01 Power of Infinite Series

1.1 Multiple Cauchy Product

The multinomial theorem is useless for the power calculation of infinite series. This is because the polynomial theorem depends on the number of terms, so it can not be applied to a series where the number of terms is infinite. For the power calculation of infinite series, multiple Cauchy product is useful.

Formula 1.1.1 (Multiple Cauchy product of infinite series)

The following expressions hold for two or more convergent infinite series.

$$\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty} b_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r-s} b_s \tag{1.2}$$

$$\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty} b_r\right)\left(\sum_{r=0}^{\infty} c_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s} b_{s-t} c_t \tag{1.3}$$

$$\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty} b_r\right)\left(\sum_{r=0}^{\infty} c_r\right)\left(\sum_{r=0}^{\infty} d_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} a_{r-s} b_{s-t} c_{t-u} d_u \tag{1.4}$$

$$\vdots$$

$$\left(\sum_{r=0}^{\infty} a_{1,r}\right)\left(\sum_{r=0}^{\infty} a_{2,r}\right)\cdots\left(\sum_{r=0}^{\infty} a_{n,r}\right) = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{1,r_1-r_2} a_{2,r_2-r_3}\cdots a_{n-1,r_{n-1}-r_n} a_{n,r_n}$$

(1.n)

Especially,

$$\left(\sum_{r=0}^{\infty} a_r\right)\left\{\sum_{r=0}^{\infty} (-1)^r a_r\right\} = \sum_{r=0}^{\infty}\sum_{s=0}^{2r} (-1)^s a_s a_{2r-s} \tag{1.2±}$$

Proof

The Cauchy product of two series is expressed as follows.

$$\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty} b_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_s b_{r-s} \tag{2*}$$

Making a,b reverse order, we obtain (1.2) .

Next, let

$$\sum_{s=0}^{r} a_s b_{r-s} = B_r \tag{B}$$

Then

$$\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty} b_r\right)\left(\sum_{r=0}^{\infty} c_r\right) = \left(\sum_{r=0}^{\infty} B_r\right)\left(\sum_{r=0}^{\infty} c_r\right)$$

According to (2*) ,

$$\left(\sum_{r=0}^{\infty} B_r\right)\left(\sum_{r=0}^{\infty} c_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} B_s c_{r-s}$$

And replacing $s \to t$, $r \to s$ in (B) ,

$$B_s = \sum_{t=0}^{s} a_t b_{s-t}$$

Substituting this for the above,

$$\left(\sum_{r=0}^{\infty} B_r\right)\left(\sum_{r=0}^{\infty} c_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_t b_{s-t} c_{r-s}$$

i.e.

$$\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty} b_r\right)\left(\sum_{r=0}^{\infty} c_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_t b_{s-t} c_{r-s}$$

Making a,b,c reverse order, we obtain (1.3) .

Next, let

$$\sum_{s=0}^{r}\sum_{t=0}^{s} a_t b_{s-t} c_{r-s} = C_r \tag{C}$$

Then

$$\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty} b_r\right)\left(\sum_{r=0}^{\infty} c_r\right)\left(\sum_{r=0}^{\infty} d_r\right) = \left(\sum_{r=0}^{\infty} C_r\right)\left(\sum_{r=0}^{\infty} d_r\right)$$

According to (**2***) ,

$$\left(\sum_{r=0}^{\infty} C_r\right)\left(\sum_{r=0}^{\infty} d_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} C_s d_{r-s}$$

And replacing $t \to u$, $s \to t$, $r \to s$ in (C) ,

$$C_s = \sum_{t=0}^{s}\sum_{u=0}^{t} a_u b_{t-u} c_{s-t}$$

Substituting this for the above,

$$\left(\sum_{r=0}^{\infty} C_r\right)\left(\sum_{r=0}^{\infty} d_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} a_u b_{t-u} c_{s-t} d_{r-s}$$

i.e.

$$\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty} b_r\right)\left(\sum_{r=0}^{\infty} c_r\right)\left(\sum_{r=0}^{\infty} d_r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} a_u b_{t-u} c_{s-t} d_{r-s}$$

Making a,b,c,d reverse order, we obtain (1.4) . Hereafter, in a similar way, we obtain (1.n) .

Let $b_r = (-1)^r a_r$ in (1.2) , then

$$\left(\sum_{r=0}^{\infty} a_r\right)\left\{\sum_{r=0}^{\infty} (-1)^r a_r\right\} = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r-s}(-1)^s a_s = \sum_{r=0}^{\infty}\sum_{s=0}^{r} (-1)^s a_s a_{r-s}$$

$$= \sum_{s=0}^{0}(-1)^s a_s a_{0-s} + \sum_{s=0}^{1}(-1)^s a_s a_{1-s} + \sum_{s=0}^{2}(-1)^s a_s a_{2-s} + \sum_{s=0}^{3}(-1)^s a_s a_{3-s} + \cdots$$

$$= \sum_{s=0}^{0}(-1)^s a_s a_{0-s} + \sum_{s=0}^{2}(-1)^s a_s a_{2-s} + \sum_{s=0}^{4}(-1)^s a_s a_{4-s} + \cdots$$

$$\left(\because \sum_{s=0}^{2n-1}(-1)^s a_s a_{2n-1-s} = 0 \quad for\ n = 1, 2, 3, \cdots \right)$$

