2 Multiple Series & Exponetial Function

2.1 Multiple Series & Half Multiple Series

Formula 2.1.0

When a multiple series $\sum_{r_1,r_2,\cdots,r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n}$ is absolutely convergent, the following expressions hold.

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{r,s} = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r-s,s} \tag{0.2}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} a_{r,s,t} = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s,s-t,t} \tag{0.3}$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_n} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{r_1-r_2,r_2-r_3,\cdots,r_{n-1}-r_n,r_n} \tag{0.n}$$

Proof

$$\begin{aligned}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{r,s} &= a_{0,0} + a_{0,1} + a_{0,2} + \cdots \\ &+ a_{1,0} + a_{1,1} + a_{1,2} + \cdots \\ &+ a_{2,0} + a_{2,1} + a_{2,2} + \cdots \\ &\vdots \\ &= a_{0,0} + (a_{1,0} + a_{0,1}) + (a_{2,0} + a_{1,1} + a_{0,2} +) \cdots \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r-s,s}\end{aligned}$$

Next,

$$\begin{aligned}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} a_{r,s,t} &= a_{0,0,0} + a_{0,0,1} + a_{0,0,2} + \cdots \quad + a_{1,0,0} + a_{1,0,1} + \cdots \quad + a_{2,0,0} + \cdots \\ &+ a_{0,1,0} + a_{0,1,1} + a_{0,1,2} + \cdots \quad + a_{1,1,0} + a_{1,1,1} + \cdots \quad \vdots \\ &+ a_{0,2,0} + a_{0,2,1} + a_{0,2,2} + \cdots \quad \vdots \\ &\vdots \\ &= a_{0,0,0} \\ &+ a_{1,0,0} + (a_{0,1,0} + a_{0,0,1}) \\ &+ a_{2,0,0} + (a_{1,1,0} + a_{1,0,1}) + (a_{0,2,0} + a_{0,1,1} + a_{0,0,2}) \\ &\vdots \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s,s-t,t}\end{aligned}$$

Hereafter, summarizing the same degree terms, we obtain (0.n) by induction.

Note

In short, we should just perform the following operations to $\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a_{r_1,r_2,\cdots,r_{n-1},r_n}$ ..

Replace r_{n-1} with $r_{n-1}-r_n$, and replace the 1st ∞ with r_{n-1} from the right.

Replace r_{n-2} with $r_{n-2}-r_{n-1}$, and replace the 2nd ∞ with r_{n-2} from the right.

⋮

Replace r_1 with r_1-r_2, and replace the $(n-1)$th ∞ with r_1 from the right.

Formula 2.1.1

When m is a non-negative integer, the following expressions hold.

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}a^r b^s\frac{x^{m+r+s}}{(m+r+s)!}=\sum_{r=0}^{\infty}\sum_{s=0}^{r}a^{r-s}b^s\frac{x^{m+r}}{(m+r)!} \tag{1.2}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}a^r b^s c^t\frac{x^{m+r+s+t}}{(m+r+s+t)!}=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}a^{r-s}b^{s-t}c^t\frac{x^{m+r}}{(m+r)!} \tag{1.3}$$

⋮

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\prod_{k=1}^{n}a_k^{r_k}\frac{x^{m+\sum_{k=1}^{n}r_k}}{\left(m+\sum_{k=1}^{n}r_k\right)!}=\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}}a_1^{r_1-r_2}a_2^{r_2-r_3}\cdots a_n^{r_n}\frac{x^{m+r_1}}{(m+r_1)!} \tag{1.n}$$

Proof

Performing the operation mentioned in the above **Note** to the left-hand side of (1.2) , we obtain the right-hand side immediately. ,

Performing the operation mentioned in the above **Note** to the left-hand side of (1.3) step by step,

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}a^r b^s c^t\frac{x^{m+r+s+t}}{(m+r+s+t)!}=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{s}a^r b^{s-t}c^t\frac{x^{m+r+s}}{(m+r+s)!}$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}a^{r-s}b^{s-t}c^t\frac{x^{m+r}}{(m+r)!}$$

Hereafter, in a similar way, we obtain (1.n) .

Formula 2.1.2

$$\sum_{s=0}^{r}2^s=\frac{2^{1+r}-1}{2-1} \tag{2.2}$$

$$\sum_{s=0}^{r}\sum_{t=0}^{s}2^{s-t}3^t=3\cdot\frac{3^{1+r}-1}{3-1}-2\cdot\frac{2^{1+r}-1}{2-1} \tag{2.3}$$

$$\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}2^{s-t}3^{t-u}4^u=8\cdot\frac{4^{1+r}-1}{4-1}-9\cdot\frac{3^{1+r}-1}{3-1}+2\cdot\frac{2^{1+r}-1}{2-1} \tag{2.4}$$

⋮

$$\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\cdots\sum_{r_n=0}^{r_{n-1}}2^{r_2-r_3}3^{r_3-r_4}\cdots n^{r_n}=\sum_{s=0}^{n-2}(-1)^s\frac{{}_nC_s\,(n-s)^{n-1}(n-s-1)}{n!}\,\frac{(n-s)^{1+r_1}-1}{n-s-1} \tag{2.n}$$

Proof

$$\sum_{s=0}^{r} 2^s = \frac{2^{1+r}-1}{2-1} = \frac{{}_2C_0\, 2^1(2-1)}{2!}\,\frac{2^{1+r}-1}{2-1} \tag{2.2}$$

Next,

$$\sum_{t=0}^{s} 2^{s-t}3^t = 2^s\sum_{t=0}^{s}\left(\frac{3}{2}\right)^t = 2^s\cdot\frac{(3/2)^{1+s}-1}{3/2\,-\,1} = 3^{1+s}-2^{1+s}$$

$$\sum_{s=0}^{r}\sum_{t=0}^{s} 2^{s-t}3^t = \sum_{s=0}^{r}\left(3^{1+s}-2^{1+s}\right) = 3\cdot\frac{3^{1+r}-1}{3-1} - 2\cdot\frac{2^{1+r}-1}{2-1}$$

$$= \frac{{}_3C_0\, 3^2(3-1)}{3!}\,\frac{3^{1+r}-1}{3-1} - \frac{{}_3C_1\, 2^2(2-1)}{3!}\,\frac{2^{1+r}-1}{2-1} \tag{2.3}$$

Next,

$$\sum_{u=0}^{t} 3^{t-u}4^u = 3^t\sum_{u=0}^{t}\left(\frac{4}{3}\right)^u = 3^t\cdot\frac{(4/3)^{1+t}-1}{4/3\,-\,1} = 4^{1+t}-3^{1+t}$$

$$\sum_{t=0}^{s}\sum_{u=0}^{t} 2^{s-t}3^{t-u}4^u = \sum_{t=0}^{s} 2^{s-t}\left(4^{1+t}-3^{1+t}\right) = 4\cdot 2^s\sum_{t=0}^{s}\left(\frac{4}{2}\right)^t - 3\cdot 2^s\sum_{t=0}^{s}\left(\frac{3}{2}\right)^t$$

$$= 4\cdot 2^s\frac{(4/2)^{1+s}-1}{4/2\,-\,1} - 3\cdot 2^s\frac{(3/2)^{1+s}-1}{3/2\,-\,1}$$

$$= 4\cdot\frac{4^{1+s}-2^{1+s}}{4\,-\,2} - 3\cdot\frac{3^{1+s}-2^{1+s}}{3\,-\,2} = 2\cdot 4^{1+s}-3\cdot 3^{1+s}+2^{1+s}$$

$$\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} 2^{s-t}3^{t-u}4^u = \sum_{s=0}^{r}\left(2\cdot 4^{1+s}-3\cdot 3^{1+s}+2^{1+s}\right) = 8\sum_{s=0}^{r}4^s - 9\sum_{s=0}^{r}3^s + 2\sum_{s=0}^{r}2^s$$

$$= 8\cdot\frac{4^{1+r}-1}{4-1} - 9\cdot\frac{3^{1+r}-1}{3-1} + 2\cdot\frac{2^{1+r}-1}{2-1}$$

$$= \frac{{}_4C_0\, 4^3(4-1)}{4!}\,\frac{4^{1+r}-1}{4-1} - \frac{{}_4C_1\, 3^3(3-1)}{4!}\,\frac{3^{1+r}-1}{3-1} + \frac{{}_4C_2\, 2^3(2-1)}{4!}\,\frac{2^{1+r}-1}{2-1} \tag{2.4}$$

Hereafter by induction, we obtain (2.n).

