07 New Formula for the Sum of Powers

7.1 Expression with binomial coefficients of a factorial

Formula 7.1.1 (S. Ruiz)

When n is a natural number, the following expression holds for arbitrary x.

$$n! = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} (x+r)^n$$

Proof

n^2 and the differences sequence are as follows.

$$\begin{array}{lccccccc} n^2: & 1 & 4 & 9 & 16 & 25 & \cdots \\ d_1: & & 3 & 5 & 7 & 9 & \cdots \\ d_2: & & & 2 & 2 & 2 & \cdots \end{array}$$

Each difference is calculated as follows.

$$d_1:\ 3 = 2^2 - 1^2,\ 5 = 3^2 - 2^2,\ 7 = 4^2 - 3^2,\ \cdots$$

$$d_2:\ 2 = 3^2 - 2\cdot 2^2 + 1^2,\ 2 = 4^2 - 2\cdot 3^2 + 2^2,\ \cdots$$

If we focus on d_2, we find that this can be expressed as follows for arbitrary x.

$$2! = {}_2C_2(x+2)^2 - {}_2C_1(x+1)^2 + {}_2C_0 x^2$$

Next, n^3 and the differences sequence are as follows.

$$\begin{array}{lccccccc} n^3: & 1 & 8 & 27 & 64 & 125 & 21 & \cdots \\ d_1: & & 7 & 19 & 37 & 61 & 91 & \cdots \\ d_2: & & & 12 & 18 & 24 & 30 & \cdots \\ d_3: & & & & 6 & 6 & 6 & \cdots \end{array}$$

These differences are also calculated as follows.

$$d_1:\ 7 = 2^3 - 1^3,\ 19 = 3^3 - 2^3,\ 37 = 4^3 - 3^3,\ 61 = 5^3 - 4^3,\ \cdots$$

$$d_2:\ 12 = 3^3 - 2\cdot 2^3 + 1^3,\ 18 = 4^3 - 2\cdot 3^3 + 2^3,\ 24 = 5^3 - 2\cdot 4^3 + 3^3,\ \cdots$$

$$d_3:\ 6 = 4^3 - 3\cdot 3^3 + 3\cdot 2^3 - 1^3,\ 6 = 5^3 - 3\cdot 4^3 + 3\cdot 3^3 - 2^3,\ \cdots$$

If we focus on d_3, we find that this can be also expressed as follows for arbitrary x.

$$3! = {}_3C_3(x+3)^3 - {}_3C_2(x+2)^3 + {}_3C_1(x+1)^3 - {}_3C_0 x^3$$

Next, n^4 and the differences sequence of are as follows.

$$\begin{array}{lccccccc} n^4: & 1 & 16 & 81 & 256 & 625 & 1296 & 2401 & \cdots \\ d_1: & & 15 & 65 & 175 & 369 & 671 & 1105 & \cdots \\ d_2: & & & 50 & 110 & 194 & 302 & 434 & \cdots \\ d_3: & & & & 60 & 84 & 108 & 132 & \cdots \\ d_4: & & & & & 24 & 24 & 24 & \cdots \end{array}$$

These differences are also calculated as follows.

d_1: $15 = 2^4 - 1^4$, $65 = 3^4 - 2^4$, $175 = 4^4 - 3^4$, $369 = 5^4 - 4^4$, $671 = 6^4 - 5^4$, $\cdots$

d_2: $50 = 3^4 - 2\cdot 2^4 + 1^4$, $110 = 4^4 - 2\cdot 3^4 + 2^4$, $175 = 5^4 - 2\cdot 4^4 + 3^4$, $\cdots$

d_3: $60 = 4^4 - 3\cdot 3^4 + 3\cdot 2^4 - 1^4$, $84 = 5^4 - 3\cdot 4^4 + 3\cdot 3^4 - 2^4$, $\cdots$

d_4: $24 = 5^4 - 4\cdot 4^4 + 6\cdot 3^4 - 4\cdot 2^4 + 1^4$, $24 = 6^4 - 4\cdot 5^4 + 6\cdot 4^4 - 4\cdot 3^4 + 2^4$, $\cdots$

And d_4 can be also expressed as follows for arbitrary x.

$$4! = {}_4C_4(x+4)^4 - {}_4C_3(x+3)^4 + {}_4C_2(x+2)^4 - {}_4C_1(x+1)^4 + {}_4C_0 x^4$$

Thus, in general, we obtain the following.