Therfore,

$$\left(\sum_{r=0}^{\infty} a_r\right)\left\{\sum_{r=0}^{\infty} (-1)^r a_r\right\} = \sum_{r=0}^{\infty}\sum_{s=0}^{2r} (-1)^s a_s a_{2r-s}$$

Example Triple Cauchy product

When (1.3) is expanded up to $m=4$ by using formula manipulation software ***Mathematica***, it is as follows

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p3[m_] := Sum[Sum[Sum[a_{r-s} b_{s-t} c_t, {t,0,s}], {s,0,r}], {r,0,m}]
```

$$p3[m\_] := \sum_{r=0}^{m}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s}\, b_{s-t}\, c_t$$

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Expand[p3[4]]
```

$a_0 b_0 c_0 + a_1 b_0 c_0 + a_2 b_0 c_0 + a_3 b_0 c_0 + a_4 b_0 c_0 + a_0 b_1 c_0 + a_1 b_1 c_0 + a_2 b_1 c_0 + a_3 b_1 c_0 + a_0 b_2 c_0 + a_1 b_2 c_0 + a_2 b_2 c_0 + a_0 b_3 c_0 + a_1 b_3 c_0 + a_0 b_4 c_0 + a_0 b_0 c_1 + a_1 b_0 c_1 + a_2 b_0 c_1 + a_3 b_0 c_1 + a_0 b_1 c_1 + a_1 b_1 c_1 + a_2 b_1 c_1 + a_0 b_2 c_1 + a_1 b_2 c_1 + a_0 b_3 c_1 + a_0 b_0 c_2 + a_1 b_0 c_2 + a_2 b_0 c_2 + a_0 b_1 c_2 + a_1 b_1 c_2 + a_0 b_2 c_2 + a_0 b_0 c_3 + a_1 b_0 c_3 + a_0 b_1 c_3 + a_0 b_0 c_4$

Especially, when $a_0 = 1$, $b_0 = 1$, $c_0 = 1$,

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ReplaceAll[%, {a_0 → 1, b_0 → 1, c_0 → 1}]
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$1 + a_1 + a_2 + a_3 + a_4 + b_1 + a_1 b_1 + a_2 b_1 + a_3 b_1 + b_2 + a_1 b_2 + a_2 b_2 + b_3 + a_1 b_3 + b_4 + c_1 + a_1 c_1 + a_2 c_1 + a_3 c_1 + b_1 c_1 + a_1 b_1 c_1 + a_2 b_1 c_1 + b_2 c_1 + a_1 b_2 c_1 + b_3 c_1 + c_2 + a_1 c_2 + a_2 c_2 + b_1 c_2 + a_1 b_1 c_2 + b_2 c_2 + c_3 + a_1 c_3 + b_1 c_3 + c_4$

By replacing $a_r , b_r , c_r , \cdots$ with $a_r z^r , b_r z^r , c_r z^r , \cdots$ in Formula 1.1.1 , the following formula is obtained immediately.

Formula 1.2.1 (Multiple Cauchy product of power series)

The following expressions hold for two or more convergent power series.

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{r=0}^{\infty} b_r z^r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r-s} b_s z^r \tag{1.2'}$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{r=0}^{\infty} b_r z^r\right)\left(\sum_{r=0}^{\infty} c_r z^r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s} b_{s-t} c_t z^r \tag{1.3'}$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{r=0}^{\infty} b_r z^r\right)\left(\sum_{r=0}^{\infty} c_r z^r\right)\left(\sum_{r=0}^{\infty} d_r z^r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} a_{r-s} b_{s-t} c_{t-u} d_u z^r \tag{1.4'}$$

$$\vdots$$

$$\left(\sum_{r=0}^{\infty} a_{1,r} z^r\right)\left(\sum_{r=0}^{\infty} a_{2,r} z^r\right)\cdots\left(\sum_{r=0}^{\infty} a_{n,r} z^r\right)$$

$$= \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{1,r_1-r_2}\, a_{2,r_2-r_3}\cdots a_{n-1,r_{n-1}-r_n}\, a_{n,r_n} z^{r_1} \tag{1.n'}$$

Especially,

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)\left\{\sum_{r=0}^{\infty} (-1)^r a_r z^r\right\} = \sum_{r=0}^{\infty}\sum_{s=0}^{2r} (-1)^s a_s a_{2r-s} z^{2r} \tag{1.2'±}$$

Example1 symbolic calculation

When (1.3') is expanded up to $m = 4$ by using formula manipulation software ***Mathematica*** , it is as follows.

$$p3[z\_ , m\_] := \sum_{r=0}^{m}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s}\, b_{s-t}\, c_t\, z^r$$

