
The first few of (2.2) are

$$\Xi(z) = \Xi(0)\left\{1 + \left(-\frac{\log^1\pi}{2^1 1!} + \frac{g_1(5/4)}{2^1 1!} + c_1\right)z^1\right.$$
$$+ \left(\frac{\log^2\pi}{2^2 2!} + \frac{g_2(5/4)}{2^2 2!} + c_2 - \frac{\log^1\pi}{2^1 1!}\frac{g_1(5/4)}{2^1 1!} + \frac{g_1(5/4)}{2^1 1!}c_1 - \frac{\log^1\pi}{2^1 1!}c_1\right)z^2$$
$$+ \left(-\frac{\log^3\pi}{2^3 3!} + \frac{g_3(5/4)}{2^3 3!} + c_3 + \frac{\log^2\pi}{2^2 2!}\frac{g_1(5/4)}{2^1 1!} + \frac{\log^2\pi}{2^2 2!}c_1 + \frac{g_2(5/4)}{2^2 2!}c_1\right.$$
$$\left. - \frac{\log^1\pi}{2^1 1!}\frac{g_2(5/4)}{2^2 2!} - \frac{\log^1\pi}{2^1 1!}c_2 + \frac{g_1(5/4)}{2^1 1!}c_2 - \frac{\log^1\pi}{2^1 1!}\frac{g_1(5/4)}{2^1 1!}c_1\right)z^3$$
$$\left. + \cdots \right\}$$

$$= 0.994242\{1. + 4.44089\times 10^{-16}z + 0.023105z^2 + 1.38778\times 10^{-16}z^3$$
$$+ 0.000248334z^4 + 2.08167\times 10^{-17}z^5 + 1.67435\times 10^{-6}z^6$$
$$+ 7.37257\times 10^{-18}z^7 + 8.0307\times 10^{-9}z^8 + 1.0842\times 10^{-18}z^9$$
$$+ 2.94014\times 10^{-11}z^{10}\}$$

We can see that the coefficients of the odd degree are almost zero.

Both sides of (2.2) are drawn as follows. The left side is blue and the right side is orange. Although the right side is calculated up to z^{14}, both sides are overlapped and the left side (blue) is almost invisible. The singular points of the gamma function (2.g) are offset by the trivial zeros of Riemann zeta (2.z) and are disappeared. so the convergence radius is ∞.



However, very high calculation precision is required at a point far from the origin. The following is the function value at the first zero point $i\,14.1347\cdots$ of $\xi(z)$. It is calculated up to z^{26} at 26 digit precision and is barely matched up to the 4th decimal place.

$$\Xi[z\_] := -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\,\text{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}+z\right)\right]\,\text{Zeta}\left[\frac{1}{2}+z\right]$$

$$\text{Tbl}\psi[r\_, z\_] := \text{Table}[\text{PolyGamma}[k, z], \{k, 0, r-1\}]$$

$$g_{r\_}\left[\frac{5}{4}\right] := \text{If}\left[r == 0, 1, \sum_{k=1}^{r}\text{BellY}\left[r, k, \text{Tbl}\psi\left[r, \frac{5}{4}\right]\right]\right]$$

$$\gamma_{s\_} := \text{StieltjesGamma}[s]$$

$$c_{r\_} := \text{If}\left[r == 0, 1, \frac{2}{\text{Zeta}[1/2]}\sum_{s=r}^{1000}(-1)^{r}\frac{\gamma_{s-1}}{(s-1)\,!}\,\text{Binomial}[s, r]\left(\frac{1}{2}\right)^{s-r}\right]$$

$$f[z\_, m\_] := \Xi[0]\sum_{r=0}^{m}\sum_{s=0}^{r}\sum_{t=0}^{s}(-1)^{r-s}\frac{\text{Log}[\pi]^{r-s}}{2^{r-s}\,(r-s)\,!}\,\frac{g_{s-t}[5/4]}{2^{s-t}\,(s-t)\,!}\,c_{t}\,z^{r}$$

$$\text{SetPrecision}\left[\left\{\Xi\left[\text{ZetaZero}[1]-\frac{1}{2}\right], f\left[\text{ZetaZero}[1]-\frac{1}{2}, 26\right]\right\}, 26\right]$$

$$\left\{0\ ,\ 0.\times 10^{-4} + 0.\times 10^{-4}\,i\right\}$$

9.3 Appendix

9.1.0 Higher order differential coefficient of $\zeta(z)$

Formula 26.1.2h in " **26 Higher and Super Calculus of Zeta Function etc** " (**SuperCalculus**) was as follows.