Example

$$\sum_{s=0}^{7}\sum_{t=0}^{s}\sum_{u=0}^{t} 2^{s-t}3^{t-u}4^u = 8\cdot\frac{4^{1+7}-1}{4-1} - 9\cdot\frac{3^{1+7}-1}{3-1} + 2\cdot\frac{2^{1+7}-1}{2-1} = 145750$$

Formula 2.1.3

$$\sum_{s=0}^{r} 2^s = \frac{2^{2+r}-2\cdot 1^{2+r}+0^{2+r}}{2!} \tag{3.2}$$

$$\sum_{s=0}^{r}\sum_{t=0}^{s} 2^{s-t}3^t = \frac{3^{3+r}-3\cdot 2^{3+r}+3\cdot 1^{3+r}-0^{3+r}}{3!} \tag{3.3}$$

$$\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} 2^{s-t}3^{t-u}4^u = \frac{4^{4+r}-4\cdot 3^{4+r}+6\cdot 2^{4+r}-4\cdot 1^{4+r}+0^{4+r}}{4!} \tag{3.4}$$

$$\vdots$$

$$\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\cdots\sum_{r_n=0}^{r_{n-1}} 2^{r_2-r_3}3^{r_3-r_4}\cdots n^{r_n} = \frac{1}{n!}\sum_{s=0}^{n}(-1)^s {}_nC_s (n-s)^{n+r_1} \tag{3.n}$$

Proof

Using Formula 2.1.2 ,

$$\sum_{s=0}^{r} 2^s = \sum_{s=0}^{r} 2^s = \frac{2^{1+r}-1}{2-1} = \frac{2^{2+r}-2\cdot 1^{2+r}+0^{2+r}}{2!} \tag{3.2}$$

Next,

$$\sum_{s=0}^{r}\sum_{t=0}^{s} 2^{s-t}3^t = 3\cdot\frac{3^{1+r}-1}{3-1} - 2\cdot\frac{2^{1+r}-1}{2-1}$$

$$= \frac{3^{2+r}-3}{2} - \frac{2^{2+r}-2}{1} = \frac{3^{2+r}}{2} - \frac{3}{2} - 2^{2+r} + 2$$

$$= \frac{3^{3+r}-3\cdot 2^{3+r}+3\cdot 1^{3+r}-0^{3+r}}{3!} \tag{3.3}$$

Next,

$$\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} 2^{s-t}3^{t-u}4^u = 8\cdot\frac{4^{1+r}-1}{4-1} - 9\cdot\frac{3^{1+r}-1}{3-1} + 2\cdot\frac{2^{1+r}-1}{2-1}$$

$$= \frac{8}{3}4^{1+r} - \frac{9}{2}3^{1+r} + 2\cdot 2^{1+r} - \frac{1}{6}$$

$$= \frac{64\cdot 4^{1+r} - 108\cdot 3^{1+r} + 48\cdot 2^{1+r} + 4\cdot 1^{1+r}}{4!}$$

$$= \frac{4^{4+r}-4\cdot 3^{4+r}+6\cdot 2^{4+r}+4\cdot 1^{4+r}-0^{3+r}}{4!} \tag{3.4}$$

Hereafter by induction, we obtain (3.n).

Example

$$\sum_{s=0}^{4}\sum_{t=0}^{s} 2^{s-t}3^t = 2^0 3^0 + (2^1 3^0 + 2^0 3^1) + (2^2 3^0 + 2^1 3^1 + 2^0 3^2) + (2^3 3^0 + 2^2 3^1 + 2^1 3^2 + 2^0 3^3)$$

$$= \frac{3^{3+4}-3\cdot 2^{3+4}+3\cdot 1^{3+4}-0^{3+4}}{3!} = 301$$

Formula 2.1.4

$$\sum_{s=0}^{n}(-1)^s {}_nC_s (n-s)^{n-1} = 0 \qquad n=1,2,3,\cdots \tag{4.n-1}$$

$$\sum_{s=0}^{n}(-1)^s {}_nC_s (n-s)^{n} = n! \qquad n=0,1,2,\cdots \tag{4.n}$$

$$\sum_{s=0}^{n}(-1)^s {}_nC_s (n-s)^{n+1} = {}_{n+1}C_2\, n! \qquad n=1,2,3,\cdots \tag{4.n+1}$$

Proof

From Formula 2.1.3 ,

$$\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\cdots\sum_{r_n=0}^{r_{n-1}} 2^{r_2-r_3}3^{r_3-r_4}\cdots n^{r_n} = \frac{1}{n!}\sum_{s=0}^{n}(-1)^s {}_nC_s (n-s)^{n+r_1}$$

$$= \sum_{s=0}^{n-2} \frac{(-1)^s}{n!} {}_nC_s (n-s)^{n+r_1} + \frac{(-1)^{n-1}}{n!} {}_nC_{n-1}$$

On the other hand, from Formula 2.1.2 ,

$$\sum_{r_2=0}^{r_1} \sum_{r_3=0}^{r_2} \cdots \sum_{r_n=0}^{r_{n-1}} 2^{r_2-r_3} 3^{r_3-r_4} \cdots n^{r_n} = \sum_{s=0}^{n-2} (-1)^s \frac{{}_nC_s (n-s)^{n-1} (n-s-1)}{n!} \frac{(n-s)^{1+r_1} - 1}{n-s-1}$$

$$= \sum_{s=0}^{n-2} (-1)^s \frac{{}_nC_s (n-s)^{n+r_1}}{n!} - \sum_{s=0}^{n-2} (-1)^s \frac{{}_nC_s (n-s)^{n-1}}{n!}$$

From both,

$$\frac{(-1)^{n-1}}{n!} {}_nC_{n-1} = -\sum_{s=0}^{n-2} (-1)^s \frac{{}_nC_s (n-s)^{n-1}}{n!}$$

This right side is

$$\sum_{s=0}^{n-2} (-1)^s \frac{{}_nC_s (n-s)^{n-1}}{n!} = \sum_{s=0}^{n} (-1)^s \frac{{}_nC_s (n-s)^{n-1}}{n!} - (-1)^{n-1} \frac{{}_nC_{n-1} 1^{n-1}}{n!}$$

Then,

$$\frac{(-1)^{n-1}}{n!} {}_nC_{n-1} = -\sum_{s=0}^{n} (-1)^s \frac{{}_nC_s (n-s)^{n-1}}{n!} + (-1)^{n-1} \frac{{}_nC_{n-1}}{n!}$$

From this, we obtain (4.n-1) .

Next, substituting $r_1 = 0$ for Formula 2.1.3 ,

$$\sum_{r_2=0}^{0} \sum_{r_3=0}^{r_2} \cdots \sum_{r_n=0}^{r_{n-1}} 2^{r_2-r_3} 3^{r_3-r_4} \cdots n^{r_n} = \frac{1}{n!} \sum_{s=0}^{n} (-1)^s {}_nC_s (n-s)^n$$

This left side is

$$\sum_{r_2=0}^{0} \sum_{r_3=0}^{r_2} \cdots \sum_{r_n=0}^{r_{n-1}} 2^{r_2-r_3} 3^{r_3-r_4} \cdots n^{r_n} = 2^0 3^0 \cdots n^0 = 1$$

Therefore, we obtain (4.n) .

Last, substituting $r_1 = 1$ for Formula 2.1.3 ,

$$\sum_{r_2=0}^{1} \sum_{r_3=0}^{r_2} \cdots \sum_{r_n=0}^{r_{n-1}} 2^{r_2-r_3} 3^{r_3-r_4} \cdots n^{r_n} = \frac{1}{n!} \sum_{s=0}^{n} (-1)^s {}_nC_s (n-s)^{n+1}$$

This left side is

$$\sum_{r_2=0}^{1} \sum_{r_3=0}^{r_2} \cdots \sum_{r_n=0}^{r_{n-1}} 2^{r_2-r_3} 3^{r_3-r_4} \cdots n^{r_n} = 2^0 3^0 \cdots n^0 + \sum_{r_3=0}^{1} \sum_{r_4=0}^{r_3} \cdots \sum_{r_n=0}^{r_{n-1}} 2^{1-r_3} 3^{r_3-r_4} \cdots n^{r_n}$$

$$= 2^0 3^0 \cdots n^0 + 2^1 3^0 \cdots n^0 + \sum_{r_4=0}^{1} \sum_{r_5=0}^{r_4} \cdots \sum_{r_n=0}^{r_{n-1}} 2^{1-1} 3^{1-r_4} \cdots n^{r_n}$$

$$= 2^0 3^0 \cdots n^0 + 2^1 3^0 \cdots n^0 + 2^0 3^1 4^0 \cdots n^0 + \sum_{r_5=0}^{1} \cdots \sum_{r_n=0}^{r_{n-1}} 2^0 3^0 4^{1-r_5} \cdots n^{r_n}$$

$$\vdots$$

$$= 2^0 3^0 \cdots n^0 + 2^1 3^0 \cdots n^0 + 2^0 3^1 4^0 \cdots n^0 + \cdots + 2^0 3^0 4^0 \cdots n^1$$

$$= \frac{(n+1)n}{2} = \frac{(n+1)n(n-1)!}{2(n-1)!} = \frac{(n+1)!}{2!(n-1)!}$$

$$= {}_{n+1}C_2$$

Thus, we obtain (4.n+1) .

