3 Generalized Multinomial Theorem

3.1 Binomial Theorem

Theorem 3.1.1

If x_1, x_2 are real numbers and n is a positive integer, then

$$\left(x_1+x_2\right)^n = \sum_{r=0}^{n} {}_nC_r\, x_1^{n-r} x_2^{r} \qquad (1.1)$$

Binomial Coefficients

Binomial Coefficient in (1.1) is a positive number and is described as ${}_nC_r$. Here, n and r are both non-negative integer. ${}_nC_r$ is the number of ways of picking r unordered outcomes from n possibilities and is calculated as follows.

$$ {}_nC_r = \frac{n!}{(n-r)!\; r!}$$

Pascal's Triangle

About ${}_nC_r$, what arranged n in the row and r in the column is called ***Pascal's Triangle***.

$$\begin{array}{ccccccccc} & & & & {}_0C_0 & & & & \\ & & & {}_1C_0 & & {}_1C_1 & & & \\ & & {}_2C_0 & & {}_2C_1 & & {}_2C_2 & & \\ & {}_3C_0 & & {}_3C_1 & & {}_3C_2 & & {}_3C_3 & \\ {}_4C_0 & & {}_4C_1 & & {}_4C_2 & & {}_4C_3 & & {}_4C_4 \\ & & & & \vdots & & & & \end{array} \qquad \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \vdots & & & & \end{array}$$

Propertis of the Binomial Coefficient

Although a lot of properties of the binomial coefficient are known, fundamental (understood from Pascal's Triangle immediately) some are as follows. Among these, **ii** is used for step-by-step calculation of ${}_nC_r$.

i ${}_nC_r = {}_nC_{n-r}$

ii ${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$

iii ${}_nC_r = \sum_{k=r-1}^{n-1} {}_kC_{r-1}$

iv $\sum_{r=0}^{n} {}_nC_r = 2^n$

v $\sum_{r=0}^{n} (-1)^r {}_nC_r = 0$

3.2 Generalized Binomial Theorem

3.2.1 Newton's Generalized Binomial Theorem

Theorem 3.2.1

The following formulas hold for a real number α .

$$(1+x)^{\alpha} = \begin{cases} \displaystyle\sum_{r=0}^{\infty} \binom{\alpha}{r} x^{r} & |x| \leq 1 \quad (\ |x| = 1 \text{ is allowed at } \alpha > 0\ ) \quad (1.1) \\ \displaystyle\sum_{r=0}^{\infty} \binom{\alpha}{\alpha-r} x^{\alpha-r} & |x| > 1 \quad (1.2) \end{cases}$$

Proof

When n is a natural number, the following expression holds from the binomial theorem.

$$(1+x)^{n} = \sum_{r=0}^{n} {}_{n}C_{r} x^{r}$$

Since ${}_{n}C_{r} = 0 \ \ for \ \ r > n$, this can be written as follows.

$$(1+x)^{n} = \sum_{r=0}^{\infty} {}_{n}C_{r} x^{r}$$

Extending n to real number α ,

$$(1+x)^{\alpha} = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^{r} = \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1)\ r!} x^{r} \tag{1.1}$$

Here, let

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1)\ r!} x^{r} \equiv a_{r} .$$

Then

$$\frac{a_{r+1}}{a_{r}} = \frac{\dfrac{\Gamma(p+1)}{\Gamma(p-r)\ (r+1)!} x^{r+1}}{\dfrac{\Gamma(p+1)}{\Gamma(p-r+1)\ r!} x^{r}} = \frac{(p-r)x}{r+1} .$$

From this,

$$\lim_{r\to\infty} \left| \frac{a_{r+1}}{a_{r}} \right| = \lim_{r\to\infty} \left| \frac{(p-r)x}{r+1} \right| = \lim_{r\to\infty} \left| \frac{(p/r-1)x}{1+1/r} \right| = \left| \frac{-x}{1} \right| = |x|$$

According to d'Alembert's ratio test, (1.1) converges absolutely if $|x| < 1$.

Next, if $|x| > 1$ then $|x^{-1}| < 1$. Therefore, from (1.1) ,

$$\left(1+x^{-1}\right)^{\alpha} = \sum_{r=0}^{\infty} \binom{\alpha}{r} \left(x^{-1}\right)^{r}$$

Multiplying by x^{α} both sides,

$$(x+1)^{\alpha} = x^{\alpha} \sum_{r=0}^{\infty} \binom{\alpha}{r} x^{-r} = \sum_{r=0}^{\infty} \binom{\alpha}{\alpha-r} x^{\alpha-r} \tag{1.2}$$

The proof in case of $|x| = 1$ is accomplished in the following sub section..

3.2.2 General Binomial Coefficient

The coefficient $\binom{\alpha}{r}$ in Theorem 3.2.1 is called ***General Binomial Coefficient*** and is as follows.

$$\binom{\alpha}{r} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1)\Gamma(r+1)} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-r+1)}{r!} \tag{2.0}$$

The first few are as follows.

$$\binom{\alpha}{0} = 1,\ \binom{\alpha}{1} = \frac{\alpha}{1!},\ \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2!},\ \binom{\alpha}{3} = \frac{\alpha(\alpha-1)(\alpha-2)}{3!},\ \cdots$$

Although properties similar to binomial coefficient also about general binomial coefficient are known, especially an important thing is sum of the general binomial coefficient. We prepare some Lemma, in order to obtain this.