When $\zeta(z)$ is Riemann zeta function, $\zeta^{(n)}(z)$ is the lineal n -th order derivative and γ_r is Stieltjes constant, the following expression holds on the whole complex plane except $z=1$.

$$\zeta^{(n)}(z) = \frac{(-1)^{-n} n!}{(z-1)^{n+1}} + \sum_{s=0}^{\infty} (-1)^s \gamma_s \frac{(z-1)^{s-n}}{\Gamma(1+s-n)}$$

Given $z=a \ (\neq 1)$ in this equation, the n th order differential coefficient at a is obtained. That is

$$\zeta^{(n)}(a) = \frac{(-1)^{-n} n!}{(a-1)^{n+1}} + \sum_{s=0}^{\infty} (-1)^s \gamma_s \frac{(a-1)^{s-n}}{\Gamma(1+s-n)} \tag{3.a}$$

(1) Higher order differential coefficient at 0

In particular, puttig $a=0$ in (3.a) , the n th order differential coefficient at 0 is obtained. That is

$$\zeta^{(n)}(0) = (-1)^n \sum_{s=0}^{\infty} \frac{\gamma_s}{(s-n)!} - n! \qquad n=0,1,2,\cdots \tag{3.0}$$

The first few are as follows. We can see that all the differential coefficients are negative and the differential coefficient of the first or higher order is close to $-n!$.

$$\zeta^{(0)}(0) = -0.5$$
$$\zeta^{(1)}(0) = -0.91893853320467\cdots \approx -1!$$
$$\zeta^{(2)}(0) = -2.00635645590858\cdots \approx -2!$$
$$\zeta^{(3)}(0) = -6.00471116686225\cdots \approx -3!$$
$$\zeta^{(4)}(0) = -23.9971031880137\cdots \approx -4!$$
$$\zeta^{(5)}(0) = -120.000232907558\cdots \approx -5!$$
$$\vdots$$

cf.

(3.0) is equivalent to the following equation.

$$\zeta^{(n)}(0) = (-1)^n \sum_{s=0}^{\infty} \frac{\gamma_{s+n}}{s!} - n! \qquad n=0,1,2,\cdots$$

From this, formula of ***O.Marichev*** (http://mathworld.wolfram.com/StieltjesConstants.html) is obtained. That is

$$\sum_{s=0}^{\infty} \frac{\gamma_{s+n}}{s!} = (-1)^n \left\{ n! + \zeta^{(n)}(0) \right\} \qquad n=0,1,2,\cdots$$

(2) Higher order differential coefficient at 1/2

In particular, when $a=1/2$ in (1.a) , the n th order differential coefficient at $1/2$ is obtained as follows.

$$\zeta^{(n)}\left(\frac{1}{2}\right) = \frac{(-1)^{-n} n!}{(1/2-1)^{n+1}} + \sum_{s=0}^{\infty}(-1)^s \gamma_s \frac{(1/2-1)^{s-n}}{\Gamma(1+s-n)}$$

$$= \frac{(-1)^{-n} n!}{(-1/2)^{n+1}} + \sum_{s=0}^{\infty}(-1)^s \gamma_s \frac{(-1/2)^{s-n}}{\Gamma(1+s-n)}$$

$$= \frac{2^n n!}{(-1)^n (-1)^{n+1}} + \sum_{s=0}^{\infty}\gamma_s \frac{2^n (-1)^s (-1)^{s-n}}{2^s (s-n)!}$$

i.e.

$$\zeta^{(n)}(1/2) = 2^n \left\{ (-1)^n \sum_{s=0}^{\infty} \frac{\gamma_s}{2^s (s-n)!} - 2n! \right\} \tag{3.h}$$

The first few are as follows. We can see that all the differential coefficients are negative and the differential coefficient of the first or higher order is close to $-2^{n+1} n!$.

$$\zeta^{(0)}(1/2) = -1.46035450880958\cdots$$

$$\zeta^{(1)}(1/2) = -3.92264613920915\cdots \approx -2^2 1!$$

$$\zeta^{(2)}(1/2) = -16.0083570139286\cdots \approx -2^3 2!$$

$$\zeta^{(3)}(1/2) = -96.0033092453190\cdots \approx -2^4 3!$$

$$\zeta^{(4)}(1/2) = -767.997319720447\cdots \approx -2^5 4!$$

$$\zeta^{(5)}(1/2) = -7680.00060066206\cdots \approx -2^6 5!$$

$$\vdots$$

9.3.1 Maclaurin Series of Riemann Zeta

Using the higher order differential coefficiets, the Maclaurin series of Riemann zeta can be easily obtained.