$$n! = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} (x+r)^n$$

Example: When $x = -0.3$, $x = \pi$

```
f[n_, x_] := Sum[(-1)^(n-r) Binomial[n, r] (x + r)^n, {r, 0, n}]

Table[f[n, x], {n, 2, 5}]

{ x^2 - 2 (1 + x)^2 + (2 + x)^2,
 -x^3 - 3 (1 + x)^3 - 3 (2 + x)^3 - (3 + x)^3,
  x^4 - 4 (1 + x)^4 - 6 (2 + x)^4 - 4 (3 + x)^4 - (4 + x)^4,
 -x^5 - 5 (1 + x)^5 - 10 (2 + x)^5 + 10 (3 + x)^5 - 5 (4 + x)^5 + (5 + x)^5 }

f[4, 0.3]            24.
Simplify[f[5, π]]    120
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7.2 Expression with binomial coefficients of a power (Part1)

Formula 7.2.1

When m,n are natural numbers, the following expression holds.

$$n^m = \sum_{r=0}^{m} {}_mB_r \, {}_nC_r$$

Where,

$${}_mB_r = \sum_{s=0}^{r} (-1)^{r-s} {}_rC_s s^m \qquad r=0,1,2,\cdots,m$$

Proof

n^2 and the difference sequence are as follows.

$$\begin{array}{lcccccc} n^2: & 0 & 1 & 4 & 9 & 16 & 25 \cdots \\ d_1: & & 1 & 3 & 5 & 7 & 9 \cdots \\ d_2: & & & 2 & 2 & 2 & 2 \cdots \end{array}$$

On the other hand, 2 times of Pascal's triangle is as follows.

$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 2 & & 2 & & & \\ & & 2 & & 4 & & 2 & & \\ & 2 & & 6 & & 6 & & 2 & \\ 2 & & 8 & & 12 & & 8 & & 2 \\ & & & & \vdots & & & & \end{array}$$

Comparing both, we obtain the followings.

$$d_2(n) = 2\,{}_nC_0$$

$$d_1(n) = 1 + 2\sum_{r=0}^{n-1} {}_rC_0 = 1 + 2\,{}_nC_1 \qquad \left(\because \sum_{r=0}^{n-1} {}_rC_0 = {}_nC_1\right)$$

Then

$$n^2 = 0 + \sum_{r=0}^{n-1}(1 + 2\,{}_rC_1) = \sum_{r=0}^{n-1} 1 + 2\sum_{r=0}^{n-1} {}_rC_1$$

Here, $\sum_{r=0}^{n-1} 1 = {}_nC_1$, $\sum_{r=0}^{n-1} {}_rC_1 = {}_nC_2$ $\left({}_0C_1 = 0\right)$. Then

$$n^2 = 1\,{}_nC_1 + 2\,{}_nC_2 \qquad (2.n)$$

Coefficients (blue and magenta) of the right hand side are corresponding with the first terms of the difference sequence of n^2. And the first terms of the difference sequences are given as follows.

$$d_1: \ 1 = 1^2 - 0^2 , \ 3 = 2^2 - 1^2 , \ \cdots$$

$$d_2: \ 2 = 2^2 - 2\cdot 1^2 + 0^2 , \ \cdots$$

That is, the r-th coefficient ${}_2B_r$ of the right side of (2.n) can be expressed as follows.

$${}_2B_r = \sum_{s=0}^{r} (-1)^{r-s} {}_rC_s s^2$$

Next, n^3 and the difference sequence are as follows.

$$
\begin{array}{lccccccccc}
n^3: & 0 & 1 & 8 & 27 & 64 & 125 & 216 & \cdots \\
d_1: & & 1 & 7 & 19 & 37 & 61 & 91 & \cdots \\
d_2: & & & 6 & 12 & 18 & 24 & 30 & \cdots \\
d_3: & & & & 6 & 6 & 6 & 6 & \cdots
\end{array}
$$