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Collect[p3[z, 4], z]
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$$a_0 b_0 c_0 + z (a_1 b_0 c_0 + a_0 b_1 c_0 + a_0 b_0 c_1) + z^2 (a_2 b_0 c_0 + a_1 b_1 c_0 + a_0 b_2 c_0 + a_1 b_0 c_1 + a_0 b_1 c_1 + a_0 b_0 c_2) + z^3 (a_3 b_0 c_0 + a_2 b_1 c_0 + a_1 b_2 c_0 + a_0 b_3 c_0 + a_2 b_0 c_1 + a_1 b_1 c_1 + a_0 b_2 c_1 + a_1 b_0 c_2 + a_0 b_1 c_2 + a_0 b_0 c_3) + z^4 (a_4 b_0 c_0 + a_3 b_1 c_0 + a_2 b_2 c_0 + a_1 b_3 c_0 + a_0 b_4 c_0 + a_3 b_0 c_1 + a_2 b_1 c_1 + a_1 b_2 c_1 + a_0 b_3 c_1 + a_2 b_0 c_2 + a_1 b_1 c_2 + a_0 b_2 c_2 + a_1 b_0 c_3 + a_0 b_1 c_3 + a_0 b_0 c_4)$$

Especially, when $a_0 = 1$, $b_0 = 1$, $c_0 = 1$,

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ReplaceAll[%, {a0 → 1, b0 → 1, c0 → 1}]
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$$1 + z (a_1 + b_1 + c_1) + z^2 (a_2 + a_1 b_1 + b_2 + a_1 c_1 + b_1 c_1 + c_2) + z^3 (a_3 + a_2 b_1 + a_1 b_2 + b_3 + a_2 c_1 + a_1 b_1 c_1 + b_2 c_1 + a_1 c_2 + b_1 c_2 + c_3) + z^4 (a_4 + a_3 b_1 + a_2 b_2 + a_1 b_3 + b_4 + a_3 c_1 + a_2 b_1 c_1 + a_1 b_2 c_1 + b_3 c_1 + a_2 c_2 + a_1 b_1 c_2 + b_2 c_2 + a_1 c_3 + b_1 c_3 + c_4)$$

Example2 numeric calculation

$$f(z) = \frac{5}{5+z} = 1 + \left(-\frac{1}{5}\right)^1 z^1 + \left(-\frac{1}{5}\right)^2 z^2 + \left(-\frac{1}{5}\right)^3 z^3 + \cdots \qquad |z| < 5$$

$$g(z) = e^{\frac{z}{3}} = 1 + \frac{1}{1!3^1} z^1 + \frac{1}{2!3^2} z^2 + \frac{1}{3!3^3} z^3 + \cdots$$

$$h(z) = \cos z = 1 + \frac{z^1}{1!} \cos\frac{1\pi}{2} + \frac{z^2}{2!} \cos\frac{2\pi}{2} + \frac{z^3}{3!} \cos\frac{3\pi}{2} + \cdots$$

When the product of these functions is calculated up to $m = 12$ by using formula manipulation software ***Mathematica***, it is as follows.

$$f[z\_] := \frac{5}{5+z} \qquad g[z\_] := e^{\frac{z}{3}} \qquad h[z\_] := \mathrm{Cos}[z]$$

$$a_{r\_} := \left(-\frac{1}{5}\right)^r \qquad b_{r\_} := \frac{1}{r!\, 3^r} \qquad c_{r\_} := \frac{1}{r!} \mathrm{Cos}\left[\frac{r\pi}{2}\right]$$

$$p3[z\_, m\_] := \sum_{r=0}^{m} \sum_{s=0}^{r} \sum_{t=0}^{s} a_{r-s}\, b_{s-t}\, c_t\, z^r$$

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p3[z, 12]
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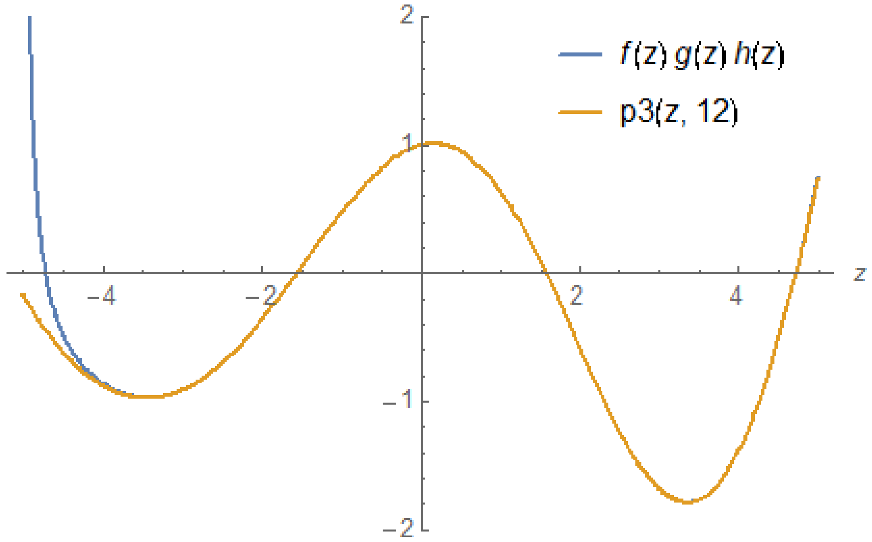
$$1 + \frac{2z}{15} - \frac{106 z^2}{225} - \frac{671 z^3}{10125} + \frac{8401 z^4}{303750} + \frac{12086 z^5}{2278125} - \frac{40024 z^6}{102515625} - \frac{3115867 z^7}{21528281250} - \frac{3781471 z^8}{1291696875000} + \frac{144995171 z^9}{87189539062500} + \frac{173608237 z^{10}}{1307843085937500} - \frac{3983534267 z^{11}}{431588218359375000} - \frac{43071773819 z^{12}}{38842939652343750000}$$

And if this is illustrated with $f(z)g(z)h(z)$, it is as follows. Both are almost identical except for the vicinity of -4.