Lemma 3.2.2

When α is not positive integer, binomial series $\sum_{r=0}^{\infty} a_r \binom{\alpha-1}{r}$ converges or diverges simultaneously with Dirichlet series $\sum_{r=1}^{\infty} (-1)^r \frac{a_r}{r^{\alpha}}$.

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Lemma 3.2.3

$\sum_{r=0}^{\infty} \binom{\alpha}{r}$ converges absolutely for non-integer $\alpha > 0$.

Proof

$$\binom{\alpha}{r} = \frac{\alpha}{\alpha-r}\binom{\alpha-1}{r}$$

Then

$$\sum_{r=0}^{\infty}\binom{\alpha}{r} = \sum_{r=0}^{\infty}\frac{\alpha}{\alpha-r}\binom{\alpha-1}{r}$$

Let $a_r = \frac{\alpha}{\alpha-r}$. Then

$$\sum_{r=0}^{\infty} a_r \binom{\alpha-1}{r} = \sum_{r=0}^{\infty}\binom{\alpha}{r} \tag{s1}$$

$$\sum_{r=1}^{\infty} (-1)^r \frac{a_r}{r^{\alpha}} = \sum_{r=1}^{\infty} (-1)^r \frac{\alpha}{r^{\alpha}(\alpha-r)} \tag{s2}$$

Here, let

$$(-1)^r \frac{\alpha}{r^{\alpha}(\alpha-r)} \equiv b_r$$

Then

$$\frac{b_{r+1}}{b_r} = \frac{(-1)^{r+1}\dfrac{\alpha}{(r+1)^{\alpha}(\alpha-r-1)}}{(-1)^{r}\dfrac{\alpha}{r^{\alpha}(\alpha-r)}} = -\frac{r^{\alpha}(\alpha-r)}{(r+1)^{\alpha}(\alpha-r-1)}$$

From this

$$\lim_{r\to\infty}\left|\frac{b_{r+1}}{b_r}\right| = \lim_{r\to\infty}\left|\frac{r^{\alpha}(\alpha-r)}{(r+1)^{\alpha}(\alpha-r-1)}\right| = \lim_{r\to\infty}\left|\frac{\alpha r^{\alpha}-r^{\alpha+1}}{\alpha(r+1)^{\alpha}-(r+1)^{\alpha+1}}\right|$$

$$= \lim_{r\to\infty}\left|\frac{r^{\alpha+1}}{(r+1)^{\alpha+1}}\frac{\dfrac{\alpha r^{\alpha}}{r^{\alpha+1}}-1}{\dfrac{\alpha(r+1)^{\alpha}}{(r+1)^{\alpha+1}}-1}\right| = \lim_{r\to\infty}\left|\left(\frac{1}{1+\dfrac{1}{r}}\right)^{\alpha+1}\frac{\dfrac{\alpha}{r}-1}{\dfrac{\alpha}{r+1}-1}\right| = 1$$

Since the judgment is impossible, we try Raabe's test for convergence.

$$\lim_{r\to\infty} r\left(\left|\frac{b_r}{b_{r+1}}\right|-1\right) = \lim_{r\to\infty} r\left(\left|\frac{(r+1)^{\alpha}(\alpha-r-1)}{r^{\alpha}(\alpha-r)}\right|-1\right)$$

$$= \lim_{r\to\infty}\left(\left|\frac{r(r+1)^{\alpha}}{r^{\alpha}}-\frac{r(r+1)^{\alpha}}{r^{\alpha}(\alpha-r)}\right|-r\right)$$

$$= \lim_{r\to\infty}\left(\left|r\left(1+\frac{1}{r}\right)^{\alpha}-\left(1+\frac{1}{r}\right)^{\alpha}\frac{r}{\alpha-r}\right|-r\right)$$

$$= \lim_{r\to\infty}\left(\left(1+\frac{1}{r}\right)^{\alpha}\left|r+\frac{1}{1-\alpha/r}\right|-r\right)$$

Since $1-\alpha/r>0$ for sufficiently large r ,

$$\lim_{r\to\infty} r\left(\left|\frac{b_r}{b_{r+1}}\right|-1\right) = \lim_{r\to\infty}\left(\left(1+\frac{1}{r}\right)^{\alpha}r+\left(1+\frac{1}{r}\right)^{\alpha}\frac{1}{1-\alpha/r}-r\right)$$

$$= \lim_{r\to\infty} r\left\{\left(1+\frac{1}{r}\right)^{\alpha}-1\right\}+\lim_{r\to\infty}\left(1+\frac{1}{r}\right)^{\alpha}\frac{1}{1-\alpha/r}$$

Here,

$$\left(1+\frac{1}{r}\right)^{\alpha} = \sum_{s=0}^{\infty}\binom{\alpha}{s}\frac{1}{r^{s}} = 1+\frac{\alpha}{1!}\frac{1}{r^{1}}+\frac{\alpha(\alpha-1)}{2!}\frac{1}{r^{2}}+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\frac{1}{r^{3}}+\cdots$$