On the other hand, 6 times of Pascal's triangle is as follows.

$$6\,{}_nC_0: \quad 6 \quad 6 \quad 6 \quad 6 \quad \cdots$$

Comparing both, we obtain the followings.

$$d_3(n) = 6\,{}_nC_0$$

$$d_2(n) = 6 + 6\sum_{r=0}^{n-1} {}_rC_0 = 6 + 6\,{}_nC_1 \qquad \left(\because \ \sum_{r=0}^{n-1} {}_rC_0 = {}_nC_1\right)$$

$$d_1(n) = 1 + \sum_{r=0}^{n-1}(6 + 6\,{}_rC_1)$$

$$= 1 + 6\sum_{r=0}^{n-1} 1 + 6\sum_{r=0}^{n-1} {}_rC_1$$

Here, $\sum_{r=0}^{n-1} 1 = {}_nC_1$, $\sum_{r=0}^{n-1} {}_rC_1 = {}_nC_2$ $\quad ({}_0C_1 = 0)$. Then,

$$d_1(n) = 1 + 6\,{}_nC_1 + 6\,{}_nC_2$$

Then

$$n^3 = 0 + \sum_{r=0}^{n-1}(1 + 6\,{}_rC_1 + 6\,{}_rC_2)$$

$$= \sum_{r=0}^{n-1} 1 + 6\sum_{r=0}^{n-1} {}_rC_1 + 6\sum_{r=0}^{n-1} {}_rC_2$$

Furthermore,

$$\sum_{r=0}^{n-1} 1 = {}_nC_1 \ , \quad \sum_{r=0}^{n-1} {}_rC_1 = {}_nC_2 \ , \quad \sum_{r=0}^{n-1} {}_rC_2 = {}_nC_3 \qquad ({}_rC_s = 0 \quad r < s)$$

Using these,

$$n^3 = 1\,{}_nC_1 + 6\,{}_nC_2 + 6\,{}_nC_3 \tag{3.n}$$

Coefficients (blue and magenta) of the right hand side are corresponding with the first terms of the difference sequence of n^3. And the first terms of the difference sequences are given as follows.

$$d_1: \ 1 = 1^3 - 0^3 \ , \ 7 = 2^3 - 1^3 \ , \ 19 = 3^3 - 2^3 \ , \quad \cdots$$

$$d_2: \ 6 = 2^3 - 2\cdot 1^3 + 0^3 \ , \ 12 = 3^3 - 2\cdot 2^3 + 1^3 \ , \quad \cdots$$

$$d_3: \ 6 = 3^3 - 3\cdot 2^3 + 3\cdot 1^3 - 0^3 \ , \quad \cdots$$

That is, the r-th coefficient ${}_3B_r$ of the right side of (3.n) can be expressed as follows.

$${}_3B_r = \sum_{s=0}^{r} (-1)^{r-s}\,{}_rC_s\, s^3$$

Hereafter by induction, we obtain the desired expression.

Example: 3^4

$$ {}_4B_0 = \sum_{s=0}^{0} (-1)^{0-s}\,{}_1C_s s^4 = 0 \quad , \qquad {}_4B_1 = \sum_{s=0}^{1} (-1)^{1-s}\,{}_1C_s s^4 = 1 $$

$$ {}_4B_2 = \sum_{s=0}^{2} (-1)^{2-s}\,{}_2C_s s^4 = 14 \ , \qquad {}_4B_3 = \sum_{s=0}^{3} (-1)^{3-s}\,{}_3C_s s^4 = 36 $$

$$ {}_4B_4 = \sum_{s=0}^{4} (-1)^{4-s}\,{}_4C_s s^4 = 24 $$

From these,

$$ n^4 = 0\,{}_nC_0 + 1\,{}_nC_1 + 14\,{}_nC_2 + 36\,{}_nC_3 + 24\,{}_nC_4 $$

Substituting $n=3$ for this,

$$ 3^4 = 0\,{}_3C_0 + 1\,{}_3C_1 + 14\,{}_3C_2 + 36\,{}_3C_3 + 24\,{}_3C_4 = 81 $$

Formula 7.2.1 is extensible to the real number for n using general binomial coefficients.

Formula7.2.2

When m is a natural number and x is a positive number, the following expression holds.

$$ x^m = \sum_{r=0}^{m} {}_mB_r \binom{x}{r} $$

Where,

$$ {}_mB_r = \sum_{s=0}^{r} (-1)^{r-s}\,{}_rC_s s^m \qquad r=0,1,2,\cdots,m $$

Example: 0.5^3