Formula 9.3.1 (Maclaurin series of $\zeta(z)$)

$$\zeta(z) = \sum_{r=0}^{\infty} \frac{\zeta^{(r)}(0)}{r!} z^r \qquad |z| < 1 \tag{3.1}$$

Where,

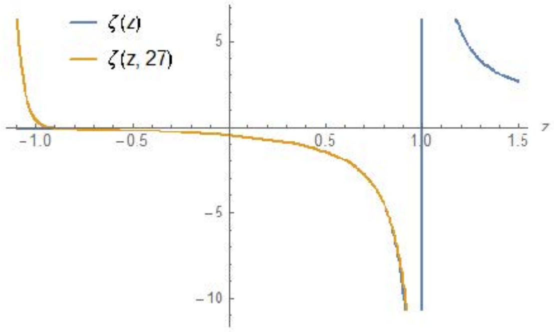
$$\zeta^{(r)}(0) = (-1)^r \sum_{s=0}^{\infty} \frac{\gamma_s}{(s-r)!} - r! \qquad r=0,1,2,\cdots$$

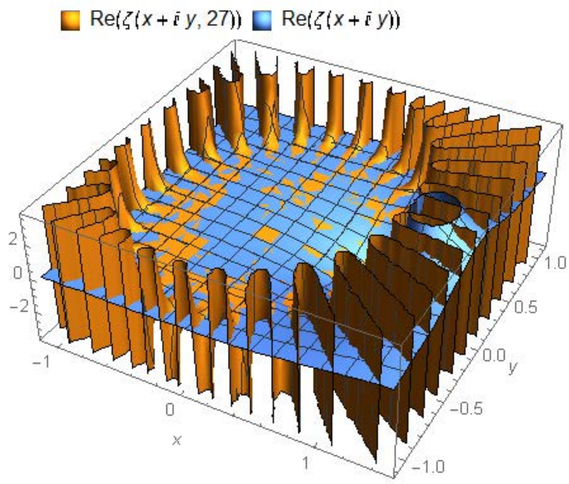
The first few terms of (3.1) are as follows. Because the absolute values of the numerator and denominator are very close, the coefficients of the 5th and higher order are almost -1.

```
N[ζ[z, 8]]
```

$$-0.5 - 0.918939\, z - 1.00318\, z^2 - 1.00079\, z^3 - 0.999879\, z^4 - 1.\, z^5 - 1.\, z^6 - 1.\, z^7 - 1.\, z^8$$

Moreover, 2D figure and 3D figure of the real part of (3.1) are as follows. The left side is blue and the right side is orange. This series (orange) can represent only the inside of the circle of 3D figure.





Therefore, this series can not represent the zeros of $\zeta(z)$. This is the reason I was hesitating the Maclaurin expansion of $\zeta(z)$ until now.

Formula 9.3.1' (Maclaurin series of $\zeta(z+1/2)$)

$$\zeta\left(z+\frac{1}{2}\right) = \sum_{r=0}^{\infty} \frac{\zeta^{(r)}(1/2)}{r\,!} z^r \qquad |z| < \frac{1}{2} \tag{3.1'}$$

Where,

$$\zeta^{(r)}(1/2) = 2^r \left\{ (-1)^r \sum_{s=0}^{\infty} \frac{\gamma_s}{2^s\,(s-r)\,!} - 2\,r! \right\} \qquad r=0,1,2,\cdots$$

The first few terms of (3.1') are as follows. Since the numerator is close to $-2^{n+1}n\,!$ and the denominator is $n\,!$, the coefficients of the 5th and higher degree are almost -2^{n+1} .

```
N[f[z, 8]]
-1.46035 - 3.92265 z - 8.00418 z^2 - 16.0006 z^3 - 31.9999 z^4
                           - 64. z^5 - 128. z^6 - 256. z^7 - 512. z^8
```

In addition, the convergence region of the series of (3.1') is within a circle with a radius of $1/2$ centered on the origin of the complex plane. Therefore, this series also can not represent the zeros of $\zeta(z+1/2)$.