Then,

$$r\left\{\left(1+\frac{1}{r}\right)^{\alpha}-1\right\} = r\sum_{s=1}^{\infty}\binom{\alpha}{s}\frac{1}{r^{s}} = \sum_{s=1}^{\infty}\binom{\alpha}{s}\frac{1}{r^{s-1}} = \alpha+\sum_{s=2}^{\infty}\binom{\alpha}{s}\frac{1}{r^{s-1}}$$

Therfore,

$$\lim_{r\to\infty} r\left\{\left(1+\frac{1}{r}\right)^{\alpha}-1\right\} = \alpha+\lim_{r\to\infty}\sum_{s=2}^{\infty}\binom{\alpha}{s}\frac{1}{r^{s-1}} = \alpha$$

Moreover,

$$\lim_{r\to\infty}\left(1+\frac{1}{r}\right)^{\alpha}\frac{1}{1-\alpha/r} = 1$$

After all,

$$\lim_{r \to \infty} r\left(\left| \frac{b_r}{b_{r+1}} \right| -1 \right) = \alpha+1$$

Thus, if $\alpha>0$, (s2) converges absolutely. Then, (s1) also converges absolutely according to **Lemma 3.2.2** .

Theorem 3.2.4

The following expressions hold for arbitrary real number $\alpha>0$.

$$\sum_{r=0}^{\infty} \binom{\alpha}{r} = 2^{\alpha} \tag{2.1}$$

$$\sum_{r=0}^{\infty} (-1)^r \binom{\alpha}{r} = 0 \tag{2.2}$$

Proof

According to **Lemma 3.2.3 ,** $\sum_{r=0}^{\infty} \binom{\alpha}{r}$ converges absolutely for non-integer $\alpha>0$.

Therefore, from Theorem 3.2.1 (1.1) ,

$$\sum_{r=0}^{\infty} \binom{\alpha}{r} 1^r = (1+1)^{\alpha} = 2^{\alpha} \tag{2.1}$$

$$\sum_{r=0}^{\infty} \binom{\alpha}{r} (-1)^r = (1-1)^{\alpha} = 0 \tag{2.2}$$

Note

In fact, it is known that (2.1) holds if $\alpha>-1$. (Where, it is conditional convergence.)

For example, when $\alpha=-0.9$, the right side is $2^{-0.9}=0.53588673\cdots$ and the left side seems to converge to this value. However, the confirmation is difficult as the convergence is very slow. So, we apply Knopp Transformation to this and accelerate the convergence. It is as follows.

```
fa[α_, q_, m_] := Sum[Sum[q^(k-r)/(q+1)^(k+1) Binomial[k, r] Binomial[α, r], {r, 0, k}], {k, 0, m}]

SetPrecision[fa[-0.9, 3/5, 15], 10]          0.5358867304
```

We can see that (2.1) converges to $2^{-0.9}$.

3.2.3 Generalized Binomial Theorem

Theorem 3.2.1 can be further generalized.

Theorem 3.2.5

When α is a real number, the following expression holds for x_1, x_2 s.t. $|x_1| \ge |x_2|$.

$$(x_1+x_2)^{\alpha} = \sum_{r=0}^{\infty} \binom{\alpha}{r} x_1^{\alpha-r} x_2^{r} \qquad (x_1 = x_2 \text{ is allowed at } \alpha>0) \tag{3.1}$$

Proof

If $|x_1| \ge |x_2|$ then $1 > \frac{|x_2|}{|x_1|} = \left| \frac{x_2}{x_1} \right|$. Therefore, using Theorem 3.2.1 (1.1) ,

$$
\begin{aligned}
(x_1+x_2)^{\alpha} &= \left\{ x_1\left(1+\frac{x_2}{x_1}\right)\right\}^{\alpha} = x_1^{\alpha}\left(1+\frac{x_2}{x_1}\right)^{\alpha} \\
&= x_1^{\alpha}\left\{ 1 + \binom{\alpha}{1}\frac{x_2}{x_1} + \binom{\alpha}{2}\left(\frac{x_2}{x_1}\right)^2 + \binom{\alpha}{3}\left(\frac{x_2}{x_1}\right)^3 + \cdots \right\} \\
&= \binom{\alpha}{0}x_1^{\alpha} + \binom{\alpha}{1}x_1^{\alpha-1}x_2^1 + \binom{\alpha}{2}x_1^{\alpha-2}x_2^2 + \binom{\alpha}{3}x_1^{\alpha-3}x_2^3 + \cdots \\
&= \sum_{r=0}^{\infty}\binom{\alpha}{r}x_1^{\alpha-r}x_2^{r}
\end{aligned}
$$

Note

As is clear from the process of the proof, if $|x_1| \ge |x_2|$ then (3.1) converges absolutely.

Where, $|x_1| = |x_2|$ is allowed at $\alpha > 0$.

This becomes important in Generalized Multinomial Theorem.

3.3 Multinomial Theorem

Theorem 3.3.0

For real numbers $x_1, x_2, \cdots, x_m$ and non negative integers $n, r_1, r_2, \cdots, r_m$, the followings hold.

$$\left(x_1+x_2+\cdots+x_m\right)^n = \sum \frac{n!}{r_1!\ r_2! \cdots r_m!} x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m} \tag{0.1}$$

where Σ denotes the sum of all combinations of $\left(r_1, r_2, \cdots, r_m\right)$ s.t. $r_1+r_2+\cdots+r_m = n$.

$$\left(x_1+x_2+\cdots+x_m\right)^n = \sum \frac{n!}{\left(n-r_1 \cdots -r_{m-1}\right)!\ r_1! \cdots r_{m-1}!} x_1^{n-r_1 \cdots -r_{m-1}} x_2^{r_1} \cdots x_m^{r_{m-1}} \tag{0.2}$$

where Σ denotes the sum of all possible combinations of $\left(n, r_1, r_2, \cdots, r_{m-1}\right)$.