Example $n=2$

$$\sum_{s=0}^{2}(-1)^s {}_2C_s(2-s)^{2-1} = {}_2C_0\,2^1 - {}_2C_1\,1^1 + {}_2C_2\,0^1 = 0$$

$$\sum_{s=0}^{2}(-1)^s {}_2C_s(2-s)^{2} = {}_2C_0\,2^2 - {}_2C_1\,1^2 + {}_2C_2\,0^2 = 2!$$

$$\sum_{s=0}^{2}(-1)^s {}_2C_s(2-s)^{2+1} = {}_2C_0\,2^3 - {}_2C_1\,1^3 + {}_2C_2\,0^3 = {}_3C_2\,2!$$

Formula 2.1.5

$$\left(e^x-1\right)^1 = \sum_{r=0}^{\infty}\left(1^{1+r} - 0^{1+r}\right)\frac{x^{1+r}}{(1+r)!} \tag{5.1}$$

$$\left(e^x-1\right)^2 = \sum_{r=0}^{\infty}\left(2^{2+r} - 2\cdot 1^{2+r} + 0^{2+r}\right)\frac{x^{2+r}}{(2+r)!} \tag{5.2}$$

$$\left(e^x-1\right)^3 = \sum_{r=0}^{\infty}\left(3^{3+r} - 3\cdot 2^{3+r} + 3\cdot 1^{3+r} - 0^{3+r}\right)\frac{x^{3+r}}{(3+r)!} \tag{5.3}$$

$$\vdots$$

$$\left(e^x-1\right)^n = \sum_{r=0}^{\infty}\sum_{s=0}^{n}(-1)^{n-s}{}_nC_s\, s^{n+r}\frac{x^{n+r}}{(n+r)!} \tag{5.n}$$

Proof

$$\left(e^x-1\right)^n = \sum_{s=0}^{n}{}_nC_s\left(e^x\right)^s(-1)^{n-s} = \sum_{s=0}^{n}(-1)^{n-s}{}_nC_s\sum_{r=0}^{\infty}\frac{(sx)^r}{r!}$$

Here,

$$\left(e^x-1\right)^n = \left(\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)^n = \left(\frac{1}{1!}\right)^n x^n + \cdots$$

Since the terms from the 0th degree to the (n-1)-th degree with respect to x does not exist, these terms in the right side of the above expression are offset and have to become 0. i.e.

$$\sum_{s=0}^{n}(-1)^{n-s}{}_nC_s\sum_{r=0}^{\infty}\frac{(sx)^r}{r!} = \sum_{s=0}^{n}(-1)^{n-s}{}_nC_s\sum_{r=0}^{\infty}\frac{(sx)^{n+r}}{(n+r)!}$$

Thus we obtain (5.n) .

2.2 Multiple Series and e^x (Part1)

If we put $a_1 = a_2 = \cdots = a_n = 1$ in Formula 2.1.1 , the following formula is obtained.

Formula 2.2.1

When m is a non-negative integer, the following expressions hold.

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^{m+r+s}}{(m+r+s)!} = \sum_{r=0}^{\infty}\frac{{}_{1+r}C_1\, x^{m+r}}{(m+r)!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{m+r+s+t}}{(m+r+s+t)!} = \sum_{r=0}^{\infty}\frac{{}_{2+r}C_2\, x^{m+r}}{(m+r)!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\frac{x^{m+r+s+t+u}}{(m+r+s+t+u)!} = \sum_{r=0}^{\infty}\frac{{}_{3+r}C_3\, x^{m+r}}{(m+r)!}$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{x^{m+\sum_{k=1}^{n} r_k}}{\left(m+\sum_{k=1}^{n} r_k\right)!} = \sum_{r=0}^{\infty}\frac{{}_{n-1+r}C_{n-1}\, x^{m+r}}{(m+r)!} \tag{1.n}$$

Proof

Formula 2.1.1 was as follows.

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\prod_{k=1}^{n} a_k^{r_k}\frac{x^{m+\sum_{k=1}^{n} r_k}}{\left(m+\sum_{k=1}^{n} r_k\right)!} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_1^{r_1-r_2} a_2^{r_2-r_3}\cdots a_n^{r_n}\frac{x^{m+r_1}}{(m+r_1)!}$$

Substituting $a_k = 1 \quad , \quad k = 1, 2, \cdots, n$ for this,

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{x^{m+\sum_{k=1}^{n} r_k}}{\left(m+\sum_{k=1}^{n} r_k\right)!} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}}\frac{x^{m+r_1}}{(m+r_1)!} = \sum_{r_1=0}^{\infty}\frac{x^{m+r_1}}{(m+r_1)!}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} 1$$

Here,

$$\sum_{s=0}^{r} 1 = \frac{1+r}{1!} = {}_{1+r}C_1$$

$$\sum_{s=0}^{r}\sum_{t=0}^{s} 1 = \sum_{s=0}^{r}\frac{1+r}{1!} = \sum_{s=1}^{1+r} s = \frac{(1+s)(2+s)}{2!} = {}_{2+r}C_2$$

$$\vdots$$

$$\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} 1 = \frac{(1+r_1)(2+r_1)\cdots(n-1+r_1)}{(n-1)!} = {}_{n-1+r_1}C_{n-1}$$

Substituting these for the above, we obtain the desired expression.

Formula 2.2.2

$$\frac{1x^1}{1!}+\frac{2x^2}{2!}+\frac{3x^3}{3!}+\frac{4x^4}{4!}+\cdots = e^x\cdot\frac{x^1}{1!}$$

$$\frac{1x^2}{2!}+\frac{3x^3}{3!}+\frac{6x^4}{4!}+\frac{10x^5}{5!}+\cdots = e^x\cdot\frac{x^2}{2!}$$

$$\frac{1x^3}{3!}+\frac{4x^4}{4!}+\frac{10x^5}{5!}+\frac{20x^6}{6!}+\cdots = e^x\cdot\frac{x^3}{3!}$$

$$\vdots$$

$$\sum_{r=0}^{\infty}\frac{{}_{n+r}C_n\, x^{n+r}}{(n+r)!} = e^x\cdot\frac{x^n}{n!} \qquad (2.n)$$

Proof

$$\sum_{r=0}^{\infty}\frac{{}_{n+r}C_n\, x^{n+r}}{(n+r)!} = \sum_{r=0}^{\infty}\frac{(n+r)!}{n!\,r!}\frac{x^{n+r}}{(n+r)!} = \frac{x^n}{n!}\sum_{r=0}^{\infty}\frac{x^r}{r!} = e^x\cdot\frac{x^n}{n!}$$

From Formula 2.2.1 and Formula 2.2.2 , we obtain the following formula immediately.

Formula 2.2.3

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^{1+r+s}}{(1+r+s)!} = e^x\cdot\frac{x^1}{1!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{2+r+s+t}}{(2+r+s+t)!} = e^x\cdot\frac{x^2}{2!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\frac{x^{3+r+s+t+u}}{(3+r+s+t+u)!} = e^x\cdot\frac{x^3}{3!}$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{x^{n-1+\sum_{k=1}^{n}r_k}}{\left(n-1+\sum_{k=1}^{n}r_k\right)!} = e^x\cdot\frac{x^{n-1}}{(n-1)!} \qquad (3.n)$$

Formula 2.2.3 can be generalized further by differentiating or integrating the both sides.