9.3.2 Maclaurin Series of Holomorphized Riemann Zeta

These formulas are almost useless because the convergence range is narrow. The cause is the existence of a singular point. Because, the convergence radius of Maclaurin series is the distance from the origin to the singular point.

If so, if the singularity is removed from $\zeta(z)$, the convergence radius of the series must be infinite. To do so, we may simply multiply $\zeta(z)$ by $(z-1)$, and multiply $\zeta(z+1/2)$ by $(z-1/2)$.

Formula 9.3.2 (Maclaurin series of $(z-1)\zeta(z)$)

$$(z-1)\zeta(z) = \frac{1}{2} + \sum_{r=1}^{\infty} \frac{r\zeta^{(r-1)}(0) - \zeta^{(r)}(0)}{r!} z^r \tag{3.2}$$

Where,

$$\zeta^{(r)}(0) = (-1)^r \sum_{s=0}^{\infty} \frac{\gamma_s}{(s-r)!} - r! \qquad r=0,1,2,\cdots$$

Proof

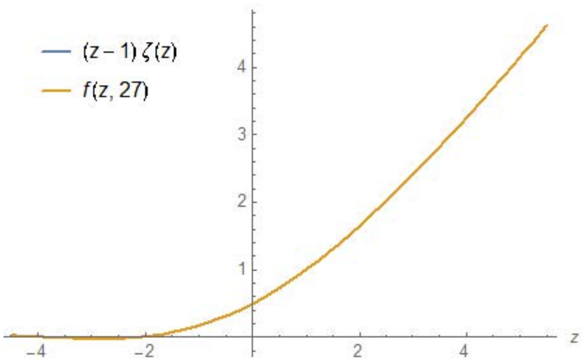
From Formula 9.3.1 (3.1) ,

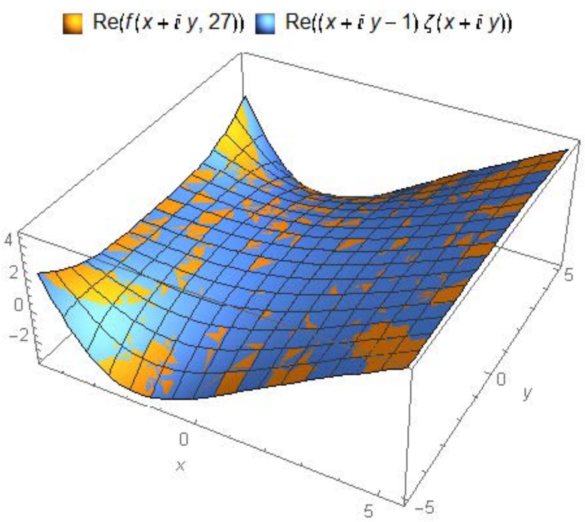
$$z\zeta(z) - \zeta(z) = \sum_{r=0}^{\infty} \frac{\zeta^{(r)}(0)}{r!} z^{r+1} - \sum_{r=0}^{\infty} \frac{\zeta^{(r)}(0)}{r!} z^r$$

$$= \sum_{r=1}^{\infty} \frac{\zeta^{(r-1)}(0)}{(r-1)!} z^r - \left\{ \frac{\zeta^{(0)}(0)}{0!} z^0 + \sum_{r=1}^{\infty} \frac{\zeta^{(r)}(0)}{r!} z^r \right\}$$

$$= \frac{1}{2} + \sum_{r=1}^{\infty} \frac{r\zeta^{(r-1)}(0) - \zeta^{(r)}(0)}{r!} z^r \tag{3.2}$$

2D figure and 3D figure of the real part of (3.2) are as follows. The left side is blue and the right side is orange. No convergence circle can be found in both figures. It seems that (3.2) holds on the whole complex plane.





In a similar way, the following is obtain from (3.1') . This also holds on the whole complex plane.

Formula 9.3.2' (Maclaurin series of $(z-1/2)\zeta(z+1/2)$)

$$\left(z-\frac{1}{2}\right)\zeta\left(z+\frac{1}{2}\right) = -\frac{1}{2}\zeta\left(\frac{1}{2}\right)\left\{1 + \sum_{r=1}^{\infty} \frac{h_r - 2rh_{r-1}}{r!} z^r\right\} \tag{3.2'}$$

Where,

$$h_r = \frac{2^r}{\zeta(1/2)}\left\{(-1)^r \sum_{s=0}^{\infty} \frac{\gamma_s}{2^s (s-r)!} - 2r!\right\} \qquad r=0,1,2,\cdots$$

9.3.3 Maclaurin Series of Completed Riemann Zeta

In a similar way to **9.1** and **9.2** , the followings are obtained using (3.2) and (3.2') .