Example3 (1.2±)

$$\left(\sum_{r=0}^{\infty}\frac{1}{r!}z^r\right)\left\{\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}z^r\right\} = \sum_{r=0}^{\infty}\sum_{s=0}^{2r}(-1)^s\frac{1}{s!}\frac{1}{(2r-s)!}z^{2r}$$

$$= \sum_{r=0}^{\infty}\frac{1}{(2r)!}\left\{\sum_{s=0}^{2r}(-1)^s{}_{2r}C_s\right\}z^{2r}$$

$$= \frac{1}{0!}{}_0C_0\,z^0 + \sum_{r=1}^{\infty}\frac{1}{(2r)!}\left\{\sum_{s=0}^{2r}(-1)^s{}_{2r}C_s\right\}z^{2r}$$

$$= 1 \qquad \left(\because \quad \sum_{s=0}^{2r}(-1)^s{}_{2r}C_s = 0\right)$$

This is consistent with $e^z e^{-z} = 1$.

1.2 Power of Infinite Series (Part1)

By setting $a = b = c = \cdots = a$ in Formula 1.1.1 , we obtain the following formula immediately.

Formula 1.2.1 (Power of infinite series)

The following expressions hold for convergent infinite series.

$$\left(\sum_{r=0}^{\infty} a_r\right)^2 = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r-s}\,a_s \tag{2.2}$$

$$\left(\sum_{r=0}^{\infty} a_r\right)^3 = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s}\,a_{s-t}\,a_t \tag{2.3}$$

$$\left(\sum_{r=0}^{\infty} a_r\right)^4 = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} a_{r-s}\,a_{s-t}\,a_{t-u}\,a_u \tag{2.4}$$

$$\vdots$$

$$\left(\sum_{r=0}^{\infty} a_r\right)^n = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{r_1-r_2}\,a_{r_2-r_3}\cdots\,a_{r_{n-1}-r_n}\,a_{r_n} \tag{2.n}$$

Example The 3rd power of zeta series

When (2.3) is expanded up to $m=7$ by using formula manipulation software ***Mathematica*** , it is as follows

$$\texttt{p3[}m\_\texttt{] :=} \sum_{r=0}^{m}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s}\,a_{s-t}\,a_t$$

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Expand[p3[7]]
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$$\begin{aligned}&a_0^3 + 3\,a_0^2\,a_1 + 3\,a_0\,a_1^2 + a_1^3 + 3\,a_0^2\,a_2 + 6\,a_0\,a_1\,a_2 + 3\,a_1^2\,a_2 + 3\,a_0\,a_2^2 + 3\,a_1\,a_2^2\\&+ a_2^3 + 3\,a_0^2\,a_3 + 6\,a_0\,a_1\,a_3 + 3\,a_1^2\,a_3 + 6\,a_0\,a_2\,a_3 + 6\,a_1\,a_2\,a_3 + 3\,a_2^2\,a_3\\&+ 3\,a_0\,a_3^2 + 3\,a_1\,a_3^2 + 3\,a_0^2\,a_4 + 6\,a_0\,a_1\,a_4 + 3\,a_1^2\,a_4 + 6\,a_0\,a_2\,a_4 + 6\,a_1\,a_2\,a_4\\&+ 6\,a_0\,a_3\,a_4 + 3\,a_0^2\,a_5 + 6\,a_0\,a_1\,a_5 + 3\,a_1^2\,a_5 + 6\,a_0\,a_2\,a_5 + 3\,a_0^2\,a_6 + 6\,a_0\,a_1\,a_6\\&+ 3\,a_0^2\,a_7\end{aligned}$$

Zeta series is as follows.

$$\zeta(z) = \sum_{r=0}^{\infty}\frac{1}{(r+1)^z} = 1+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\cdots \qquad Re(z)>1$$

So, by replacing $a_0 \to 1$, $a_r \to "1/(r+1)^z"$ $r=1,2,3,\cdots$, it becomes as follows.

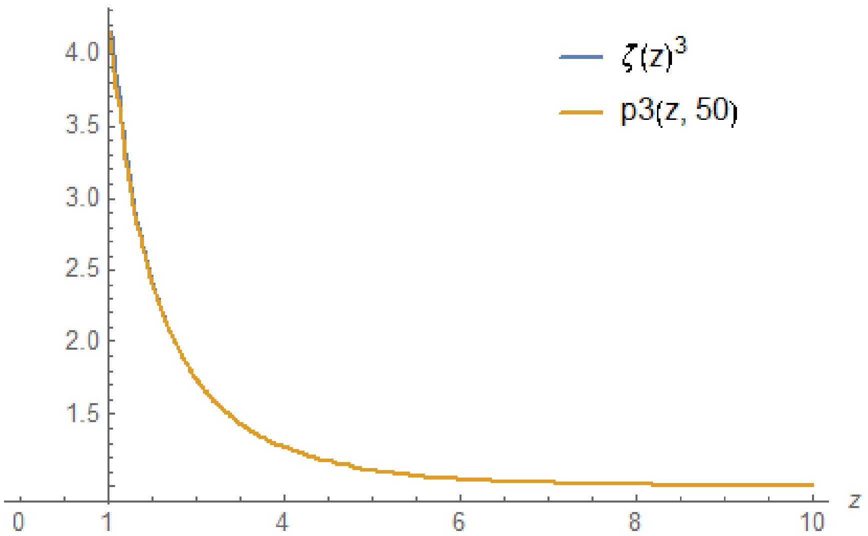
$$\texttt{ReplaceAll}\Big[\%,\ \Big\{a_0\to 1,\ a_1\to "\tfrac{1}{2^z}",\ a_2\to "\tfrac{1}{3^z}",\ a_3\to "\tfrac{1}{4^z}",\ a_4\to "\tfrac{1}{5^z}",\ a_5\to "\tfrac{1}{6^z}",\ a_6\to "\tfrac{1}{7^z}",\ a_7\to "\tfrac{1}{8^z}"\Big\}\Big]$$