Formula 2.2.4

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}\frac{x^{n-m+\sum_{k=1}^{n+1}r_k}}{\left(n-m+\sum_{k=1}^{n+1}r_k\right)!} = e^x\sum_{r=0}^{n}\binom{m}{r}\frac{x^{n-r}}{(n-r)!} \qquad (n\geqq m) \qquad (4.d)$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}\frac{x^{n+m+\sum_{k=1}^{n+1}r_k}}{\left(n+m+\sum_{k=1}^{n+1}r_k\right)!} = e^x\sum_{r=0}^{n}\binom{-m}{r}\frac{x^{n-r}}{(n-r)!} - \sum_{s=0}^{m-1}\binom{-m+s}{n}\frac{x^s}{s!} \qquad (4.s)$$

Proof

From Formula 2.2.3 ,

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}\frac{x^{n+\sum_{k=1}^{n+1}r_k}}{\left(n+\sum_{k=1}^{n+1}r_k\right)!} = e^x\cdot\frac{x^n}{n!} \qquad (4.n+1)$$

Let us differentiate both sides m times with respect to x . Then the left side is as follows.

$$\frac{d^m}{dx^m}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}\frac{x^{n+\sum_{k=1}^{n+1}r_k}}{\left(n+\sum_{k=1}^{n+1}r_k\right)!}=\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}\frac{x^{n-m+\sum_{k=1}^{n+1}r_k}}{\left(n-m+\sum_{k=1}^{n+1}r_k\right)!}$$

According to Formula 18.4.1 (*Super Calculus* **18**) ,

$$\left(e^x x^n\right)^{(m)}=e^x\sum_{r=0}^{n}\binom{m}{r}\frac{\Gamma(1+n)}{\Gamma(1+n-r)}x^{n-r}$$

Using this, the right side becomes as follows.

$$\frac{d^m}{dx^m}e^x\cdot\frac{x^n}{n!}=\frac{e^x}{n!}\sum_{r=0}^{n}\binom{m}{r}\frac{\Gamma(1+n)}{\Gamma(1+n-r)}x^{n-r}=e^x\sum_{r=0}^{n}\binom{m}{r}\frac{x^{n-r}}{(n-r)!}$$

From both,

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}\frac{x^{n-m+\sum_{k=1}^{n+1}r_k}}{\left(n-m+\sum_{k=1}^{n+1}r_k\right)!}=e^x\sum_{r=0}^{n}\binom{m}{r}\frac{x^{n-r}}{(n-r)!} \tag{4.d}$$

Next, let us integrate both sides of (4.n+1) m times with respect to x from 0 to x .
Then the left side is as follows.

$$\int_0^x\!\!\cdots\!\int_0^x\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}\frac{x^{n+\sum_{k=1}^{n+1}r_k}}{\left(n+\sum_{k=1}^{n+1}r_k\right)!}dx^m=\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}\frac{x^{n+m+\sum_{k=1}^{n+1}r_k}}{\left(n+m+\sum_{k=1}^{n+1}r_k\right)!}$$

According to Formula 16.5.1' (*Super Calculus* **16**) ,

$$\int_{-\infty}^x\!\!\cdots\!\int_{-\infty}^x e^x x^n dx^m=e^x\sum_{r=0}^{n}\binom{-m}{r}\frac{\Gamma(1+n)}{\Gamma(1+n-r)}x^{n-r}$$

Using this,

$$\int_0^x e^x\cdot\frac{x^n}{n!}dx=\frac{e^x}{n!}\sum_{r=0}^{n}\binom{-1}{r}\frac{\Gamma(1+n)}{\Gamma(1+n-r)}x^{n-r}-\frac{e^0}{n!}\sum_{r=0}^{n}\binom{-1}{r}\frac{\Gamma(1+n)}{\Gamma(1+n-r)}0^{n-r}$$

$$=e^x\sum_{r=0}^{n}\binom{-1}{r}\frac{x^{n-r}}{\Gamma(1+n-r)}-e^0\sum_{r=0}^{n}\binom{-1}{r}\frac{0^{n-r}}{\Gamma(1+n-r)}$$

$$=e^x\sum_{r=0}^{n}\binom{-1}{r}\frac{x^{n-r}}{(n-r)!}-\binom{-1}{n}\frac{x^0}{0!}$$

$$\int_0^x\!\!\int_0^x e^x\cdot\frac{x^n}{n!}dx^2=e^x\sum_{r=0}^{n}\binom{-2}{r}\frac{x^{n-r}}{(n-r)!}-\binom{-2}{n}\frac{x^0}{0!}-\binom{-1}{n}\frac{x^1}{1!}$$

$$\vdots$$

$$\int_0^x\!\!\cdots\!\int_0^x e^x\cdot\frac{x^n}{n!}dx^m=e^x\sum_{r=0}^{n}\binom{-m}{r}\frac{x^{n-r}}{(n-r)!}-\sum_{s=0}^{m-1}\binom{-m+s}{n}\frac{x^s}{s!}$$

From both,

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}\frac{x^{n+m+\sum_{k=1}^{n+1}r_k}}{\left(n+m+\sum_{k=1}^{n+1}r_k\right)!}=e^x\sum_{r=0}^{n}\binom{-m}{r}\frac{x^{n-r}}{(n-r)!}-\sum_{s=0}^{m-1}\binom{-m+s}{n}\frac{x^s}{s!} \tag{4.s}$$

Example: n=2, m=1, 2

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{0+r+s+t}}{(0+r+s+t)!}=e^{x}\sum_{r=0}^{2}\binom{2}{r}\frac{x^{2-r}}{(2-r)!}=e^{x}\left(1\cdot\frac{x^{2}}{2!}+2\cdot\frac{x^{1}}{1!}+1\cdot\frac{x^{0}}{0!}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{1+r+s+t}}{(1+r+s+t)!}=e^{x}\sum_{r=0}^{2}\binom{1}{r}\frac{x^{2-r}}{(2-r)!}=e^{x}\left(1\cdot\frac{x^{2}}{2!}+1\cdot\frac{x^{1}}{1!}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{2+r+s+t}}{(2+r+s+t)!}=e^{x}\cdot\frac{x^{2}}{2!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{3+r+s+t}}{(3+r+s+t)!}=e^{x}\sum_{r=0}^{2}\binom{-1}{r}\frac{x^{2-r}}{(2-r)!}-\sum_{s=0}^{1-1}\binom{-1+s}{2}\frac{x^{s}}{s!}$$

$$=e^{x}\left\{\binom{-1}{0}\frac{x^{2}}{2!}+\binom{-1}{1}\frac{x^{1}}{1!}+\binom{-1}{2}\frac{x^{0}}{0!}\right\}-\binom{-1}{2}\frac{x^{0}}{0!}$$

$$=e^{x}\left(\frac{x^{2}}{2!}-\frac{x^{1}}{1!}+\frac{x^{0}}{0!}\right)-\frac{x^{0}}{0!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{4+r+s+t}}{(4+r+s+t)!}=e^{x}\sum_{r=0}^{2}\binom{-2}{r}\frac{x^{2-r}}{(2-r)!}-\sum_{s=0}^{2-1}\binom{-2+s}{2}\frac{x^{s}}{s!}$$

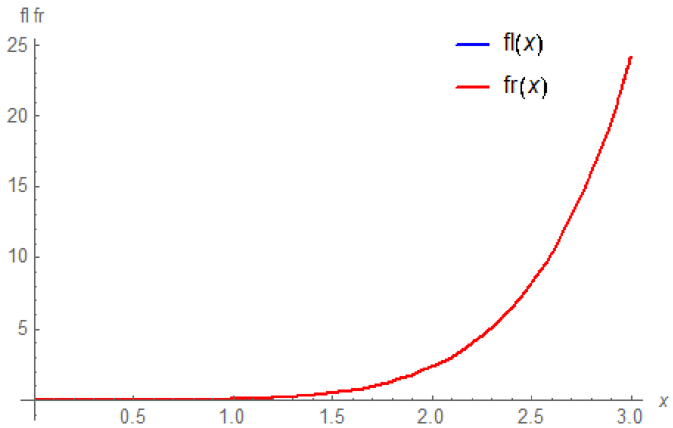
$$=e^{x}\left\{\binom{-2}{0}\frac{x^{2}}{2!}+\binom{-2}{1}\frac{x^{1}}{1!}+\binom{-2}{2}\frac{x^{0}}{0!}\right\}-\left\{\binom{-2}{2}\frac{x^{0}}{0!}+\binom{-1}{2}\frac{x^{1}}{1!}\right\}$$

$$=e^{x}\left(1\cdot\frac{x^{2}}{2!}-2\cdot\frac{x^{1}}{1!}+3\cdot\frac{x^{0}}{0!}\right)-\left(3\cdot\frac{x^{0}}{0!}+1\cdot\frac{x^{1}}{1!}\right)$$

When both sides of the last expression are shown in the figure, it is as follows. Both sids have overlapped exactly and the left side (blue) cannot be seen.

$$\text{fl}[x\_]:=\sum_{r=0}^{30}\sum_{s=0}^{30}\sum_{t=0}^{30}\frac{x^{4+r+s+t}}{(4+r+s+t)!}$$

$$\text{fr}[x\_]:=e^{x}\left(1\times\frac{x^{2}}{2!}-2\times\frac{x^{1}}{1!}+3\times\frac{x^{0}}{0!}\right)-\left(3\times\frac{x^{0}}{0!}+1\times\frac{x^{1}}{1!}\right)$$



The following formula is obtained as a special case of Formula 2.2.4.