Since (0.1) is well known, the proof is omitted. In addition, it is also clear (0.2) and (0.1) are synonymous. These are near a definition rather than a theorem.

How to generate multinomial coefficients

Theorem 3.3.0 is not difficult in theory. Difficulty is its proviso. This is to actually generate combinations (m choose n) with repetition. But this is not easy when it becomes more than 3 terms. Since I found out the formulas which generates these without leak, I present it here as a theorem. (1.2) realizes the provis by an iterated series (multiple series) and (1.1) realizes it by a diagonal series (half-multiple series).

Theorem 3.3.1

For real numbers $x_1, x_2, \cdots, x_m$ and a natural number n , the following expressions hold.

$$\left(x_1+x_2+\cdots+x_m\right)^n = \sum_{r_1=0}^{n} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{m-1}=0}^{r_{m-2}} \binom{n}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{m-2}}{r_{m-1}} x_1^{n-r_1} x_2^{r_1-r_2} \cdots x_{m-1}^{r_{m-2}-r_{m-1}} x_m^{r_{m-1}} \tag{1.1}$$

$$= \sum_{r_1=0}^{n} \sum_{r_2=0}^{n} \cdots \sum_{r_{m-1}=0}^{n} \binom{n}{r_1+r_2+\cdots+r_{m-1}} \binom{r_1+r_2+\cdots+r_{m-1}}{r_2+\cdots+r_{m-1}} \cdots \binom{r_{m-2}+r_{m-1}}{r_{m-1}}$$

$$\times\ x_1^{n-r_1-\cdots-r_{m-1}} x_2^{r_1} x_3^{r_2} \cdots x_m^{r_{m-1}} \tag{1.2}$$

Proof

According to Theorem 3.1.1 , the following expressions hold.

$$\left(x_1+x_2+x_3+x_4+\cdots+x_m\right)^n = \sum_{r_1=0}^{n} {}_nC_{r_1} x_1^{n-r_1} \left(x_2+x_3+x_4+\cdots+x_m\right)^{r_1} \tag{1}$$

$$\left(x_2+x_3+x_4+\cdots+x_m\right)^{r_1} = \sum_{r_2=0}^{r_1} {}_{r_1}C_{r_2} x_2^{r_1-r_2} \left(x_3+x_4+\cdots+x_m\right)^{r_2} \tag{2}$$

$$\left(x_3+x_4+\cdots+x_m\right)^{r_2} = \sum_{r_3=0}^{r_2} {}_{r_2}C_{r_3} x_3^{r_2-r_3} \left(x_4+\cdots+x_m\right)^{r_3} \tag{3}$$

$$\vdots$$

$$\left(x_{m-2}+x_{m-1}+x_m\right)^{r_{m-3}} = \sum_{r_{m-2}=0}^{r_{m-3}} {}_{r_{m-3}}C_{r_{m-2}} x_{m-2}^{r_{m-3}-r_{m-2}} \left(x_{m-1}+x_m\right)^{r_{m-2}} \tag{m-2}$$

$$\left(x_{m-1}+x_m\right)^{r_{m-2}} = \sum_{r_{m-1}=0}^{r_{m-2}} {}_{r_{m-2}}C_{r_{m-1}} x_{m-1}^{r_{m-2}-r_{m-1}} x_m^{r_{m-1}} \qquad (m-1)$$

Substituting (2), (3), $\cdots$, $(m-2)$, $(m-1)$ for (1) one by one, we obtain (1.1) .

Next, according to Theorem 3.4.1 (later) , when $|x_1| \ge |x_2+x_3+\cdots+x_m|$,

$$\left(x_1+x_2+\cdots+x_m\right)^{\alpha} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{m-1}=0}^{\infty}\binom{\alpha}{r_1+r_2+\cdots+r_{m-1}}\binom{r_1+r_2+\cdots+r_{m-1}}{r_2+\cdots+r_{m-1}}\cdots\binom{r_{m-2}+r_{m-1}}{r_{m-1}}$$
$$\times\, x_1^{\alpha-r_1-\cdots-r_{m-1}} x_2^{r_1} x_3^{r_2}\cdots x_m^{r_{m-1}}$$

Replacing the real number α with non-negative integer n ,

$$\left(x_1+x_2+\cdots+x_m\right)^{n} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{m-1}=0}^{\infty}\binom{\alpha}{r_1+r_2+\cdots+r_{m-1}}\binom{r_1+r_2+\cdots+r_{m-1}}{r_2+\cdots+r_{m-1}}\cdots\binom{r_{m-2}+r_{m-1}}{r_{m-1}}$$
$$\times\, x_1^{n-r_1-\cdots-r_{m-1}} x_2^{r_1} x_3^{r_2}\cdots x_m^{r_{m-1}}$$

Since $\binom{n}{r} = 0 \;\; for \;\; r>n, \;\; r=1,2,3,\cdots$, this is a definite multiple series. Therefore, the condition $|x_1| \ge |x_2+x_3+\cdots+x_m|$ is unnecessary. Although this is not bad as it is, replacing ∞ on the Σ with n , we obtain (1.2) .

cf.