Theorem 9.3.3 (Maclaurin series of $\xi(z)$)

Let completed Riemann zeta be

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Then, the following expression holds on the whole complex plane.

$$\xi(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}(-1)^{r-s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{\Gamma^{(s-t)}(1)}{2^{s-t}(s-t)!}\frac{2t\zeta_0^{(t-1)}-2\zeta_0^{(t)}}{t!}z^r$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$\Gamma^{(r)}(1) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) & r=1,2,3,\cdots \end{cases}$$

$$\zeta_0^{(r)} = \begin{cases} 0 & r=-1 \\ (-1)^r\sum_{s=0}^{\infty}\frac{\gamma_s}{(s-r)!} - r! & r=0,1,2,\cdots \end{cases}$$

cf.

Theorem 9.1.3 was as follows.

$$\xi(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{(-1)^{s-t}g_{s-t}(3/2)}{2^{s-t}(s-t)!}c_t z^r$$

Where,

$$c_r = \begin{cases} 1 & r=0 \\ -\frac{\gamma_{r-1}}{(r-1)!} & r=1,2,3,\cdots \end{cases}$$

This expression is a Maclaurin expansion of

$$\xi(z) = \xi(1-z) = \pi^{\frac{z}{2}}\left\{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3-z}{2}\right)\right\}\{-z\zeta(1-z)\}$$

This is a rear attack. The weak point is a little hard to overlook the whole. The storong point is that the differential coefficient of $-z\zeta(1-z)$ is obtained from only one γ_s . Therefore, although high precision is required, it is possible to calculate these zeros even on notebook computers.

On the contrary, Theorem 9.3.3 is Maclaurin expansion of

$$\xi(z) = \pi^{-\frac{z}{2}}\Gamma\left(1+\frac{z}{2}\right)\{2(z-1)\zeta(z)\}$$

This is a frontal attack. The strong point is easy to overlook the whole. The weak point is that γ_s has to be accumulated infinitely to obtain the differential coefficient of $2(z-1)\zeta(z)$. For this purpose, high precision and a large calculation amount are required. Therefore, it is almost impossible to calculate these zeros on note book computers according to Theorem 9.3.3 .

Theorem 9.3.3' (Maclaurin series of $\Xi(z)$)

Let completed Riemann zeta be

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right)$$

Then, the following expression holds on the whole complex plane.

$$\Xi(z) = \Xi(0)\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}(-1)^{r-s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{g_{s-t}}{2^{s-t}(s-t)!}\frac{h_t-2t\,h_{t-1}}{t!}z^r$$

$$\Xi(0) = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.9942415563\cdots$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{5}{4}\right) = \begin{cases} 1 & r=0 \\ \displaystyle\sum_{k=1}^{r}B_{r,k}\left(\psi_0\left(\frac{5}{4}\right), \psi_1\left(\frac{5}{4}\right), \ldots, \psi_{r-1}\left(\frac{5}{4}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$h_r = \begin{cases} 0 & r=-1 \\ \displaystyle\frac{2^r}{\zeta(1/2)}\left\{(-1)^r\sum_{s=0}^{\infty}\frac{\gamma_s}{2^s(s-r)!} - 2r!\right\} & r=0,1,2,\cdots \end{cases}$$

cf.

Theorem 9.2.3 was as follows.

$$\Xi(z) = \Xi(0)\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}(-1)^{r-s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{g_{s-t}(5/4)}{2^{s-t}(s-t)!}c_t z^r$$

Where,

$$c_r = \begin{cases} 1 & r=0 \\ \displaystyle\frac{2}{\zeta(1/2)}\sum_{s=r}^{\infty}(-1)^r\frac{\gamma_{s-1}}{(s-1)!}\binom{s}{r}\left(\frac{1}{2}\right)^{s-r} & r=1,2,3,\cdots \end{cases}$$

This theorem and Theorem 9.3.3' are almost the same. Both are orthodox approaches. The difference is the the formula for the differential coefficient of holomorphized Riemann zeta. Comparing c_r and h_r, we see that the former is half of the latter at the amount of calculation. This difference is not small. When calculating zeros by using a notebook computer, the calculation by Thorem 9.2.3 is somehow possible, but the calculation by Thorem 9.3.3' is almost impossible.

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