Formula 2.2.4'

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^{2+r+s}}{(2+r+s)!} = e^x\left(\frac{x^1}{1!}-\frac{x^0}{0!}\right)+1$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{3+r+s+t}}{(3+r+s+t)!} = e^x\left(\frac{x^2}{2!}-\frac{x^1}{1!}+\frac{x^0}{0!}\right)-1$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\frac{x^{4+r+s+t+u}}{(4+r+s+t+u)!} = e^x\left(\frac{x^3}{3!}-\frac{x^2}{2!}+\frac{x^1}{1!}-\frac{x^0}{0!}\right)+1$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{x^{n+\sum_{k=1}^{n} r_k}}{\left(n+\sum_{k=1}^{n} r_k\right)!} = e^x\sum_{r=0}^{n-1}(-1)^r\frac{x^{n-1-r}}{(n-1-r)!}+(-1)^n \tag{4.n'}$$

Proof

Replacing n with $n-1$ and substituting $m=1$ for (4.s) in Formula 2.2.4 ,

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{x^{n+\sum_{k=1}^{n} r_k}}{\left(n+\sum_{k=1}^{n} r_k\right)!} = e^x\sum_{r=0}^{n-1}\binom{-1}{r}\frac{x^{n-1-r}}{(n-1-r)!}-\sum_{s=0}^{0}\binom{-1+s}{n-1}\frac{x^s}{s!}$$

$$= e^x\sum_{r=0}^{n-1}(-1)^r\frac{x^{n-1-r}}{(n-1-r)!}-\binom{-1}{n-1}\frac{x^0}{0!}$$

$$= e^x\sum_{r=0}^{n-1}(-1)^r\frac{x^{n-1-r}}{(n-1-r)!}+(-1)^n \tag{4.n'}$$

2.3 Multiple Series and e^x (Part2)

Replacing x with $-x$ in formulas in the previous section, we obtain the following formulas.

Formula 2.3.1

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{x^{m+r+s}}{(m+r+s)!} = \sum_{r=0}^{\infty}(-1)^{r}\frac{{}_{1+r}C_{1}\,x^{m+r}}{(m+r)!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{x^{m+r+s+t}}{(m+r+s+t)!} = \sum_{r=0}^{\infty}(-1)^{r}\frac{{}_{2+r}C_{2}\,x^{m+r}}{(m+r)!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}(-1)^{r+s+t+u}\frac{x^{m+r+s+t+u}}{(m+r+s+t+u)!} = \sum_{r=0}^{\infty}(-1)^{r}\frac{{}_{3+r}C_{3}\,x^{m+r}}{(m+r)!}$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{x^{m+\sum_{k=1}^{n}r_k}}{\left(m+\sum_{k=1}^{n}r_k\right)!} = \sum_{r=0}^{\infty}(-1)^{r}\frac{{}_{n-1+r}C_{n-1}\,x^{m+r}}{(m+r)!}$$

Formula 2.3.2

$$\frac{1x^1}{1!}-\frac{2x^2}{2!}+\frac{3x^3}{3!}-\frac{4x^4}{4!}+-\cdots = \frac{1}{e^x}\frac{x^1}{1!}$$

$$\frac{1x^2}{2!}-\frac{3x^3}{3!}+\frac{6x^4}{4!}-\frac{10x^5}{5!}+-\cdots = \frac{1}{e^x}\frac{x^2}{2!}$$

$$\frac{1x^3}{3!}-\frac{4x^4}{4!}+\frac{10x^5}{5!}-\frac{20x^6}{6!}+-\cdots = \frac{1}{e^x}\frac{x^3}{3!}$$

$$\vdots$$

$$\sum_{r=0}^{\infty}(-1)^{r}\frac{{}_{n+r}C_{n}\,x^{n+r}}{(n+r)!} = \frac{1}{e^x}\frac{x^n}{n!}$$

Formula 2.3.3

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{x^{1+r+s}}{(1+r+s)!} = \frac{1}{e^x}\frac{x^1}{1!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{x^{2+r+s+t}}{(2+r+s+t)!} = \frac{1}{e^x}\frac{x^2}{2!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}(-1)^{r+s+t+u}\frac{x^{3+r+s+t+u}}{(3+r+s+t+u)!} = \frac{1}{e^x}\frac{x^3}{3!}$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{x^{n-1+\sum_{k=1}^{n}r_k}}{\left(n-1+\sum_{k=1}^{n}r_k\right)!} = \frac{1}{e^x}\frac{x^{n-1}}{(n-1)!}$$

Formula 2.3.4

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}(-1)^{\sum_{k=1}^{n+1}r_k}\frac{x^{n-m+\sum_{k=1}^{n+1}r_k}}{\left(n-m+\sum_{k=1}^{n+1}r_k\right)!}=\frac{(-1)^m}{e^x}\sum_{r=0}^{n}\binom{m}{r}\frac{(-1)^r x^{n-r}}{(n-r)!} \qquad \text{(4.d)}$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{n+1}=0}^{\infty}(-1)^{\sum_{k=1}^{n+1}r_k}\frac{x^{n+m+\sum_{k=1}^{n+1}r_k}}{\left(n+m+\sum_{k=1}^{n+1}r_k\right)!}=\frac{(-1)^m}{e^x}\sum_{r=0}^{n}\binom{-m}{r}\frac{(-1)^r x^{n-r}}{(n-r)!}$$
$$-(-1)^m\sum_{s=0}^{m-1}\binom{-m+s}{n}\frac{(-1)^s x^s}{s!} \qquad \text{(4.s)}$$

Example: $n=2$, $m=1$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{x^{1+r+s+t}}{(1+r+s+t)!}=\frac{(-1)^1}{e^x}\sum_{r=0}^{2}\binom{1}{r}\frac{(-1)^r x^{2-r}}{(2-r)!}$$
$$=-\frac{1}{e^x}\left(1\cdot\frac{x^2}{2!}-1\cdot\frac{x^1}{1!}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{x^{2+r+s+t}}{(2+r+s+t)!}=\frac{1}{e^x}\frac{x^2}{2!}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{x^{3+r+s+t}}{(3+r+s+t)!}=\frac{(-1)^1}{e^x}\sum_{r=0}^{1}\binom{-1}{r}\frac{(-1)^r x^{2-r}}{(2-r)!}$$
$$-(-1)^1\sum_{s=0}^{1-1}\binom{-1+s}{2}\frac{(-1)^s x^s}{s!}$$
$$=-\frac{1}{e^x}\left\{\binom{-1}{0}\frac{x^2}{2!}-\binom{-1}{1}\frac{x^1}{1!}+\binom{-1}{2}\frac{x^0}{0!}\right\}+\binom{-1}{2}\frac{x^0}{0!}$$
$$=-\frac{1}{e^x}\left(\frac{x^2}{2!}+\frac{x^1}{1!}+\frac{x^0}{0!}\right)+\frac{x^0}{0!}$$

Formula 2.3.4'

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{x^{2+r+s}}{(2+r+s)!}=1-\frac{1}{e^x}\left(\frac{x^1}{1!}+\frac{x^0}{0!}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{x^{3+r+s+t}}{(3+r+s+t)!}=1-\frac{1}{e^x}\left(\frac{x^2}{2!}+\frac{x^1}{1!}+\frac{x^0}{0!}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}(-1)^{r+s+t+u}\frac{x^{4+r+s+t+u}}{(4+r+s+t+u)!}=1-\frac{1}{e^x}\left(\frac{x^3}{3!}+\frac{x^2}{2!}+\frac{x^1}{1!}+\frac{x^0}{0!}\right)$$

$\vdots$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{x^{n+\sum_{k=1}^{n}r_k}}{\left(n+\sum_{k=1}^{n}r_k\right)!}=1-\frac{1}{e^x}\sum_{r=0}^{n-1}\frac{x^{n-1-r}}{(n-1-r)!}$$

2.4 Multiple Series and e^x (Part3)

If we put $a_1, a_2, \cdots a_n = 1, 2, \cdots, n$ in Formula 2.1.1 , the following formula is obtained.

Formula 2.4.1

$$\sum_{r=0}^{\infty} \frac{1^r}{(1+r)!} x^{1+r} = \frac{1}{1!}\left(e^x - 1\right)^1$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{1^r 2^s}{(2+r+s)!} x^{2+r+s} = \frac{1}{2!}\left(e^x - 1\right)^2$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{1^r 2^s 3^t}{(3+r+s+t)!} x^{3+r+s+t} = \frac{1}{3!}\left(e^x - 1\right)^3$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} \frac{\prod_{k=1}^{n} k^{r_k}}{\left(n+\sum_{k=1}^{n} r_k\right)!} x^{n+\sum_{k=1}^{n} r_k} = \frac{1}{n!}\left(e^x - 1\right)^n \qquad (1.n)$$

Proof

Formula 2.1.1 was as follows.