(1.2) results in (0.2) . Because,

$$\binom{n}{r_1+r_2+\cdots+r_{m-1}}\binom{r_1+r_2+\cdots+r_{m-1}}{r_2+\cdots+r_{m-1}}\cdots\binom{r_{m-2}+r_{m-1}}{r_{m-1}} = \frac{n!}{(n-r_1-\cdots-r_{m-1})!\,r_1!+\cdots+r_{m-1}!}$$

Example 1: The expasion of $(x_1+x_2+x_3)^4$

Using (1.1) ,

$$(x_1+x_2+x_3)^4 = \sum_{r=0}^{4}\sum_{s=0}^{r}\binom{4}{r}\binom{r}{s}x_1^{4-r}x_2^{r-s}x_3^{s}$$
$$= \binom{4}{0}x_1^4\sum_{s=0}^{0}\binom{0}{s}x_2^{0-s}x_3^{s}$$
$$+\binom{4}{1}x_1^3\sum_{s=0}^{1}\binom{1}{s}x_2^{1-s}x_3^{s} + \binom{4}{2}x_1^2\sum_{s=0}^{2}\binom{2}{s}x_2^{2-s}x_3^{s}$$
$$+\binom{4}{3}x_1^1\sum_{s=0}^{3}\binom{3}{s}x_2^{3-s}x_3^{s} + \binom{4}{4}x_1^0\sum_{s=0}^{4}\binom{4}{s}x_2^{4-s}x_3^{s}$$
$$= x_1^4$$
$$+\,4x_1^3(x_2+x_3)$$
$$+\,6x_1^2(x_2^2+2x_2x_3+x_3^2)$$
$$+\,4x_1(x_2^3+3x_2^2x_3+3x_2x_3^2+x_3^3)$$
$$+\,x_2^4+4x_2^3x_3+6x_2^2x_3^2+4x_2x_3^3+x_3^4$$

Using (1.2) ,

$$(x_1+x_2+x_3)^4 = \sum_{r=0}^{4}\sum_{s=0}^{4}\binom{4}{r+s}\binom{r+s}{s}x_1^{4-r-s}x_2^{r}x_3^{s}$$

$$= \sum_{s=0}^{4}\binom{4}{0+s}\binom{0+s}{s}x_1^{4-s}x_2^{0}x_3^{s}$$

$$+ \sum_{s=0}^{4}\binom{4}{1+s}\binom{1+s}{s}x_1^{3-s}x_2^{1}x_3^{s} + \sum_{s=0}^{4}\binom{4}{2+s}\binom{2+s}{s}x_1^{2-s}x_2^{2}x_3^{s}$$

$$+ \sum_{s=0}^{4}\binom{4}{3+s}\binom{3+s}{s}x_1^{1-s}x_2^{3}x_3^{s} + \sum_{s=0}^{4}\binom{4}{4+s}\binom{4+s}{s}x_1^{0-s}x_2^{4}x_3^{s}$$

$$\begin{aligned} = \quad & x_1^4 + 4x_1^3x_3 + 6x_1^2x_3^2 + 4x_1x_3^3 + x_3^4 \\ & + 4x_1^3x_2 + 12x_1^2x_2x_3 + 12x_1x_2x_3^2 + 4x_2x_3^3 + 0 \\ & + 6x_1^2x_2^2 + 12x_1x_2^2x_3 + 6x_2^2x_3^2 + 0 + 0 \\ & + 4x_1x_2^3 + 4x_2^3x_3 + 0 + 0 + 0 \\ & + x_2^4 + 0 + 0 + 0 + 0 \end{aligned}$$

we can see that what totaled this along the diagonal line is equal to the above.

Example 2: The expasion of $(x_1+x_2+x_3+x_4)^3$

Now, the formulas of the theorem are expanded using mathematical software. (1.1) and (1.2) are expanded and are verified respectively.

```
fl[n_] := (x1 + x2 + x3 + x4)^n
Expand[fl[3]]
```

$x_1^3+3x_1^2x_2+3x_1x_2^2+x_2^3+3x_1^2x_3+6x_1x_2x_3+3x_2^2x_3+3x_1x_3^2+3x_2x_3^2+x_3^3+3x_1^2x_4$
$+6x_1x_2x_4+3x_2^2x_4+6x_1x_3x_4+6x_2x_3x_4+3x_3^2x_4+3x_1x_4^2+3x_2x_4^2+3x_3x_4^2+x_4^3$

$$\text{fr}[n\_] := \sum_{r=0}^{n}\sum_{s=0}^{r}\sum_{t=0}^{s}\text{Binomial}[n, r]\,\text{Binomial}[r, s]\,\text{Binomial}[s, t] \times x_1^{n-r}x_2^{r-s}x_3^{s-t}x_4^{t}$$

```
fr[3]
```