$$ {}_3B_0 = \sum_{s=0}^{0} (-1)^{0-s}\,{}_1C_s s^3 = 0 \quad , \qquad {}_3B_1 = \sum_{s=0}^{1} (-1)^{1-s}\,{}_1C_s s^3 = 1 $$

$$ {}_3B_2 = \sum_{s=0}^{2} (-1)^{2-s}\,{}_2C_s s^3 = 6 \quad , \qquad {}_3B_3 = \sum_{s=0}^{3} (-1)^{3-s}\,{}_3C_s s^3 = 6 $$

Then,

$$ 0.5^3 = 0\binom{0.5}{0} + 1\binom{0.5}{1} + 6\binom{0.5}{2} + 6\binom{0.5}{3} = 0.125 $$

Note

The first few steps of the triangle of these coefficients are as follows.

$$
\begin{array}{ccc}
{}_1B_1 & & 1 \\
{}_2B_1 \ \ {}_2B_2 & & 1 \quad 2 \\
{}_3B_1 \ \ {}_3B_2 \ \ {}_3B_3 & & 1 \quad 6 \quad 6 \\
{}_4B_1 \ \ {}_4B_2 \ \ {}_4B_3 \ \ {}_4B_4 & = & 1 \quad 14 \quad 36 \quad 24 \\
{}_5B_1 \ \ {}_5B_2 \ \ {}_5B_3 \ \ {}_5B_4 \ \ {}_5B_5 & & 1 \quad 30 \quad 150 \quad 240 \quad 120 \\
{}_6B_1 \ \ {}_6B_2 \ \ {}_6B_3 \ \ {}_6B_4 \ \ {}_6B_5 \ \ {}_6B_6 & & 1 \quad 62 \quad 540 \quad 1560 \quad 1800 \quad 720 \\
{}_7B_1 \ \ {}_7B_2 \ \ {}_7B_3 \ \ {}_7B_4 \ \ {}_7B_5 \ \ {}_7B_6 \ \ {}_7B_7 & & 1 \quad 126 \quad 1806 \quad 8400 \quad 16800 \quad 15120 \quad 5040 \\
\vdots & & \vdots
\end{array}
$$

According to **On-Line Encyclopedia of Integer Sequences** , this coefficient ${}_mB_r$ is already known. (A019538) And this coefficient has the following relation to Stirling numbers of the 2nd kind.

$${}_mB_r = r!\ S_2(m,r)$$

7.3 Expression with binomial coefficients of a power (Part2)

Formula 7.3.1

When m,n are natural numbers, the following expression holds.

$$n^m = \sum_{r=0}^{m-1} {}_mD_{r+1}\,{}_{n+r}C_m$$

Where, ${}_mD_r$ $r=1,2,\cdots,m$ are **Eulerian Numbers** which are given by

$${}_mD_r = \sum_{s=0}^{r-1} (-1)^s\,{}_{m+1}C_s (r-s)^m \qquad m=1,2,3,\cdots$$

Proof

From Formula 7.2.1 ,

$$\begin{aligned}
n^2 &= {}_nC_1 + 2\,{}_nC_2 \\
&= {}_nC_1 + {}_nC_2 + {}_nC_2 \\
&= {}_{n+1}C_2 + {}_nC_2 \\
n^3 &= {}_nC_1 + 6\,{}_nC_2 + 6\,{}_nC_3 \\
&= ({}_nC_1 + {}_nC_2) + 5\,{}_nC_2 + 6\,{}_nC_3 \\
&= {}_{n+1}C_2 + 5\,{}_nC_2 + 6\,{}_nC_3 \\
&= {}_{n+1}C_2 + 5({}_nC_2 + {}_nC_3) + {}_nC_3 \\
&= {}_{n+1}C_2 + 5\,{}_{n+1}C_3 + {}_nC_3 \\
&= ({}_{n+1}C_2 + {}_{n+1}C_3) + 4\,{}_{n+1}C_3 + {}_nC_3 \\
&= {}_{n+2}C_3 + 4\,{}_{n+1}C_3 + {}_nC_3 \\
n^4 &= {}_nC_1 + 14\,{}_nC_2 + 36\,{}_nC_3 + 24\,{}_nC_4 \\
&= ({}_nC_1 + {}_nC_2) + 13\,{}_nC_2 + 36\,{}_nC_3 + 24\,{}_nC_4 \\
&= {}_{n+1}C_2 + 13\,{}_nC_2 + 36\,{}_nC_3 + 24\,{}_nC_4 \\
&= {}_{n+1}C_2 + 13({}_nC_2 + {}_nC_3) + 23\,{}_nC_3 + 24\,{}_nC_4 \\
&= {}_{n+1}C_2 + 13\,{}_{n+1}C_3 + 23\,{}_nC_3 + 24\,{}_nC_4 \\