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\prod_{k=1}^{n} a_k^{r_k} \frac{x^{m+\sum_{k=1}^{n} r_k}}{\left(m+\sum_{k=1}^{n} r_k\right)!} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_1^{r_1-r_2} a_2^{r_2-r_3}\cdots a_n^{r_n} \frac{x^{m+r_1}}{(m+r_1)!}$$

Substituting $m = n$, $a_k = k \quad (k=1,2,\cdots,n)$ for this,

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\prod_{k=1}^{n} k^{r_k} \frac{x^{n+\sum_{k=1}^{n} r_k}}{\left(n+\sum_{k=1}^{n} r_k\right)!} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} 1^{r_1-r_2} 2^{r_2-r_3}\cdots n^{r_n} \frac{x^{n+r_1}}{(n+r_1)!} \qquad (a)$$

Next, Formula 2.1.3 was as follows.

$$\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\cdots\sum_{r_n=0}^{r_{n-1}} 2^{r_2-r_3} 3^{r_3-r_4}\cdots n^{r_n} = \frac{\sum_{s=0}^{n} (-1)^s {}_nC_s (n-s)^{n+r_1}}{n!}$$

Multiplying by $\dfrac{x^{n+r_1}}{(n+r_1)!}$ the both sides and adding from 0 to ∞ for r_1 ,

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} 1^{r_1-r_2} 2^{r_2-r_3}\cdots n^{r_n} \frac{x^{n+r_1}}{(n+r_1)!} = \frac{\sum_{r_1=0}^{\infty}\sum_{s=0}^{n} (-1)^s {}_nC_s (n-s)^{n+r_1}}{n!} \frac{x^{n+r_1}}{(n+r_1)!} \qquad (b)$$

Last, dividing both sides of Formula 2.1.5 by $n!$ and replacing r with r_1 ,

$$\frac{1}{n!}\left(e^x - 1\right)^n = \frac{\sum_{r_1=0}^{\infty}\sum_{s=0}^{n} (-1)^s {}_nC_s (n-s)^{n+r_1}}{n!} \frac{x^{n+r_1}}{(n+r_1)!} \qquad (c)$$

Thus, from (a),(b) and (c) , we obtain the desired expression

Formula 2.4.1 can be generalized further by differentiating or integrating the both sides.

Formula 2.4.2

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n-m+\sum_{k=1}^{n}r_k\right)!}x^{n-m+\sum_{k=1}^{n}r_k}$$

$$=\frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\,(n-r)^{m-1}e^{(n-r)x}\qquad n\geqq m \tag{2.d}$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+m+\sum_{k=1}^{n}r_k\right)!}x^{n+m+\sum_{k=1}^{n}r_k}$$

$$=\frac{1}{n!}\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^m}e^{(n-s)x}-\frac{1}{n!}\sum_{r=0}^{m}\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^{m-r}}\frac{x^r}{r!} \tag{2.s}$$

Proof

From Formula 2.4.1 ,

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+\sum_{k=1}^{n}r_k\right)!}x^{n+\sum_{k=1}^{n}r_k}=\frac{1}{n!}\left(e^x-1\right)^n \tag{2.n}$$

Let us differentiate both sides m times with respect to x . Then the left side is as follows.

$$\frac{d^m}{dx^m}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+\sum_{k=1}^{n}r_k\right)!}x^{n+\sum_{k=1}^{n}r_k}=\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n-m+\sum_{k=1}^{n}r_k\right)!}x^{n-m+\sum_{k=1}^{n}r_k}$$

The Right side is as follows.

$$\frac{d}{dx}\frac{1}{n!}\left(e^x-1\right)^n=\frac{e^x}{(n-1)!}\left(e^x-1\right)^{n-1}=\frac{e^x}{(n-1)!}\sum_{r=0}^{n-1}{}_{n-1}C_r\,(-1)^r\left(e^x\right)^{n-1-r}$$

$$=\frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\,(n-r)^0e^{(n-r)x}$$

$$\frac{d^2}{dx^2}\frac{1}{n!}\left(e^x-1\right)^n=\frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\,(n-r)^1e^{(n-r)x}$$

$$\frac{d^3}{dx^3}\frac{1}{n!}\left(e^x-1\right)^n=\frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\,(n-r)^2e^{(n-r)x}$$

$$\vdots$$

$$\frac{d^m}{dx^m}\frac{1}{n!}\left(e^x-1\right)^n=\frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\,(n-r)^{m-1}e^{(n-r)x}$$

Thus, we obtain (2.d) .

Next, let us integrate both sides of (2.n) m times with respect to x from 0 to x . Then the left side is as follows.

$$\int_0^x \cdots \int_0^x \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \frac{\prod_{k=1}^{n} k^{r_k}}{\left(n+\sum_{k=1}^{n} r_k\right)!} x^{n+\sum_{k=1}^{n} r_k} dx^m$$

$$= \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \frac{\prod_{k=1}^{n} k^{r_k}}{\left(n+m+\sum_{k=1}^{n} r_k\right)!} x^{n+m+\sum_{k=1}^{n} r_k}$$

The right side is as follows.

$$\frac{1}{n!}\left(e^x-1\right)^n = \frac{1}{n!} \sum_{s=0}^{n} {}_nC_s \left(e^x\right)^{n-s} (-1)^s = \frac{1}{n!} \sum_{s=0}^{n} {}_nC_s (-1)^s e^{(n-s)x}$$

$$= \frac{1}{n!}\left\{ \sum_{s=0}^{n-1} {}_nC_s (-1)^s e^{(n-s)x} + {}_nC_n (-1)^n e^0 \right\}$$

From this,

$$\int_0^x \frac{1}{n!}\left(e^x-1\right)^n dx = \frac{1}{n!}\left\{ \int_0^x \sum_{s=0}^{n-1} {}_nC_s (-1)^s e^{(n-s)x} dx + (-1)^n \int_0^x dx \right\}$$

$$= \frac{1}{n!}\left[\sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^1} e^{(n-s)x} \right]_0^x + \frac{(-1)^n}{n!}\left[\frac{x^1}{1!} \right]_0^x$$

$$= \frac{1}{n!} \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^1} e^{(n-s)x} - \frac{1}{n!} \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^1} \frac{x^0}{0!} - \frac{(-1)^n}{n!} \frac{x^1}{1!}$$

Since $(-1)^n = -\sum_{s=0}^{n-1} (-1)^s {}_nC_s$,

$$\int_0^x \frac{1}{n!}\left(e^x-1\right)^n dx = \frac{1}{n!} \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^1} e^{(n-s)x}$$

$$- \frac{1}{n!}\left\{ \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^1} \frac{x^0}{0!} + \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^0} \frac{x^1}{1!} \right\}$$

$$\int_0^x\int_0^x \frac{1}{n!}\left(e^x-1\right)^n dx^2 = \frac{1}{n!} \int_0^x \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^1} e^{(n-s)x} dx$$

$$- \frac{1}{n!} \int_0^x \left\{ \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^1} \frac{x^0}{0!} + \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^0} \frac{x^1}{1!} \right\} dx$$

$$= \frac{1}{n!} \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^2} e^{(n-s)x} - \frac{1}{n!}\left\{ \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^2} \frac{x^0}{0!} \right.$$

$$\left. + \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^1} \frac{x^1}{1!} + \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^0} \frac{x^2}{2!} \right\}$$

$$\vdots$$

$$\int_0^x \cdots \int_0^x \frac{1}{n!}\left(e^x-1\right)^n dx^\lambda = \frac{1}{n!} \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^m} e^{(n-s)x} - \frac{1}{n!} \sum_{r=0}^{m} \sum_{s=0}^{n-1} \frac{(-1)^s {}_nC_s}{(n-s)^{m-r}} \frac{x^r}{r!}$$

From both, we obtain (2.s) .

Example: n**=3,** m**=1, 2**

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r2^s3^t}{(1+r+s+t)!}x^{1+r+s+t}=\frac{1}{(3-1)!}\sum_{r=0}^{3-1}(-1)^r{}_{3-1}C_r(3-r)^{2-1}e^{(3-r)x}$$

$$=\frac{1}{2!}\left(3^1e^{3x}-2\cdot2^1e^{2x}+1^1e^x\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r2^s3^t}{(2+r+s+t)!}x^{2+r+s+t}=\frac{1}{(3-1)!}\sum_{r=0}^{3-1}(-1)^r{}_{3-1}C_r(3-r)^{1-1}e^{(3-r)x}$$

$$=\frac{1}{2!}\left(e^{3x}-2e^{2x}+e^x\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r2^s3^t}{(3+r+s+t)!}x^{3+r+s+t}=\frac{1}{3!}\left(e^x-1\right)^3$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r2^s3^t}{(4+r+s+t)!}x^{4+r+s+t}=\frac{1}{3!}\sum_{s=0}^{3-1}\frac{(-1)^s{}_3C_s}{(3-s)^1}e^{(3-s)x}-\frac{1}{3!}\sum_{r=0}^{1}\sum_{s=0}^{3-1}\frac{(-1)^s{}_3C_s}{(3-s)^{1-r}}\frac{x^r}{r!}$$

$$=\frac{1}{3!}\left(\frac{1}{3^1}e^{3x}-\frac{3}{2^1}e^{2x}+\frac{3}{1^1}e^x\right)$$

$$-\frac{1}{3!}\left\{\left(\frac{1}{3^1}-\frac{3}{2^1}+\frac{3}{1^1}\right)\frac{x^0}{0!}+\left(\frac{1}{3^0}-\frac{3}{2^0}+\frac{3}{1^0}\right)\frac{x^1}{1!}\right\}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r2^s3^t}{(5+r+s+t)!}x^{5+r+s+t}=\frac{1}{3!}\sum_{s=0}^{3-1}\frac{(-1)^s{}_3C_s}{(3-s)^2}e^{(3-s)x}-\frac{1}{3!}\sum_{r=0}^{2}\sum_{s=0}^{3-1}\frac{(-1)^s{}_3C_s}{(3-s)^{2-r}}\frac{x^r}{r!}$$