$x_1^3+3x_1^2x_2+3x_1x_2^2+x_2^3+3x_1^2x_3+6x_1x_2x_3+3x_2^2x_3+3x_1x_3^2+3x_2x_3^2+x_3^3+3x_1^2x_4$
$+6x_1x_2x_4+3x_2^2x_4+6x_1x_3x_4+6x_2x_3x_4+3x_3^2x_4+3x_1x_4^2+3x_2x_4^2+3x_3x_4^2+x_4^3$

```
Expand[fl[3]] == fr[3]
True
```

$$\text{fs}[n\_] := \sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{t=0}^{n}\text{Binomial}[n, r+s+t]\,\text{Binomial}[r+s+t, s+t] \times \text{Binomial}[s+t, t]\,x_1^{n-r-s-t}x_2^{r}x_3^{s}x_4^{t}$$

```
N[fs[3]]
```

$x_1^3+3.x_1^2x_2+3.x_1x_2^2+x_2^3+3.x_1^2x_3+6.x_1x_2x_3+3.x_2^2x_3+3.x_1x_3^2+3.x_2x_3^2+x_3^3+3.x_1^2x_4$
$+6.x_1x_2x_4+3.x_2^2x_4+6.x_1x_3x_4+6.x_2x_3x_4+3.x_3^2x_4+3.x_1x_4^2+3.x_2x_4^2+3.x_3x_4^2+x_4^3$

```
Expand[fl[3]] == fs[3]
True
```

Sum of multinomial coefficients

$$\sum_{r_1=0}^{n}\sum_{r_2=0}^{r_1}\cdots\sum_{r_{m-1}=0}^{r_{m-2}} {}_nC_{r_1}\,{}_{r_1}C_{r_2}\cdots{}_{r_{m-2}}C_{r_{m-1}} = m^n \tag{1.1"}$$

Proof

$$\sum_{r=0}^{n} {}_nC_r = \sum_{r=0}^{n} {}_nC_r 1^{n-r}1^r = (1+1)^n = 2^n$$

$$\sum_{r=0}^{n}\sum_{s=0}^{r} {}_nC_r\,{}_rC_s = \sum_{r=0}^{n} {}_nC_r\left(\sum_{s=0}^{r} {}_rC_s\right) = \sum_{r=0}^{n} {}_nC_r 1^{n-r}2^r = (1+2)^n = 3^n$$

$$\sum_{r=0}^{n}\sum_{s=0}^{r}\sum_{t=0}^{s} {}_nC_r\,{}_rC_s\,{}_sC_t = \sum_{r=0}^{n} {}_nC_r\left(\sum_{s=0}^{r}\sum_{t=0}^{s} {}_rC_s\,{}_sC_t\right) = \sum_{r=0}^{n} {}_nC_r 1^{n-r}3^r = (1+3)^n = 4^n$$

Hereafter, by induction we obtain the desired expression.

Example: Sum of multinomial coefficients of $(x_1+x_2+x_3)^4$

Let's calculate sum of multinomial coefficients in **Example 1** . Then it is as follows.

$$1+(4+4)+(6+12+6)+(4+12+12+4)+(1+4+6+4+1) = 81 = 3^4$$

3.4 Generalized Multinomial Theorem

Although I do not know whether the theorem like generalized multinomial theorem exists or not , since this is essential for Higher Calculus of Function Product, I present this here.

Theorem 3.4.1

The following expressions hold for real numbers α and $x_1, x_2, \cdots, x_m$ $s.t.$ $|x_1| \ge |x_2+x_3+\cdots+x_m|$.

$$\left(x_1+x_2+\cdots+x_m\right)^{\alpha} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_{m-1}=0}^{r_{m-2}}\binom{\alpha}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{m-2}}{r_{m-1}} x_1^{\alpha-r_1}x_2^{r_1-r_2}\cdots x_{m-1}^{r_{m-2}-r_{m-1}}x_m^{r_{m-1}} \tag{1.1}$$

$$= \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{m-1}=0}^{\infty}\binom{\alpha}{r_1+r_2+\cdots+r_{m-1}}\binom{r_1+r_2+\cdots+r_{m-1}}{r_2+\cdots+r_{m-1}}\cdots\binom{r_{m-2}+r_{m-1}}{r_{m-1}} \times x_1^{\alpha-r_1-\cdots-r_{m-1}}x_2^{r_1}x_3^{r_2}\cdots x_m^{r_{m-1}} \tag{1.2}$$

Where, $|x_1| = |x_2+x_3+\cdots+x_m|$ is allowed at $\alpha>0$.

Proof

From Theorem 3.2.5 , when $|x_1| \ge |x_2+x_3+\cdots+x_m|$, the following expression holds.

$$\left(x_1+x_2+x_3+x_4+\cdots+x_m\right)^{\alpha} = \sum_{r_1=0}^{\infty}\binom{\alpha}{r_1}x_1^{\alpha-r_1}\left(x_2+x_3+x_4+\cdots+x_m\right)^{r_1}$$

Here, the right side converges absolutely.