$$\begin{aligned}&1 + 3\,\frac{1}{2^z} + 3\,\frac{1}{2^z}^2 + \frac{1}{2^z}^3 + 3\,\frac{1}{3^z} + 6\,\frac{1}{2^z}\,\frac{1}{3^z} + 3\,\frac{1}{2^z}^2\,\frac{1}{3^z} + 3\,\frac{1}{3^z}^2 + 3\,\frac{1}{2^z}\,\frac{1}{3^z}^2\\&+ \frac{1}{3^z}^3 + 3\,\frac{1}{4^z} + 6\,\frac{1}{2^z}\,\frac{1}{4^z} + 3\,\frac{1}{2^z}^2\,\frac{1}{4^z} + 6\,\frac{1}{3^z}\,\frac{1}{4^z} + 6\,\frac{1}{2^z}\,\frac{1}{3^z}\,\frac{1}{4^z} + 3\,\frac{1}{3^z}^2\,\frac{1}{4^z}\\&+ 3\,\frac{1}{4^z}^2 + 3\,\frac{1}{2^z}\,\frac{1}{4^z}^2 + 3\,\frac{1}{5^z} + 6\,\frac{1}{2^z}\,\frac{1}{5^z} + 3\,\frac{1}{2^z}^2\,\frac{1}{5^z} + 6\,\frac{1}{3^z}\,\frac{1}{5^z} + 6\,\frac{1}{2^z}\,\frac{1}{3^z}\,\frac{1}{5^z}\\&+ 6\,\frac{1}{4^z}\,\frac{1}{5^z} + 3\,\frac{1}{6^z} + 6\,\frac{1}{2^z}\,\frac{1}{6^z} + 3\,\frac{1}{2^z}^2\,\frac{1}{6^z} + 6\,\frac{1}{3^z}\,\frac{1}{6^z} + 3\,\frac{1}{7^z} + 6\,\frac{1}{2^z}\,\frac{1}{7^z} + 3\,\frac{1}{8^z}\end{aligned}$$

Though the above was symbolic calculation, if it is numerically calculated and illustrated with $\zeta^3(z)$, it is as follows. Though it is calculated up to $m=50$, both are almost overlapped and blue $\left(\zeta^3(z)\right)$ cannot be seen.

$$a_{r\_}[z\_] := \frac{1}{(r+1)^z}$$

$$p3[z\_, m\_] := \sum_{r=0}^{m}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s}[z]\, a_{s-t}[z]\, a_t[z]$$



By replacing a_r with $a_r z^r$ in Formula 1.2.1 , the following formula is obtained immediately.

Formula 1.2.2 (Power of power series)

The following expressions hold for convergent power series.

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^2 = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{r-s}\, a_s\, z^r \tag{2.2'}$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^3 = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s}\, a_{s-t}\, a_t\, z^r \tag{2.3'}$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^4 = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t} a_{r-s}\, a_{s-t}\, a_{t-u}\, a_u\, z^r \tag{2.4'}$$

$$\vdots$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^n = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}} a_{r_1-r_2}\, a_{r_2-r_3}\cdots a_{r_{n-1}-r_n}\, a_{r_n}\, z^r \tag{2.n'}$$

Example All 3 power series

Let us calculate the 3 power of the series as follows.

$$\sum_{r=0}^{\infty}\frac{z^{3r}}{(3r)!} = \frac{1}{3}\left\{e^z + 2e^{-z/2}\cos\frac{\sqrt{3}z}{2}\right\} \equiv f(z)$$

The Maclaurin expansion of the 3rd power of the right side is possible, but it is not so easy. So, we obediently calculate the cube of the left side. As first, if (2.3') is expanded up to $m=6$ by using formula manipulation software ***Mathematica,*** it is as follows.