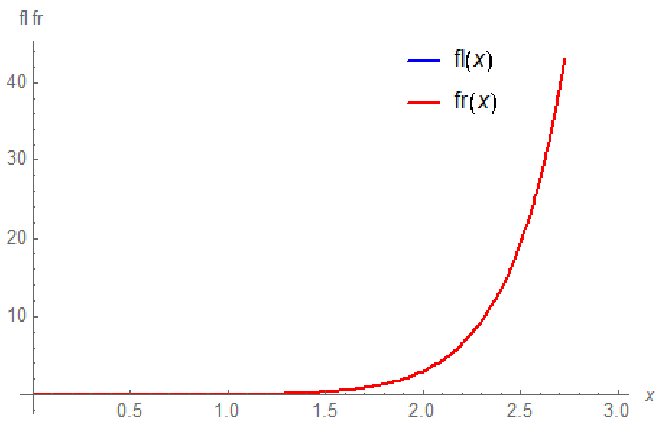
$$=\frac{1}{3!}\left(\frac{1}{3^2}e^{3x}-\frac{3}{2^2}e^{2x}+\frac{3}{1^2}e^x\right)$$

$$-\frac{1}{3!}\left\{\left(\frac{1}{3^2}-\frac{3}{2^2}+\frac{3}{1^2}\right)\frac{x^0}{0!}+\left(\frac{1}{3^1}-\frac{3}{2^1}+\frac{3}{1^1}\right)\frac{x^1}{1!}+\left(\frac{1}{3^0}-\frac{3}{2^0}+\frac{3}{1^0}\right)\frac{x^2}{2!}\right\}$$

When both sides of the last expression are shown in the figure, it is as follows. Both sids have overlapped exactly and the left side (blue) cannot be seen.

$$\text{fl}[x\_]:=\sum_{r=0}^{30}\sum_{s=0}^{30}\sum_{t=0}^{30}\frac{1^r\,2^s\,3^t}{(5+r+s+t)!}\,x^{5+r+s+t}$$

$$\text{fr}[x\_]:=\frac{1}{3!}\sum_{s=0}^{2}\frac{(-1)^s\,\text{Binomial}[3,s]}{(3-s)^2}\,e^{(3-s)\,x}-\frac{1}{3!}\sum_{r=0}^{2}\sum_{s=0}^{2}\frac{(-1)^s\,\text{Binomial}[3,s]}{(3-s)^{2-r}}\,\frac{x^r}{r!}$$



Replacing x with $\log x$ in Formula 2.4.2 , we obtain the following formula.

Formula 2.4.3

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n-m+\sum_{k=1}^{n}r_k\right)!}(\log x)^{n-m+\sum_{k=1}^{n}r_k}$$
$$=\frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\,(n-r)^{m-1}x^{(n-r)} \qquad n\geqq m \qquad (3.d)$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+m+\sum_{k=1}^{n}r_k\right)!}(\log x)^{n+m+\sum_{k=1}^{n}r_k}$$
$$=\frac{1}{n!}\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^m}x^{(n-s)}-\frac{1}{n!}\sum_{r=0}^{m}\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^{m-r}}\frac{(\log x)^r}{r!} \qquad (3.s)$$

The following formula is obtained as a special case of Formula 2.4.3.

Formula 2.4.3'

$$\sum_{r=0}^{\infty}\frac{1^r}{(1+r)!}(\log x)^{1+r}=\frac{1}{1!}(x-1)^1$$
$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{1^r2^s}{(2+r+s)!}(\log x)^{2+r+s}=\frac{1}{2!}(x-1)^2$$
$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{1^r2^s3^t}{(3+r+s+t)!}(\log x)^{3+r+s+t}=\frac{1}{3!}(x-1)^3$$
$$\vdots$$
$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+\sum_{k=1}^{n}r_k\right)!}(\log x)^{n+\sum_{k=1}^{n}r_k}=\frac{1}{n!}(x-1)^n$$

Proof

Substituting $m=0$ for (3.s) in Formula 2.4.3 ,

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+\sum_{k=1}^{n}r_k\right)!}(\log x)^{n+\sum_{k=1}^{n}r_k}$$
$$=\frac{1}{n!}\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^0}x^{(n-s)}-\frac{1}{n!}\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^0}\frac{(\log x)^0}{0!}$$
$$=\frac{1}{n!}\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^0}x^{(n-s)}-\frac{1}{n!}\left\{\sum_{s=0}^{n}(-1)^s{}_nC_s-(-1)^n{}_nC_n\right\}$$
$$=\frac{1}{n!}\sum_{s=0}^{n}{}_nC_s\,x^{(n-s)}(-1)^s=\frac{1}{n!}(x-1)^n$$

Needless to say, we can obtain this also by replacing x with $\log x$ in Formula 2.4.1 .

2.5 Multiple Series and e^x (Part4)

Replacing x with $-x$ in formulas in the previous section, we obtain the following formulas.

Formula 2.5.1

$$\sum_{r=0}^{\infty}(-1)^r\frac{1^r}{(1+r)!}x^{1+r}=\frac{1}{1!}\left(1-\frac{1}{e^x}\right)^1$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{1^r2^s}{(2+r+s)!}x^{2+r+s}=\frac{1}{2!}\left(1-\frac{1}{e^x}\right)^2$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r2^s3^t}{(3+r+s+t)!}x^{3+r+s+t}=\frac{1}{3!}\left(1-\frac{1}{e^x}\right)^3$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+\sum_{k=1}^{n}r_k\right)!}x^{n+\sum_{k=1}^{n}r_k}=\frac{1}{n!}\left(1-\frac{1}{e^x}\right)^n$$

Formula 2.5.2

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n-m+\sum_{k=1}^{n}r_k\right)!}x^{\sum_{k=1}^{n}r_k}$$

$$=\frac{(-1)^{n-m}}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r{}_{n-1}C_r\frac{(n-r)^{m-1}}{e^{(n-r)x}}\qquad n\geqq m \qquad \text{(2.d)}$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\frac{\prod_{k=1}^{n}k^{r_k}}{\left(n+m+\sum_{k=1}^{n}r_k\right)!}x^{\sum_{k=1}^{n}r_k}$$

$$=\frac{(-1)^{n+m}}{n!}\left\{\sum_{s=0}^{n-1}\frac{(-1)^s{}_nC_s}{(n-s)^m e^{(n-s)x}}-\sum_{r=0}^{m}\sum_{s=0}^{n-1}\frac{(-1)^{r+s}{}_nC_s}{(n-s)^{m-r}}\frac{x^r}{r!}\right\} \qquad \text{(2.s)}$$

Example: $n=2$, $m=1$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r2^s3^t}{(2+r+s+t)!}x^{2+r+s+t}=\frac{(-1)^{3-1}}{(3-1)!}\sum_{r=0}^{3-1}(-1)^r{}_{3-1}C_r\frac{(3-r)^{1-1}}{e^{(3-r)x}}$$

$$=\frac{1}{2!}\left(\frac{1}{e^{3x}}-\frac{2}{e^{2x}}+\frac{1}{e^x}\right)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r2^s3^t}{(3+r+s+t)!}x^{3+r+s+t}=\frac{(-1)^{3-0}}{3!}\left(\frac{1}{e^x}-1\right)^3$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r2^s3^t}{(4+r+s+t)!}x^{4+r+s+t}$$

$$=\frac{(-1)^{3+1}}{3!}\left\{\sum_{s=0}^{3-1}\frac{(-1)^s{}_3C_s}{(3-s)^1e^{(3-s)x}}-\sum_{r=0}^{1}\sum_{s=0}^{3-1}\frac{(-1)^{r+s}{}_3C_s}{(3-s)^{1-r}}\frac{x^r}{r!}\right\}$$

$$= \frac{1}{3!}\left(\frac{1}{3^1 e^{3x}} - \frac{3}{2^1 e^{2x}} + \frac{3}{1^1 e^{x}}\right)$$

$$- \frac{1}{3!}\left\{\left(\frac{1}{3^1} - \frac{3}{2^1} + \frac{3}{1^1}\right)\frac{x^0}{0!} - \left(\frac{1}{3^0} - \frac{3}{2^0} + \frac{3}{1^0}\right)\frac{x^1}{1!}\right\}$$