On the other hand, from Theorem 3.3.1 (1.1) , the following expression holds.

$$\left(x_1+x_2+\cdots+x_m\right)^{r_1} = \sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\cdots\sum_{r_{m-1}=0}^{r_{m-2}}\binom{r_1}{r_2}\binom{r_2}{r_3}\cdots\binom{r_{m-2}}{r_{m-1}} x_2^{r_1-r_2}x_3^{r_2-r_3}\cdots x_{m-1}^{r_{m-2}-r_{m-1}}x_m^{r_{m-1}}$$

Substituting the latter for the former ,

$$\left(x_1+x_2+\cdots+x_m\right)^{\alpha} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_{m-1}=0}^{r_{m-2}}\binom{\alpha}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{m-2}}{r_{m-1}} x_1^{\alpha-r_1}x_2^{r_1-r_2}\cdots x_{m-1}^{r_{m-2}-r_{m-1}}x_m^{r_{m-1}} \tag{1.1}$$

Naturally, the right side also converges absolutely.

Next, let us describe a multiple series and its iterated series as follows respectively.

$$\sum_{r_1,r_2,\cdots,r_m=0}^{\infty} a_{r_1,r_2,\cdots,r_m} \quad , \quad \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_m=0}^{\infty} a_{r_1,r_2,\cdots,r_{m-1},r_m}$$

In order to convert the iterated series to its diagonal series, we should just perform the following operations. (See " **02 Multiple Series & Exponential Function** ").

Replace r_{m-1} with $r_{m-1}-r_m$, and replace the 1st ∞ with r_{m-1} from the right.

Replace r_{m-2} with $r_{m-2}-r_{m-1}$, and replace the 2nd ∞ with r_{m-2} from the right.

⋮

Replace r_1 with r_1-r_2 , and replace the $(m-1)$th ∞ with r_1 from the right.

If so, in order to return the diagonal series to the original iterated series, we should just perform this opposite operation. That is,

Replace r_1 with r_1+r_2 , and replace r_1 on the 2nd Σ with ∞ from the left.

Replace r_2 with r_2+r_3 , and replace r_2 on the 3rd Σ with ∞ from the left.

⋮

Replace r_{m-1} with $r_{m-1}+r_m$, and replace r_{m-1} on the $(m-1)$th Σ with ∞ from the left.
For example,

$$(x_1+x_2+x_3+x_4)^{\alpha} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\binom{\alpha}{r_1}\binom{r_1}{r_2}\binom{r_2}{r_3} x_1^{\alpha-r_1}x_2^{r_1-r_2}x_3^{r_2-r_3}x_4^{r_3}$$

$$r_1 \to r_1+r_2,\ \sum^{r_1} \to \sum^{\infty};\quad = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{r_2}\binom{\alpha}{r_1+r_2}\binom{r_1+r_2}{r_2}\binom{r_2}{r_3} x_1^{\alpha-r_1-r_2}x_2^{r_1}x_3^{r_2-r_3}x_4^{r_3}$$

$$r_2 \to r_2+r_3,\ \sum^{r_2} \to \sum^{\infty};\quad = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\binom{\alpha}{r_1+r_2+r_3}\binom{r_1+r_2+r_3}{r_2+r_3}\binom{r_2+r_3}{r_3} x_1^{\alpha-r_1-r_2-r_3}x_2^{r_1}x_3^{r_2}x_4^{r_3}$$

Thus, performing this operation to (1.1), we obtain the following.

$$(x_1+x_2+\cdots+x_m)^{\alpha} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{m-1}=0}^{\infty}\binom{\alpha}{r_1+r_2+\cdots+r_{m-1}}\binom{r_1+r_2+\cdots+r_{m-1}}{r_2+\cdots+r_{m-1}}\cdots\binom{r_{m-2}+r_{m-1}}{r_{m-1}}$$

$$\times\ x_1^{\alpha-(r_1+\cdots+r_{m-1})}x_2^{r_1}x_3^{r_2}\cdots x_m^{r_{m-1}} \tag{1.2}$$

Since (1.1) converges absolutely, this rearrangement is allowed.

Example 1: The expasion of $(x_1+x_2+x_3)^{3.9}$

Using (1.1),

$$(x_1+x_2+x_3)^{3.9} = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{3.9}{r}\binom{r}{s}x_1^{3.9-r}x_2^{r-s}x_3^{s}$$

$$= \binom{3.9}{0}x_1^{3.9}\sum_{s=0}^{0}\binom{0}{s}x_2^{0-s}x_3^{s} + \binom{3.9}{1}x_1^{2.9}\sum_{s=0}^{1}\binom{1}{s}x_2^{1-s}x_3^{s}$$

$$+ \binom{3.9}{2}x_1^{1.9}\sum_{s=0}^{2}\binom{2}{s}x_2^{2-s}x_3^{s} + \binom{3.9}{3}x_1^{0.9}\sum_{s=0}^{3}\binom{3}{s}x_2^{3-s}x_3^{s}$$

$$+ \binom{3.9}{4}\frac{1}{x_1^{0.1}}\sum_{s=0}^{4}\binom{4}{s}x_2^{4-s}x_3^{s} + \binom{4.1}{5}\frac{1}{x_1^{1.1}}\sum_{s=0}^{5}\binom{5}{s}x_2^{5-s}x_3^{s} + \cdots$$