$$\texttt{p3[z\_, m\_]} := \sum_{r=0}^{m}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s}\, a_{s-t}\, a_t\, z^r$$

$$\texttt{Collect[p3[z, 6], z]}$$

$$a_0^3 + 3 z a_0^2 a_1 + z^2 \left(3 a_0 a_1^2 + 3 a_0^2 a_2\right) + z^3 \left(a_1^3 + 6 a_0 a_1 a_2 + 3 a_0^2 a_3\right)$$
$$+ z^4 \left(3 a_1^2 a_2 + 3 a_0 a_2^2 + 6 a_0 a_1 a_3 + 3 a_0^2 a_4\right)$$
$$+ z^5 \left(3 a_1 a_2^2 + 3 a_1^2 a_3 + 6 a_0 a_2 a_3 + 6 a_0 a_1 a_4 + 3 a_0^2 a_5\right)$$
$$+ z^6 \left(a_2^3 + 6 a_1 a_2 a_3 + 3 a_0 a_3^2 + 3 a_1^2 a_4 + 6 a_0 a_2 a_4 + 6 a_0 a_1 a_5 + 3 a_0^2 a_6\right)$$

Here, by replacing $a_0 \to 1$, $a_r \to "1/(3r)!"$ $(r=1,2,3,\cdots)$, $z \to z^3$, it becomes as follows.

$$\texttt{ReplaceAll}\left[\%, \left\{a_0 \to 1,\ a_1 \to "\frac{1}{3\,!}",\ a_2 \to "\frac{1}{6\,!}",\ a_3 \to "\frac{1}{9\,!}",\ a_4 \to "\frac{1}{12\,!}",\ a_5 \to "\frac{1}{15\,!}",\ a_6 \to "\frac{1}{18\,!}"\right\}\right];$$

$$\texttt{ReplaceAll}\left[\%, z \to z^3\right]$$

$$1 + 3 \frac{1}{3\,!} z^3 + \left(3 \frac{1}{3\,!}^2 + 3 \frac{1}{6\,!}\right) z^6 + \left(\frac{1}{3\,!}^3 + 6 \frac{1}{3\,!} \frac{1}{6\,!} + 3 \frac{1}{9\,!}\right) z^9 +$$
$$\left(3 \frac{1}{12\,!} + 3 \frac{1}{3\,!}^2 \frac{1}{6\,!} + 3 \frac{1}{6\,!}^2 + 6 \frac{1}{3\,!} \frac{1}{9\,!}\right) z^{12} +$$
$$\left(3 \frac{1}{15\,!} + 6 \frac{1}{12\,!} \frac{1}{3\,!} + 3 \frac{1}{3\,!} \frac{1}{6\,!}^2 + 3 \frac{1}{3\,!}^2 \frac{1}{9\,!} + 6 \frac{1}{6\,!} \frac{1}{9\,!}\right) z^{15}$$
$$\left(3 \frac{1}{18\,!} + 6 \frac{1}{15\,!} \frac{1}{3\,!} + 3 \frac{1}{12\,!} \frac{1}{3\,!}^2 + 6 \frac{1}{12\,!} \frac{1}{6\,!} + \frac{1}{6\,!}^3 + 6 \frac{1}{3\,!} \frac{1}{6\,!} \frac{1}{9\,!} + 3 \frac{1}{9\,!}^2\right) z^{18}$$

Although the above was symbolic calculation, when this is numerically calculated, it is as follows.

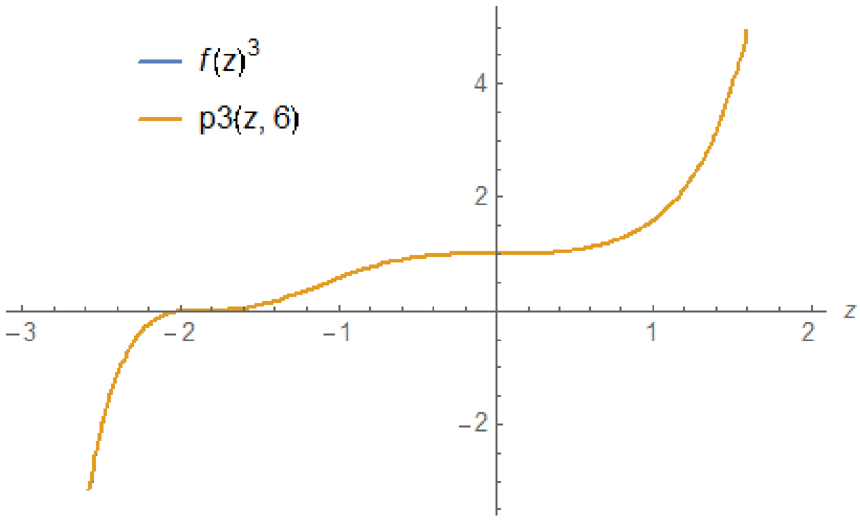
$$a_{r\_} := \frac{1}{(3\, r)\,!}$$

$$\texttt{p3[z\_, m\_]} := \sum_{r=0}^{m}\sum_{s=0}^{r}\sum_{t=0}^{s} a_{r-s}\, a_{s-t}\, a_t\, z^{3r}$$

$$\texttt{p3[z, 6]}$$

$$1 + \frac{z^3}{2} + \frac{7 z^6}{80} + \frac{27 z^9}{4480} + \frac{7 z^{12}}{56320} + \frac{2187 z^{15}}{1793792000} + \frac{937 z^{18}}{139403264000}$$

If this is illustrated with $f^3(z)$, it is as follows. Both are overlapped exactly and blue $\left(f^3(z)\right)$ can not be seen.