&= ({}_{n+1}C_2 + {}_{n+1}C_3) + 12\,{}_{n+1}C_3 + 23({}_nC_3 + {}_nC_4) + {}_nC_4 \\
&= {}_{n+2}C_3 + 12\,{}_{n+1}C_3 + 23\,{}_{n+1}C_4 + {}_nC_4 \\
&= {}_{n+2}C_3 + 12({}_{n+1}C_3 + {}_{n+1}C_4) + 11\,{}_{n+1}C_4 + {}_nC_4 \\
&= {}_{n+2}C_3 + 12\,{}_{n+2}C_4 + 11\,{}_{n+1}C_4 + {}_nC_4 \\
&= ({}_{n+2}C_3 + {}_{n+2}C_4) + 11\,{}_{n+2}C_4 + 11\,{}_{n+1}C_4 + {}_nC_4 \\
&= {}_{n+3}C_4 + 11\,{}_{n+2}C_4 + 11\,{}_{n+1}C_4 + {}_nC_4 \\
n^5 &= {}_nC_1 + 30\,{}_nC_2 + 150\,{}_nC_3 + 240\,{}_nC_4 + 120\,{}_nC_5 \\
&= ({}_nC_1 + {}_nC_2) + 29\,{}_nC_2 + 150\,{}_nC_3 + 240\,{}_nC_4 + 120\,{}_nC_5 \\
&= {}_{n+1}C_2 + 29\,{}_nC_2 + 150\,{}_nC_3 + 240\,{}_nC_4 + 120\,{}_nC_5 \\
&= {}_{n+1}C_2 + 29({}_nC_2 + {}_nC_3) + 121\,{}_nC_3 + 240\,{}_nC_4 + 120\,{}_nC_5 \\
&= {}_{n+1}C_2 + 29\,{}_{n+1}C_3 + 121\,{}_nC_3 + 240\,{}_nC_4 + 120\,{}_nC_5 \\
&= ({}_{n+1}C_2 + {}_{n+1}C_3) + 28\,{}_{n+1}C_3 + 121({}_nC_3 + {}_nC_4) + 119\,{}_nC_4 + 120\,{}_nC_5
\end{aligned}$$

$$= {}_{n+2}C_3 + 28\,{}_{n+1}C_3 + 121\,{}_{n+1}C_4 + 119\,{}_nC_4 + 120\,{}_nC_5$$
$$= {}_{n+2}C_3 + 28({}_{n+1}C_3 + {}_{n+1}C_4) + 93\,{}_{n+1}C_4 + 119({}_nC_4 + {}_nC_5) + {}_nC_5$$
$$= {}_{n+2}C_3 + 28\,{}_{n+2}C_4 + 93\,{}_{n+1}C_4 + 119\,{}_{n+1}C_5 + {}_nC_5$$
$$= ({}_{n+2}C_3 + {}_{n+2}C_4) + 27\,{}_{n+2}C_4 + 93({}_{n+1}C_4 + {}_{n+1}C_5) + 26\,{}_{n+1}C_5 + {}_nC_5$$
$$= {}_{n+3}C_4 + 27\,{}_{n+2}C_4 + 93\,{}_{n+2}C_5 + 26\,{}_{n+1}C_5 + {}_nC_5$$
$$= {}_{n+3}C_4 + 27({}_{n+2}C_4 + {}_{n+2}C_5) + 66\,{}_{n+2}C_5 + 26\,{}_{n+1}C_5 + {}_nC_5$$
$$= {}_{n+3}C_4 + 27\,{}_{n+3}C_5 + 66\,{}_{n+2}C_5 + 26\,{}_{n+1}C_5 + {}_nC_5$$
$$= ({}_{n+3}C_4 + {}_{n+3}C_5) + 26\,{}_{n+3}C_5 + 66\,{}_{n+2}C_5 + 26\,{}_{n+1}C_5 + {}_nC_5$$
$$= {}_{n+4}C_5 + 26\,{}_{n+3}C_5 + 66\,{}_{n+2}C_5 + 26\,{}_{n+1}C_5 + {}_nC_5$$
$$\vdots$$

These coefficients are "**Coefficients that make powers constant (Eulerian Numbers)** " which are mentioned in " **01 Zeta Generating Functions** ". Therefore, by induction, we obtain the desired expression.

Example: 7^3, 3^4

$$7^3 = {}_7C_3 + 4\,{}_8C_3 + {}_9C_3 \qquad = 35 + 4\cdot 56 + 84 \qquad = 343$$
$$3^4 = {}_3C_4 + 11\,{}_4C_4 + 11\,{}_5C_4 + {}_6C_4 \quad = 0 + 11\cdot 1 + 11\cdot 5 + 15 = \ 81$$

Formula 7.3.1 is also extensible to the real number for n .

Formula 7.3.2

When m is a natural number and x is a positive number, the following expression holds.

$$x^m = \sum_{r=0}^{m-1} {}_mD_{r+1} \binom{x+r}{m}$$

Where, ${}_mD_r$ $r = 1, 2, \cdots, m$ are Eulerian Numbers which are given by