$$= x_1^{3.9}$$

$$+\ 3.9\,x_1^{2.9}(x_2+x_3)$$

$$+\ 5.655\,x_1^{1.9}(x_2^2+2x_2x_3+x_3^2)$$

$$+\ 3.5815\,x_1^{0.9}(x_2^3+3x_2^2x_3+3x_2x_3^2+x_3^3)$$

$$+\ \frac{0.805838}{x_1^{0.1}}(x_2^4+4x_2^3x_3+6x_2^2x_3^2+4x_2x_3^3+x_3^4)$$

$$-\ \frac{0.0161168}{x_1^{1.1}}(x_2^5+5x_2^4x_3+10x_2^3x_3^2+10x_2^3x_3^2+5x_2x_3^4+x_3^5)$$

$$\vdots$$

Using (1.2),

$$(x_1+x_2+x_3)^{3.9} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{3.9}{r+s}\binom{r+s}{s}x_1^{3.9-r-s}x_2^{r}x_3^{s}$$

$$= \sum_{s=0}^{\infty} \binom{3.9}{0+s} \binom{0+s}{s} x_1^{3.9-s} x_2^{0} x_3^{s} + \sum_{s=0}^{\infty} \binom{3.9}{1+s} \binom{1+s}{s} x_1^{2.9-s} x_2^{1} x_3^{s}$$
$$+ \sum_{s=0}^{\infty} \binom{3.9}{2+s} \binom{2+s}{s} x_1^{1.9-s} x_2^{2} x_3^{s} + \sum_{s=0}^{\infty} \binom{3.9}{3+s} \binom{3+s}{s} x_1^{0.9-s} x_2^{3} x_3^{s}$$
$$+ \sum_{s=0}^{\infty} \binom{3.9}{4+s} \binom{4+s}{s} x_1^{-0.1-s} x_2^{4} x_3^{s} + \cdots$$

$$= x_1^{39} + 3.9 x_1^{29} x_3 + 5.655 x_1^{19} x_3^2 + 3.5815 x_1^{09} x_3^3 + \frac{0.805838 x_3^4}{x_1^{0.1}} + \cdots$$
$$+ 3.9 x_1^{29} x_2 + 11.31 x_1^{19} x_2 x_3 + 10.7445 x_1^{09} x_2 x_3^2 + \frac{3.22335 x_2 x_3^3}{x_1^{0.1}} - \frac{0.0805838}{x_1^{1.1}}$$
$$+ 5.655 x_1^{19} x_2^2 + 10.7445 x_1^{09} x_2^2 x_3 + \frac{4.83503 x_2^2 x_3^2}{x_1^{0.1}} - \frac{0.161168 x_2^2 x_3^3}{x_1^{1.1}} + \cdots$$
$$+ 3.5815 x_1^{09} x_2^3 + \frac{3.22335 x_2^3 x_3}{x_1^{0.1}} - \frac{0.161168 x_2^3 x_3^2}{x_1^{1.1}} + \frac{0.0590948 x_2^3 x_3^3}{x_1^{2.1}} + \cdots$$
$$+ \frac{0.805838 x_2^4}{x_1^{0.1}} - \frac{0.0805838 x_2^4 x_3}{x_1^{1.1}} + \frac{0.0443211 x_2^4 x_3^2}{x_1^{2.1}} - \frac{0.0310247 x_2^4 x_3^3}{x_1^{3.1}} + \cdots$$
$$\vdots$$

we can see that what totaled this along the diagonal line is equal to the above.

Example 2: The expasion of $(a+b+c+d)^{2.9}$

```
Clear[p]; p = 2.9;
f1[a_, b_, c_, d_] := (a + b + c + d)^p
f1[5, -2, 3, 4]
794.328
```

$$fr[a\_, b\_, c\_, d\_] := \sum_{r=0}^{\infty} \sum_{s=0}^{r} \sum_{t=0}^{s} \text{Binomial}[p, r]\ \text{Binomial}[r, s]\ \text{Binomial}[s, t] \times a^{p-r} b^{r-s} c^{s-t} d^{t}$$

```
N[fr[5, -2, 3, 4]]
794.328
```

$$fs[a\_, b\_, c\_, d\_] := \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \text{Binomial}[p, r+s+t]\ \text{Binomial}[r+s+t, s+t] \times \text{Binomial}[s+t, t]\ a^{p-r-s-t} b^{r} c^{s} d^{t}$$

```
fs[5, -2, 3, 4]
794.328
```

Although $a = b+c+d$ in this numerical example, (1.1) and (1.2) are consistent.

Sum of General Multinomial Coefficients

$$\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{m-1}=0}^{r_{m-2}} \binom{\alpha}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{m-2}}{r_{m-1}} \Leftarrow m^{\alpha} \qquad (1.1")$$

As expected, the following expression does not hold.

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_{m-1}=0}^{r_{m-2}}\binom{\alpha}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{m-2}}{r_{m-1}} = m^{\alpha}$$

It is because $x_1=x_2=\cdots=x_m=1$ does not satisfy the condition $|x_1| \ge |x_2+x_3+\cdots+x_m|$.
Let its partial sum be

$$S_n = \sum_{r_1=0}^{n}\sum_{r_2=0}^{r_1}\cdots\sum_{r_{m-1}=0}^{r_{m-2}}\binom{\alpha}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{m-2}}{r_{m-1}}$$

Then, when $n\to\infty$, S_n oscillates and diverges. And m^{α} is the median of this oscillating divergent series.
In fact, applying Knopp Transformation to S_n, we can obtain the approximate value of m^{α} with high precision.
However, it does not become a series but becomes an asymptotic expansion.

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