$f(z)^3$
p3(z, 6)
4
2
−2
−3
−2
−1
1
2
z

1.3 Power of Infinite Series (Part2)

Formula 1.3.1 (Power of infinite series)

The following expressions hold for convergent infinite series.

$$\left(\sum_{r=0}^{\infty} a_r\right)^2 = \sum_{r=0}^{\infty} a_r^2 + 2\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s \tag{3.2}$$

$$\left(\sum_{r=0}^{\infty} a_r\right)^3 = \sum_{r=0}^{\infty} a_r^3 + 3\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s\right) - 3\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty} a_r a_s a_t \tag{3.3}$$

$$\begin{aligned}\left(\sum_{r=0}^{\infty} a_r\right)^4 = &\sum_{r=0}^{\infty} a_r^4 - 2\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r^2 a_s^2 + 4\left(\sum_{r=0}^{\infty} a_r\right)^2\left(\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s\right)\\ &- 8\left(\sum_{r=0}^{\infty} a_r\right)\left(\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty} a_r a_s a_t\right) + 8\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty} a_r a_s a_t a_u\end{aligned} \tag{3.4}$$

Proof

$$(a+b)^2 = a^2+b^2+2ab$$

Performing the following substitution to this, we obtain (3.2) .

$$a+b \to \sum_{r=0}^{\infty} a_r \quad , \quad a^2+b^2 \to \sum_{r=0}^{\infty} a_r^2 \quad , \quad ab \to \sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s$$

Next,

$$\begin{aligned}(a+b+c)^3 &= a^3+b^3+c^3+3a^2b+3ab^2+3a^2c+3ac^2+3b^2c+3bc^2+6abc\\ &= a^3+b^3+c^3+3\left(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2+3abc\right) - 3abc\end{aligned}$$

Here,

$$a^2b+ab^2+a^2c+ac^2+b^2c+bc^2+3abc = (a+b+c)(ab+bc+ca)$$

Substituting this for the above,

$$(a+b+c)^3 = a^3+b^3+c^3+3(a+b+c)(ab+bc+ca) - 3abc$$

Performing the following substitution to this, we obtain (3.3) .

$$a^3+b^3+c^3 \to \sum_{r=0}^{\infty} a_r^3 \quad , \quad a+b+c \to \sum_{r=0}^{\infty} a_r \quad , \quad ab+bc+ca \to \sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s$$

$$, \quad abc \to \sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty} a_r a_s a_t$$

In case of the 4th degree, the following is obtained by similar calculation.

$$\begin{aligned}(a+b+c+d)^4 = a^4+b^4+c^4+d^4 &- 2\left(a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2\right)\\ &+ 4(a+b+c+d)^2(ab+ac+ad+bc+bd+cd)\\ &- 8(abc+abd+acd+bcd)(a+b+c+d)\\ &+ 8abcd\end{aligned}$$

Performing the following substitution to this, we obtain (3.4) .

$$a+b+c+d \to \sum_{r=0}^{\infty} a_r \quad , \quad a^4+b^4+c^4+d^4 \to \sum_{r=0}^{\infty} a_r^4$$

$$a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2 \to \sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r^2 a_s^2$$

$$ab+ac+ad+bc+bd+cd \rightarrow \sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s$$

$$abc+abd+acd+bcd \rightarrow \sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty} a_r a_s a_t \ , \quad abcd \rightarrow \sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty} a_r a_s a_t a_u$$

In addition, in the case of the 5th or higher degree, a simple expression can not be obtained.

Example The 3rd power of infinite series

When (3.3) is expanded up to $m=5$ by using formula manipulation software ***Mathematica***, it is as follows Since it was long, the 4th and subsequent lines were omitted. Instead, the equivalence was verified, and the equality was also verified in the case of $m=100$. As the result, it was confirmed that both sides are equal in both cases.

$$\text{f3}[m\_] := \left(\sum_{r=0}^{m} a_r\right)^3$$

$$\text{g3}[m\_] := \sum_{r=0}^{m} a_r^3 + 3\left(\sum_{r=0}^{m} a_r\right)\left(\sum_{r=0}^{m}\sum_{s=r+1}^{m} a_r a_s\right) - 3\sum_{r=0}^{m}\sum_{s=r+1}^{m}\sum_{t=s+1}^{m} a_r a_s a_t$$

```
Expand[f3[5]]
```

$a_0^3 + 3 a_0^2 a_1 + 3 a_0 a_1^2 + a_1^3 + 3 a_0^2 a_2 + 6 a_0 a_1 a_2 + 3 a_1^2 a_2 + 3 a_0 a_2^2 + 3 a_1 a_2^2 +$
$a_2^3 + 3 a_0^2 a_3 + 6 a_0 a_1 a_3 + 3 a_1^2 a_3 + 6 a_0 a_2 a_3 + 6 a_1 a_2 a_3 + 3 a_2^2 a_3 +$
$3 a_0 a_3^2 + 3 a_1 a_3^2 + 3 a_2 a_3^2 + a_3^3 + 3 a_0^2 a_4 + 6 a_0 a_1 a_4 + 3 a_1^2 a_4 + 6 a_0 a_2 a_4 +$