Formula 2.5.3

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\frac{\prod_{k=1}^{n} k^{r_k}}{\left(n-m+\sum_{k=1}^{n} r_k\right)!}(\log x)^{\sum_{k=1}^{n} r_k}$$

$$= \frac{(-1)^{n-m}}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r {}_{n-1}C_r \frac{(n-r)^{m-1}}{x^{(n-r)}} \qquad n \geqq m \qquad (3.d)$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\frac{\prod_{k=1}^{n} k^{r_k}}{\left(n+m+\sum_{k=1}^{n} r_k\right)!}(\log x)^{\sum_{k=1}^{n} r_k}$$

$$= \frac{(-1)^{n+m}}{n!}\left\{\sum_{s=0}^{n-1}\frac{(-1)^s {}_nC_s}{(n-s)^m x^{(n-s)}} - \sum_{r=0}^{m}\sum_{s=0}^{n-1}\frac{(-1)^{r+s} {}_nC_s}{(n-s)^{m-r}}\frac{(\log x)^r}{r!}\right\} \qquad (3.s)$$

Formula 2.5.3'

$$\sum_{r=0}^{\infty}(-1)^r\frac{1^r}{(1+r)!}(\log x)^{1+r} = \frac{1}{1!}\left(1-\frac{1}{x}\right)^1$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{1^r 2^s}{(2+r+s)!}(\log x)^{2+r+s} = \frac{1}{2!}\left(1-\frac{1}{x}\right)^2$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{r+s+t}\frac{1^r 2^s 3^t}{(3+r+s+t)!}(\log x)^{3+r+s+t} = \frac{1}{3!}\left(1-\frac{1}{x}\right)^3$$

$$\vdots$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n} r_k}\frac{\prod_{k=1}^{n} k^{r_k}}{\left(n+\sum_{k=1}^{n} r_k\right)!}(\log x)^{n+\sum_{k=1}^{n} r_k} = \frac{1}{n!}\left(1-\frac{1}{x}\right)^n$$

2.6 Euler-Mascheroni Constant

Formula 2.6.1

When γ is the Euler-Mascheroni Constant $(=0.57721566\cdots)$, the following expressions hold.

$$1-\gamma=\sum_{r=2}^{\infty}\sum_{s=2}^{\infty}\frac{1}{sr^{s}} \tag{1.1}$$

$$=\sum_{r=2}^{\infty}\sum_{s=2}^{r}\frac{1}{s(2+r-s)^{s}} \tag{1.1'}$$

$$=\sum_{s=2}^{\infty}\frac{\zeta(s)-1}{s} \tag{1.1''}$$

$$\gamma=\sum_{r=1}^{\infty}\sum_{s=2}^{\infty}\frac{(-1)^{s}}{s\,r^{s}} \tag{1.2}$$

$$=\sum_{r=2}^{\infty}\sum_{s=2}^{r}\frac{(-1)^{s}}{s(1+r-s)^{s}} \tag{1.2'}$$

$$=\sum_{s=2}^{\infty}(-1)^{s}\frac{\zeta(s)}{s} \tag{1.2''}$$

Proof

$$\gamma=\sum_{r=1}^{\infty}\frac{1}{r}-log\,\infty=\sum_{r=1}^{\infty}\frac{1}{r}-log\left(\frac{2}{1}\frac{3}{2}\frac{4}{3}\cdots\right)$$

$$=\sum_{r=1}^{\infty}\frac{1}{r}-\left(log\frac{2}{1}+log\frac{3}{2}+log\frac{4}{3}+\cdots\right)$$

$$=1+\sum_{r=2}^{\infty}\frac{1}{r}-\sum_{r=2}^{\infty}log\frac{r}{r-1}$$

$$=1+\sum_{r=2}^{\infty}\frac{1}{r}-\sum_{r=2}^{\infty}\sum_{s=1}^{\infty}\frac{1}{sr^{s}}=1+\sum_{r=2}^{\infty}\frac{1}{r}-\sum_{r=2}^{\infty}\left(\frac{1}{r}+\sum_{s=2}^{\infty}\frac{1}{sr^{s}}\right)$$

$$=1-\sum_{r=2}^{\infty}\sum_{s=2}^{\infty}\frac{1}{sr^{s}}$$

$$\therefore\quad 1-\gamma=\sum_{r=2}^{\infty}\sum_{s=2}^{\infty}\frac{1}{sr^{s}} \tag{1.1}$$

Next, rearranging (1.1) along the diagonal line,

$$\sum_{r=2}^{\infty}\sum_{s=2}^{\infty}\frac{1}{sr^{s}}=\frac{1}{2\cdot2^{2}}+\frac{1}{3\cdot2^{3}}+\frac{1}{4\cdot2^{4}}+\cdots$$

$$+\frac{1}{2\cdot3^{2}}+\frac{1}{3\cdot3^{3}}+\frac{1}{4\cdot3^{4}}+\cdots$$

$$+\frac{1}{2\cdot4^{2}}+\frac{1}{3\cdot4^{3}}+\frac{1}{4\cdot4^{4}}+\cdots$$

$$=\frac{1}{2\cdot2^{2}}+\left(\frac{1}{2\cdot3^{2}}+\frac{1}{3\cdot2^{3}}\right)+\left(\frac{1}{2\cdot4^{2}}+\frac{1}{3\cdot3^{3}}+\frac{1}{4\cdot2^{4}}\right)+\cdots$$

$$= \sum_{s=2}^{2} \frac{1}{s(2+2-s)^s} + \sum_{s=2}^{3} \frac{1}{s(2+3-s)^s} + \sum_{s=2}^{4} \frac{1}{s(2+4-s)^s} + \cdots$$

$$= \sum_{r=2}^{\infty} \sum_{s=2}^{r} \frac{1}{s(2+r-s)^s} \tag{1.1'}$$

Next,

$$\gamma = \sum_{r=1}^{\infty} \frac{1}{r} - \left(log\frac{2}{1} + log\frac{3}{2} + log\frac{4}{3} + \cdots \right)$$

$$= \sum_{r=1}^{\infty} \frac{1}{r} - \sum_{r=1}^{\infty} log\left(1+\frac{1}{r}\right)$$

$$= \sum_{r=1}^{\infty} \frac{1}{r} - \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{s\,r^s} = \sum_{r=1}^{\infty} \frac{1}{r} - \sum_{r=1}^{\infty} \left\{ \frac{(-1)^{1+1}}{1\,r^1} + \sum_{s=2}^{\infty} \frac{(-1)^{s+1}}{s\,r^s} \right\}$$

$$= \sum_{r=1}^{\infty} \sum_{s=2}^{\infty} \frac{(-1)^s}{s\,r^s} \tag{1.2}$$

And rearranging (1.2) along the diagonal line, we obtain (1.2') .

Last,

$$1-\gamma = \sum_{r=2}^{\infty} \sum_{s=2}^{\infty} \frac{1}{sr^s} = \sum_{s=2}^{\infty} \frac{1}{s} \left(\sum_{r=1}^{\infty} \frac{1}{r^s} - 1 \right) = \sum_{s=2}^{\infty} \frac{\zeta(s)-1}{s} \tag{1.1''}$$

$$\gamma = \sum_{s=2}^{\infty} \frac{(-1)^s}{s} \sum_{r=1}^{\infty} \frac{1}{r^s} = \sum_{s=2}^{\infty} \frac{(-1)^s}{s} \zeta(s) \tag{1.2''}$$

Let us assume the precision aimed at to be 4 digits below the decimal point. When these formulas are calculated with mathematical software, it is as follows.

1 - γ , γ

```
N[{1 - EulerGamma, EulerGamma}]
{0.422784, 0.577216}
```

Zeta Series

$$f[m\_] := \sum_{s=2}^{m} \frac{Zeta[s]-1}{s}$$

```
N[f[10]]
0.422701
```

Alternating Double Series

$$g[m\_] := \sum_{s=2}^{m} \frac{(-1)^s}{s} \sum_{r=1}^{m} \frac{1}{r^s}$$

```
N[g[44]]
0.577298
```

Calculation Result

If γ is calculated according to the definitional identity, the 6000th terms are required to obtain the 4 digits below the decimal point.

On the other hand, Zeta Series (1.1") reached the precision aimed at in term only 9, the 2nd was the Alternating Double Series (1.2), which reached the precision aimed at in term 44×43 . Other formulas were farther slower than the definitional identity.

2007.12.10
2011.07.09 renewed
2016.02.05 updated

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Alien's Mathematics