$${}_mD_r = \sum_{s=0}^{r-1} (-1)^s\, {}_{m+1}C_s (r-s)^m \qquad m = 1, 2, 3, \cdots$$

Example: 0.7^3, 0.9^4

$$0.7^3 = \binom{0.7}{3} + 4\binom{1.7}{3} + \binom{2.7}{3} \qquad = 0.343$$
$$0.9^4 = \binom{0.9}{4} + 11\binom{1.9}{4} + 11\binom{2.9}{4} + \binom{3.9}{4} = 0.6561$$

Note: Eulerian Triangle

The first 7 steps of Eulerian Triangle are as follows.

$$\begin{array}{c} {}_1D_1 \\ {}_2D_1\ {}_2D_2 \\ {}_3D_1\ {}_3D_2\ {}_3D_3 \\ {}_4D_1\ {}_4D_2\ {}_4D_3\ {}_4D_4 \\ {}_5D_1\ {}_5D_2\ {}_5D_3\ {}_5D_4\ {}_5D_5 \\ {}_6D_1\ {}_6D_2\ {}_6D_3\ {}_6D_4\ {}_6D_5\ {}_6D_6 \\ {}_7D_1\ {}_7D_2\ {}_7D_3\ {}_7D_4\ {}_7D_5\ {}_7D_6\ {}_7D_7 \end{array} = \begin{array}{c} 1 \\ 1\quad 1 \\ 1\quad 4\quad 1 \\ 1\quad 11\quad 11\quad 1 \\ 1\quad 26\quad 66\quad 26\quad 1 \\ 1\quad 57\quad 302\quad 302\quad 57\quad 1 \\ 1\quad 120\ 1191\ 2416\ 1191\ 120\quad 1 \end{array}$$

7.4 New formula for the sum of powers

Formula 7.4.1

When m,n are natural numbers, the following expression holds.

$$\sum_{k=1}^{n} k^m = \sum_{r=1}^{m} {}_mD_r \, {}_{n+r}C_{m+1}$$

Where, ${}_mD_r$ $r=1,2,\cdots,m$ are **Eulerian Numbers** which are given by

$${}_mD_r = \sum_{s=0}^{r-1} (-1)^s \, {}_{m+1}C_s (r-s)^m \qquad m=1,2,3,\cdots$$

Proof

From Formula 7.3.1 ,

$$n^2 = {}_{n+1}C_2 + {}_nC_2$$

Then,

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} {}_{k+1}C_2 + \sum_{k=1}^{n} {}_kC_2$$

Here,

$$\sum_{k=1}^{n} {}_{k+1}C_2 = {}_{n+2}C_3 \quad , \quad \sum_{k=1}^{n} {}_kC_2 = \sum_{k=1}^{n-1} {}_{k+1}C_2 = {}_{n+1}C_3$$

Substituting these for the above,

$$\sum_{k=1}^{n} k^2 = {}_{n+2}C_3 + {}_{n+1}C_3$$

This is what the all subscripts of n^2 are increased one. .

Next, from Formula 7.3.1 ,

$$n^3 = {}_{n+2}C_3 + 4\,{}_{n+1}C_3 + {}_nC_3$$

Then,

$$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n} {}_{k+2}C_3 + 4\sum_{k=1}^{n} {}_{k+1}C_3 + \sum_{k=1}^{n} {}_kC_3$$

Here,

$$\sum_{k=1}^{n} {}_{k+2}C_3 = {}_{n+3}C_4 \quad , \quad \sum_{k=1}^{n} {}_{k+1}C_3 = {}_{n+2}C_4 \quad , \quad \sum_{k=1}^{n} {}_kC_3 = {}_{n+1}C_4$$

Substituting these for the above,

$$\sum_{k=1}^{n} k^3 = {}_{n+3}C_4 + 4\,{}_{n+2}C_4 + {}_{n+1}C_4$$

This is also what the all subscripts of n^3 are increased one.

When the 4th degree, we obtain the following in a similar way.

$$\sum_{k=1}^{n} k^4 = {}_{n+4}C_5 + 11\,{}_{n+3}C_5 + 11\,{}_{n+2}C_5 + {}_{n+1}C_5$$

Hereafter by induction, we obtain the desired expression.

Example: $\displaystyle\sum_{k=1}^{100} k^5$