The rest is omitted

```
Expand[g3[5]]
```

$a_0^3 + 3 a_0^2 a_1 + 3 a_0 a_1^2 + a_1^3 + 3 a_0^2 a_2 + 6 a_0 a_1 a_2 + 3 a_1^2 a_2 + 3 a_0 a_2^2 + 3 a_1 a_2^2 +$
$a_2^3 + 3 a_0^2 a_3 + 6 a_0 a_1 a_3 + 3 a_1^2 a_3 + 6 a_0 a_2 a_3 + 6 a_1 a_2 a_3 + 3 a_2^2 a_3 +$
$3 a_0 a_3^2 + 3 a_1 a_3^2 + 3 a_2 a_3^2 + a_3^3 + 3 a_0^2 a_4 + 6 a_0 a_1 a_4 + 3 a_1^2 a_4 + 6 a_0 a_2 a_4 +$

The rest is omitted

```
Expand[f3[5]] == Expand[g3[5]]
True
Expand[f3[100]] == Expand[g3[100]]
True
```

By replacing a_r with $a_r z^r$ in Formula 1.3.1 , the following formula is obtained immediately.

Formula 1.3.2 (Power of power series)

The following expressions hold for convergent power series.

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^2 = \sum_{r=0}^{\infty} a_r^2 z^{2r} + 2\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s z^{r+s} \tag{3.2'}$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^3 = \sum_{r=0}^{\infty} a_r^3 z^{3r} + 3\left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s z^{r+s}\right) - 3\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty} a_r a_s a_t z^{r+s+t} \tag{3.3'}$$

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)^4 = \sum_{r=0}^{\infty} a_r^4 z^{4r} - 2\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r^2 a_s^2 z^{2r+2s} + 4\left(\sum_{r=0}^{\infty} a_r z^r\right)^2\left(\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s z^{r+s}\right)$$
$$-8\left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty} a_r a_s a_t z^{r+s+t}\right) + 8\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty} a_r a_s a_t a_u z^{r+s+t+u} \tag{3.4'}$$

Example The 4th power of infinite series

When (3.4') is expanded up to $m=5$ by using formula manipulation software ***Mathematica***, it is as follows Since it was long, the 4th and subsequent lines were omitted. Instead, the equivalence was verified, and the equality was also verified in the case of $m=50$. As the result, it was confirmed that both sides are equal in both cases.

$$\text{f4}[z\_, m\_] := \left(\sum_{r=0}^{m} a_r z^r\right)^4$$

$$\text{g4}[z\_, m\_] := \sum_{r=0}^{m} a_r^4 z^{4r} - 2\sum_{r=0}^{m}\sum_{s=r+1}^{m} a_r^2 a_s^2 z^{2r+2s} + 4\left(\sum_{r=0}^{m} a_r z^r\right)^2\left(\sum_{r=0}^{m}\sum_{s=r+1}^{m} a_r a_s z^{r+s}\right)$$
$$-8\left(\sum_{r=0}^{m} a_r z^r\right)\left(\sum_{r=0}^{m}\sum_{s=r+1}^{m}\sum_{t=s+1}^{m} a_r a_s a_t z^{r+s+t}\right)$$
$$+8\sum_{r=0}^{m}\sum_{s=r+1}^{m}\sum_{t=s+1}^{m}\sum_{u=t+1}^{m} a_r a_s a_t a_u z^{r+s+t+u}$$

Collect[f4[z, 5], z]

$$a_0^4 + 4 z a_0^3 a_1 + z^2 \left(6 a_0^2 a_1^2 + 4 a_0^3 a_2\right) + z^3 \left(4 a_0 a_1^3 + 12 a_0^2 a_1 a_2 + 4 a_0^3 a_3\right) +$$
$$z^4 \left(a_1^4 + 12 a_0 a_1^2 a_2 + 6 a_0^2 a_2^2 + 12 a_0^2 a_1 a_3 + 4 a_0^3 a_4\right) + 4 z^{19} a_4 a_5^3 + z^{20} a_5^4 +$$
$$z^5 \left(4 a_1^3 a_2 + 12 a_0 a_1 a_2^2 + 12 a_0 a_1^2 a_3 + 12 a_0^2 a_2 a_3 + 12 a_0^2 a_1 a_4 + 4 a_0^3 a_5\right) +$$

The rest is omitted

Collect[g4[z, 5], z]

$$a_0^4 + 4 z a_0^3 a_1 + z^2 \left(6 a_0^2 a_1^2 + 4 a_0^3 a_2\right) + z^3 \left(4 a_0 a_1^3 + 12 a_0^2 a_1 a_2 + 4 a_0^3 a_3\right) +$$
$$z^4 \left(a_1^4 + 12 a_0 a_1^2 a_2 + 6 a_0^2 a_2^2 + 12 a_0^2 a_1 a_3 + 4 a_0^3 a_4\right) + 4 z^{19} a_4 a_5^3 + z^{20} a_5^4 +$$
$$z^5 \left(4 a_1^3 a_2 + 12 a_0 a_1 a_2^2 + 12 a_0 a_1^2 a_3 + 12 a_0^2 a_2 a_3 + 12 a_0^2 a_1 a_4 + 4 a_0^3 a_5\right) +$$

The rest is omitted

```
Collect[f4[z, 5], z] == Collect[g4[z, 5], z]
True
Collect[f4[z, 50], z] == Collect[g4[z, 50], z]
True
```

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Alien's Mathematics