$${}_{101}C_6 + 26\,{}_{102}C_6 + 66\,{}_{103}C_6 + 26\,{}_{104}C_6 + {}_{105}C_6 = 171708332500$$

$$\frac{2\cdot 100^6 + 6\cdot 100^5 + 5\cdot 100^4 - 100^2}{12} = 171708332500$$

Formula 7.4.1 is also extensible to the real number for n .

Formula 7.4.2

When m is a natural number and x is a positive number, the following expression holds.

$$\sum_{k=0}^{x} k^m = \sum_{r=1}^{m} {}_mD_r \binom{x+r}{m+1}$$

Where, ${}_mD_r$ $r=1,2,\cdots,m$ are Eulerian Numbers which are given by

$${}_mD_r = \sum_{s=0}^{r-1} (-1)^s \, {}_{m+1}C_s (r-s)^m \qquad m=1,2,3,\cdots$$

Example: $\sum_{k=0}^{0.9} k^4$

$$\binom{1.9}{5} + 11\binom{2.9}{5} + 11\binom{3.9}{5} + \binom{4.9}{5} = 0.659148$$

$$\frac{6\cdot 0.9^5 + 15\cdot 0.9^4 + 10\cdot 0.9^3 - 0.9^1}{30} = 0.659148$$

Note: Calculation method one by one for Eulerian Numbers

Eulerian Number can be calculated also by the algorithm one by one as follows.

		Calculating formula
2	1 1	
	2,2	
3	1 4 1	4 = 1×2 + 1×2
	3,2 2,3	
4	1 11 11 1	11 = 1×3 + 4×2
	4,2 3,3 2,4	
5	1 26 66 26 1	26 = 1×4 +11×2 , 66 = 11×3 + 11×3
	5,2 4,3 3,4 2,5	
6	1 57 302 302 57 1	57 = 1×5 +26×2 , 302 = 26×4 + 66×3
	⋮	⋮

7.5 Formula for the sum of powers of real numbers

In this section, we calculate the following the sum of powers. Where, m,n are natural numbers and a,b are real numbers.

$$S_{m,n} = (1a+b)^m + (2a+b)^m + (3a+b)^m + \cdots + (na+b)^m$$

Formula 7.5.1

When m,n are natural numbers and a,b are real numbers, the following expression holds.

$$\sum_{k=1}^{n}(ak+b)^m = b^m{}_nC_1 + \sum_{r=1}^{m}{}_mC_r a^r b^{m-r}\sum_{s=1}^{r}{}_rD_s\,{}_{n+s}C_{r+1}$$

Where, ${}_rD_s$ $s=1,2,\cdots,r$ are **Eulerian Numbers** which are given by

$${}_rD_s = \sum_{t=0}^{s-1}(-1)^t{}_{r+1}C_t(s-t)^r \qquad r=1,2,\cdots,m$$

Proof

$$(ak+b)^m = \sum_{r=0}^{m}{}_mC_r(ak)^r b^{m-r} = \sum_{r=0}^{m}{}_mC_r k^r a^r b^{m-r}$$

Then

$$\sum_{k=1}^{n}(ak+b)^m = \sum_{k=1}^{n}\sum_{r=0}^{m}{}_mC_r k^r a^r b^{m-r} = \sum_{r=0}^{m}{}_mC_r a^r b^{m-r}\sum_{k=1}^{n}k^r$$

$$= {}_mC_0 a^0 b^m \sum_{k=1}^{n}k^0 + \sum_{r=1}^{m}{}_mC_r a^r b^{m-r}\sum_{k=1}^{n}k^r$$

i.e.

$$\sum_{k=1}^{n}(ak+b)^m = b^m{}_nC_1 + \sum_{r=1}^{m}{}_mC_r a^r b^{m-r}\sum_{k=1}^{n}k^r \qquad \left(\because \sum_{k=1}^{n}k^0 = {}_nC_1\right)$$

From Formula 7.4.1 in the previous section,

$$\sum_{k=1}^{n}k^r = \sum_{s=1}^{r}{}_rD_s\,{}_{n+s}C_{r+1}$$

$${}_rD_s = \sum_{t=0}^{s-1}(-1)^t{}_{r+1}C_t(s-t)^r \qquad r=1,2,3,\cdots,m$$

Substituting these for the above,

$$\sum_{k=1}^{n}(ak+b)^m = b^m{}_nC_1 + \sum_{r=1}^{m}{}_mC_r a^r b^{m-r}\sum_{s=1}^{r}{}_rD_s\,{}_{n+s}C_{r+1}$$

Although this seems a complicated formula, if this is written down about $m=1,\sim,5$, it is as follows. The coefficients of a,b are the m-th step of Pascal's Triangle and the coefficients of ${}_{n+s}C_{r+1}$ are the upper m steps of the Eulerian Triangle.

$$\sum_{k=1}^{n}(ak+b)^1 = b\,{}_nC_1 + a\,{}_{n+1}C_2$$

$$\sum_{k=1}^{n}(ak+b)^2 = b^2{}_nC_1 + 2ab\,{}_{n+1}C_2 + a^2({}_{n+1}C_3 + {}_{n+2}C_3)$$

$$\sum_{k=1}^{n}(ak+b)^3 = b^3{}_nC_1 + 3ab^2{}_{n+1}C_2 + 3a^2b({}_{n+1}C_3 + {}_{n+2}C_3) + a^3({}_{n+1}C_4 + 4\,{}_{n+2}C_4 + {}_{n+3}C_4)$$

$$\sum_{k=1}^{n}(ak+b)^4 = b^4{}_nC_1 + 4ab^3{}_{n+1}C_2 + 6a^2b^2({}_{n+1}C_3 + {}_{n+2}C_3) + 4a^3b({}_{n+1}C_4 + 4\,{}_{n+2}C_4 + {}_{n+3}C_4) + a^4({}_{n+1}C_5 + 11\,{}_{n+2}C_5 + 11\,{}_{n+3}C_5 + {}_{n+4}C_5)$$

$$\sum_{k=1}^{n}(ak+b)^5 = b^5{}_nC_1 + 5ab^4{}_{n+1}C_2 + 10a^2b^3({}_{n+1}C_3 + {}_{n+2}C_3) + 10a^3b^2({}_{n+1}C_4 + 4\,{}_{n+2}C_4 + {}_{n+3}C_4) + 5a^4b({}_{n+1}C_5 + 11\,{}_{n+2}C_5 + 11\,{}_{n+3}C_5 + {}_{n+4}C_5) + a^5({}_{n+1}C_6 + 26\,{}_{n+2}C_6 + 66\,{}_{n+3}C_6 + 26\,{}_{n+4}C_6 + {}_{n+5}C_6)$$

The strong point of this formula is that the calculations are essentially performed by integers. And if there are non-integer calculations, the result becomes easier. We show two examples below.

Example 1

$$\begin{aligned}\sum_{k=1}^{50}(\pi k-e)^4 &= e^4{}_{50}C_1 - 4\pi e^3{}_{51}C_2 + 6\pi^2e^2({}_{51}C_3 + {}_{52}C_3) - 4\pi^3e({}_{51}C_4 + 4\,{}_{52}C_4 + {}_{53}C_4) + \pi^4({}_{51}C_5 + 11\,{}_{52}C_5 + 11\,{}_{53}C_5 + {}_{54}C_5)\\ &= 50e^4 - 4\cdot 1275\pi e^3 + 6(20825 + 22100)\pi^2e^2 - 4(249900 + 4\cdot 270725 + 292825)\pi^3e + (2349060 + 11\cdot 2598960 + 11\cdot 2869685 + 3162510)\pi^4\end{aligned}$$

i.e.

$$\sum_{k=1}^{50}(\pi k-e)^4 = 50e^4 - 5100\pi e^3 + 257550\pi^2e^2 - 6502500\pi^3e + 65666665\pi^4$$

Example 2

$$\begin{aligned}\sum_{k=1}^{100}(\sqrt{2}k+\sqrt{3})^3 &= (\sqrt{3})^3{}_{100}C_1 + 3\sqrt{2}(\sqrt{3})^2{}_{101}C_2 + 3(\sqrt{2})^2\sqrt{3}({}_{101}C_3 + {}_{102}C_3) + (\sqrt{2})^3({}_{101}C_4 + 4\,{}_{102}C_4 + {}_{103}C_4)\\ &= 100\cdot 3\sqrt{3} + 5050\cdot 9\sqrt{2} + (166650 + 171700)\cdot 6\sqrt{3} + (4082925 + 4\times 4249575 + 4421275)\cdot 2\sqrt{2}\end{aligned}$$

i.e.

$$\sum_{k=1}^{100}(\sqrt{2}k+\sqrt{3})^3 = 51050450\sqrt{2} + 2030400\sqrt{3}$$

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